AMATH 732: Lab Book

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Preface

This lab book is an assessing component of AMATH 732: Asymptotic Analysis and Perturbation Theory taught by Professor Marek Stastna. The book is written by Kaixin Zheng, master student in the applied mathematics department at University of Waterloo.

I am grateful for Professor Stastna's passion towards teaching, and unwavering support towards his students.

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1 Mechanical Models

1.1 2D Simple Harmonic Oscillation: Case 1

Time spent: 0.5 hour.

Similar to 1D simple harmonic oscillation (SHO) defined in lectures:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x, \omega = \sqrt{\frac{k}{m}} \tag{1.1}$$

where m is mass, k is Hooke's law constant of string connected to the mass.

We have already known that solution to Equation 1.1 is

$$x = x(0)\cos\omega t + \frac{x'(0)}{\omega}\sin\omega t \tag{1.2}$$

Let's consider a modified case: on a friction-less 2D plane, a mass m is connected to a 0-mass string fixed at origin o, Hooke's constant is k, as shown in Figure 1.1.

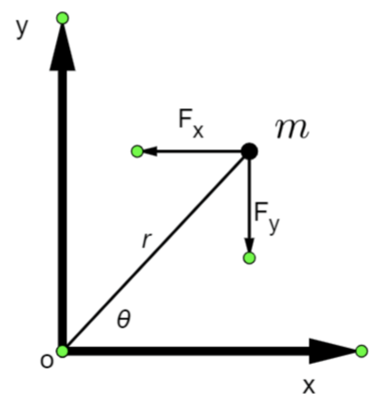


Figure 1.1: Schematic diagram of 2D SHO. F_x, F_y denotes force along x,y direction respectively. θ is angle between positive x-axis and string. r is the length of string

To simplify the dynamics of the mass, we assume the length of string is 0 when no external force is applied. We can write governing equations of mass location (x,y):

$$m\frac{d^2x}{dt^2} = -k\sqrt{x^2+y^2}\cos\theta = -kx \tag{1.3}$$

Similarly for y, we have

$$m\frac{d^2y}{dt^2} = -ky \tag{1.4}$$

Denote $\omega = \sqrt{\frac{k}{m}}$, we can instantly have mass is doing 1D SHO in both x and y directions with angular frequency ω.

Denote $\vec{x}(t) = (x,y)^T$

We instantly have

$$\frac{d^2\vec{x}}{dt^2} = -\omega^2\vec{x} \tag{1.5}$$

$$\vec{x}(t) = \vec{x}(0)\cos\omega t + \vec{x}'(0)\sin\omega t \tag{1.6}$$

We can conclude if original length of string is 0, 2D SHO is equivalent to the combination of 2 1D SHO in x and y directions respectively. 2D SHO can also be seens as a 2D vector having analogous 1D SHO behavior.

1.2 2D Simple Harmonic Oscillation: Case 2

Time spent: 4.5 hour.

Obviously, assumption of 0 original string length can be improved. What if we let the original length be non-zero, say r_0?

Before writing new governing equation, we can notice if we still try to write acceleration under x-y coordinate, the governing equations will have terms like $\sqrt{x^2+y^2}-r_0$ which will result in difficulties in solving. We may first convert acceleration into polar coordinate and decompose string force (which is only along negative r direction). The governing equations will be simplified and predictably easier to solve.

For polar coordinate (r, θ), velocity is straightforward:

$$\vec{v} = r\frac{d\theta}{dt}\hat{\theta} + \frac{dr}{dt}\hat{r}, \text{ where } \hat{\theta}, \hat{r} \text{ are unit vectors along } \theta, r \text{ directions} \tag{1.7}$$

The the acceleration $\vec{a}$:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dr}{dt}\frac{d\theta}{dt}\hat{\theta} + r\frac{d^2\theta}{dt^2}\hat{\theta} + r\frac{d\theta}{dt}\frac{d\hat{\theta}}{dt} + \frac{d^2r}{dt^2}\hat{r} + \frac{dr}{dt}\frac{d\hat{r}}{dt} \tag{1.8}$$

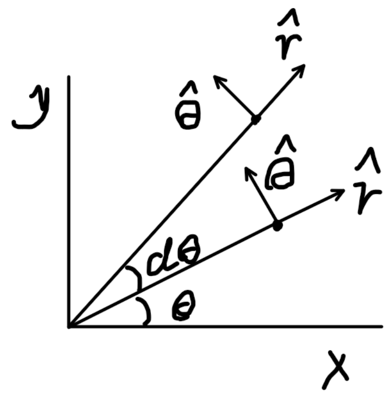


Figure 1.2: The rotation of unit vectors under polar coordinates.

As shown in Figure 1.2, unit vectors does not change length, which is constantly 1, so they only rotate. We can straightforwardly see $\hat{\theta}$ is rotating towards negative r direction and $\hat{r}$ is rotating to positive θ direction with rate the same θ increases. We can write

$$\frac{d\hat{\theta}}{dt} = -\frac{d\theta}{dt}\hat{r}, \frac{d\hat{r}}{dt} = \frac{d\theta}{dt}\hat{\theta} \tag{1.9}$$

Plug Equation 1.9 into Equation 1.8, we have:

$$\vec{a} = (r\frac{d^2\theta}{dt} + 2\frac{dr}{dt}\frac{d\theta}{dt})\hat{\theta} + (\frac{d^2r}{dt} - r(\frac{d\theta}{dt})^2)\hat{r} \tag{1.10}$$

Therefore we have new governing equations:

$$m(\frac{d^2r}{dt} - r(\frac{d\theta}{dt})^2) = -k(r - r_0) \tag{1.11}$$

$$m(r\frac{d^2\theta}{dt} + 2\frac{dr}{dt}\frac{d\theta}{dt}) = 0 \tag{1.12}$$

Notice for Equation 1.12, multiple both sides with $\frac{r}{m}$ we have:

$$\frac{d}{dt}(r^2\theta') = 0 \to r^2\theta' = C_1 = r^2(0)\theta'(0) \tag{1.13}$$

Denote $\omega = \sqrt{\frac{k}{m}}$ we can plug Equation 1.13 into Equation 1.11:

$$r'' = -\omega^2(r - r_0) + \frac{C_1^2}{r^3} \tag{1.14}$$

Along with θ governing equation in term of r:

$$\theta' = \frac{C_1}{r^2} \tag{1.15}$$

To further understand behavior of mass motion, we first do perturbation to have an approximation solution. Assume $r(t) = r_0 + \delta r(t)$ and $\delta r << r_0$.
We have

$$\delta r'' = r'' = -\omega^2\delta r + \frac{(r_0 + \delta r(0))^4\theta'(0)^2}{(r_0 + \delta r)^3} \approx -\omega^2\delta + \theta'(0)^2\frac{r_0^4(1 + 4\frac{\delta r(0)}{r_0})}{r_0^3(1 + 3\frac{\delta r}{r_0})} \tag{1.16}$$

$$\approx -\omega^2\delta r + (\theta(0)')^2 r_0(1 + 4\frac{\delta r(0)}{r_0})(1 - 3\frac{\delta r}{r_0}) = -(\omega^2 + 3(\theta(0)')^2)\delta r + (\theta(0)')^2 r_0(1 + 4\frac{\delta r(0)}{r_0}) \tag{1.17}$$

$$= -\Omega^2(\delta r - \Delta_r) \tag{1.18}$$

Here $\Omega = \sqrt{\omega^2 + 3(\theta'(0))^2}$, $\Delta_r = \frac{1}{\Omega}(\theta(0)')^2 r_0(1 + 4\frac{\delta r(0)}{r_0})$
We can find δr is doing a 1D SHO with angular frequency Ω and original point Δ_r
We can solve δr

$$\delta r(t) = (\delta r(0) - \Delta_r)\cos\Omega t + \frac{\delta r'(0)}{\Omega}\sin\Omega t + \Delta_r \tag{1.19}$$

For angle we have

$$\theta' = \frac{(r_0 + \delta r(0))^2\theta'(0)}{(r_0 + \delta r)^2} \approx \theta'(0)\frac{r_0^2}{r_0^2}\frac{1 + 2\frac{\delta r(0)}{r_0}}{1 + 2\frac{\delta r}{r_0}} \approx \theta'(0)(1 + 2\frac{\delta r(0)}{r_0})(1 - 2\frac{\delta r}{r_0}) \tag{1.20}$$

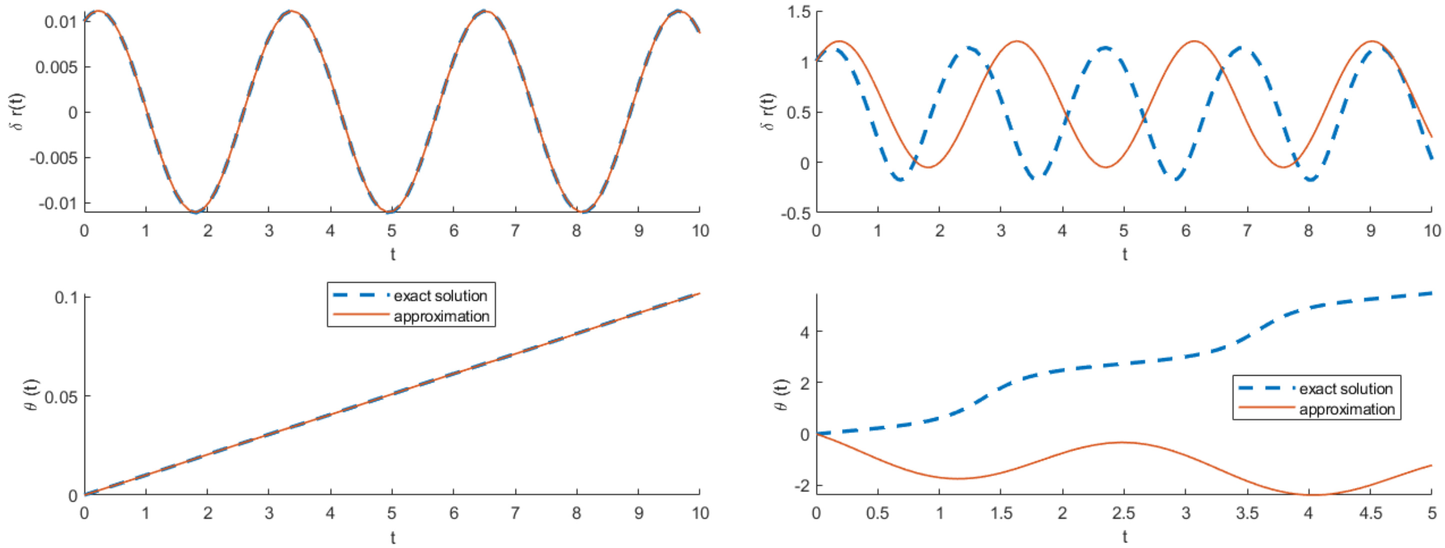
$$= \theta'(0)(1 + 2\frac{\delta r(0)}{r_0}) - \frac{2}{r_0}\theta'(0)(1 + 2\frac{\delta r(0)}{r_0})\delta r \tag{1.21}$$

If we do integral to both sides, to simply solution let $\theta(0) = 0$, we can have:

$$\theta(t) = \theta'(0)(1 + \frac{2\delta r(0)}{r_0})((1 - \frac{2\Delta_r}{r_0})t - \frac{2(\delta r(0) - \Delta_r)}{\Omega r_0}\sin\Omega t - \frac{2\delta r'(0)}{\Omega^2 r_0}(1 - \cos\Omega t)) \tag{1.22}$$

In other words, θ is oscillation around a function linear to t. The motion on angle direction without constant motion is oscillation with the same frequency as r and constant increasing rate.

Then we also give numerical simulations to compare approximation solution with exact solution under $r \approx r_0$. The implementation is given in the appendix.



(a) $\delta r(0) = 0.01, \delta r'(0) = 0.01, \theta(0) = 0, \theta'(0) = 0.01$ (b) $\delta r(0) = 1, \delta r'(0) = 1, \theta(0) = 0, \theta'(0) = 0.5$

Figure 1.3: Exact solution and approximation under two initial conditions with same $r_0 = 1, \omega = 2$

As shown in Figure 1.3a, we can clearly see that under conditions of $r(t) \approx r$, our approximation fits exact solution almost perfectly but the oscillation of $\theta(t)$ is in-obvious because amplitude of θ, by directly calculation from Equation 1.22 $= \sqrt{(\frac{2(\delta r(0)-\Delta_r)}{\Omega r_0})^2 + (\frac{2\delta r'(0)}{\Omega^2 r_0})^2} \approx 0.0012$. In r direction, mass is doing a 1D SHO.

Interestingly, as shown in Figure 1.3b, if the mass can move relatively far away from (r_0, θ), though our approximation show strong discrepancy compared with exact solution, the exact solution still has sustaining periodic oscillation along r direction. Meantime, the exact solution motion in θ direction is also similar to the combination of an periodic oscillation plus a constant motion.

Physically, we may conclude the motion of mass on the plane is the combination of two motions: 1, a rotation around the point string length is r_0; 2, meantime, the point where the rotation is around is rotating around the origin of the coordinate where $(x, y) = (0, 0)$. The behavior of the mass is exactly like the moon rotating around the earth and meantime the earth is rotating around the sun.

1.3 The Model with Variable Mass: the Rocket

Time spent: 1.5 hour.

One motivation to investigate variable mass system is to understand the motion of a rocket when it is consuming the rocket fuel to provide additional kinetics on the body of the rocket.

Before we model the rocket motion, let us consider a simple mass has 1D motion with variable mass, as shown in Figure 1.4.

The variable mass is denoted as $m(t)$ with x-direction velocity v(t) and external x-direction force F. We are assuming mass conservation in the system because the speed of the rocket like escape velocity (around 40,250 km per hour) is significantly smaller than light speed [1].

We use momentum law and have:

$$Fdt = udm + (m(t) - dm)(v(t) + dv) - m(t)v(t) \tag{1.23}$$

$$F = (u - v(t))\frac{dm}{dt} + m(t)\frac{dv(t)}{dt} \tag{1.24}$$

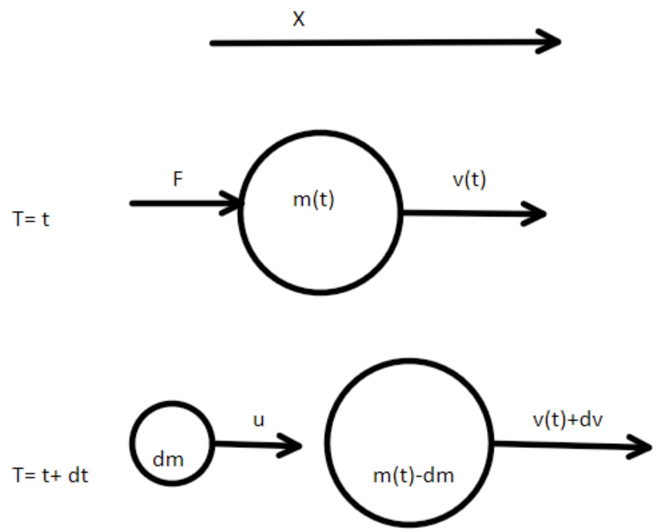


Figure 1.4: The motion of variable mass at time T=t and T=dt

It is reasonable to assume there is no external forces F in the environment the rocket is moving, because it is well known that there is almost no air outside the the Earth's atmosphere. Then the governing equation becomes:

$$0 = (u - v)\frac{dm}{dt} + m\frac{dv}{dt} \tag{1.25}$$

To determine the motion of rocket $x(t)$ we only need to determine $u - v$ and $\frac{dm}{dt}$ with certain initial conditions.

One assumption is that the mass leaving the rocket has a constant velocity relative to the rocket body. In other words, $u - v = v_e$

Equation 1.25 becomes

$$-\frac{1}{m}\frac{dm}{dt} = \frac{1}{v_e}\frac{dv}{dt} \tag{1.26}$$

$$-\ln\frac{m(t)}{m(0)} = \frac{v(t) - v(0)}{v_e} \tag{1.27}$$

$$v(t) = v(0) + v_e \ln\frac{m(0)}{m(t)} \tag{1.28}$$

This is a famous result named **Tsiolkovsky rocket equation** [2].

Equation 1.28 gives us implications that the most important aspects in rocket design to increase rocket velocity: 1. increasing relative exhaust velocity v_e; 2. increasing the rate of mass consumption to obtain higher $ln\frac{m(0)}{m(t)}$ at a given time t. This is why rocket engineers try to develop nuclear powered rocket because the nuclear fuel can produce much higher v_e than chemical fuel [3].

2 Approximation

I want to first use two sub-setctions to introduce the most exciting and interesting result I found when I self-study some Fourier analysis from Stein's famous book [4].

2.1 Parseval's Theorem

Time spent: 1 hour.

We first give the form of Fourier series we use in the following sub-sections.

For a periodic function f(t) with period $T = 2\pi$: $\mathbb{R} \to \mathbb{R}$, its Fourier series is written as

$$f(t) = u(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt \tag{2.1}$$

The coefficients are given:

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\tau)d\tau, a_n = \frac{1}{\pi}\int_0^{2\pi} \cos^2(kt)dt, b_n = \frac{1}{\pi}\int_0^{2\pi} \sin^2(kt)dt \tag{2.2}$$

In 1799, French mathematician Marc-Antoine Parseval derived a result which is later applied to Fourier series [5]:

$$\frac{1}{2\pi}\int_0^{2\pi} f^2(\tau)d\tau = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \tag{2.3}$$

Proof:

In our lecture, we learned orthogonality of function collection $\{1, sin(t), sin(2t), ...cos(t), cos(2t)...\}$.

We denote $A_n(t) = \cos nt, B_n(t) = \sin nt$ then we have

$$\int_0^{2\pi} A_n(\tau)B_n(\tau)d\tau = \int_0^{2\pi} A_n(\tau)B_m(\tau)d\tau = \int_0^{2\pi} A_n(\tau)A_m(\tau)d\tau = 0, \forall m \neq n \tag{2.4}$$

Therefore, LHS in Equation 2.3 can be written as (we can switch summation and integral here):

$$LHS = \frac{1}{2\pi}\int_0^{2\pi} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} A_i(\tau)B_j(\tau)d\tau = \frac{1}{2\pi}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_0^{2\pi} A_i(\tau)B_j(\tau)d\tau \tag{2.5}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sum_{i=1} A_i^2(\tau) + \sum_{i=1} B_i^2(\tau) = \frac{1}{2\pi}(2\pi a_0^2 + \pi \sum_{n=1}^{\infty} a_n^2 + b_n^2) = RHS \tag{2.6}$$

2.2 One interesting application of Fourier series

Time spent: 3 hour.

One interesting problem is if we have a rope with length L, how can we let the rope has the maximum inter area A if we put the rope on a surface. Or in other words, if the farmer bought some meters of fences with all budget he can afford, what shape should the sheep pen have to keep the most sheep? This is a famous problem with an answer, isoperimetric inequality,

$$A \leq \frac{L^2}{4\pi} \tag{2.7}$$

The maximum condition is when the rope forms a circle.

Before formal discussion of the problem, we should notice the shape should have some reasonable properties as shown in Figure 2.1: 1. the shape is convex, or there obviously exists a larger area. 2. the rope does not intersect with itself or we can also easily find a bigger inner area.

In 1902, Hurwitz, A. gave an beautiful direct proof of the inequality Equation 2.7 and the proof is written in a more modern style by Elias M. Stein in his famous textbook [4].

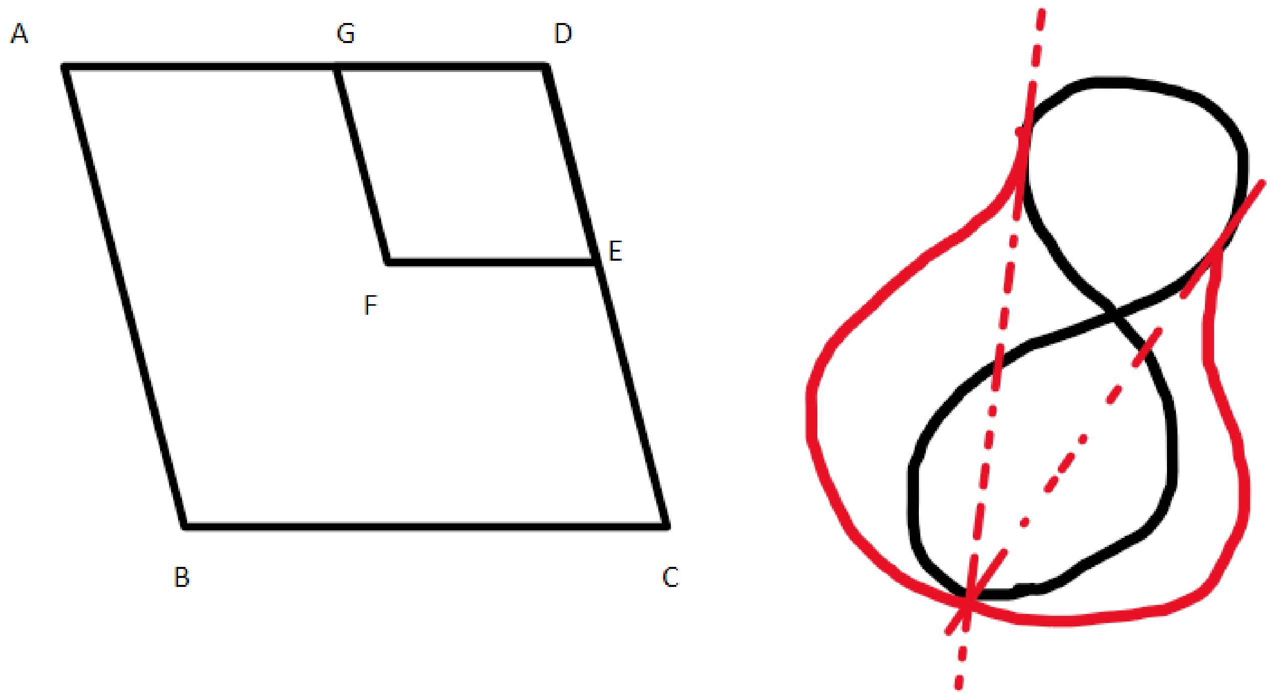


Figure 2.1: 1. ABCD and ABCEFG have the same perimeter but because ABCEFG is not convex, ABCD has a larger area. 2.If the rope intersects with itself, we can consider another shape with partial segments unfolded by symmetry to the dashed line (red parts). The shape with red lines obviously has a larger area than the black line shape.

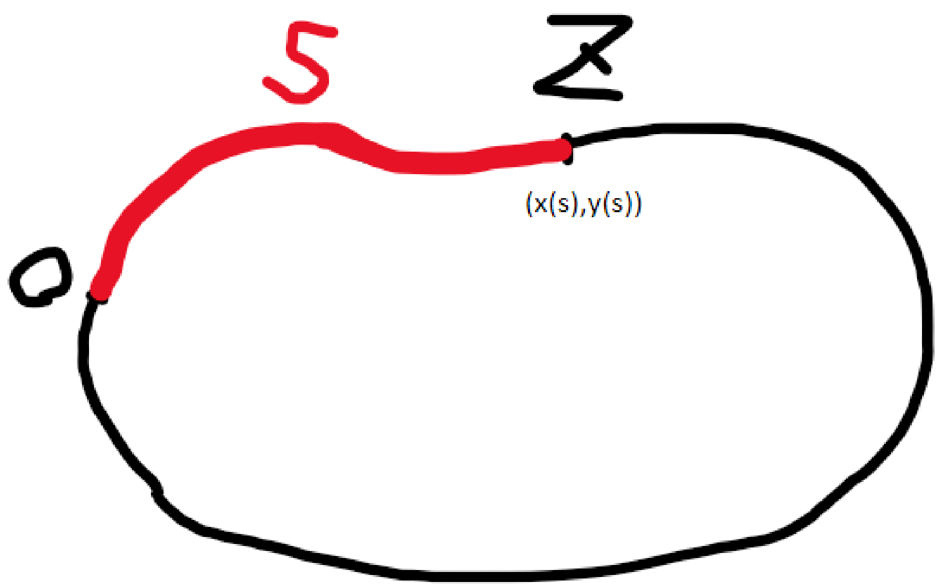


Figure 2.2: Schematic diagram of the coordinate system.

Interestingly, this proof starts with parameterization and also uses non-dimensionalization.

Consider in a 2D coordinate system shown in Figure 2.2, we pick some point on the closed rope as the origin, say O. There exists one segment of the rope between any point on the rope, Z, and O. The location of any point Z, (x,y) can be uniquely expressed as (x(s),y(s)) where s is the length of the segment.

For the parameterized coordinates to describe the rope (x,y), we introduce the new parameter $t = \frac{L}{2\pi}$ and write down (x(t),y(t)):

$$x(0) = x(2\pi), y(0) = y(2\pi), (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = (\frac{L}{2\pi})^2 \tag{2.8}$$

Therefore, we can expand x,y to be Fourier series:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt \tag{2.9}$$

$$y(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos nt + d_n \sin nt \tag{2.10}$$

To get their derivatives, we assume we can switch order of derivative calculation and summation (this is a non-trivial process but we do not prove it is valid here.):

$$x'(t) = \sum_{n=1}^{\infty} n(-a_n \sin nt + b_n \cos nt) \tag{2.11}$$

$$y'(t) = \sum_{n=1}^{\infty} n(-c_n \sin nt + d_n \cos nt) \tag{2.12}$$

Notice we have:

$$\frac{L^2}{2\pi} = \int_0^{2\pi} (\frac{L}{2\pi})^2 dt = \int_0^{2\pi} (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 dt = \pi \sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2 + c_n^2 + d_n^2) \tag{2.13}$$

We used third equality in Equation 2.8 and Parseval's Theorem Equation 2.3 here.

We now use Green's theorem to calculate the area inside the rope and orthogonality:

$$A = \int_0^{2\pi} x\frac{dy}{dt} dt = \sum_{n=1}^{\infty} \pi n(a_n d_n - b_n c_n) \tag{2.14}$$

Finally, we have

$$L^2 - 4\pi A = 2\pi^2 \sum_{n=1}^{\infty} (n^2(a_n^2 + b_n^2 + c_n^2 + d_n^2) - 2n(a_n d_n - b_n c_n)) \tag{2.15}$$

$$= 2\pi^2 \sum_{n=1}^{\infty} n^2(d_n^2 + c_n^2) + (na_n - d_n)^2 - d_n^2 + (nb_n + c_n)^2 - c_n^2 \tag{2.16}$$

$$= 2\pi^2 \sum_{n=1}^{\infty} (n^2 - 1)(d_n^2 + c_n^2) + (na_n - d_n)^2 + (nb_n + c_n)^2 \geq 0 \tag{2.17}$$

Fortunately, we know $=$ is reachable when the rope forms a circle!

This proof still has some gaps such as validation of switching integrals and summations, existence of the maximum area, uniqueness of the solution (is it possible there's another shape has the same area as the circle?), but the proof has given a great framework.

2.3 Padé Approximation: Motivation and Example

Time spent: 1hr

Taylor series for periodic function usually converges slowly for obvious reason: single polynomial is not periodic. Therefore, we try Padé approximation for periodic function.

We define Padé approximation of degree n+m to be:

$$R_{n,m}(x) = \frac{\sum_{i=0}^{n} a_i x^i}{1 + \sum_{j=1}^{m} b_j x^j} \tag{2.18}$$

Usually we need $m \geq n$ then

$$x \to \infty, R_{n,m} \to \begin{cases} 0 & \text{if } n < m \\ \frac{a_n}{b_m} & \text{if } n = m \end{cases} \tag{2.19}$$

We use $R_{2,2}$ to approximate $\cos x$ near x=0

$$(1 - x^2/2 + x^4/24) \approx \cos x \approx \frac{\sum_{i=0}^{2} a_i x^i}{1 + \sum_{j=1}^{2} b_j x^j} \tag{2.20}$$

$$(1 - x^2/2 + x^4/24)(1 + \sum_{j=1}^{2} b_j x^j) = \sum_{i=0}^{2} a_i x^i \tag{2.21}$$

Rearrange by x order and omit terms $O(x^5)$ and higher:

$$1 : 1 - a_0 = 0 \tag{2.22}$$

$$x : b_1 = a_1 \tag{2.23}$$

$$x^2 : b_2 - a_2 - 1/2 = 0 \tag{2.24}$$

$$x^3 : b_1/2 = 0 \tag{2.25}$$

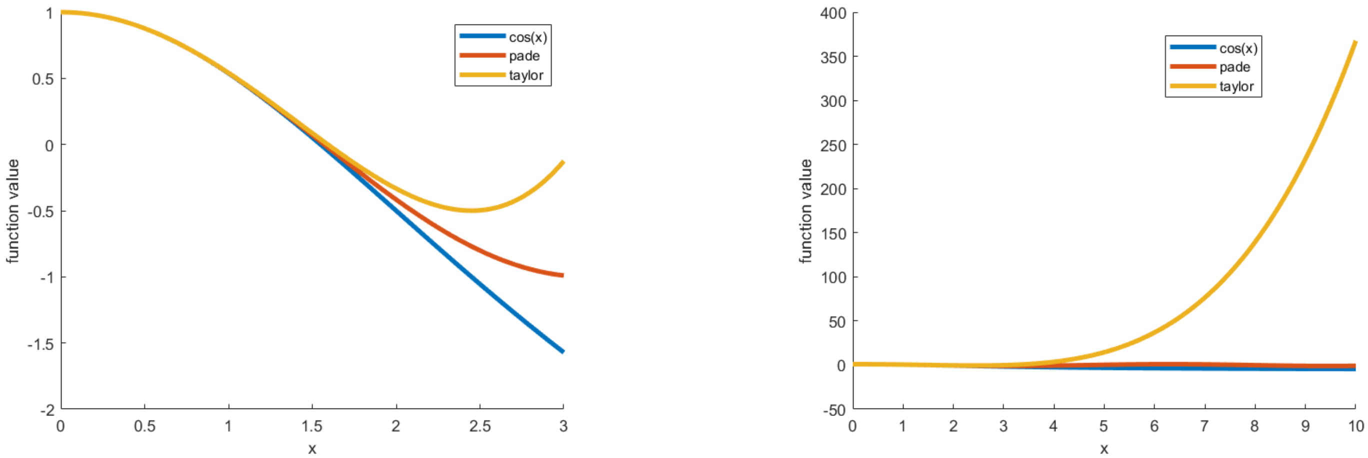
$$x^4 : 1/24 - b_2/2 = 0 \tag{2.26}$$

$$\tag{2.27}$$

We have:

$$R_{2,2}(x) = \frac{1 - \frac{5}{12}x^2}{1 + \frac{1}{12}x^2} \tag{2.28}$$

We compare $\cos(x)$, $R_{2,2}(x)$ and Taylor expansion $T(x) = 1 - x^2/2 + x^4/24$ at two time scales as shown in Figure 2.3. We can clearly see Pade approximation is more accurate with the same total x degree 2+2=4 as Taylor expansion in both small and time scale.



(a) $0 \le x \le 3$ (b) $0 \le x \le 10$

Figure 2.3: Comparison of exact function, Padé approximation, and Taylor expansion.

3 Population model

3.1 What is Mathematical oncology?

Time spent: 1 hour.

Mathematical oncology is an important field in mathematical biology and quantitative biology. It is investigating tumors and cancers in mathematical approaches. One major goal of mathematical oncology is using the present biological data to challenge the established mathematical models to predict the progression of cancer [6]. In practice, clinical physicians are using mathematical models to personalize and optimize cancer treatment [7]. The mathematical oncology also theoretically assist drug development [8].

This term, I am taking the course Mathematical Oncology by Professor Kohandel. In my study, majority of cancer models can be summarized as a population model of certain cancer cells [9] interacting with other types of cancer cells [10], related biological population like immune cells [11], and nutrients [12].

A common approach in mathematical oncology is to utilize stochastic process [13], ordinary differential equations [14], or partial differential equations [15] to simulate the growth and progression of cancer cells inside a tumor. Like in many other interdisciplinary studies, a mathematical oncology model is established and baseline parameters are fitted. Then the model is validated if the model can successfully predict some observed phenomenon after changing the corresponding components in the model.

As we will see in the following sub-section, even just using the simplest two ODEs, we can have clinical insights and further understanding of the cancer treatment. The topic I introduced is not a part of the lectures I took this term but I found it interesting.

3.2 The Framework Model of Drug-resistance Cancer Cells

Time spent: 6 hour. I implemented my version of code and tested different treatment compared with the original paper. I also refitted the baseline parameters. This chapter is written for AMATH 732, not a part for AMATH 881 project.

One major cause of cancer treatment failure is drug-resistance of cancer cells [16]. In other words, most of cancer treatments fail because the drugs are eventually ineffective to kill cancer cells and patients die. One major goal in personalized cancer treatment is to minimize the resistance of cancer cells and extend patients' lives.

There are two main challenges to tackle with drug-resistance. Drug resistance can be classified to drug-selected type or drug-induced type [17]. Drug-selected type means that the drug resistance is pre-existing in the cancer cells. Drug-induced type means that the administration induced the mutation of the cancer cells which leads to resistance. Additionally, most of established mathematical models to investigate drug-resistance do not incorporate the drug dosage as an explicit term.

Differentiating these two types is non-trivial work. It is easy to verify that the administration of cancer drugs leads to the resistance of cancer cells, as long as we can observe at some certain time point, the population of cancer cells or the size of tumor increases again after the administration of the drug initially effective. However, it is difficult to distinguish these two cases: 1. the drug kills non-resistance cancer cells, while pre-existing type cancer cells grow freely and leads to the growth of the tumor; 2. the drug induces resistance in cancer cells and cancer cells acquired resistance grow. Some experimental studies demonstrate that certain cases previously attributed to drug-induced effects are, in fact, drug-selective and caused by pre-existing gene mutations within the cancer cells [18].

In 2019, Greene et al. proposed a framework mathematical model to differentiate spontaneous and induced resistance in the cancer treatment with an explicit term of the drug dosage [19]. In brief, the system consists of two ODEs of the drug-sensitive cancer cell population, $S(t)$, and the drug-resistance cancer cell population, $R(t)$.

The non-dimensionalization version of the system can be written as:

$$\frac{dS}{dt} = (1 - (S + R))S - (\epsilon + \alpha u(t))S - du(t)S \tag{3.1}$$

$$\frac{dR}{dt} = p_r(1 - (S + R))R + (\epsilon + \alpha u(t))S \tag{3.2}$$

The first term means that each type is assumed to have logistic growth if stands alone. $0 \leq p_r \leq 1$ is a constant to present the ratio of logistic growth rate between two types. The term $(\epsilon + \alpha u(t))S$ represents the transition from the sensitive type to the resistant type. The transition term consists of spontaneous resistance, ϵS, and induced resistance, $\alpha u(t)S$, where ϵ and α are constants. $u(t)$ is a control input, the effective dosage of the drug. The last term in Equation 3.1 represents that the drug kills sensitive cancer cells, where d is a constant.

To compare how the treatment, $u(t)$, will affect the patients' survival time, we need to first set a treatment protocol. We first assume the tumor size, $V(t)$, is proportional to the total population of both types. We denote $V(t) = R(t) + S(t)$. The treatment starts when we can practically test the existence of cancer. In other words, the tumor size must be visible under medical imaging equipment like MRI. We denote the detectable size of the tumor as V_d and this time point to be t=0. The treatment will be first effective and eventually fail and the tumor size grow to some lethal size denoted as V_c at time $t = T_c$. The longer T_c is, the better our treatment strategy is. The protocol is shown in Figure 3.1.

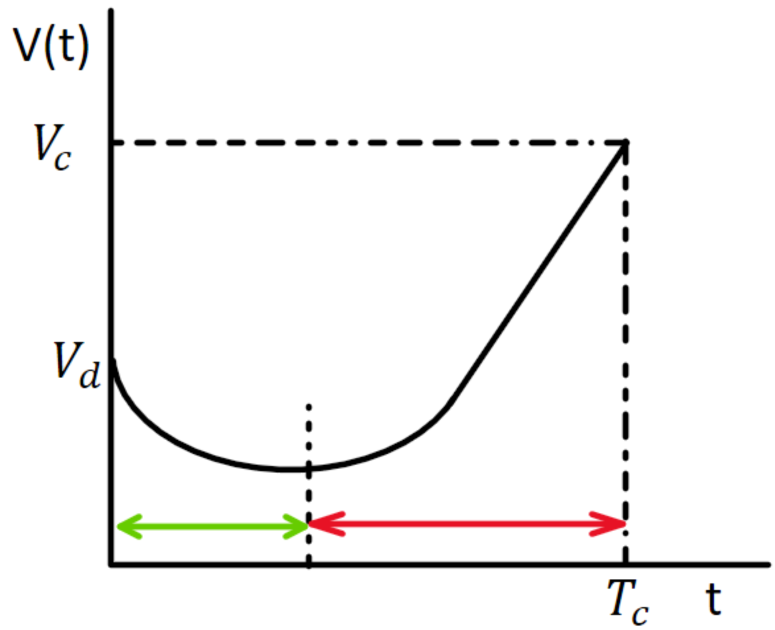


Figure 3.1: The schematic diagram of tumor protocol. At t=0, $V = V_d$. At time $t = T_c, V = V_c$ and the patient dies. In the green arrow time interval, the treatment is effective, while in the red arrow interval, the resistance is dominant.

Mathematically, we can prove if initial condition $S(0) = S_0, R(0) = R_0$ satisfies $S_0 + R_0 < 1$, $(S(t), R(t))$ converges to (0,1) asymptotically, so V_c is guaranteed to be reached. The proof is a bit complicated and will be provided in the next subsection.

In this subsection, we try to compare two types of treatments: 1. $u(t) = u_0$ is a constant function; 2. u(t) is an oscillation function, $u(t) = \max\{0, u_0(0.2 + 0.8\sin(2\pi\frac{t}{T}))\}$ with period T (remember in our discussion both volume size and treatment time are dimensionless). Both our treatment protocol and treatment strategy are different from the original paper and the results are different too. The Python implementation is given in section 9.2. The parameters chosen are given in Table 1. Two treatments are compared under two conditions: 1. the drug is phenotype-preserving ($\alpha = 0$); 2. the drug is resistance-inducing ($\alpha = 10^{-2}$).

Parameter	Value
S_0	0.09
R_0	0.01
V_d	0.1
V_c	0.9
ϵ	1e-6
d	1
p_r	0.2
u_0	1.5

Table 1: Parameters in Figure 3.2, Figure 3.3.

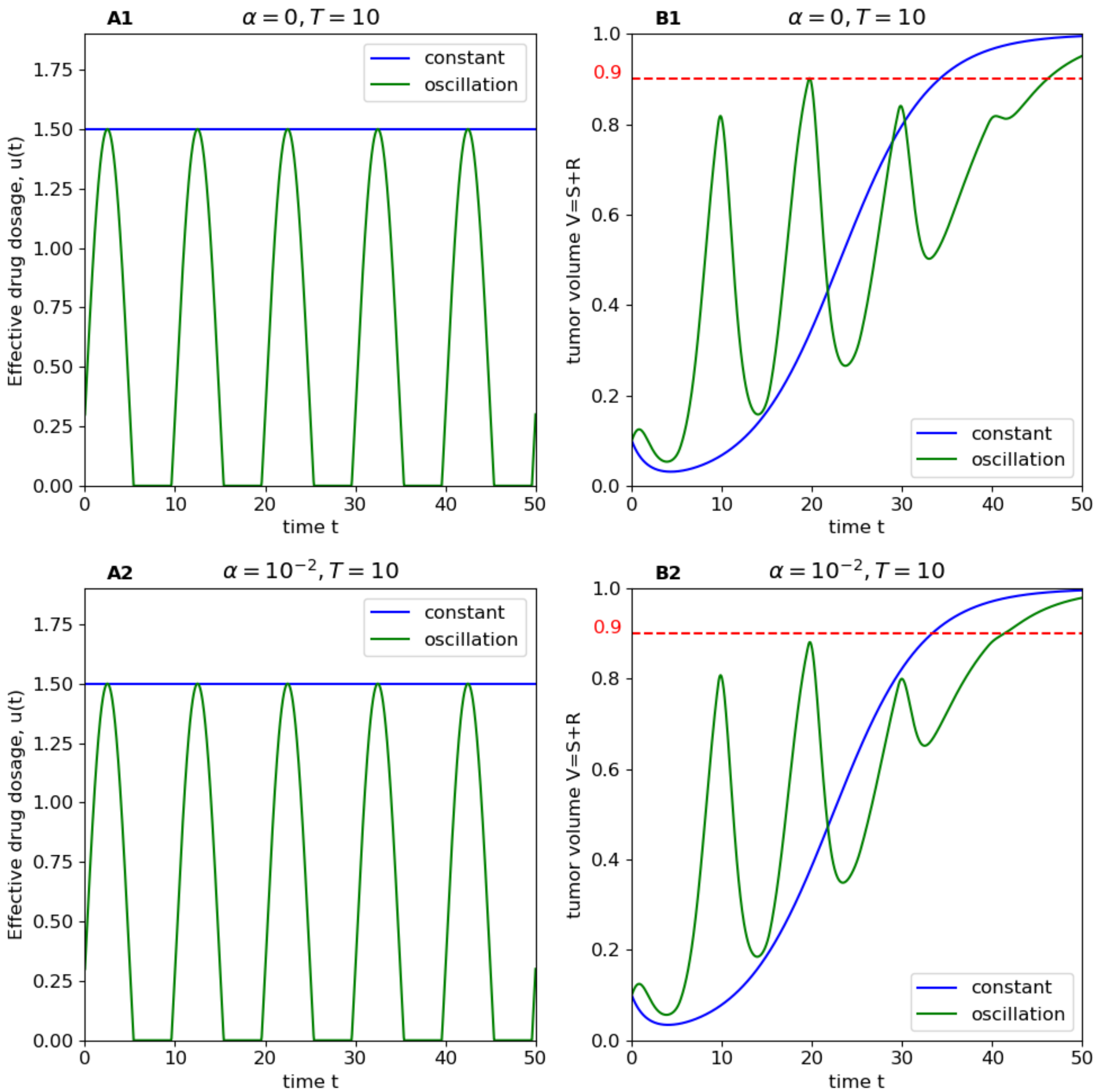


Figure 3.2: Tumor volume, V(t), progression overtime. In A1-B1, $\alpha = 0$, the drug is phenotype-preserving, while in A2-B2, $\alpha = 10^{-2}$, the drug is resistance-inducing. In both cases, the period of oscillation drug administration has period T=10. The red dashed line represents $V_c = 0.9$, the lethal tumor size.

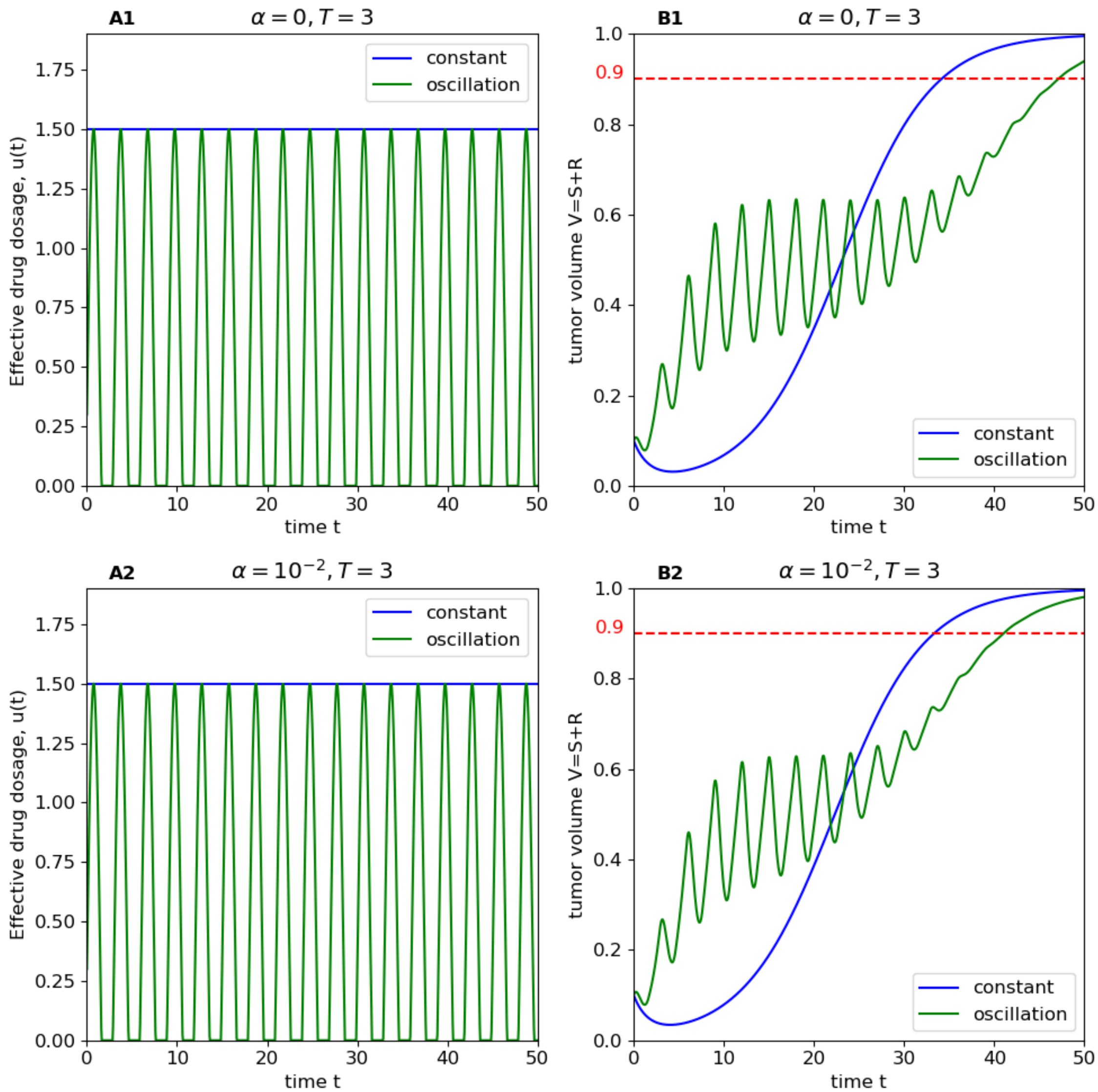


Figure 3.3: Tumor volume, V(t), progression overtime. In A1-B1, $\alpha = 0$, the drug is phenotype-preserving, while in A2-B2, $\alpha = 10^{-2}$, the drug is resistance-inducing. In both cases, the period of oscillation drug administration has period T=3. The red dashed line represents $V_c = 0.9$, the lethal tumor size.

As shown in 3.2B1, if the drug is phenotype-preserving the oscillation treatment results in a much shorter survival time $T_c = 20$ than $T_c = 34$ under the constant treatment. The reason is that the oscillation treatment administration is too slow to inhibit the growth of sensitive cancer cells. Interestingly, when the drug induces resistance, the oscillation treatment is better than the constant one, because the resistant cells compete with sensitive cells, which results in the growth at the early treatment stage is slightly slower. However, the oscillation treatment slows the population phenotype transition from partially resistant to completely resistant, which leads to a longer survival time for the patient. We can also notice in Figs. 3.2B1,3.2B2, the survival time of the constant treatment under two type of drugs are almost identical because the constant therapy quickly eliminates the sensitive cells and the resistant type dominates the whole process under each drug.

As shown in 3.3, if we increase the frequency of oscillation treatment, the oscillation treatment can bypass the peak volume at the early stage of the treatment. For both types of drugs, the oscillation treatment results in a longer survival time compared to the constant treatment. We can also notice at the late stage of the oscillation treatment, the tumor volume progression is close to lower frequency case in 3.2, because the resistant type is dominant in the population and the drug has minimal efficacy.

3.3 Proof of Globally Asymptotic Convergence: $\lim_{t\to\infty}(S,R)=(0,1)$

Time spent: 2 hour.

With initial value $S(0)=S_0\geq 0, V(0)=V_0\geq 0, 0<S_0+V_0<1$, we want to prove convergence of Equation 3.1,Equation 3.2. We assume $u(t)<\infty$.

Theorem 1:

The solution $(S(t),R(t))$ to the IVP problem exists and it is unique.

Proof:

Consider function

$$F=\begin{pmatrix} dS/dt \\ dR/dt \end{pmatrix} \tag{3.3}$$

According to Equation 3.1,Equation 3.2, $F(\cdot,R,S)$ is measurable for fixed (R,S) and $F(t,\cdot)$ is continuous for fixed t, as shown in the theorem 54, Appendix C, in Sontag's textbook [20], the existence of maximal solution guarantees the existence and uniqueness of $(S(t),R(t))$

Theorem 2:

$\forall t\geq 0, S\geq 0, R\geq 0, V=S+R\leq 1$

Proof: If S=0, $\frac{dS}{dt}=0, \frac{dR}{dt}=p_r(1-R)R$ there is trajectory, S=0 and R converges to 1 (R is a simple logistic growth now). Because F is bounded, the trajectories cannot intersect, no trajectory can approach R-axis, $R\geq 0$

Similarly if R=0, $\frac{dR}{dt}\geq 0$, no trajectory can approach S-axis, $S\geq 0$.

Moreover, $\frac{dV}{dt}=\alpha(t)(1-V)-\beta(t)$ where $\alpha(t)=S+p_rR\geq 0, \beta(t)=dus\geq 0$, so $\frac{dV}{dt}\leq 0$ if $V\leq 1$ and V cannot exceed 1.

Theorem 3: If u(t) is measurable, then $\lim_{t\to\infty}(S,R)=(0,1)$.

Proof:

From Theorem 2, we know $\frac{dR(t)}{dt}\geq 0$ and $0\leq R\leq R+S\leq 1$. R is monotone and bounded, R converges:

$$\lim_{t\to\infty} R=R^* \tag{3.4}$$

We now show

$$\rho=\liminf_{t\to\infty} S=0 \tag{3.5}$$

For sake of contradiction, assume $\rho>0$. Notice Equation 3.2 implies

$$\frac{dR}{dt}\geq(\epsilon+\alpha u(t))S\geq\epsilon S \tag{3.6}$$

By definition of $\rho, \exists t_0, S(t_0)\geq\frac{\rho}{2}$ because liminf is monotone we have for

$$R(T)-R(t_0)\geq\int_{t_0}^{T}\frac{\epsilon\rho}{2}d\tau \xrightarrow{T\to\infty}\infty \tag{3.7}$$

This is impossible, so we have $\rho=0$. The last inequality is from Equation 3.6.

Then we prove

$$\sigma=\liminf_{T\to\infty} R+S=1 \tag{3.8}$$

For the sake of contradiction, if $\sigma<1, \exists\delta>0, \sigma=1-\delta$.

Notice from Equation 3.2, we have

$$\frac{dR}{dt}=p_r(1-(S+R))R+(\epsilon+\alpha u(t))S\geq p_r(1-(S+R))R\geq p_r\delta R \tag{3.9}$$

We do integral again like we did for Equation 3.7. The contradiction requires $\delta=0,\rho=1$.

Because R converges, we have

$$\lim_{T\to\infty} R=\liminf_{T\to\infty} R=\liminf_{T\to\infty}(R+S)-\liminf_{T\to\infty} S=1-0=1 \tag{3.10}$$

By definition of convergence and $R \le 1$, we know

$$\exists t_1 > 0, t > t_1, 1 - \epsilon/2 \le R \le 1 \tag{3.11}$$

$$\frac{dS}{dt} = (1-(S+R))S - (\epsilon + \alpha u(t))S - du(t)S \le (1-R)S - \epsilon S \le (\frac{\epsilon}{2} - \epsilon)S = -\frac{\epsilon S}{2} \le 0 \tag{3.12}$$

Therefore, S is monotone decreasing and bounded when $t > t_1$ and convergent.

$$\lim_{T\to\infty} S = \liminf_{T\to\infty} S = 0 \tag{3.13}$$

We proved if the transition from sensitive to resistant is irreversible and the resistant cells are completely resistant to the drug, the population will asymptotically convergent to resistant.

4 Asymptotic Series

4.1 Important Concepts

Time spent: 1hr
Big-oh notation, O()
For two functions f,g with domain $D \subset R^n$ or $\subset C^n$, big-oh notation

$$f(x) = O(g(x)), x \in D \tag{4.1}$$

is to indicate the following fact:

$$\exists k \in R, k \ge 0, |f(x)| \le k|g(x)| \tag{4.2}$$

An alternative notation is to compare f and g in a neighbor of point x_0:

$$f(x) = O(g(x)), x \to x_0, \text{ means } \exists u \subset D, x_0 \in u, \exists k \in R, k \ge 0, |f(x)| \le k|g(x)|, \forall x \in u \tag{4.3}$$

Important notes:
Equal sign here does not have transitivity. For example, $\forall x > 1, x = O(x^2), 1 = O(x^2)$ but $x \neq 1$. In other words, big-oh is only giving an upper bound of the magnitude of left side function. $\exists k, |f(x)| \le k|g(x)|$ is not equivalent to $\exists k', |g(x)| \le k'|f(x)|$
Some examples:
1. if f,g are bounded on D, or D is compact and f,g are continuous, we can trivially have $f = O(g), g = O(f)$ because both $|f| < \infty, |g| < \infty$. This gives us insights that the main purpose of big-oh notation is to compare magnitude of function on unbounded domain D.
2. if $n > m \ge 0, \forall x \in R, x^m = O(x^n)$. Notice if $x = 0, |0^m| \le 1 \cdot |0^n|$; if $0 < |x| \le 1, k = |x|^{-n} < \infty, |x^m| < |1^m| = 1 = k|x^n|$; if $|x| > 1, |x^m| < 1 \cdot |x^n|$
3. $\forall x \in R, n \in N, x^n = O(e^x)$. $n!|e^x| = n!\sum_{i=0}^{\infty} \frac{x^i}{i!} \ge n!\frac{x^n}{n!} \ge |x^n|$
Small-oh notation, o()
Small-oh denotes that for limit point x_0 of domain D,

$$f(x) = o(g(x)) \text{as } x \to x_0, \text{ indicates } \frac{f(x)}{g(x)} \to 0 \text{ as } x \to x_0 \tag{4.4}$$

Important notes:
For point-wise condition, small-oh is a stronger condition because big-oh just need k to be finite but small-oh requires k to be 0. When domain D is R, we usually consider extended real line which means $\pm\infty$ can be selected.
Asymptotic Equivalence

Two functions f,g are asymptotic equivalence, $f \sim g$ at limit point x_0 if

$$\frac{f(x)}{g(x)} \to 1, \text{if } x \to x_0 \tag{4.5}$$

Asymptotic Expansion
We say $\sum_{n=0}^{\infty} a_n \phi_n(x)$ is an asymptotic expansion of function $f(x)$ if it fits following properties [21]:

$$\forall n \in \mathbf{N}, \phi_{n+1}(x) = o(\phi_n(x)), x \to \infty \tag{4.6}$$

$$f(x) - \sum_{k=0}^{M-1} a_n \phi_n(x) = O(\phi_M(x)), x \to \infty \tag{4.7}$$

4.2 Examples of Asymptotic Expansion

Time spent: 1hr
Example 1:
We know:

$$|p| < 1, \sum_{n=0}^{\infty} p^n = \frac{1}{1-p} \tag{4.8}$$

Therefore we have:

$$x > 1, f(x) = \frac{1}{1 - 1/x} = \sum_{n=0}^{\infty} (\frac{1}{x})^n \tag{4.9}$$

We can show this is an asymptotic expansion if $x \to \infty$:

$$x \to \infty \tag{4.10}$$

$$\frac{\frac{1}{x^{n+1}}}{\frac{1}{x^n}} = \frac{1}{x} \to 0 \tag{4.11}$$

$$\frac{f(x) - 1}{\frac{1}{x}} = \frac{\frac{1}{x} f(x)}{\frac{1}{x}} \to 1 \to f(x) - 1 = O(1/x) \tag{4.12}$$

$$\frac{f(x) - \sum_{n=0}^{N-1} x^{-n}}{x^{-N}} = x^N \sum_{n=N}^{\infty} x^{-n} = f(x) \to 1 \to f(x) - \sum_{n=0}^{N-1} x^{-n} = O(x^{-N}) \tag{4.13}$$

We have:

$$f(x) \sim \sum_{n=0}^{N} x^{-n} + O(\frac{1}{x^{N+1}}) \tag{4.14}$$

Example 2:
We want to find asymptotic expansion of

$$f(x) = x \sin \frac{1}{x}, x \to \infty \tag{4.15}$$

Because f(x) is even, we expand by assuming:

$$f(x) = \sum_{n=0}^{N} a_n (\frac{1}{x})^{2n} + O(\frac{1}{x^{2(N+1)}}) \tag{4.16}$$

$$a_0 = \lim_{x\to\infty} f(x) = \lim_{y\to 0} \frac{\sin y}{y} = 1 \tag{4.17}$$

$$a_1 = \lim_{x\to\infty} \frac{f(x)-1}{\frac{1}{x^2}} \tag{4.18}$$

To calculate limit Equation 4.18, we first denote $y = 1/x$ and we need to use L'Hôpital's rule twice:

$$\begin{aligned} a_1 &= \lim_{y\to 0} \frac{\frac{\sin y}{y} - 1}{y^2} = \lim_{y\to 0} \frac{\frac{y\cos y - \sin y}{y^2}}{2y} = \lim_{y\to 0} \frac{y\cos y - \sin y}{2y^3} \\ &= \lim_{y\to 0} \frac{\cos y - y\sin y - \cos y}{6y^2} = \lim_{y\to 0} -\frac{\sin y}{6y} = \frac{-1}{6} \end{aligned} \tag{4.19}$$

We have an asymptotic expansion:

$$f(x) = x\sin\frac{1}{x} \sim 1 - \frac{1}{6}x^{-2} + O(x^{-4}), x\to\infty \tag{4.20}$$

Our derivation is consistent with Taylor series:

$$f(y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} y^{2n}, y = \frac{1}{x} \tag{4.21}$$

5 Regular Perturbation Theory

5.1 RPT for First Order ODEs

Time spent: 2.5 hour.

Case 1: perturbed initial condition

Let us consider a very simple condition, a friction proportional to velocity (with constant γ) is applied to a 1-D mass point m. The governing equation is:

$$m\frac{dv}{dt} = -\gamma v^2 \tag{5.1}$$

We first do non-dimensionalization on the Equation 5.1: $v = v_0 V, t = \frac{m}{\gamma v_0}\tau$ where y, V are dimensionless:

$$\frac{dV}{d\tau} = -V^2 \tag{5.2}$$

The perturbed initial condition is:

$$V(0) = 1 + 2\epsilon, \epsilon \ll 1 \tag{5.3}$$

Do regular perturbation theory for $V(\tau)$ to first order, we have:

$$V(\tau) = V^{(0)}(\tau) + \epsilon V^{(1)}(\tau) + O(\epsilon^2) \tag{5.4}$$

Plug Equation 5.4 into Equation 5.2:

$$V^{(0)\prime} + \epsilon V^{(1)\prime} + O(\epsilon^2) = -((V^{(0)})^2 + 2\epsilon V^{(1)}V^{(0)} + O(\epsilon^2)) \tag{5.5}$$

$$V^{(0)\prime} = -(V^{(0)})^2 \tag{5.6}$$

$$\epsilon V^{(1)\prime} = -2\epsilon V^{(1)}V^{(0)} \tag{5.7}$$

The initial condition Equation 5.3 is expanded as:

$$V^{(0)}(0) + \epsilon V^{(1)}(0) = 1 + 2\epsilon \rightarrow V^{(0)}(0) = 1, V^{(1)}(0) = 2 \tag{5.8}$$

We can easily solve Equation 5.6 and Equation 5.7:

$$V^{(0)} = \frac{1}{1+\tau} \tag{5.9}$$

$$V^{(1)} = \frac{2}{(1+\tau)^2} \tag{5.10}$$

The perturbation solution is:

$$V(\tau) = \frac{1}{1+\tau} + \frac{2\epsilon}{(1+\tau)^2} \tag{5.11}$$

The exact solution to Equation 5.2 is:

$$V(\tau) = \frac{V(0)}{V(t)\tau + 1} = \frac{1+2\epsilon}{(1+2\epsilon)\tau + 1} \tag{5.12}$$

The sketch and comparison under different ϵ is shown in Figure 5.1. We can clearly see the error of perturbation is obvious when τ is small and increases as ϵ increases. The sketch code is given in section 9.2.

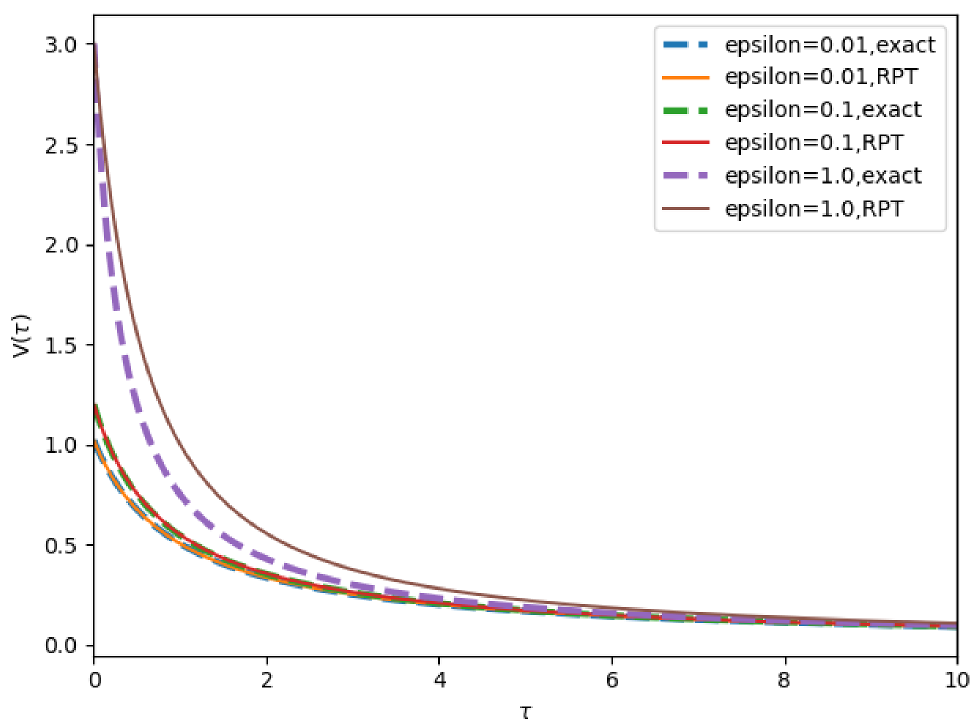


Figure 5.1: Comparison of exact solution and RPT solution under different ϵ values for perturbed initial condition.

Case 2: perturbed ODE

Consider perturbed Equation 5.2:

$$\frac{dV}{d\tau} = -(1+2\epsilon)V^2 \tag{5.13}$$

Initial condition is

$$V^{(0)} + \epsilon V^{(1)} + O(\epsilon^2) = V(0) = 1 \rightarrow V^{(0)} = 1, V^{(1)} = 0 \tag{5.14}$$

Plug Equation 5.4 into Equation 5.13:

$$V^{(0)\prime} + \epsilon V^{(1)\prime} + O(\epsilon^2) = -(1+2\epsilon)((V^{(0)})^2 + 2\epsilon V^{(1)}V^{(0)} + O(\epsilon^2)) \tag{5.15}$$

$$V^{(0)\prime} = -(V^{(0)})^2 \tag{5.16}$$

$$\epsilon V^{(1)\prime} = -2\epsilon V^{(1)}V^{(0)} - 2\epsilon(V^{(0)})^2 \tag{5.17}$$

We stil have:

$$V^{(0)} = \frac{1}{\tau+1} \tag{5.18}$$

For $V^{(1)}$:

$$V^{(1)\prime} = -2\frac{1}{\tau+1}V^{(1)} - 2(\frac{1}{\tau+1})^2 \tag{5.19}$$

$$V^{(1)\prime} + 2\frac{1}{\tau+1}V^{(1)} = -2(\frac{1}{\tau+1})^2 \tag{5.20}$$

It is easy to see general solution $V_g^{(1)}$

$$V_g^{(1)\prime} + 2\frac{1}{\tau+1}V_g^{(1)} = 0 \tag{5.21}$$

$$\frac{dV_g^{(1)}}{V_g^{(1)}} = -\frac{2}{\tau+1}d\tau \tag{5.22}$$

$$d\ln V_g^{(1)} = d(-2\ln(\tau+1)) \tag{5.23}$$

$$V_g^{(1)} = \frac{C_1}{(\tau+1)^2} \tag{5.24}$$

We can guess the particular solution is $V_p^{(1)} = C_2(\tau+1)^n$. Plug into Equation 5.20:

$$C_2(\tau+1)^{n-1}(n+2) = -2(\tau+1)^{-2} \rightarrow n-1 = -2, n = -1 \rightarrow C_2(n+2) = -2 \rightarrow C_2 = -2 \tag{5.25}$$

Use IC, we have:

$$C_1 - 2 = 0 \rightarrow C_1 = 2, V(\tau) = \frac{1}{\tau+1} + \epsilon(\frac{2}{(\tau+1)^2} - \frac{2}{\tau+1}) \tag{5.26}$$

We compare the exact and RPT solutions again as shown in Figure 5.2. Similar to perturbed initial condition, the larger ϵ is, the larger the difference between exact and RPT solutions is, while the difference is larger under the same order of magnitude of ϵ. The error of RPT solution is more sensitive to the direct perturbation for the ODE.

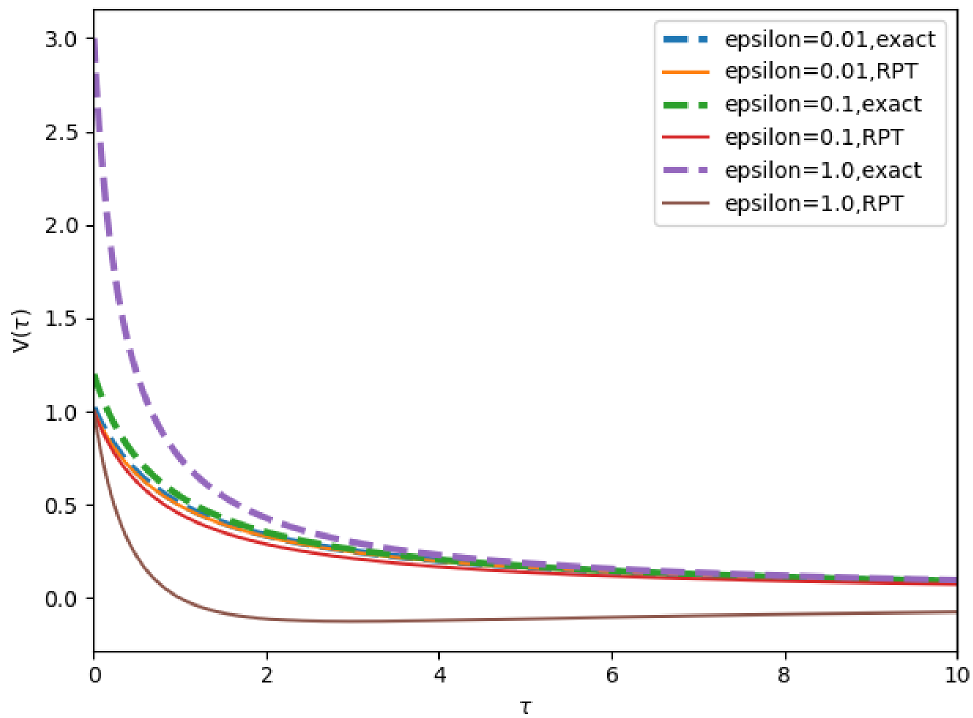


Figure 5.2: Comparison of exact solution and RPT solution under different ϵ values for perturbed ODE.

5.2 RPT for transcendental equations

Time spent: 1hr

We want to solve by RPT.

$$\epsilon\cos(x) = 1 - x^2 \tag{5.27}$$

Expand root x in RPT

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + O(\epsilon^3) \tag{5.28}$$

Use Taylor expansion:

$$LHS = \epsilon(\cos(x_0) - \sin(x_0)(\epsilon x_1 + \epsilon^2 x_2) - \frac{2\cos(x_0)}{2}(\epsilon x_1 + \epsilon^2 x_2)^2 + O(\epsilon^3)) \tag{5.29}$$

$$= \epsilon\cos(x_0) + \epsilon^2(-\sin(x_0)x_1) + O(\epsilon^3) \tag{5.30}$$

$$RHS = 1 - (x_0^2 + \epsilon^2 x_1^2 + 2\epsilon x_0 x_1 + 2\epsilon^2 x_0 x_2 + O(\epsilon^3)) \tag{5.31}$$

We have:

$$1 - x_0^2 = 0 \tag{5.32}$$

$$\cos(x_0) = -2x_0 x_1 \tag{5.33}$$

$$-\sin(x_0)x_1 = -x_1^2 - 2x_0 x_2 \tag{5.34}$$

We have:

$$x_0 = \pm 1 \tag{5.35}$$

$$x_1 = \frac{\cos(x_0)}{2x_0} = \pm\frac{\cos 1}{2} \tag{5.36}$$

$$x_2 = \frac{1}{2x_0}(x_1^2 - \sin(x_0)x_1) = \pm\frac{(\frac{\cos 1}{2})^2 - \frac{\sin(1)\cos(1)}{2}}{2} \tag{5.37}$$

We can notice for two roots solved, they have same absolute value but different signs which is consistent with Equation 5.27 has even functions on both sides.

Define error (Notice it is the same for both positive and negative roots):

$$ERR = \epsilon \cos(x) - 1 + x^2, x = 1 + \epsilon\frac{\cos 1}{2} + \epsilon^2\frac{(\frac{\cos 1}{2})^2 - \frac{\sin(1)\cos(1)}{2}}{2} \tag{5.38}$$

ERR versus ϵ is shown in Figure 5.3 which is small enough when ϵ is small and the curve is almost straight because when ϵ is small, first order error is dominant.

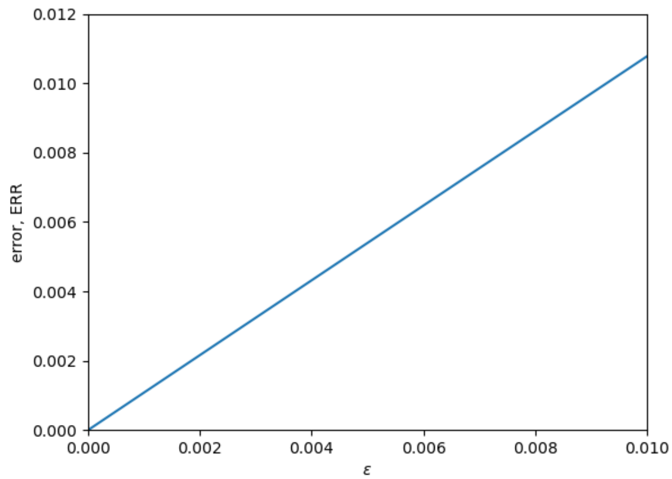


Figure 5.3: Error versus ϵ.

6 The Method of Multiple Scales (MMS)

6.1 Unforced Van der Pol Oscillator

Time spent: 3hr

Consider dimensionless unforced Van der Pol Oscillator we explored in our lab 2 but damping constant is small $\epsilon \ll 1$:

$$\frac{d^2x}{dt^2} + \epsilon(1 - x^2)\frac{dx}{dt} + x = 0 \tag{6.1}$$

Do asymptotic expansion on x with two time scale $t, \tau = \epsilon t$

$$x = x_0(t, \tau) + \epsilon x_1(t, \tau) + O(\epsilon^2) \tag{6.2}$$

Then we have by chain rule:

$$\frac{dx}{dt} = \frac{\partial x}{\partial t} + \frac{\partial x}{\partial \tau}\frac{d\tau}{dt} = \frac{\partial x_0}{\partial t} + \epsilon(\frac{\partial x_0}{\partial \tau} + \frac{\partial x_1}{\partial t}) + O(\epsilon^2) \tag{6.3}$$

Similarly, use chain rule:

$$\frac{d^2x}{dt^2} = \frac{\partial^2 x_0}{\partial t^2} + \epsilon(2\frac{\partial^2 x_0}{\partial t \partial \tau} + \frac{\partial^2 x_1}{\partial t^2}) + O(\epsilon^2) \tag{6.4}$$

Notice

$$1 - x^2 = 1 - (x_0^2 + 2\epsilon x_0 x_1 + O(\epsilon^2)) \tag{6.5}$$

Plug back into Equation 6.1 and remove term of order $O(\epsilon^2)$:

$$\frac{\partial^2 x_0}{\partial t^2} + \epsilon(2\frac{\partial^2 x_0}{\partial t \partial \tau} + \frac{\partial^2 x_1}{\partial t^2}) + \epsilon(1 - x_0^2)\frac{\partial x_0}{\partial t} + x_0 + \epsilon x_1 = 0 \tag{6.6}$$

We have:

$$\frac{\partial^2 x_0}{\partial t^2} + x_0 = 0 \tag{6.7}$$

$$\frac{\partial^2 x_0}{\partial t \partial \tau} + \frac{\partial^2 x_1}{\partial t^2} + (1 - x_0^2)\frac{\partial x_0}{\partial t} + x_1 = 0 \tag{6.8}$$

Solve Equation 6.7 is easy:

$$x_0 = a(\tau)\cos t + b(\tau)\sin t \tag{6.9}$$

$$\tag{6.10}$$

We can rewrite by defining:

$$r(\tau) = \sqrt{a(\tau)^2 + b(\tau)^2} \tag{6.11}$$

$$\phi(\tau) = \frac{a(\tau)}{r(\tau)} \tag{6.12}$$

$$x_0 = r(\tau)\sin(t + \phi(\tau)) \tag{6.13}$$

The motivation to do so is that we know Van der Pol has a limit cycle, convert leading term to be in one triangle function should simplify the final solution. Another advantage is to determine secular term, we will show later.

We then have:

$$\frac{\partial x_0}{\partial t} = r(\tau)\cos(t + \phi(\tau)) \tag{6.14}$$

$$\frac{\partial^2 x_0}{\partial \tau \partial t} = r'(\tau)\cos(t + \phi(\tau)) - r(\tau)\phi'(\tau)\sin(t + \phi(\tau)) \tag{6.15}$$

Plug into Equation 6.8, denote $\theta = t + \phi(\tau)$, notice $\frac{\partial \theta}{\partial t} = 1$:

$$\frac{\partial^2 x_1}{\partial t^2} + x_1 = -2r'(\tau)\cos\theta + 2r(\tau)\phi'(\tau)\sin\theta - (1 - r^2\sin^2\theta)r\cos\theta \tag{6.16}$$

LHS is undamped SHO with period 2π for t. Any term on RHS in form of $g(\tau)\sin\theta, h(\tau)\cos\theta$ will cause resonance and they are secular terms. Remember $\frac{\partial \theta}{\partial t} = 1$, θ and t has only phase difference in the context of partial derivative.

We can notice:

$$\begin{aligned}
\cos 3\theta &= \cos\theta\cos 2\theta - \sin\theta\sin 2\theta \\
&= \cos\theta(2\cos^2\theta - 1) - 2\sin^2\theta\cos\theta \\
&= 2\cos^3\theta - \cos\theta - 2(1 - \cos^2\theta)\cos\theta \\
&= 4\cos^3\theta - 3\cos\theta \\
&\to \cos^3\theta = \frac{\cos 3\theta + 3\cos\theta}{4} \\
\sin^2\theta\cos\theta &= \cos\theta - \cos^3\theta = \frac{\cos\theta - \cos 3\theta}{4}
\end{aligned} \tag{6.17}$$

In Equation 6.16 RHS, we have:

$$\begin{aligned} RHS &= -2r'(\tau)\cos\theta + 2r(\tau)\phi'(\tau)\sin\theta - (1 - r^2\sin^2\theta)r\cos\theta \\ &= (-2r' - r + \frac{r^3}{4})\cos\theta + 2r'\phi'\sin\theta - \frac{r^3}{4}\cos 3\theta \end{aligned} \tag{6.18}$$

We have known that $\cos 3\theta$ terms is non-secular (no resonance).

To do secular elimination, we have:

$$-2r' - r + \frac{1}{4}r^3 = 0 \to r' = \frac{1}{8}r(r^2 - 4) \tag{6.19}$$

$$r\phi' = 0 \to \phi' = 0 \to \phi(\tau) = Constant = \phi(0) \tag{6.20}$$

Unfortunately, we cannot analytically solve Equation 6.19, but we can observe that:

$$r' = \begin{cases} 0 & \text{if } r = 2, 0, -2 \\ < 0 & \text{if } 0 < r < 2, r < -2 \\ > 0 & \text{if } r > 2, -2 < r < 0 \end{cases} \tag{6.21}$$

If $|r(0)| < 2$, the leading order converges to $x = 0$. If $|r(0)| > 2$, the leading order diverges to $\pm\infty$.

We have (denote $\phi(0) = \phi_0$):

$$x(0) = r(0)\sin(\phi_0), x'(0) = r(0)\cos(\phi_0) \to r(0)^2 = x(0)^2 + x'(0)^2 \tag{6.22}$$

In Equation 6.22, first term is proportional to potential, the second term is proportional to kinetic. In conclusion, unforced perturbed Van der Pol will converges to origin if the initial system energy is smaller than threshold, while the system will diverge to infinity if the initial system energy is higher than threshold.

6.2 General Damped Oscillator

Time spent: 3hr

In this subsection, we want to discuss the failure of RPT and the multiple scale solution for:

$$\frac{d^2x}{dt^2} + \epsilon(\frac{dx}{dt})^N + x = 0, \text{N is a fixed positive integer} \tag{6.23}$$

We want to first answer a question: What is the condition that RPT fails for Equation 6.23? The answer is when n is odd, RPT fails and when n is even, RPT works.

Assume RPT solution exists:

$$x = x_0(t) + \epsilon x_1(t) + O(\epsilon^2) \tag{6.24}$$

We know:

$$(\frac{dx}{dt})^N = (x_0' + \epsilon x_1' + O(\epsilon^2))^N = x_0'^N(1 + \epsilon x_1'/x_0' + O(\epsilon^2))^N = x_0'^N(1 + \epsilon\frac{Nx_1'}{x_0'}) + O(\epsilon^2) \tag{6.25}$$

For different orders, we have:

$$O(1): x_0'' + x_0 = 0 \to x_0 = C_1\cos t + C_2\sin t = C\sin(t + \phi), C = \sqrt{C_1^2 + C_2^2}, \phi = sin^{-1}\frac{C_1}{C} \tag{6.26}$$

$$O(\epsilon): x_1'' + x_1 = -x_0'^N = -C^N\cos^N(t + \phi) \tag{6.27}$$

Use trigonometric identity [22] for Equation 6.27 RHS, denote $\theta = t + \phi$

$$RHS = \begin{cases} \frac{1}{2^{2n}}\sum_{k=0}^{n}\binom{2n+1}{k}\cos(2n + 1 - 2k)\theta & \text{if N is odd}, N = 2n + 1 \\ \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}}\sum_{k=1}^{n-1}\cos 2(n - k)\theta & \text{if N is even}, N = 2n \end{cases} \tag{6.28}$$

Only when N is odd, RHS will have secular term $\cos\theta$, so our answer is verified: when n is odd, RPT fails and when n is even, RPT works.

We can still do MMS for odd condition because as terms respect to scale τ maintain constant, MMS solution degenerates to RPT solution, but for the rest of discussion, we only consider N=2n+1 is odd.

$$\tau = \epsilon t, x(t,\tau) = x_0(t,\tau) + \epsilon x_1(t,\tau) + O(\epsilon^2) \tag{6.29}$$

Plug Equation 6.3,Equation 6.4 into Equation 6.23:

$$O(1): \frac{\partial^2 x_0}{\partial t^2} + x_0 = 0 \to x_0 = r(\tau)\sin(t+\phi(\tau)) \tag{6.30}$$

$$O(\epsilon): \frac{\partial^2 x_1}{\partial t^2} + \frac{\partial x_1}{\partial t} = -2\frac{\partial^2 x_0}{\partial t \partial \tau} - (\frac{\partial x}{\partial t})^N \tag{6.31}$$

Again, denote $\theta = t + \phi(\tau)$, use 6.14,6.15, the two secular terms in Equation 6.31 RHS:

$$\begin{aligned} &(-2r' - \frac{1}{2^{N-1}}\binom{N}{(N-1)/2} r^N)\cos\theta \\ &2r\phi'\sin\theta \end{aligned} \tag{6.32}$$

By eliminating secular term:

$$\begin{aligned} &\frac{dr}{d\tau} = -\frac{1}{2^N}\binom{N}{(N-1)/2} r^N \to r(\tau) = \frac{r_0}{(1+(N-1)r_0^{N-1}\frac{1}{2^N}\binom{N}{(N-1)/2}\tau)^{1/(N-1)}} \\ &2r\phi' = 0 \to \phi' = 0 \to \phi(\tau) = \phi(0) \end{aligned} \tag{6.33}$$

We can verify that when N=5, $\phi(0) = \pi/2, r(0) = 1$ the solution is consistent with section 5.4 in main notes [23]:

$$x_0 = \frac{1}{(1+\frac{5}{4}\tau)}\cos t = \frac{\sqrt{2}}{(4+5\epsilon t)^{1/4}}\cos t \tag{6.34}$$

We can also give an example when N is even, RPT can work. Let $x(t) = x_0(t) + \epsilon x_1(t) + O(\epsilon^2)$ and N=2. From Equation 6.26,Equation 6.27, we let IC fits

$$C_1 = C_2 = A \to x_1'' + x_1 = -A^2(1 + 2\sin t\cos t) = -A^2(1+\sin 2t) \tag{6.35}$$

We instant have general solution and two particular solutions:

$$x_{1,g}'' + x_{1,g} = 0 \to x_{1,g} = B_1\cos t + B_2\sin t \tag{6.36}$$

$$x_{1,p1}'' + x_{1,p1} = -A^2 \to x_{1,p1} = -A^2 \tag{6.37}$$

$$x_{1,p2}'' + x_{1,p2} = -A^2\sin 2t \to x_{1,p2} = -\frac{A^2}{3}\sin 2t \tag{6.38}$$

$$x = A(\cos t + \sin t) + \epsilon(B_1\cos t + B_2\sin t - A^2 - \frac{A^2}{3}\sin 2t) + O(\epsilon^2) \tag{6.39}$$

This RPT solution does not have issue that $O(\epsilon)$ term grows to $O(1)$.

7 Resonance

7.1 An Alternative Derivation of Resonance

Time spent: 1hr

For the simplest case of damped oscillation:

$$m\ddot{x} + \gamma\dot{x} + kx = F\cos\Omega t \tag{7.1}$$

Let us do Fourier transform $x \to X : t \to \omega$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\exp(-i\omega t), x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\exp(i\omega t) \tag{7.2}$$

Plug into Equation 7.1, denote $\omega_0 = \sqrt{k/m}, A = F/m, \gamma/m = \beta$:

$$\begin{aligned}
&(-\omega^2 m + \gamma i\omega + k)X(\omega) = F\pi(\delta(\omega - \Omega) + \delta(\omega + \Omega)) \\
&X(\omega) = \pi A\frac{\delta(\omega - \Omega) + \delta(\omega + \Omega)}{\omega_0^2 - \omega^2 + i\beta\omega} \\
&x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\exp(i\omega t) = \frac{A}{2}(\frac{\exp(i\Omega t)}{\omega_0^2 - \Omega^2 + i\beta\Omega} + \frac{\exp(-i\Omega t)}{\omega_0^2 - \Omega^2 - i\beta\Omega})
\end{aligned} \tag{7.3}$$

Amplitude of x(t) is:

$$\frac{A}{\sqrt{(\omega_2^2 - \Omega^2)^2 + \beta^2\Omega^2}} = \frac{A}{\sqrt{(\Omega^2)^2 - (2\omega_0^2 - \beta^2)\Omega^2 + \omega_0^4}} \tag{7.4}$$

This amplitude expression is consistent with lecture notes [23].

When

$$\Omega = \sqrt{\omega_0^2 - \frac{\beta^2}{2}} \tag{7.5}$$

amplitude reaches max value

$$\frac{A}{\sqrt{\omega_0^4 - (\omega_0^2 - \beta^2/2)^2}} = \frac{A}{\sqrt{\beta^2(\omega_0^2 - \beta^2/4)}} \tag{7.6}$$

This is consistent with that resonance requires the system is not over damped which means $\beta < \sqrt{2}\omega_0$

7.2 Non-linear Damped Oscillator

Time spent: 2hr

Let us consider the oscillator with non-linear damping and periodic external force:

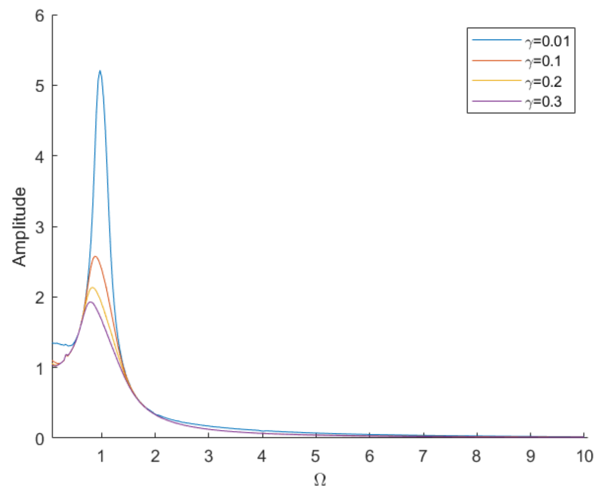
$$mx'' + \gamma x'^3 + kx = F\cos\Omega t \tag{7.7}$$

or in dimensionless version and let F=1:

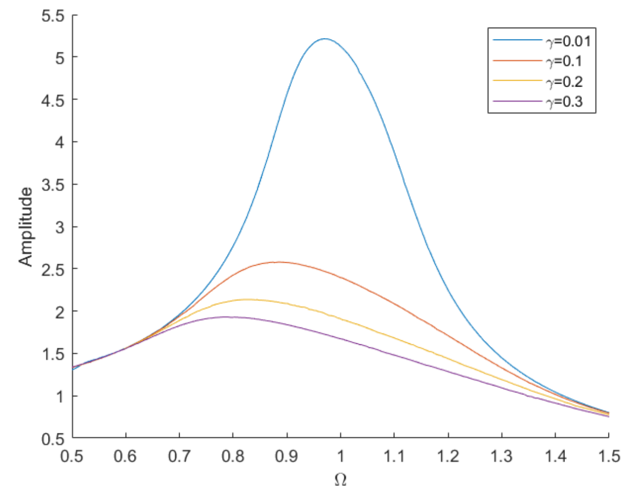
$$\ddot{x} + \gamma\dot{x}^3 + x = \cos\Omega t \tag{7.8}$$

Let IC is $\dot{x}(t) = 0, x(0) = 0$ and simulate the system in a long time interval. We can notice x has spatial symmetry which means to calculate amplitude of x, we just need find maximum of x when t is large. The implementation is given in section 9.2.

Numerical simulation is shown in Figure 7.1 with multiple γ values. We can notice for given non-negative γ, there exists only one resonance frequency. As γ increases, resonance amplitude and resonance frequency decrease. Therefore we can conclude non-linear damping results in amplitude shift to left and has similar effect on peak amplitude as linear damping.



(a) $0 \leq \Omega < 10$



(b) $0.5 \leq \Omega \leq 1.5$

Figure 7.1: Relationship between solution amplitude and external force angular frequency, Ω

8 Normal Modes

8.1 Four Coupled Oscillators

Time spent: 4hr

Figure 8.1 is a schematic diagram of N oscillators with mass $m_{1,2,...,N}$, and string on left with Hooke constant $k_{1,2,...,N}$. First mass is connected to the rigid, fixed wall on left, N-th mass is connected to N-1-th mass on left and no string on right. The displacement x_i is set to be relative to the place of mass when the system is completely still. Each oscillator also encounters resistance proportional to velocity with constant $-\gamma < 0$.

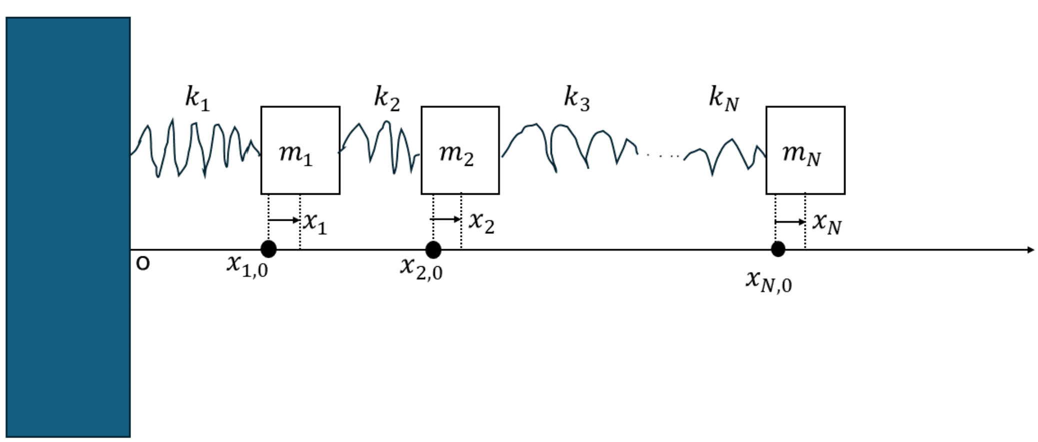


Figure 8.1: N connected oscillators.

We first consider the system $N = 4$. To simplify, we let $m_{1,2,..N} = 1, k_{1,2,...N} = k$

The governing equations can be written as:

$$\begin{aligned} x_1'' + \gamma x_1' + k(2x_1 - x_2) &= 0 \\ x_2'' + \gamma x_2' + k(-x_1 + 2x_2 - x_3) &= 0 \\ x_3'' + \gamma x_3' + k(-x_2 + 2x_3 - x_4) &= 0 \\ x_4'' + \gamma x_4' + k(x_4 - x_3) &= 0 \end{aligned} \tag{8.1}$$

Define:

$$X = (x_1, x_2, x_3, x_4)^T \tag{8.2}$$
$$Y = (y_1, y_2, y_3, y_4)^T \tag{8.3}$$
$$\tag{8.4}$$

We assume:

$$X = Y \exp(i(\Omega t - \delta)) \tag{8.5}$$

Then solving X is equivalent to solving Y:

$$AY = \vec{0}, A = \begin{pmatrix} -\Omega^2 + 2k + i\Omega\gamma & -1 & 0 & 0 \\ -1 & -\Omega^2 + 2k + i\Omega\gamma & -1 & 0 \\ 0 & -1 & -\Omega^2 + 2k + i\Omega\gamma & -1 \\ 0 & 0 & -1 & -\Omega^2 + k + i\Omega\gamma \end{pmatrix} \tag{8.6}$$

The resonance frequency results in $det(A)$=0. If not, Y is 0 (non-singular matrix null space is 0).
Charateristic polynomial is:

$$\begin{aligned} 0 = &(\Omega^2)^4 - (\Omega^2)^3\gamma k4i - 7(\Omega^2)^3 k - 6(\Omega^2)^2\gamma^2 k^2 + (\Omega^2)^2\gamma k^2 21i + 18(\Omega^2)^2 k^2 - 3(\Omega^2)^2 + \\ &(\Omega^2)\gamma^3 k^3 4i + 21(\Omega^2)\gamma^2 k^3 - (\Omega^2)\gamma k^3 36i + (\Omega^2)\gamma k6i - 20(\Omega^2)k^3 + 10(\Omega^2)k + \gamma^4 k^4 \\ &- \gamma^3 k^4 7i - 18\gamma^2 k^4 + 3\gamma^2 k^2 + \gamma k^4 20i - \gamma k^2 10i + 8k^4 - 8k^2 + 1 \end{aligned} \tag{8.7}$$

Let $\gamma = 0$, the system is undamped, let $k = 1$ we can solve Ω (only picking positive roots):

$$\Omega = 0.347296355333860697703433253538\text{63}, \tag{8.8}$$
$$1.0, \tag{8.9}$$
$$1.532088886237956070404785301110\text{8}, \tag{8.10}$$
$$1.8793852415718167681082185546495 \tag{8.11}$$

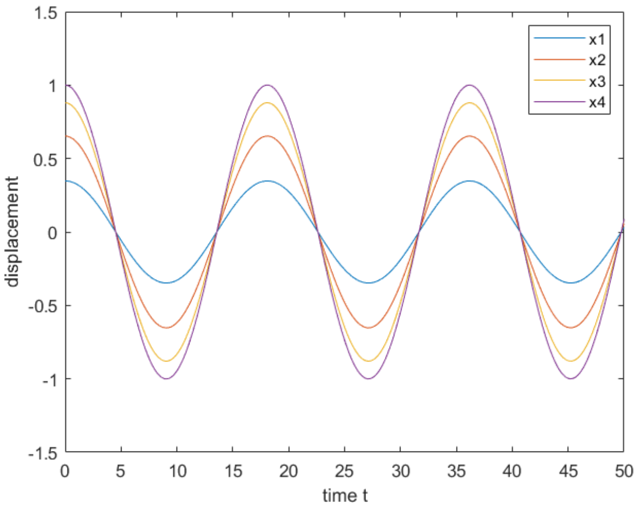
The solutions Y corresponds to first two frequencies are:

$$Y_1 = \begin{bmatrix} 0.34729635533386069770343325353863 \\ 0.65270364466613930229656674646137 \\ 0.87938524157181676810821855464946 \\ 1 \end{bmatrix}, Y_2 = \begin{bmatrix} -1.0 \\ -1.0 \\ 0 \\ 1 \end{bmatrix}, \tag{8.12}$$

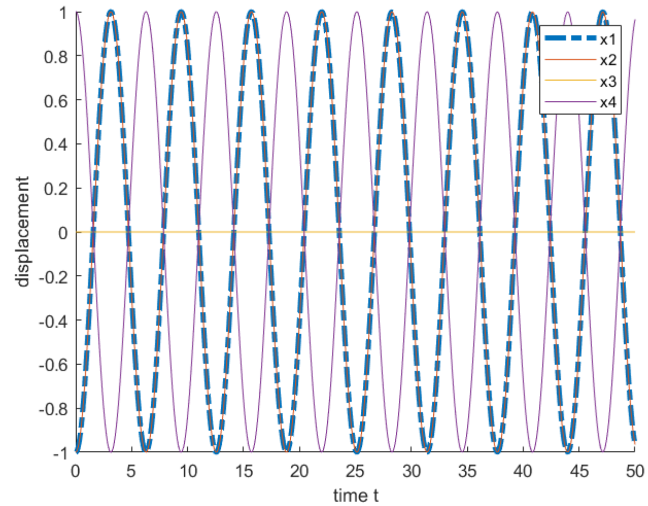
Numerically, let us test two interesting solutions in Matlab by set initial condition:

$$\text{Case 1: } \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix} = Y_1, \begin{bmatrix} x_1'(0) \\ x_2'(0) \\ x_3'(0) \\ x_4(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ Case 2: } \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix} = Y_2, \begin{bmatrix} x_1'(0) \\ x_2'(0) \\ x_3'(0) \\ x_4(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{8.13}$$

As shown in Figure 8.2, case 1 corresponds to the case all 4 oscillators have the same phase. Case 2 corresponds to the scenario that x_1, x_2 are oscillating in the same phase, while x_4 is moving in the opposite phase. The gross effect is that there is no gross force on x_3 because forces from x_2, x_4 cancel each other. If we replace x_3 with a rigid wall, the other three oscillators in the way.



(a) Case 1



(b) Case 2

Figure 8.2: Simulation of two normal modes.

8.2 General N Coupled Oscillators

Time spent: 4hr

Let's consider a general system $(x_1, x_2, ..., x_N)$ with N coupled oscillators analogous to Equation 8.1 shown in Figure 8.1:

$$\begin{aligned} &x_1'' + \gamma x_1' + k(2x_1 - x_2) = 0 \\ &x_n'' + \gamma x_n' + k(-x_{n-1} + 2x_n - x_{n+1}) = 0, \forall 2 \leq n \leq N-1 \\ &x_N'' + \gamma x_N' + k(x_N - x_{N-1}) = 0 \end{aligned} \tag{8.14}$$

Again, let $\gamma = 0, k_i = k = 1, \forall 1 \leq i \leq N, \gamma = 0$

$$\vec{X} = (x_1, x_2, ...x_n)^T, \vec{Y} = (y_1, y_2, ...y_n)^T, \vec{Y} = \vec{X}\exp(i(\Omega t - \delta)) \tag{8.15}$$

$$AY = \vec{0} \tag{8.16}$$

$$A = [a_{i,j}], a_{i,j} = \begin{cases} -1 & \text{if } |i-j| = 1 \\ -\Omega^2 + 2 & \text{if } i = j < N \\ -\Omega^2 + 1 & \text{if } i = j = N \\ 0 & \text{otherwise} \end{cases} \tag{8.17}$$

For general roots of Ω, A is a good tri-diagonal matrix with property about determinant [24].

For N oscillators, denote determinant of A as f_N, denote $z = \Omega^2$, we have

$$f_1 = -z + 1 \tag{8.18}$$

$$f_2 = z^2 - 3z + 1 \tag{8.19}$$

$$f_n = (-z + 2)f_{n-1} - f_{n-2} \tag{8.20}$$

$$\tag{8.21}$$

f_1 corresponds to the case of single SHO $x_1'' + kx_1 = 0$.

It is obvious that Equation 8.20 is a first order linear, constant coefficient difference equation, we know it has solution like:

$$f_n = c_1\alpha^{n-1} + c_2\beta^{n-1} \tag{8.22}$$

where α, β are roots

$$\begin{aligned}
&x^2 + (z-2)x + 1 = 0 \\
&f_1 = c_1 + c_2 \\
&f_2 = c_1\alpha + c_2\beta
\end{aligned} \tag{8.23}$$

We can have

$$\begin{aligned}
\alpha &= \frac{2 - z - \sqrt{z(4-z)}}{2}, \beta = \frac{2 - z + \sqrt{z(4-z)}}{2} \\
c_1 &= \frac{3z + \sqrt{z(z-4)} - z^2 - z\sqrt{z(z-4)}}{2\sqrt{z(z-4)}} \\
c_2 &= -\frac{3z - \sqrt{z(z-4)} - z^2 + z\sqrt{z(z-4)}}{2\sqrt{z(z-4)}}
\end{aligned} \tag{8.24}$$

Finally we have for N oscillators:

$$|A| = 0 \Leftrightarrow f_N = 0 \Leftrightarrow c_1\alpha^{N-1} + c_2\beta^{N-1} = 0 \tag{8.25}$$

We can solve Equation 8.25 for several N as shown in Table 2.

N	Ω
1	1
2	$\sqrt{\frac{3}{2} - \frac{\sqrt{5}}{2}}, \sqrt{\frac{\sqrt{5}}{2} + \frac{3}{2}}$
3	0.445041, 1.24697, 1.80193

Table 2: Normal mode frequencies Ω at different N

Results are consistent for N=1, one oscillator and N=2, the example in the lecture notes [23].

This messy expression to calculate z gives us a more interesting insights to N-coupled system.

If you notice, for N=1 (single oscillator) and N=4 in last subsection, $\Omega = 1$ is a normal mode frequency. After testing multiple N and rank(A), an interesting phenomenon is that $rank(A) < N$ if $N \bmod 3 = 1$.

Can we formally prove this? Yes, just use Equation 8.24 and plug in $\Omega = 1$ and use induction.

Base case is either N=1 or N=4, $f_4(\Omega = 1) = 0$

Assume for $k \in Z^+$, $f_{3k+1}(\Omega = 1) = 0 \rightarrow c_2\beta^{3k} = -c_2\alpha^{3k}$

For $k+1$ case,

$$\begin{aligned}
f_{3(k+1)+1}(\Omega = 1) &= c_1\alpha^{3k+3} + c_2\beta^{3k+3} = c_1\alpha^{3k}(\alpha^3 - \beta^3) = c_1\alpha^{3k}(\alpha - \beta)((\alpha+\beta)^2 - \alpha\beta) \\
&= c_1\alpha^{3k}(\alpha - \beta)(1^2 - 1) = 0
\end{aligned} \tag{8.26}$$

At last step, we plug in values of α, β when $\Omega = 1$

If $\Omega = 1, N \bmod 3 = 1$, we can find a non-trivial vector in Null(A):

$$\vec{S} = (s_i), 1 \le i \le N.i = 3k + r, s_i = \begin{cases} (-1)^k & \text{if } r \neq 0 \\ 0 & \text{if } r = 0 \end{cases}, A\vec{S} = \vec{0} \tag{8.27}$$

This tells us, if $N \bmod 3 = 1$ there is one possible common normal mode: all oscillators have either same non-zero amplitude or 0 amplitude (stationary) From the leftmost, oscillators are arranged in groups of three. In each group, rightmost third oscillator is stationary. Other two oscillators oscillate with the opposite phase to the oscillator next to stationary oscillator. Two non-stationary oscillators in the same group oscillate at the same phase. In the edge case, the rightmost oscillator oscillates alone with the opposite phase to $N-3$-th and $N-2$-th oscillators. An example is case 2 in Figure 8.2. In a viewpoint of physics, we can notice the string between two oscillators in the same group does not exert any force due to no stretching or compressing, the whole system are actually some oscillators oscillates around the closest stationary point if we consider leftmost rigid fixed wall is also a stationary point.

9 Data

9.1 FFT, Spectral Leakage, and Windowing

Time spent: 2hr

Assume we have time series signal $\{x_1, x_2, ..., x_N\}$ with time stamp $\{\Delta t, 2\Delta t, ...N\Delta t\}$ respectively (where $N\Delta t = T$). Because these signal values are finite and are detected in a finite time interval, we want to decompose the signal into sine functions by Fast Fourier Transform

$$x_k = \sum_{n=1}^{N} b_n \sin(\omega_n k \Delta t) \tag{9.1}$$

$$\omega_n = \frac{2\pi}{n\Delta t} \tag{9.2}$$

Then we can find the primary frequencies by the distribution of b_n to determine spectra of the signal.

In an ideal condition, we can exactly measure the signal in exact one or multiple periods. In other words, $x_1 = x_N$ but this is rare in practice. Then the signal is discontinuous and Fourier transform will have Gibbs phenomenon near discontinuous point which results in spectra distribution in a wide range which is called **Spectral Leakage**.

To solve this problem, we can use a windowing function,$w(t)$, which satisfies $w(0) = w(T) = 0$. Because of convolution theorem, $F(x(t) \cdot w(t)) = F(x(t)) \cdot F(w(t))$, in frequency domain, amplitude distribution of $x(t)$ is not shifted but only changed magnitude weighted by transform of windowing function. Meantime, we removed discontinuity to reduce spectral leakage.

For the simplest case, let us try FFT on the summation of two cosine functions. As shown in Figure 9.1, if we set both cosine function have period which divides the time interval, the original function does not have severe spectral leakage compared with windowed function. It is even more concentrated near the expected peaks, 1.05 and 2.09. However, as shown in Figure 9.2, even for the simplest cos functions if sampling time is not exact an integer multiple of periods, spectra leakage can happen and spectra in original function is significantly wider than exact distribution and windowed function. For frequencies between two peaks, original function spectra has obvious non-zero distributions.

In conclusion, we find the discontinuity $x(0) \neq x(T)$ accounts for a significant portion of spectra leakage and windowed function can effectively reduce the leakage.

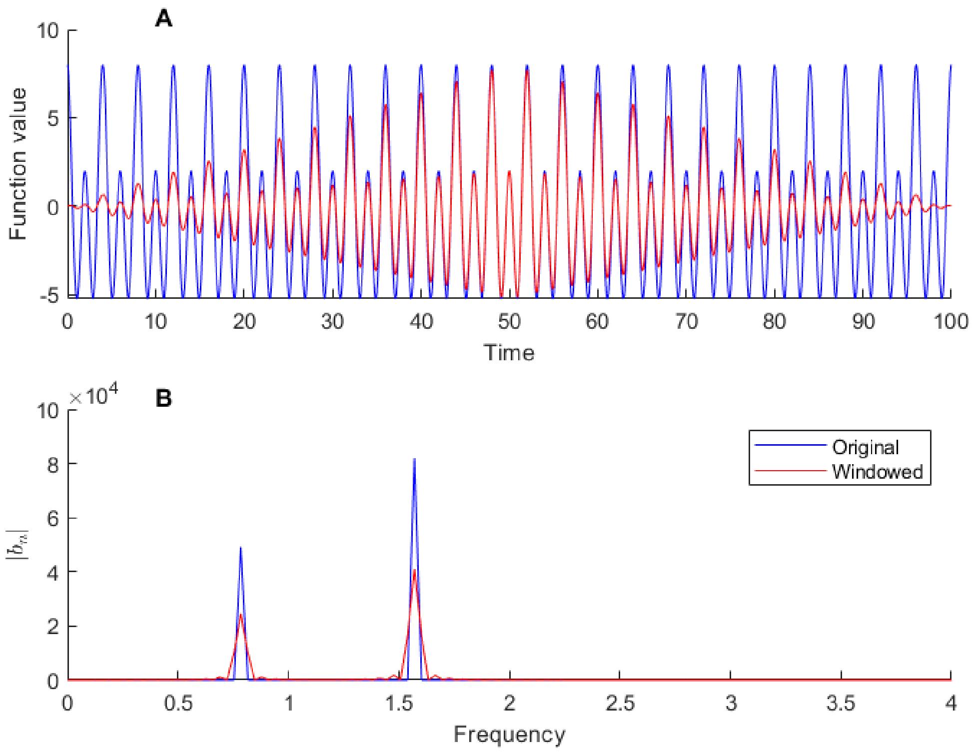


Figure 9.1: Original function=$3\cos\frac{\pi}{2}t + 5\cos\pi t$. Windowing function=$1 - \frac{|t-50|}{50}$. A: Function values respect to time. B: Absolute value of amplitude $|b_n|$ distribution respect to frequency.

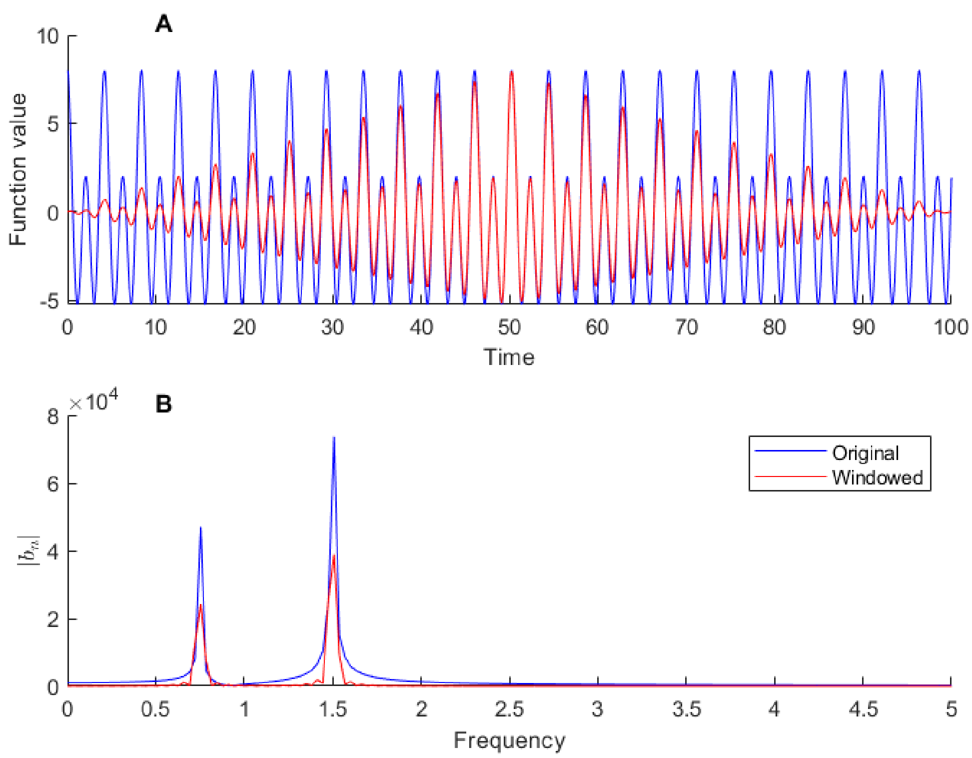


Figure 9.2: Original function=$3\cos 1.5t + 5\cos 3t$. Windowing function=$1 - \frac{|t-50|}{50}$. A: Function values respect to time. B: Absolute value of amplitude $|b_n|$ distribution respect to frequency.

9.2 Empirical Orthogonal Functions

Time spent: 2hr

In practice, we want to capture behavior of a system with various noise. We want to find a way to decompose the system and find the primary component with largest variance. Therefore we introduce Empirical Orthogonal Functions (EOF).

EOF refers to: For N time spots we have M time series $f_i^{(1))}, f_i^{(2)}, ..., f_i^{(M)}$ where $i = 1, 2, ..., N$. For covariance matrix:

$$C = \begin{bmatrix} var(f^{(1)}) & cov(f^{(1)}, f^{(2)}) & ... & cov(f^{(1)}, f^{(M)}) \\ ... & var(f^{(2)}) & ... & cov(f^{(2)}, f^{(M)}) \\ ... & ... & ... & ... \\ cov(f^{(M)}, f^{(1)}) & ... & cov(f^{(M)}, f^{(m-1)}) & var(f^{(M)}) \end{bmatrix} \tag{9.3}$$

Notice for data matrix subtracts mean value:

$$X = \begin{bmatrix} f_1^{(1)} - mean(f^{(1)}) & ... & f_N^{(1)} - mean(f^{(1)}) \\ ... & ... & ... \\ f_1^{(M)} - mean(f^{(M)}) & ... & f_N^{(M)} - mean(f^{(M)}) \end{bmatrix}, mean(f^{(k)}) = \frac{1}{N}\sum_{i=1}^{N} f_i^{(k)} \tag{9.4}$$

We have:

$$C = \frac{1}{N-1} X^T X \tag{9.5}$$

By SVD, we know

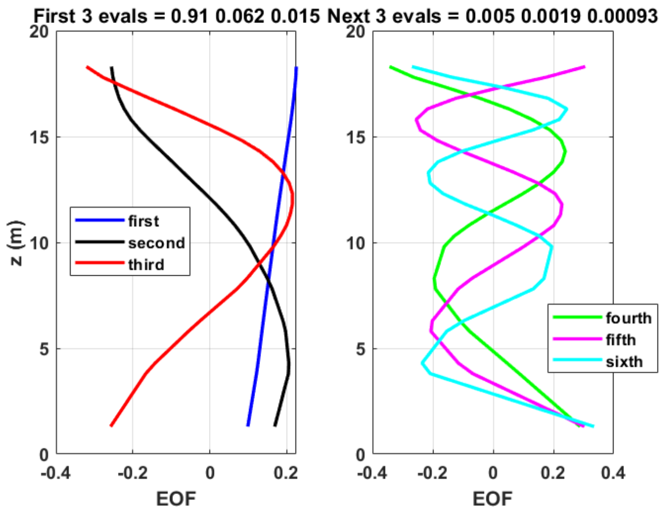
$$X = u\Sigma v^T, \text{where V,U are unitary, } \Sigma \text{ is diagonal.} \tag{9.6}$$

Therefore we have $X^T X = V\Sigma^T \Sigma V^T$ and columns of V is eigenvectors of $X^T X$. We can find the component by order of variance magnitude through multiplication $X^t V$. Here columns of V are sorted from the largest corresponding eigenvalues of $X^T X$ to smallest.

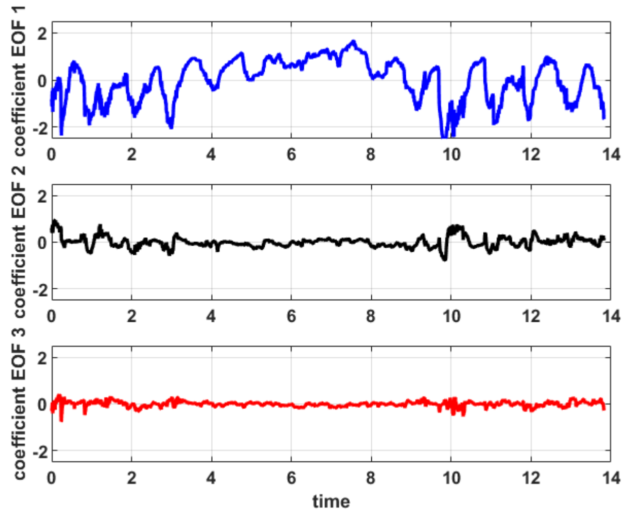
For our original Monterey Bay dataset, the baseline EOF results are given in Figure 9.3, the first EOF has dominantly larger eigenvalue and significantly larger variance compared with other EOFs. We want to know: if we add an arbitrary known turbulence to the dataset, can EOF method successfully distinguish it from the original data? Will EOF method detect the noise and separate it as a new EOF or the noise will only change existing EOFs? For simplicity, for all dataset entry, we add a normal distribution noise

$$noise = \sigma \cdot N(0, 1) \tag{9.7}$$

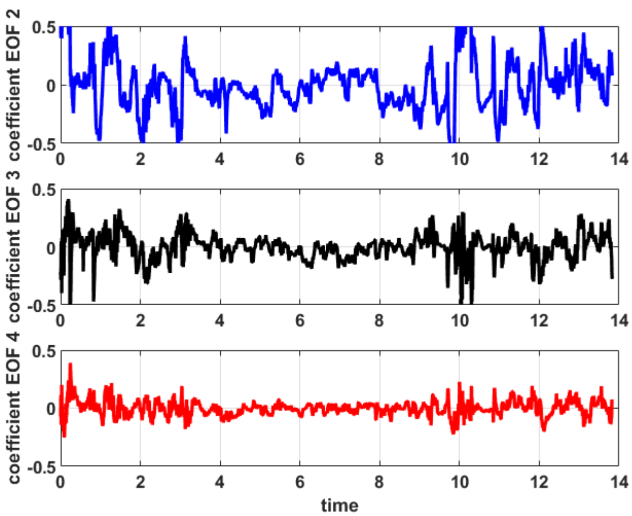
where $N(0, 1)$ is standard normal distribution. Then we explore the effects of noise variance. As shown in Figure 9.5, weak noise $\sigma = 0.1$ brings more variance to all EOFs and the principal EOF has significantly smaller eigenvalue. However, principal EOF still has obviously stronger variance than any other EOFS. As shown in Figure 9.4, if noise has high variance $\sigma = 5$, first to fourth dominant EOFs all lose their original shapes. As shown in Figure 9.4(d), in scree plot, all EOFs are not distinguishable with almost identical eigenvalues. This reveals the nature of EOF method is to sort data by variance. Though EOF method is robust with weak external noise, EOF itself is unable to separate strong noise as an individual EOF. Therefore, EOF is particularly effective to process dataset with relatively weak external noise like oceanology and climate change data.



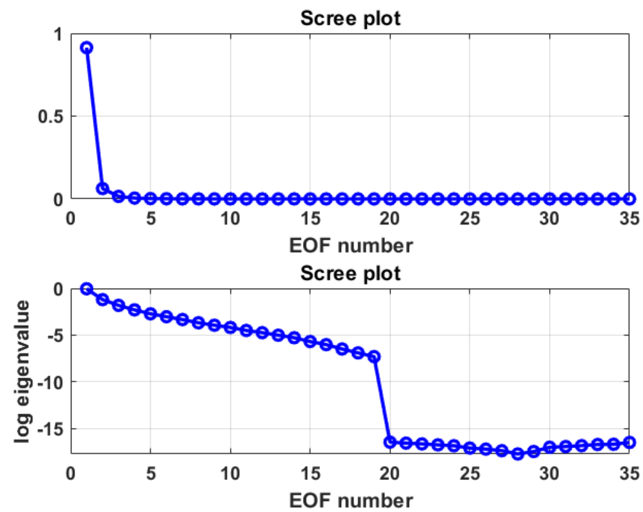
(a) EOF and z



(b) First-third EOFs vs time

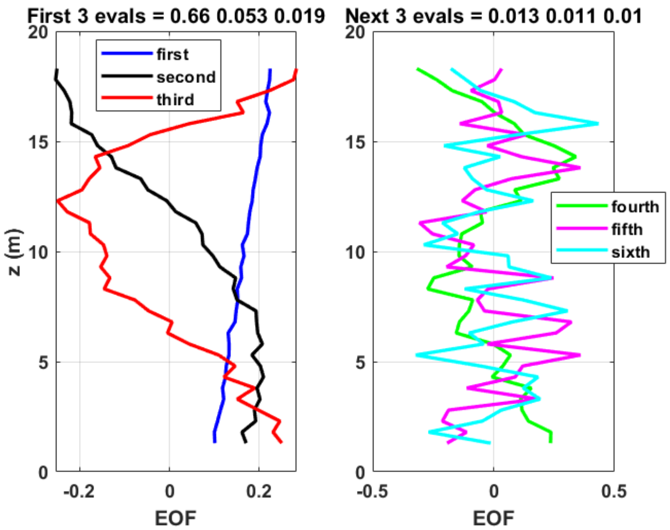


(c) Second-fourth EOFs vs time

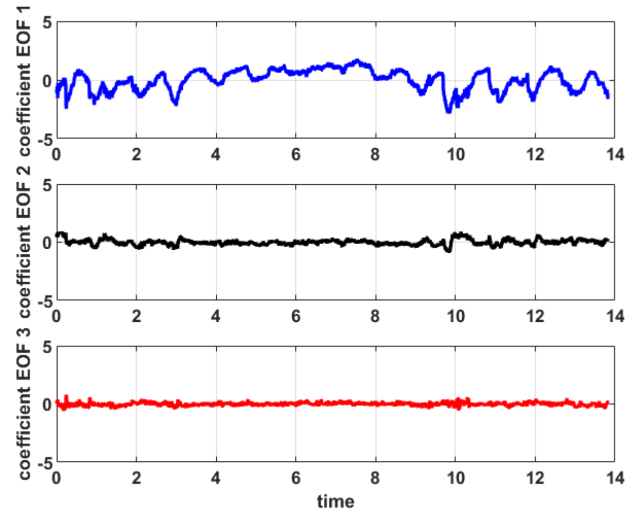


(d) Original scree plot

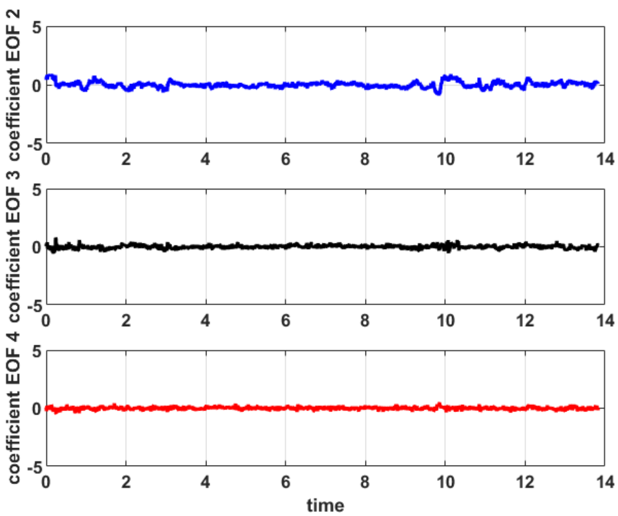
Figure 9.3: EOFs of original dataset



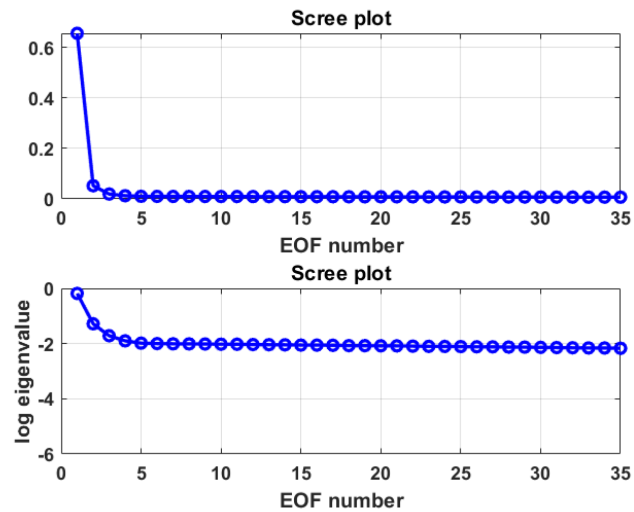
(a) EOF and z



(b) First-third EOFs vs time

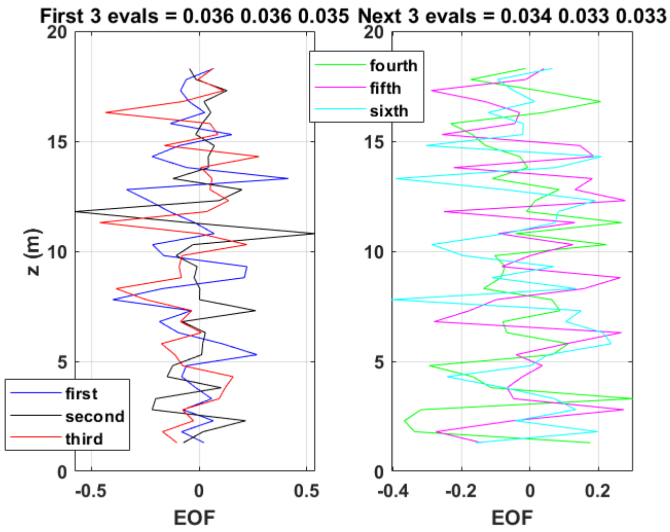


(c) Second-fourth EOFs vs time

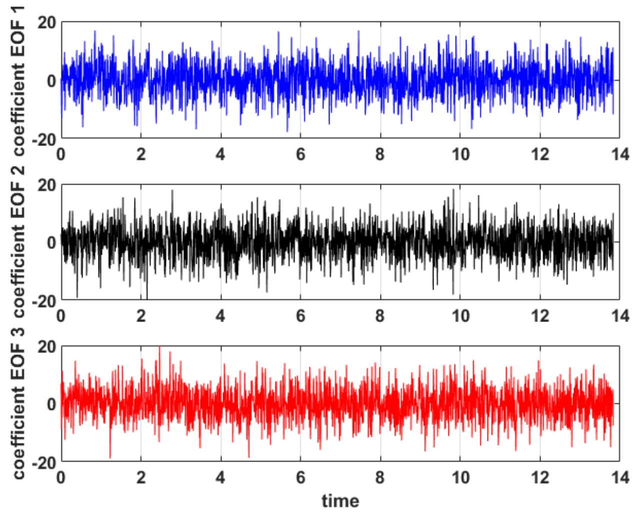


(d) Original scree plot

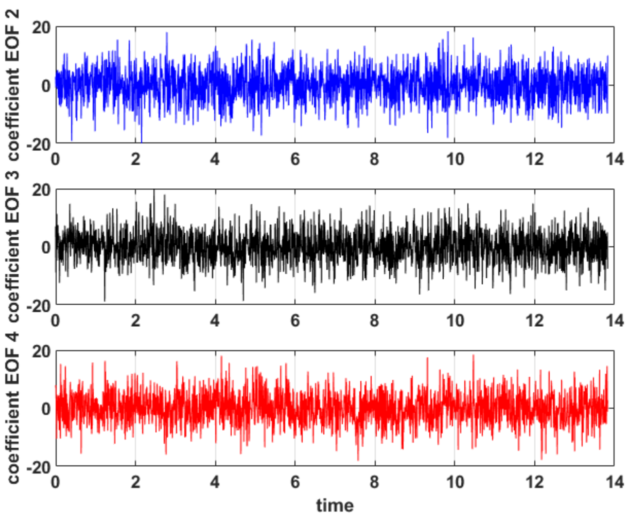
Figure 9.4: EOFs of dataset with weak $noise = 0.1 \cdot N(0, 1)$



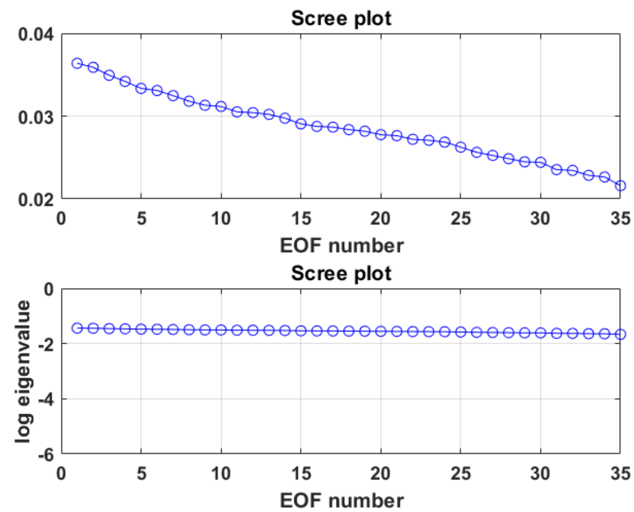
(a) EOF and z



(b) First-third EOFs vs time



(c) Second-fourth EOFs vs time



(d) Original scree plot

Figure 9.5: EOFs of dataset with strong $noise = 5 \cdot N(0, 1)$

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Appendix: Simulation Implementations

1.2: 2D SHO: Case 2 Simulation

The implementation below simulates Figure 1.3a.

```
omega=2;
r0=1;
delta_r_0=0.01;
delta_r_v_0=0.01;
theta_v_0=0.01;
C1=(1+delta_r_0)^2*theta_v_0;

%initial condition
IC=[r0+delta_r_0,delta_r_v_0,0];
timeSpan = [0, 10];

[t, y] = ode45(@(t, y) exact_gov(t, y, omega, r0, C1), timeSpan, IC);

approx_sol=zeros(length(t),2);
for i = 1:length(t)
    approx_sol(i,:)= approx(t(i), omega, r0, delta_r_0,...
        delta_r_v_0,theta_v_0);
end

R0=r0*ones(length(t),1);

% Plot the results
figure;
subplot(2, 1, 1);
hold on
plot(t,y(:,1)-R0(:,1),'LineWidth',2,'LineStyle','--')
plot(t, approx_sol(:, 1), 'LineWidth', 1);
xlabel('t');
ylabel('\delta r(t)');
hold off
legend('exact solution','approximation')

subplot(2, 1, 2);
hold on
```

```
plot(t,y(:,3),'LineWidth',2,'LineStyle','--')
plot(t, approx_sol(:, 2), 'LineWidth', 1);
xlabel('t');
ylabel('\theta (t)');
hold off
legend('exact solution','approximation')

%exact solution governing equations,y=[r,r',theta]
function dydt=exact_gov(t,y, omega, r0, C1)
dydt=zeros(3,1);
dydt(1)=y(2);
dydt(2)=-omega^2*(y(1)-r0)+C1^2/(y(1)^3);
dydt(3)=C1/(y(1)^2);
end


%approximation solution,sol=[delta r,theta]
function sol=approx(t,omega,r0,delta_r_0,delta_r_v_0,theta_v_0)
sol=zeros(2,1);
big_omega=(omega^2+3*(theta_v_0^2))^0.5;
big_delta_r=(theta_v_0^2)*r0*(1+4*delta_r_0/r0)/big_omega;
%delta r
sol(1)=(delta_r_0-big_delta_r)*cos(big_omega*t)+(delta_r_v_0/big_omega)...
    *sin(big_omega*t)+big_delta_r;
%theta
sol(2)=theta_v_0*(1+2*delta_r_0/r0)*((1-2*big_delta_r/r0)*t-...
    2*(delta_r_0-big_delta_r)*sin(big_omega*t)/(big_omega*r0)...
    -2*delta_r_v_0*(1-cos(big_omega*t))/(r0*big_omega^2));
end
```

3.2: Differentiating drug-resistance

This code is to generate Figure 3.2.

```
import numpy as np
import matplotlib.pyplot as plt

from scipy.integrate import odeint

plt.rcParams.update({'font.size': 12})

S0=0.09
R0=0.01
Vd=0.1
Vc=0.9
epsilon=1e-6
pr=0.2
d=1
u0=1.5
#initial condition and time grid
```

```
y0=np.array([S0,R0])
Tend=50
t= np.linspace(0, Tend, 1000)

def var_u(t):
    return max(0,u0*(0.2+0.8*np.sin(2*np.pi*t/10)))
#------------------------------
alpha=0
def constant1(y,t):
    def u(t):
        return u0
    S=y[0]
    R=y[1]
    dSdt=(1-(S+R))*S-(epsilon+alpha*u(t))*S-d*u(t)*S
    dRdt=pr*(1-(S+R))*R+(epsilon+alpha*u(t))*S
    return [dSdt,dRdt]

def sin1(y,t):
    def u(t):
        return var_u(t)
    S=y[0]
    R=y[1]
    dSdt=(1-(S+R))*S-(epsilon+alpha*u(t))*S-d*u(t)*S
    dRdt=pr*(1-(S+R))*R+(epsilon+alpha*u(t))*S
    return [dSdt,dRdt]

y1= odeint(constant1, y0, t)
y2= odeint(sin1, y0, t)
#---------------------
alpha=1e-2
def constant2(y,t):
    def u(t):
        return u0
    S=y[0]
    R=y[1]
    dSdt=(1-(S+R))*S-(epsilon+alpha*u(t))*S-d*u(t)*S
    dRdt=pr*(1-(S+R))*R+(epsilon+alpha*u(t))*S
    return [dSdt,dRdt]

def sin2(y,t):
    def u(t):
        return var_u(t)
    S=y[0]
    R=y[1]
    dSdt=(1-(S+R))*S-(epsilon+alpha*u(t))*S-d*u(t)*S
    dRdt=pr*(1-(S+R))*R+(epsilon+alpha*u(t))*S
    return [dSdt,dRdt]

y3= odeint(constant1, y0, t)
y4= odeint(sin1, y0, t)
```

```
#-----------------------

fig, axs = plt.subplots(2, 2, figsize=(10, 10))
for i in range(2):
    for j in range(2):
        if i==0:
            letter='A'
        elif i==1:
            letter='B'
        axs[j,i].text(0.05, 1.05,letter+str(j+1),transform=axs[j,i].transAxes
                     , fontweight='bold', va='top')
axs[0,0].plot(t,u0*np.ones(len(t)),label='constant',color='blue')
axs[0,0].plot(t,[var_u(t[i]) for i in range(len(t))],label='oscillation',
              color='green')
axs[0,0].set_xlim([0,Tend])
axs[0,0].set_ylim([0,u0+0.4])
axs[0,0].set_xlabel('time t')
axs[0,0].set_ylabel('Effective drug dosage, u(t)')
axs[0,0].set_title(r'$\alpha=0,T=10$')
axs[0,0].legend(loc='upper right')

axs[0,1].plot(t,y1[:,0]+y1[:,1],label='constant',color='blue')
axs[0,1].plot(t,y2[:,0]+y2[:,1],label='oscillation',color='green')
axs[0,1].legend(loc='lower right')
axs[0,1].axhline(0.9,color='red', linestyle='--')
axs[0,1].text(-1, 0.9, str(0.9), color='red', ha='right')
axs[0,1].set_xlim([0,Tend])
axs[0,1].set_ylim([0,1])
axs[0,1].set_xlabel('time t')
axs[0,1].set_ylabel('tumor volume V=S+R')
axs[0,1].set_title(r'$\alpha=0,T=10$')

axs[1,0].plot(t,u0*np.ones(len(t)),label='constant',color='blue')
axs[1,0].plot(t,[var_u(t[i]) for i in range(len(t))],label='oscillation',
              color='green')
axs[1,0].set_xlim([0,Tend])
axs[1,0].set_ylim([0,u0+0.4])
axs[1,0].set_xlabel('time t')
axs[1,0].set_ylabel('Effective drug dosage, u(t)')
axs[1,0].set_title(r'$\alpha=10^{-2},T=10$')
axs[1,0].legend(loc='upper right')

axs[1,1].plot(t,y3[:,0]+y3[:,1],label='constant',color='blue')
axs[1,1].plot(t,y4[:,0]+y4[:,1],label='oscillation',color='green')
axs[1,1].legend(loc='lower right')
axs[1,1].axhline(0.9,color='red', linestyle='--')
axs[1,1].text(-1, 0.9, str(0.9), color='red', ha='right')
axs[1,1].set_xlim([0,Tend])
axs[1,1].set_ylim([0,1])
axs[1,1].set_xlabel('time t')
```

```
axs[1,1].set_ylabel('tumor volume V=S+R')
axs[1,1].set_title(r'$\alpha=10^{-2},T=10$')

plt.tight_layout()
```

5.1: Comparison of exact and RPT solutions

This code is to generate Figure 5.1.

```
import numpy as np
import matplotlib.pyplot as plt

t_lin=np.linspace(0, 10,1000)
def  exact(t,epsilon):
    return (1+2*epsilon)/((1+2*epsilon)*t+1)
def est(t,epsilon):
    return 1/(1+t)+2*epsilon/np.power(1+t,2)
for i in range(3):
    epsilon=0.01*np.power(10,i)
    plt.plot(t_lin,exact(t_lin,epsilon),label='epsilon='+str(epsilon)+
             ',exact',linestyle='--',linewidth=3)
    plt.plot(t_lin,est(t_lin,epsilon),label='epsilon='+str(epsilon)+',RPT')

plt.legend()
plt.xlim([0,10])
plt.xlabel(r'$\tau$')
plt.ylabel(r'V($\tau$)')
```

7.2 Non-linear Damping Resonance

This code is to generate Figure 7.1.

```
num_pts=300;
omega_lst=linspace(0.5,1.5,num_pts);
gamma_lst=[0.01,0.1,0.2,0.3];
hold on
for j=1:length(gamma_lst)
    amp_lst=zeros(100,1);
    gamma=gamma_lst(j);
for i=1:num_pts
omega=omega_lst(i);
amp_lst(i)=amplitude(gamma,omega);
end
plot(omega_lst,amp_lst)
end
ylabel('Amplitude')
xlabel('\Omega')
legend('\gamma=0.01','\gamma=0.1','\gamma=0.2','\gamma=0.3')

function dy=myode(t,y,para)
```

```
gamma=para(1);
omega=para(2);
dy=zeros(2,1);
dy(1)=y(2);
dy(2)=-gamma*y(2)^3-y(1)+cos(omega*t);
end

function max_var=amplitude(gamma, omega)
t_span=[0 1000];
IC=[0,0];
params=[gamma,omega];
[t,y]=ode45(@(t,y) myode(t, y, params),t_span,IC);
max_var=0;
max_length=length(t);
for i=floor(0.9*max_length):max_length
if y(i,1)>max_var
    max_var=y(i,1);
end
end

end
```