AMATH 732 PRESENTATION
NORMAL MODES OF N COUPLED OSCILLATORS

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BACKGROUND AND EXAMPLE

N Coupled Oscillators and Governing Equations

As shown in Fig. 1, consider N 1-D oscillators with mass, $m_1, m_2, \ldots, m_N$. The leftmost oscillator is conneceted to a fixed wall. The rightmost has oscillator is only connected to N-1-th oscillator on left.

All strings connecting oscillators satisfy Hooke's law with string constant, $k_1, k_2, \ldots, k_N$.

Relative to original point when the whole system is stationary, displacements of oscillators $x_1, x_2, \ldots, x_N$ have following governing equations (assume all mass=1, all string constant=1):

$$x_1'' + (2x_1 - x_2) = 0$$

$$x_i'' + (-x_{i-1} + 2x_i - x_{i+1}) = 0, 2 \leq i \leq N - 1$$

$$x_N'' + (x_N - x_{N-1}) = 0$$

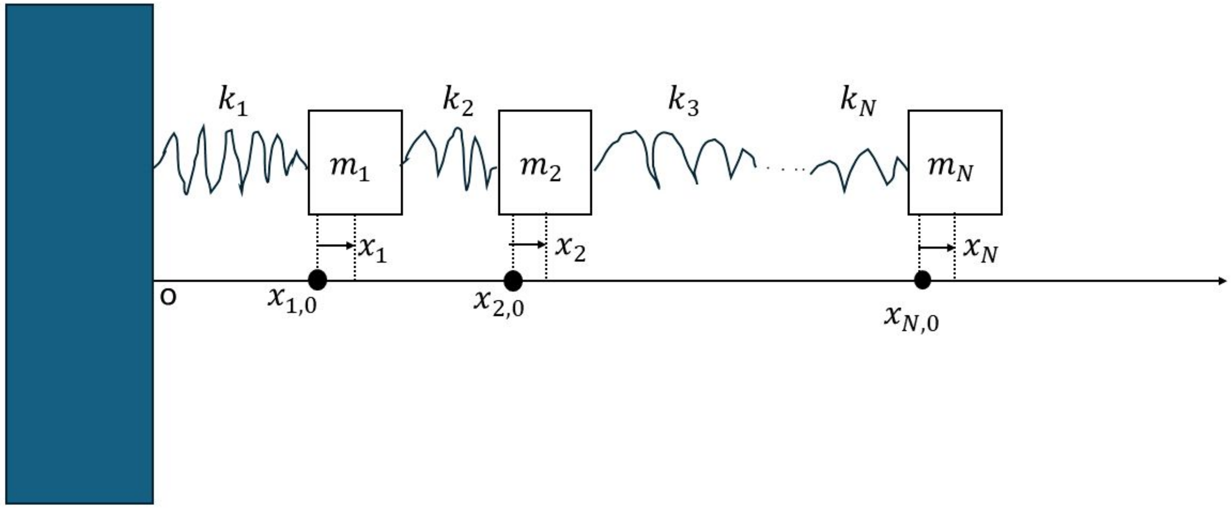


Figure 1: Schematic diagram of N coupled oscillators

General Normal Modes of N Coupled Oscillators

To find normal modes of N coupled oscillators, we assume solution for each displacement is in the form below:

$$x_i(t) = y_i\cos(\Omega t + \phi) \text{ where } \mathrm{y_i}, \Omega, \phi \text{ are constants.}$$

Then we plug in the solution into the system and rewrite the system as below (notice $\cos(\Omega t + \phi)$ is a common factor):

$$A\vec{Y} = \vec{0}, \vec{Y} = (y_1, y_2, \dots, y_N)^T, A = \begin{bmatrix} -\Omega^2+2 & -1 & & & & \\ -1 & -\Omega^2+2 & -1 & & & \\ & -1 & -\Omega^2+2 & -1 & & \\ & & \cdots & \cdots & \cdots & \\ & & & -1 & -\Omega^2+2 & -1 \\ & & & & -1 & -\Omega^2+1 \end{bmatrix}$$

Finding the original solution is equivalent to find frequency Ω and amplitude vector $\vec{Y}$.

Because we need nontrivial amplitude $\vec{Y}$, we need nontrivial null space of A. In other words, A is singular, $|A| = 0$

The problem now is simply solving characteristic polynomial of A to find frequency and find amplitude in null(A).

In later slides, I will show how to find all frequencies satisfying $|A| = 0$.

Normal Modes of Four Coupled Oscillators

If N=4, we can write down matrix A and find 4 positive roots of Ω by solving $|A| = 0$:

$$A = \begin{bmatrix} -\Omega^2+2 & -1 & & & \\ -1 & -\Omega^2+2 & -1 & & \\ & & -1 & -\Omega^2+2 & -1 \\ & & & -1 & -\Omega^2+1 \end{bmatrix}$$

$$|A| = 0 \rightarrow \Omega = 0.347, 1, 1.532, 1.879$$

Amplitude vector, $\vec{Y}$, corresponding to first two frequencies are particularly interesting:

$$\vec{Y} = \begin{bmatrix} 0.347 \\ 0.653 \\ 0.878 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

First solution (Fig. 2) corresponds to all oscillators have the same direction of motion and the amplitude is amplified from left to right, while second solution (Fig. 3) corresponds to the scenario x1 and x2 are identical, x3 is stationary and x4 has opposite motion to x1 and x2. Additionally, second solution frequency is exact 1, the same as a single oscillator alone.

We are wondering: in what condition, 1 is normal mode frequency and how the system will behave.

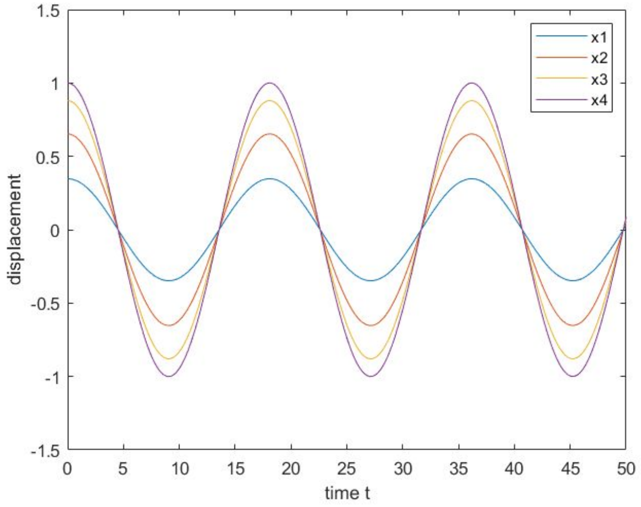


Figure 2: First normal mode

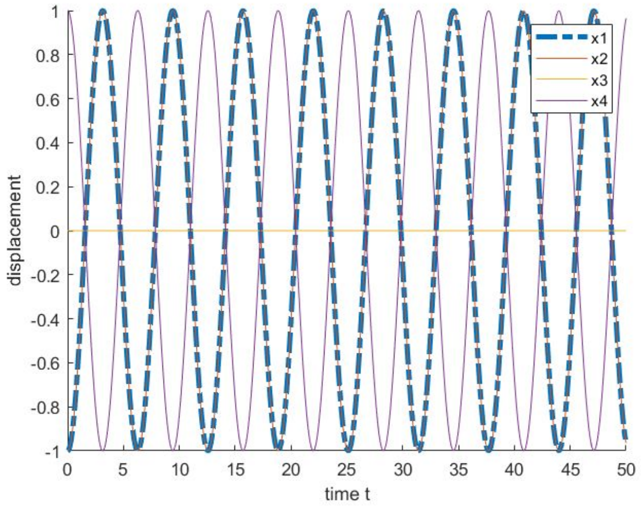


Figure 3: Second normal mode

WHEN IS 1 NORMAL MODE FREQUENCY?

General Method to Find All Normal Mode Frequencies

We first try to directly find all normal mode frequencies.

Noticing A is a tridiagonal matrix, denote $z = \Omega^2$ and f_n as characteristic polynomial of matrix A for n oscillators, we have base conditions and recursive relation:

$$x_1'' + x_1 = 0 \rightarrow f_1 = -z + 1$$

$$f_2 = z^2 - 3z + 1$$

$$f_n = (-z + 2)f_{n-1} - f_{n-2}$$

Noticing third equation is a first order linear, constant coefficient difference equation we know $f_n = C_1\alpha^{n-1} + C_2\beta^{n-1}$ where α, β are roots of: $m^2 + (z - 2)m + 1 = 0$. We can then solve C_1, C_2 by plug-in.

Finally, we can theoretically find all frequencies Ω numerically:

$$|A| = 0 \Leftrightarrow f_N = 0 \Leftrightarrow C_1\alpha^{N-1} + C_2\beta^{N-1} = 0$$

How can we use this general but messy result to find condition when 1 is normal mode frequency?

Intuition and Verification

Because we have known N=1 and N=4 have normol mode frequency=1. Intuitively, we can guess if $N = 3k + 1, k$ is non-negative integer, the system may have normal mode frequency 1.

This is easy to verify by induction, let $\Omega = 1 \rightarrow z = 1 \rightarrow C_1 = -C_2, \alpha\beta = 1, \alpha + \beta = 1$:

Base case N=1, done.

If $f_{3k+1} = C_1\alpha^{3k} + C_2\beta^{3k} \rightarrow f_{3(k+1)+1} = C_1\alpha^{3k+3} + C_2\beta^{3k+3} = \mathrm{C}_1\alpha^{3k}(\alpha^3 - \beta^3) = \mathrm{C}_1\alpha^{3k}(\alpha \quad - \beta \quad)((\alpha + \beta)^2 - \alpha\beta) = 0.$

Therefore we verified our hypothesis and avoided messy algebra.

Our next question is: what will amplitude vector $\vec{Y}$ be like? Or how will the system oscillate at frequency 1?

What is $\vec{Y}$ if $\Omega = 1$?

When $\Omega = 1$:

$$A = \begin{bmatrix} 1 & -1 & & & & \\ -1 & 1 & -1 & & & \\ & -1 & 1 & -1 & & \\ & & \cdots & \cdots & \cdots & \\ & & & -1 & 1 & -1 \\ & & & & -1 & 0 \end{bmatrix}$$

Denote $N = 3k + 1$, after some row operations we can easily find non-trivial vector $\vec{Y}$ in null(A):

If k is odd: $\vec{Y} = (1,1,0,-1,-1,0,1,1,\dots,-1,-1,0,1)^T$

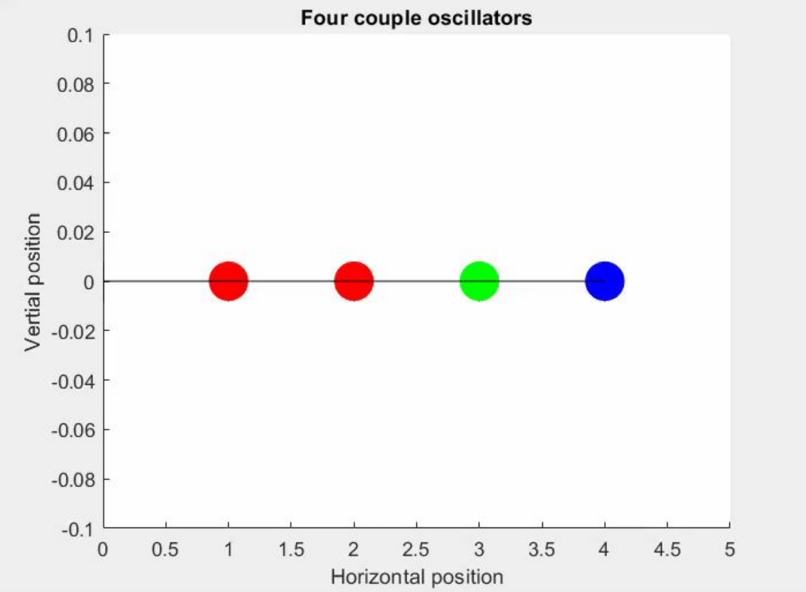
If k is even: $\vec{Y} = (1,1,0,-1,-1,0,1,1,\dots,1,1,0,-1)^T$

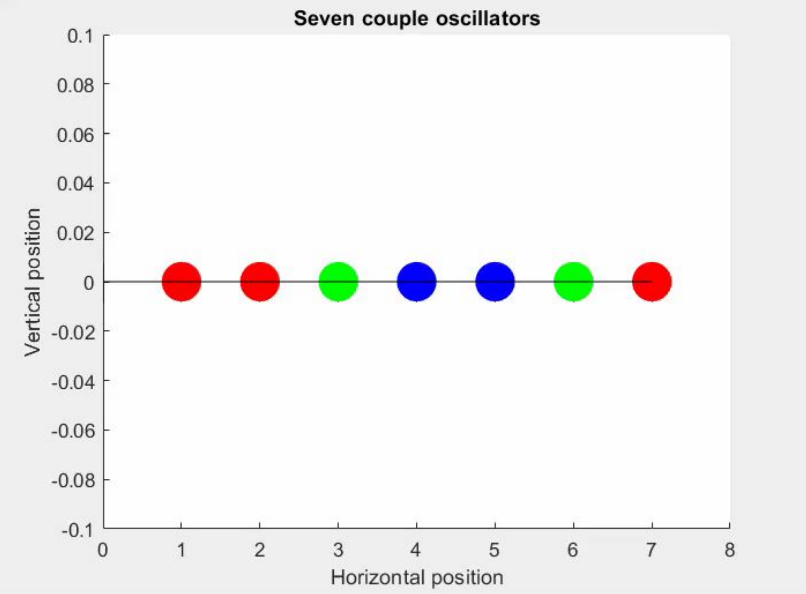
Oscillators x_n is:

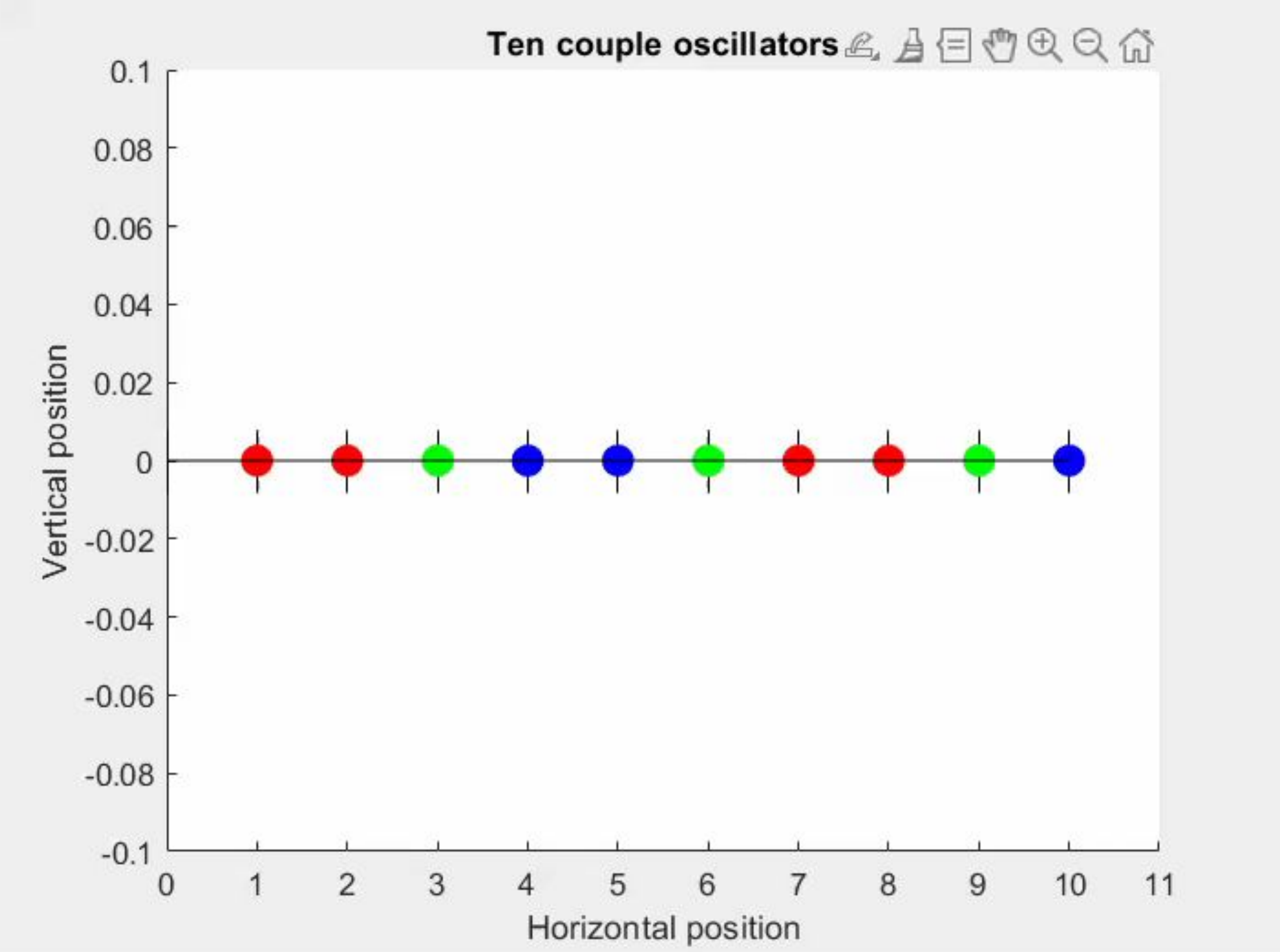
1. stationary, if $n\ Mod\ 3 \equiv 0$.
2. oscillating with a fixed amplitude consistent across all non-stationary oscillators. Between two stationary oscillators or the wall, two moving oscillators have the same velocity direction, which is opposite to the next two moving oscillators.

I know it is hard to understand by words. Let's see some Matlab movies.

Visulization of Normal Modes When $\Omega = 1$







Q&A

Off-topic Remarks

N by N tridiagonal matrix has at least N-1 linearly independent column vectors, so dim(null(A))=N-rank(A) is at most 1, which implies at given normal mode, amplitudes of all SHOs have fixed ratio to each other.

Without guessing, we can plot f_N when $\Omega = 1$ and observe even stronger results: f_N is periodic and happen to have value 0.

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THANK YOU!