第三次作业

3.1/T3(1)

$$\alpha_1=\begin{bmatrix}-1\\3\\0\\-5\end{bmatrix}\quad \alpha_2=\begin{bmatrix}2\\0\\7\\-3\end{bmatrix}\quad \alpha_3=\begin{bmatrix}-4\\1\\-2\\6\end{bmatrix}\quad \beta=\begin{bmatrix}8\\3\\-1\\-25\end{bmatrix}$$

考虑 由 $x_1\alpha_1+x_2\alpha_2+x_3\alpha_3=\beta$

系数矩阵

$$\begin{bmatrix}-1&2&-4&8\\3&0&1&3\\0&7&-2&-1\\-5&-3&6&-25\end{bmatrix}\longrightarrow\begin{bmatrix}-1&2&-4&8\\0&6&-11&27\\0&7&-2&-1\\0&-13&26&-65\end{bmatrix}$$

$$\longrightarrow\begin{bmatrix}-1&2&-4&8\\0&1&\cancel{-13}\,9&-28\\0&7&-2&-1\\0&0&13&-39\end{bmatrix}\longrightarrow\begin{bmatrix}-1&2&-4&8\\0&1&9&-28\\0&0&-65&195\\0&0&1&-3\end{bmatrix}$$

$$\longrightarrow\begin{bmatrix}-1&2&-4&8\\0&1&9&-28\\0&0&1&-3\\0&0&0&0\end{bmatrix}$$

唯一解 $\begin{cases}x_1=2x_2-4x_3-8=2\\x_2=-1\\x_3=-3\end{cases}$

可以 $\beta=2\alpha_1-2\alpha_2-3\alpha_3$ #

3.1/T5 标准基 $e_1, e_2, e_3\ e_4$

$e_1=\alpha_1$

$e_2=\alpha_2-\alpha_1$

$e_3=\alpha_3-\alpha_2$

$e_4=\alpha_4-\alpha_3$

设 $\alpha=[a_1,a_2,a_3,a_4]^T$

$=a_1e_1+a_2e_2+a_3e_3+a_4e_4$

$=(a_1-a_2)\alpha_1+(a_2-a_3)\alpha_2+(a_3-a_4)\alpha_3+a_4\alpha_4$

可线性表出且唯一

3.1/T6 $\alpha_i = 2\alpha_i$

3.2/T1 (1) 若有全0数 $k_1, k_2 \cdots k_s$，使

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s = 0$$

则向量组 $\alpha_1, \alpha_2, \cdots \alpha_s$ 线性无关

错误， 反例 $\alpha_1 = (1,0)^T$ $\alpha_2 = (1,0)^T$ $0\cdot\alpha_1 + 0\cdot\alpha_2 = 0$

α_1, α_2 线性相关

(2) 若有一组不全为0数 $k_1, \cdots k_s$ 使

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s \neq 0$$

则 $\alpha_1, \cdots \alpha_s$ 线性无关

错误，举反例： $\alpha_1 = (1,0)^T$ $\alpha_2 = (1,0)^T$

$\alpha_1 + \alpha_2 \neq 0$，但线性相关

(3) 若 $\alpha_1 \cdots \alpha_s$ 线性相关 $(s \geq 2)$，则其~~素~~每个向量均可由其余向量线性表出

错： $\alpha_1 = (1,0)$ $\alpha_2 = (1,0)$ $\alpha_3 = (0,1)$

$\alpha_1, \alpha_2, \alpha_3$ 线性相关，但 α_3 不可由 α_1, α_2 表出.

3.2/T2(3) $\alpha_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ $\alpha_2 = \begin{bmatrix} 1 \\ 5 \\ -7 \end{bmatrix}$ $\alpha_3 = \begin{bmatrix} 7 \\ -13 \\ 20 \end{bmatrix}$ $\alpha_4 = \begin{pmatrix} -2 \\ 6 \\ 1 \end{pmatrix}$

考虑 $k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + k_4\alpha_4 = 0$ 是否有非零解

系数矩阵 $\begin{bmatrix} 3 & 1 & 7 & -2 \\ -1 & 5 & -13 & 6 \\ 2 & -7 & 20 & 1 \end{bmatrix} \to \begin{bmatrix} -1 & 5 & -13 & 6 \\ 0 & 16 & -32 & 16 \\ 0 & 3 & -6 & 13 \end{bmatrix}$

$\to \begin{bmatrix} -1 & 5 & -13 & 6 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 10 \end{bmatrix}$

取 ~~k_4~~ $k_4 = 0$, $k_3 = 1$ $\Rightarrow$ $k_2 = 2$

$k_1 = 5k_2 - 13k_3 = 10 - 13 = -3$

$\Rightarrow$ ~~$-3k_1 +$~~ $-3\alpha_1 + 2\alpha_2 + \alpha_3 = 0$ 线性相关

3.2/T4: 若 $(\alpha_1+\alpha_2)k_1 + (\alpha_2+5\alpha_3)k_2 + (4\alpha_3+3\alpha_1)k_3=0$

$$\Updownarrow$$

$$(2k_1+3k_3)\alpha_1 + (k_1+k_2)\alpha_2 + (5k_2+4k_3)\alpha_3=0$$

$\Updownarrow$ $\alpha_1,\alpha_2,\alpha_3$ 线性无关

$$\begin{cases} 2k_1+3k_3=0 \\ k_1+k_2=0 \\ 5k_2+4k_3=0 \end{cases} \quad (*)$$

对应系数矩阵行列式 $\begin{vmatrix} 2 & 0 & 3 \\ 1 & 1 & 0 \\ 0 & 5 & 4 \end{vmatrix} = 8 - \begin{vmatrix} 0 & 3 \\ 5 & 4 \end{vmatrix} = 8+15 \neq 0$

Cramer法则说明 (*) 只有零解，从而 $2\alpha_1+\alpha_2, \alpha_2+5\alpha_3,$
$4\alpha_3+3\alpha_1$ 线性无关

3.2/T5

方法同T4，

判断系数矩阵行列式

$$\begin{vmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ 1 & & & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ 0 & -1 & & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & 1 & 1 \end{vmatrix} = 0$$

故线性相关

（或注意到 $\alpha_1+\alpha_4 = \alpha_1+\alpha_2-(\alpha_2+\alpha_3)+\alpha_3+\alpha_4$）

3.2/T6 $\alpha_1,\cdots\alpha_s$ 线性无关，$\beta=b_1\alpha_1+\cdots+b_s\alpha_s$

若 $\exists i$, $b_i\neq 0$，用 β 替换 α_i 得到向量组

$\alpha_1,\alpha_2,\cdots,\alpha_{i-1},\beta,\alpha_{i+1},\cdots\alpha_s$

~~同T4，对应系~~

若 $k_1\alpha_1+k_2\alpha_2+\cdots+k_{i-1}\alpha_{i-1}+k_i\beta+k_{i+1}\alpha_{i+1}+\cdots+k_s\alpha_s=0$

这等价于 $\sum_{\substack{j=1\\j\neq i}}^{s}(k_j+b_jk_i)\alpha_j+k_ib_i\alpha_i=0$

$$\Leftrightarrow\begin{cases}k_j+b_jk_i=0 & (j\neq i)\\ k_ib_i=0\end{cases}$$

$$\overset{b_i\neq0}{\Longleftrightarrow}\begin{cases}k_i=0\\ k_j=0 & \forall j\neq i\end{cases}\Leftrightarrow k_j=0\ \ \forall j=1,\cdots,s$$

从而 $\alpha_1,\alpha_2,\cdots,\alpha_{i-1},\beta,\alpha_{i+1},\cdots,\alpha_s$ 线性无关.

3.2/T7

用T4方法，~~只须~~将 $\alpha_1,\cdots,\alpha_r$ 补为 $\alpha_1,\cdots,\alpha_n$，

$\alpha_1\cdots\alpha_n$ 线性无关 $\Leftrightarrow$ $\begin{vmatrix}1 & 1 & \cdots & 1\\ a_1 & a_2 & \cdots & a_n\\ \vdots & \vdots & & \vdots\\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1}\end{vmatrix}\neq0$

$\Leftrightarrow$ ~~若~~ $a_1,\cdots a_r,\cdots a_n$ 互异.

故取 $\alpha_{r+1}\cdots\alpha_n$ 使 $a_{r+1}\cdots a_n$ 与 $a_1\cdots a_r$ 互异

此时 $\alpha_1\cdots\alpha_n$ 线性无关 $\Rightarrow$ $\alpha_1\cdots\alpha_r$ 线性无关.

3.3/T2

$\alpha_2=9\alpha_1\Rightarrow$ 秩 ≤2

又 α_1,α_3 线性无关 $\Rightarrow$ 秩 ≥2

$\Rightarrow$ 秩 $=2$，α_1,α_3 为一组极大无关组

3.3/T3：秩为 r ~~定义~~ 极大线性无关组的元素(向量)个数为 r

故任 r 个线性无关向量一定为极大线性无关组

3.3/T4 由 设 $\alpha_i = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix}$ $i=1,\dots,n+1$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} & a_{1,n+1} \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & a_{n,n+1} \end{pmatrix}$$ 列大于行，有无穷解

$\Rightarrow \sum x_i\alpha_i = 0$ 有无非零解 $\Rightarrow$ $\alpha_1 \cdots \alpha_{n+1}$ 线性相关 #

3.3/T5：由T4，$\alpha_1 \cdots \alpha_n, \beta$ 线性相关

$\Rightarrow \exists$ 不全为零 x_i, y 使 $\sum_{i=1}^{n} x_i\alpha_i + y\beta = 0$

若 $y=0 \Rightarrow \alpha_1 \cdots \alpha_n$ 线性相关，矛盾

故 $y \neq 0 \Rightarrow \beta = -\sum_{i=1}^{n} \frac{x_i}{y}\alpha_i$ 由 $\alpha_1 \cdots \alpha_n$ 表示

3.3/T6 设 $e_1 \cdots e_n$ 为标准正交基

则 ~~$\alpha_1 \cdots \alpha_n$~~ $e_1 \cdots e_n$ 可由 $\alpha_1 \cdots \alpha_n$ 表示

$\Rightarrow \underbrace{\operatorname{rank}\{e_1 \cdots e_n\}}_{=n} \le \operatorname{rank}\{\alpha_1 \cdots \alpha_n\} \le n$

$\Rightarrow \operatorname{rank}\{\alpha_1 \cdots \alpha_n\} = n \Rightarrow \alpha_1 \cdots \alpha_n$ 线性无关

T7 设 $\alpha_1 \cdots \alpha_{i_r}$ 可线性表出 $\alpha_1 \cdots \alpha_n$

$\Rightarrow \underbrace{\operatorname{rank}\{\alpha_1 \cdots \alpha_n\}}_{=r} \le \operatorname{rank}\{\alpha_{i_1}, \cdots \alpha_{i_r}\} \le r$

$\Rightarrow \operatorname{rank}\{\alpha_{i_1} \cdots \alpha_{i_r}\} = r \Rightarrow \{\alpha_{i_1} \cdots \alpha_{i_r}\}$ 有为线性无关

$\Rightarrow \{\alpha_{i_1} \cdots \alpha_{i_r}\}$ 为 $\{\alpha_1 \cdots \alpha_n\}$ 极大线性无关组

例 证明: $\text{rank}\{\alpha_1,\alpha_2,\cdots\alpha_s,\beta_1,\beta_2\cdots\beta_r\} \leq \text{rank}\{\alpha_1,\alpha_2\cdots\alpha_s\} + \text{rank}\{\beta_1,\beta_2,\cdots\beta_r\}$

设 $\alpha_{i_1}\cdots\alpha_{i_k}$ 为 $\alpha_1,\alpha_2\cdots\alpha_s$ 极大无关组

$\beta_{j_1}\cdots\beta_{j_l}$ 为 $\beta_1\cdots\beta_r$ 极大无关组

则 右边 $= k+l$

$\alpha_1\cdots\alpha_s\cdots\beta_1\cdots\beta_r$ 可由 $\{\alpha_{i_1}\cdots\alpha_{i_k},\beta_{j_1}\cdots\beta_{j_l}\}$ 表出

$$\Rightarrow \text{rank}\{\alpha_1,\cdots\alpha_s,\cdots\beta_1\cdots\beta_r\} \leq \text{rank}\{\alpha_{i_1}\cdots\alpha_{i_k},\beta_{j_1}\cdots\beta_{j_l}\}$$

$$\leq k+l$$

#.