第二次作业

P24/2.1：

1.(1)、 315462 逆序数为

$2+2+1+1=6$ 偶排列

2(1) $(n-1)(n-2)\cdots 2,1,n$ 逆序数为

$n-2+n-1+\cdots+1=\frac{(n-1)(n-2)}{2}$

5、 $j_1 j_2 \cdots j_{n-1} j_n$ 逆序数为 r，求 $j_n j_{n-1} \cdots j_2 j_1$ 逆序数

$j_n j_{n-1} \cdots j_2 j_1$ 逆序数为 $j_1 j_2 \cdots j_{n-1} j_n$ "顺序数"

为 $C_n^2 - r = \frac{n(n-1)}{2} - r$

7、 判断 $\begin{cases} 2x_1 - 3x_2 = 7 \\ 5x_1 + 4x_2 = 6 \end{cases}$ 是否唯一解！若有，求出；

$$\begin{vmatrix} 2 & -3 \\ 5 & 4 \end{vmatrix} = 8+15 = 23 \neq 0 \Rightarrow \text{唯一解}$$

$$\begin{vmatrix} 7 & -3 \\ 6 & 4 \end{vmatrix} = 28+18 = 56$$

$$\begin{vmatrix} 2 & 7 \\ 5 & 6 \end{vmatrix} = 12-35 = -23$$

Cramer 法则 $\Rightarrow$ $x_1 = 2$ $x_2 = -1$

P26/2.2

1.(1)

$$\begin{vmatrix} & & & a_{14} \\ & & a_{23} & a_{24} \\ & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{14}a_{23}a_{32}a_{41}$$

(3) $\begin{vmatrix} & b_1 & & & \\ & & b_2 & & \\ & & & \ddots & \\ & & & & b_{n-1} \\ b_n & & & & \end{vmatrix} = (-1)^{1+1} \cancel{(-1)^{1+2}(-1)^{2+3}\cdots(-1)^{n-1+n}} b_1 b_2 \cdots b_{n-1} b_n$

$= (-1)^{n+1} \cancel{(-1)^{\cdots}} b_1 b_2 \cdots b_{n-1} b_n$

$= (-1)^{n+1} b_1 b_2 \cdots b_{n-1} b_n$

3、定义计算 $\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & 0 & 0 & 0 \\ d_1 & d_2 & 0 & 0 & 0 \\ e_1 & e_2 & 0 & 0 & 0 \end{vmatrix}$

$$= \sum_{(i_1,\cdots,i_5)\in S_5} a_{i_1} b_{i_2} c_{i_3} d_{i_4} e_{i_5} (-1)^{\tau(i_1,i_2,i_3,i_4,i_5)}$$

$$= \sum_{\substack{(i_1 i_2 \cdots i_5)\in S_5 \\ i_3 \le 2 \quad i_4 \le 2 \quad i_5 \le 2}} \cdots = 0$$

4. $\begin{vmatrix} & & a_1 \\ & a_2 & \\ a_n & & \end{vmatrix} = (-1)^{\tau(n,\cdots,1)} a_1 \cdots a_n$

$= (-1)^{(n-1)n/2} a_1 \cdots a_n$

不一定

当 n 除以4 余数为 0,1 时 无负号

2,3 有负号

p34/2.3

1. (2)
$$\begin{vmatrix} -1 & 203 & \frac{1}{3} \\ 3 & 298 & \frac{1}{2} \\ 5 & 399 & \frac{2}{3} \end{vmatrix} = \begin{vmatrix} -1 & 203 & \frac{1}{3} \\ 0 & \cancel{-511}\ 907 & \frac{3}{2} \\ 0 & \cancel{-616}\ 1414 & \frac{7}{3} \end{vmatrix}$$

$$= (-1)\left(\frac{7}{3}\times 907 - \frac{3}{2}\times 1414\right)$$

$$= \frac{14}{3}$$

或者
$$= \begin{vmatrix} -1 & 200 & \frac{1}{3} \\ 3 & 300 & \frac{1}{2} \\ 5 & 400 & \frac{2}{3} \end{vmatrix} + \begin{vmatrix} -1 & 3 & \frac{1}{3} \\ 3 & -2 & \frac{1}{2} \\ 5 & -1 & \frac{2}{3} \end{vmatrix} =$$

(4)
$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix} = 10\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$$

$$= 10\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 2 & -1 \\ 3 & 1 & -2 & -1 \\ 4 & -3 & -2 & -1 \end{vmatrix}$$

$$= \cancel{10}\ 20\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ -3 & -1 & -1 \end{vmatrix} = 20\begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ -3 & 2 & -4 \end{vmatrix}$$

$$= 160.$$

2.(2)
$$\begin{vmatrix} a_1-b & a_2 & \cdots & a_n \\ a_1 & a_2-b & \cdots & a_n \\ \vdots & \vdots & & \\ a_1 & a_2 & \cdots & a_n-b \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & a_1-b & a_2 & \cdots & a_n \\ 1 & a_1 & a_2-b & \cdots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_1 & a_2 & \cdots & a_n-b \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -a_1 & -a_2 & \cdots & -a_n \\ 1 & -b & & & \\ \vdots & & -b & & \\ \vdots & & & \ddots & -b \end{vmatrix}$$

$$
= \begin{vmatrix} 1-\frac{a_1+\cdots+a_n}{b} & 0 & \cdots & 0 \\ 1 & -b & & \\ \vdots & & \ddots & \\ 1 & & & -b \end{vmatrix}
$$

$$
= (-b)^n + (-b)^{n-1}(a_1+\cdots+a_n)
$$

（这样做假设 $b\neq 0$）

$b=0$ 时 行列式为 0 符合 ✓

故 值为 $(-b)^n + (-b)^{n-1}(a_1+\cdots+a_n)$

(Remark: 每列加到第一列亦可做)

3. (1)

$$
\begin{vmatrix} a_1-b_1 & b_1-c_1 & c_1-a_1 \\ a_2-b_2 & b_2-c_2 & c_2-a_2 \\ a_3-b_3 & b_3-c_3 & c_3-a_3 \end{vmatrix}
$$

$$
= \begin{vmatrix} a_1-c_1 & b_1-c_1 & c_1-a_1 \\ a_2-c_2 & b_2-c_2 & c_2-a_2 \\ a_3-c_3 & b_3-c_3 & c_3-a_3 \end{vmatrix} = 0
$$

4、(4)

$$\begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ b_2 & 1 & & & \\ \vdots & & 1 & & \\ b_n & & & & 1 \end{vmatrix} = \begin{vmatrix} a_1 - a_2b_2 - a_3b_3 - \cdots - a_nb_n & 0 & 0 \\ b_2 & 1 & \\ \vdots & & \\ b_n & & \end{vmatrix}$$

$$= a_1 - a_2b_2 - a_3b_3 - \cdots - a_nb_n$$

习题 2-4

1(3)

$$\begin{vmatrix} \lambda-2 & -2 & 2 \\ -2 & \lambda-5 & 4 \\ 2 & 4 & \lambda-5 \end{vmatrix} = \begin{vmatrix} \lambda-2 & -2 & 0 \\ -2 & \lambda-5 & \lambda-1 \\ 2 & 4 & \lambda-1 \end{vmatrix}$$

$$= (\lambda-2)(\lambda-1)(\lambda-9) + 2\begin{vmatrix} -2 & \lambda-1 \\ 2 & \lambda-1 \end{vmatrix}$$

$$= (\lambda-2)(\lambda-1)(\lambda-9) - 8(\lambda-1)$$

$$= (\lambda-1)(\lambda^2 - 11\lambda + 18 - 8) = (\lambda-1)(\lambda^2 - 11\lambda + 10)$$

$$= (\lambda-1)^2(\lambda-10)$$

2、
$$\begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ 1 & -1 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & n-1 & 1-n \end{vmatrix}$$

$$= \begin{vmatrix} \sum_{i=1}^{n} a_i & \sum_{i=2}^{n} a_i & \sum_{i=3}^{n} a_i & \cdots & a_n \\ 0 & -1 & & & \\ \vdots & & -2 & & \\ \vdots & & & \ddots & \\ 0 & & & & 1-n \end{vmatrix}$$

$$= (-1)^{n-1}(n-1)!\sum_{i=1}^{n} a_i$$

3、用 Vandermonde
$$= \prod_{1 \le j < i \le n} (a_i - a_j)$$

6、
$$\begin{vmatrix} 2a & a^2 & & & \\ 1 & 2a & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & a^2 \\ & & & 1 & 2a \end{vmatrix}$$

例5　$(n+1)a^n$

7、
$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x & a_1 & \cdots & a_{n-1} \\ \vdots & \vdots & & \vdots \\ x^{n-1} & a_1^{n-1} & \cdots & a_{n-1}^{n-1} \end{vmatrix} = \cancel{(x-a_1)\cdots(x-a_{n-1})}\,(a_1-x)\cdots(a_{n-1}-x)\prod_{1\le j<i\le n-1}(a_i-a_j) = 0$$

$\Rightarrow$ $x=a_1$ 或 ~~a_2~~ $x=a_2, \cdots, x=a_{n-1}$

习题 2.5

2、
$$\begin{pmatrix} a_1^2 & a_2^2 & \cdots & a_n^2 \\ a_1^3 & a_2^3 & \cdots & a_n^3 \\ a_1^{n+1} & a_2^{n+1} & & a_n^{n+1} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

判断有无解，若有，多少解？其中 a_i 互异非零.

$$\Delta \triangleq \begin{vmatrix} a_1^2 & a_2^2 & \cdots & a_n^2 \\ & & & \vdots \\ a_1^{n+1} & a_2^{n+1} & \cdots & a_n^{n+1} \end{vmatrix} = \prod_{i=1}^{n} a_i^2 \prod_{1\le i<j\le n} (a_j - a_i) \neq 0$$

~~当有 $a_i = 0$ 或 $a_i = a_j$ 时 $\Delta = 0$~~

由 Cramer 法则知 唯一解.

3、考虑系数行列式

$$\begin{vmatrix} \lambda-2 & -3 & -2 \\ -1 & \lambda-8 & -2 \\ 2 & 14 & \lambda+3 \end{vmatrix} = \begin{vmatrix} \lambda-2 & -3 & -2 \\ -1 & \lambda-8 & -2 \\ 0 & 2\lambda-2 & \lambda-1 \end{vmatrix}$$

$$= (\lambda-2)(\lambda-1)(\lambda-8+4) + \begin{vmatrix} -3 & -2 \\ 2\lambda-2 & \lambda-1 \end{vmatrix}$$

$$= (\lambda-1)(\lambda^2-6\lambda+8) + (\lambda-1)(-3+4)$$

$$= (\lambda-1)(\lambda^2-6\lambda+9) = (\lambda-1)(\lambda-3)^2$$

当 $\lambda=1$ 或 3 时 方程组有非零解

5、 系数行列式

$$\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 2b & 1 \end{vmatrix} = \begin{vmatrix} a-1 & 1 & 1 \\ 0 & b & 1 \\ 0 & 2b & 1 \end{vmatrix} = (1-a)b$$

当 $a\neq 1$ 且 ~~或~~ $b\neq 0$ 时 方程组唯一解

6、 $b=0$时 显然无解

$a=1$时

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & b & 1 & 1 \\ 1 & 2b & 1 & 2 \end{array}\right) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & b-1 & 0 & -1 \\ 0 & 2b-1 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & b-1 & 0 & -1 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

当 $b-1\neq -\frac{1}{2}$ 即 $b\neq\frac{1}{2}$时 无解

$b=\frac{1}{2}$时 无穷解

综上：

~~$a\neq$~~ 唯一解， $a\neq 1$ 或 $b\neq 0$

无解 $b=0$ 或 $(a=1$ 且 $b\neq\frac{1}{2})$

无穷解 $a=1$ 且 $b=\frac{1}{2}$

课后2-6

例 1、

$$\begin{vmatrix} 2 & 3 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 & 0 \\ 37 & 85 & 1 & 2 & 0 \\ 29 & 73 & 0 & 3 & 4 \\ 19 & 67 & 1 & 0 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix} \cdot \begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= 11 \begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & -2 & 2 \end{vmatrix} = 11\times 14 = 154$$

3、

$$\begin{vmatrix} & & & a_{11} & \cdots & a_{1k} \\ & & & \vdots & & \vdots \\ & & & a_{k1} & \cdots & a_{kk} \\ b_{11} & \cdots & b_{1r} & c_{11} & \cdots & c_{1k} \\ \vdots & & \vdots & \vdots & & \vdots \\ b_{r1} & \cdots & b_{rr} & c_{r1} & \cdots & c_{rk} \end{vmatrix}$$

$$\overset{\text{Laplace}}{=} (-1)^{1+\cdots+k+(k+1)+\cdots+r+k} \cdot \begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix} \begin{vmatrix} b_{11} & \cdots & b_{1r} \\ \vdots & & \vdots \\ b_{r1} & \cdots & b_{rr} \end{vmatrix}$$

$$= (-1)^{rk} \begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix} \begin{vmatrix} b_{11} & \cdots & b_{1r} \\ \vdots & & \vdots \\ b_{r1} & \cdots & b_{rr} \end{vmatrix}$$

4、(2)

$$A\begin{pmatrix} 1\,2 \cdots n-1 \\ 1\,3 \cdots n \end{pmatrix} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 3 & 4 & \cdots & n \\ \vdots & & & & \vdots \\ 1 & 3^{n-2} & 4^{n-2} & \cdots & n^{n-2} \end{vmatrix}$$

$$\overset{\text{Vandermonde}}{=\!=} (n-1)(n-3)!\;(n-2)(n-4)!\;\cdots\;3\cdot 1 \cdot 2$$

$$= \cancel{(n-2)!}\;(n-1)!\;(n-3)!\;(n-4)!\;\cdots\;1$$

$$= (n-1)! \prod_{i=1}^{n-3} i!$$