The weak inverse mean curvature flow: existence theories and applications

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Preface

The role of this note. This is a self-formatted version of my PhD thesis at Duke University, 2025. There are two main purposes for posting this to my website: (i) I hope it to be a helpful introductory material for the weak inverse mean curvature flow, and (ii) the reader may find the official version of this thesis difficult to read (due to many of the formatting requirements from my graduate school).

The reader is welcomed to contact me for any comments or suggestions (especially if you find that some part of the text can be made more self-contained).

How to read it / source of materials. For a summary of the recent developments of weak IMCF, see Chapter 1 (especially from the end of Section 1.2 to Section 1.5). For the fundamentals of the weak IMCF, see Chapter 2 (particularly from Sections 2.1 to 2.4; they should mostly cover the preliminary sections of [113, 114]).

Chapter 1 and Sections 2.1 ∼ 2.4 are newly written. Chapter 3 (except Section 3.1) is basically a copy-paste of my paper [114] with minor edits and non-mathematical improvements. The other sections come from organizing fractions in the papers [14, 30, 113, 112] with some additional editing and minor improvements.

Some other useful information can also be found in page 8.

I would like to thank Gioacchino Antonelli and Mark Haskins for comments on the previous draft of this note. There is an error in the official acknowledgements: my first talk invitation was from Peter McGrath in North Carolina State University.

Contents

List of Figures

Chapter 1

Introduction

The inverse mean curvature flow (IMCF) is, in its classical setting, an evolution of hypersurfaces where the speed of movement is everywhere equal to 1/(mean curvature). It is formally written as

$$\frac{\partial \Sigma_t}{\partial t} = \frac{\nu_t}{H_t}, \tag{1.0.1}$$

where $\{\Sigma_t\}$ is the family of evolving hypersurfaces, ν_t is the unit normal of Σ_t, and H_t is the mean curvature of Σ_t. The IMCF is a nonlinear parabolic flow as long as $H_t > 0$.

The analytic study of IMCF started with Gerhardt [43] and Urbas [106] in the 1990s, where the IMCF was put inside a large class of inverse curvature type flows. The long-time existence and convergence results in [43, 106] had applications to Minkowski type inequalities [22, 49]. Another thread of study was initiated by Geroch [44] in the much earlier 1973, where a relation among IMCF, scalar curvature, and the Riemannian Penrose inequality, known as the monotonicity of Hawking mass, was discovered. This showed the potential of the use of IMCF in studying scalar curvature. Later, since the seminal work of Huisken-Ilmanen [53] which defined the weak flow, and followed by many other works [12, 19, 20, 21, 29, 55, 70, 93, 101], the weak IMCF has become a central tool in 3-dimensional scalar curvature geometry. In 2007, the work of Moser [89] connected the weak IMCF to the $p \to 1$ behavior of p-harmonic functions, and thus opened a new branch of research relating the IMCF to nonlinear potential theory [1, 6, 13, 42, 65, 79, 90, 91].

Analytically, there are two main issues obstructing the long-time existence of IMCF – finite-time singularity and finite-time escape. For the first issue, the hypersurface may have zero mean curvature somewhere in some finite time, which poses a serious problem in continuing the flow (since the speed of movement is infinite). The resolution of this issue is by considering the weak IMCF, as developed by Huisken-Ilmanen [53]. For the second issue, the evolving hypersurface may become noncompact or diverge to infinity within finite time, noting that the IMCF is an outward expanding flow. The escaping phenomenon affects the unique existence of solutions, as parabolic equations alone on noncompact domains are usually ill-posed. In this thesis, we propose the innermost solution as a resolution – namely, an innermost solution is the one that expands the slowest among all possible flows with the same initial data. Our main theorem will show the unique existence of such an object. As a result, we obtain (to some extent) a full existence theory to the initial value problem of IMCF.

Along this line, we also address the IMCF in bounded domains. It was previously unknown what boundary condition leads to the existence of nontrivial solutions. In this thesis, we propose an "outer obstacle condition" at the boundary, and prove a unique existence theorem for the corresponding initial value problem. Also, it turns out that the

IMCF with outer obstacle and the innermost IMCF introduced previously are the same object.

We also study when a manifold admits "non-escaping" solutions of IMCF, namely, solutions where all hypersurfaces enclose a bounded region. We prove a criterion that relates non-escaping with the isoperimetric inequality of the ambient manifold.

Having studied the existence theory, we then provide two applications of IMCF to scalar curvature. They make use of the Geroch monotonicity formula as well as the innermost solutions mentioned above.

Organization

The rest of this chapter is a more detailed and technical explanation of the content of this thesis. It is also intended to be a comprehensive introduction to the weak IMCF, where I try to incorporate my own perspective and the new developments into this subject's framework. The main results of this thesis are marked as Theorems A ∼ G.

In Chapter 2 we present more preliminary materials, and prove Theorems A, B. In Chapter 3 we introduce the IMCF with outer obstacle, and prove Theorems D, E. In Chapter 4 we discuss innermost solutions, and prove Theorem C. In Chapter 5 we discuss applications to scalar curvature problems, and prove Theorems F, G. Appendix A includes materials about sets with locally finite perimeters.

Finally, in Appendix B we include a list of standard notations, sign conventions, frequently used symbols and function spaces, for the reader's reference.

Source of materials

The majority of this thesis come from my works [112, 113, 114]. In particular: Theorems C, D, and Sections 2.1 ∼ 2.4, Chapter 3, Sections 4.1 ∼ 4.2 are extracted from [114]. Theorem A and Section 2.5, A.3, A.4 are extracted from [113]. Theorem E and Sections 5.1, 5.2 are extracted from [112].

Furthermore, Theorem B, E and Sections 2.6, 3.7 are to appear in the joint note [14] with L. Benatti, L. Mari, M. Rigoli and A. Setti. Theorem F and Sections 4.3, 5.3 are based on joint work [30] with O. Chodosh and Y. Lai.

1.1 The smooth IMCF

Definitions, first properties. A smooth family of hypersurfaces $\{\Sigma_t\}$ solves the IMCF if it satisfies the equation

$$\frac{\partial \Sigma_t}{\partial t} = \frac{\nu_t}{H_t}, \tag{1.1.1}$$

where ν_t and H_t denote the unit normal and mean curvature of Σ_t. In this thesis we adopt the following sign convention: if Σ is the boundary of a given domain, then its unit normal always points outward. Also, the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ has mean curvature $n-1$. We say that $\{\Sigma_t\}_{0\leqslant t<T}$ is a solution of IMCF with initial value Σ, if $\Sigma_0 = \Sigma$.

As a parabolic evolution equation, the IMCF is guaranteed short-time existence if Σ_0 is compact with positive mean curvature [5, Theorem 18.18]. Furthermore, the flow can always be continued as long as the mean curvature remains positive everywhere [54].

The simplest example of IMCF is the following:

Example 1.1.1. The family $\Sigma_t = \left\{|x| = e^{\frac{t}{n-1}}\right\} \subset \mathbb{R}^n$, $t \in \mathbb{R}$, is a solution of the IMCF.

In general, the IMCF in spherical symmetry is explicitly computable:

Example 1.1.2. Consider the warped product metric $\left(\mathbb{R} \times N, h(r)^2 dr^2 + f(r)^2 g_N\right)$, with $h(r) > 0$ and $f(r)$ strictly increasing in an open interval. Then the hypersurfaces

$$\Sigma_t = \left\{r = f^{-1}\left(e^{\frac{t}{n-1}} f(a)\right)\right\}, \qquad (n-1)\log\left(\frac{\inf f}{f(a)}\right) < t < (n-1)\log\left(\frac{\sup f}{f(a)}\right)$$

is a solution of the IMCF with $\Sigma_0 = \{r = a\}$. Indeed, setting $r_t = f^{-1}\left(e^{t/(n-1)} f(a)\right)$, we may compute

$$\frac{\partial \Sigma_t}{\partial t} = \frac{1}{n-1} e^{\frac{t}{n-1}} f(a) \frac{1}{f'(r_t)} \frac{\partial}{\partial r},$$

and

$$H_t = (n-1)\frac{1}{h(r_t)}\frac{f'(r_t)}{f(r_t)}, \qquad \nu_t = \frac{1}{h(r_t)}\frac{\partial}{\partial r}.$$

Hence $\partial \Sigma_t / \partial t = \nu_t / H_t$.

An important property of IMCF is the exponential growth of area:

Lemma 1.1.3. *If $\{\Sigma_t\}$ is a family of closed hypersurfaces that solve the IMCF, then*

$$|\Sigma_t| = e^{t-s}|\Sigma_s| \qquad \forall\, s, t.$$

Proof. By the first variation of area, we have

$$\frac{d}{dt}|\Sigma_t| = \int_{\Sigma_t} H \cdot \frac{1}{H} = |\Sigma_t|. \qquad \square$$

The evolution of mean curvature in an IMCF is computed as [53, (1.3)]:

$$\frac{\partial H}{\partial t} = \frac{\Delta_\Sigma H}{H^2} - 2\frac{|\nabla_\Sigma H|^2}{H^3} - \frac{|A|^2}{H} - \frac{\mathrm{Ric}(\nu, \nu)}{H}, \tag{1.1.2}$$

where $\nabla_\Sigma, \Delta_\Sigma$ are the covariant derivative and Laplacian on Σ_t. The term $-|A|^2/H$ plays the dominant role in this equation, and as a result, the mean curvature will never blow up to $+\infty$ along an IMCF. More specifically, if $\{\Sigma_t\}_{t \geqslant 0}$ is an IMCF, then we have an interior estimate of the type

$$\sup_{\Sigma_t \cap K} H \leqslant \sup_{\Sigma_0} H + C(K), \qquad \forall \text{ precompact } K. \tag{1.1.3}$$

See Theorem 2.4.3 for a precise version. The mean curvature estimate in turn implies a lower estimate on the speed of evolution; thus, a long-time existing IMCF will eventually sweep out the entire manifold. On the other hand, it is a highly nontrivial task to bound H away from zero. Such control usually requires certain star-shapedness of the evolving hypersurfaces [31, 43, 51, 54, 106]. In $\mathbb{R}^n$, star-shapedness often comes as an assumption. In general Riemannian manifolds, usually one does not have lower bounds of H.

Let $\Sigma_0 \subset \mathbb{R}^n$ be a closed, star-shaped, strictly mean-convex hypersurface (i.e. $H > 0$). The classical theorem of Gerhardt [43] and Urbas [106] states that starting from Σ_0, the smooth IMCF exists for all time $t \geqslant 0$. Furthermore, as $t \to \infty$, the rescaled hypersurfaces converge to a round sphere. Later, Huisken-Ilmanen [54] improved this result by allowing Σ_0 to be only star-shaped and mean-convex (i.e. $H \geqslant 0$). Here note that the flow may have infinite speed somewhere at $t = 0$. Nevertheless, it is shown [54] (see also [31, Appendix A]) that star-shapedness has strong enough effect so that the hypersurface becomes strictly mean-convex for any $t > 0$. As a result, the IMCF exists in the sense that the hypersurfaces move smoothly by $1/H$ for all $t > 0$ and C^0-converge to the initial condition when $t \to 0$.

In this thesis, we will mostly not use the parabolic techniques in these works.

Scaling of IMCF. We would like to emphasize the scaling property of IMCF. Here, for a set X and constant λ, we denote $\lambda X = \{\lambda x : x \in X\}$.

Fact 1.1.4. If $\{\Sigma_t\}$ solves the IMCF inside $\mathbb{R}^n$, then for any constant $\lambda > 0$, the hypersurfaces $\{\lambda\Sigma_t\}$ solves the IMCF as well.

Proof. The mean curvature of $\lambda\Sigma_t$ is λ^{-1} times the mean curvature of Σ_t, while the evolution speed of $\{\lambda\Sigma_t\}$ is λ times the evolution speed of $\{\Sigma_t\}$. □

Speaking in terms of units, this says that "time is unitless in IMCF" [53, p.362]. As a consequence, IMCF preserves dilation symmetry. Notice that cones are special objects as they are invariant under dilations. Hence, the IMCF may flow cones to cones.

Example 1.1.5 (the blossoming cones, [52])**.** For all $n \geqslant 3$, the cones

$$\Sigma_t = \big\{x_n = -|x| \cos\theta_t\big\}, \qquad \theta_t = \arcsin\left(e^{t/(n-2)}\right), \quad t \in (-\infty, 0)$$

form a solution of IMCF away from the common vertex. This is an ancient solution: Σ_t converges to the vertical half-line as $t \to -\infty$, and converges to $\{x_n = 0\}$ as $t \to 0$.

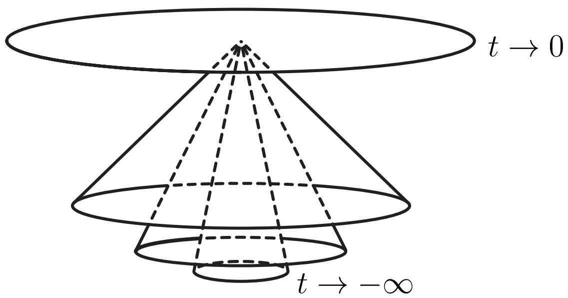

Figure 1.1: The blossoming cones

In general, we have:

Example 1.1.6. If $\{\Gamma_t\}$ solves the IMCF in $\mathbb{S}^{n-1}$, then their cones $C_t := \big\{cx : c > 0, x \in \Gamma_t\big\}$ solve the IMCF in $\mathbb{R}^n \setminus \{0\}$.

Indeed, let $x \in \Gamma_t$ and $c > 0$, thus $cx \in C_t$. Let $H_\Gamma(x)$ be the mean curvature of Γ_t at x. Then we note that: the mean curvature of C_t at cx is $c^{-1}H_\Gamma(x)$, while the flow speed of C_t at cx is $c \cdot$ (flow speed of Γ_t at x), which is exactly $cH_\Gamma(x)^{-1}$.

The existence of cone solutions implies that conical singularities are not immediately resolved by the IMCF (unlike mean curvature flow). On the other hand, cone solutions model the asymptotic behavior of solutions at infinity or near a cone point. Along this line, we briefly mention the following results:

1. Choi-Daskalopoulos [31] studied the IMCF starting from the boundary of a non-compact convex smooth domain $E_0 \subset \mathbb{R}^n$. Let C_0 be the asymptotic cone of ∂E_0 at infinity, and $\Gamma_0 \subset \mathbb{S}^{n-1}$ be the link of C_0. Let us assume for simplicity that Γ_0 is smooth and strictly convex. It is shown in [31] that as we evolve ∂E_0 through IMCF, their asymptotic cones C_t also evolve by IMCF. Hence, the links $\Gamma_t = C_t \cap \mathbb{S}^{n-1}$ evolve by IMCF as well. By Makowski-Scheuer [78], the IMCF in spheres exists until the hypersurface becomes an equator. So by the exponential growth of area, the maximal existence time of $\{\Gamma_t\}$ is equal to
$$T = \log \frac{|\mathbb{S}^{n-2}|}{|\Gamma_0|}.$$
Then by the correspondence of asymptotic cones, it is natural to see that the IMCF from ∂E_0 exists until the same time T (see [31] for a full proof).
2. Choi-Hung [32] studied the IMCF starting from the boundary of a compact convex set $E_0 \subset \mathbb{R}^n$ with cone singularities (for example, a cube). In this case, the conical singularity persists for a positive amount of time. Denoting by Γ_x the link of the tangent cone of a vertex x, the singularity at x persists for time $\log\big(|\mathbb{S}^{n-2}|/|\Gamma_x|\big)$.

The blow-up technique will be employed in Chapter 3, in showing the existence of solutions for IMCF with outer obstacle.

Obstructions to long-time existence. There are two geometric / analytic issues in the long-time existence of IMCF.

The first issue is the occurrence of finite-time singularities. As mentioned above, there is always a good estimate of H from above, but there is no general control of H away from zero, so issues do occur on the $H = 0$ side. The following examples are typical and are helpful to keep in mind.

Example 1.1.7. (i) Let M be a spherically symmetric manifold that expands then closes then expands again (Figure 1.2, left), and Σ_0 be a small geodesic ball centered at the tip. Then the IMCF exists until reaching the unstable minimal surface Σ_T, there $H \equiv 0$ and one cannot continue the flow. This example will re-appear several times later.

We generally observe that: once there exists a compact minimal surface S outside Σ_0, it is impossible to have long-time existence of IMCF. Otherwise, Σ_t would sweep out the manifold, so there will be a time t where S touches Σ_t from inside. This contradicts the maximum principle.

(ii) [53, Example 1.5] Let $M = \mathbb{R}^n$ and Σ_0 be the disjoint union of two spheres. Then the IMCF exists until the two components collide. Here we do not view "immersed solutions" as solutions, so at the colliding time, Σ_T becomes a singular hypersurface.

The second issue is finite-time escaping. As the IMCF is an outward expanding flow, there is a natural concern that Σ_t may (partially or entirely) diverge to ∞ in finite time.

Example 1.1.8. In Example 1.1.2, we take $h(r) \equiv 1$ and $f(r) = \tanh(r)$. Note that $f(+\infty) = 1$, so the manifold is asymptotically cylindrical. The smooth IMCF starting from $\{r = 1\}$ is
$$\Sigma_t = \Big\{r = \operatorname{arctanh}\big(e^{t/(n-1)}\tanh(1)\big)\Big\},$$
which escapes to $r = +\infty$ within $-(n-1)\log\tanh(1)$ seconds.

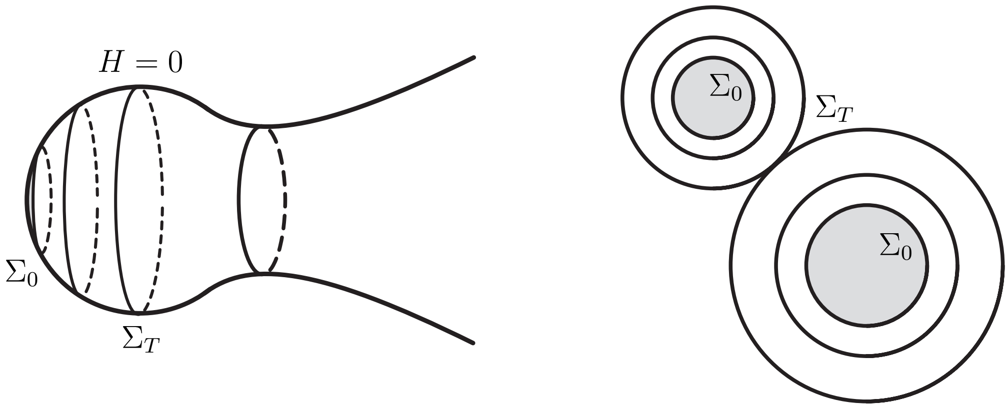

Figure 1.2: Two examples of finite-time singularities

Recall that the area grows exponentially in an IMCF. If the manifold has a “finite circumference” at infinity, then heuristically, the IMCF diverges when its area becomes larger than this circumference.

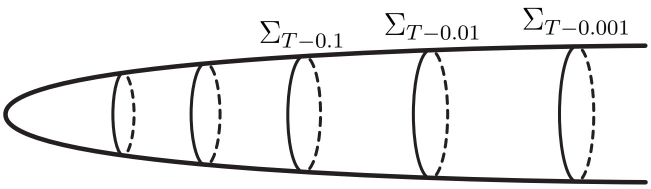

Figure 1.3: Finite-time escape

The escaping phenomenon affects the unique existence of the solution.

Example 1.1.9. Take M to be an open disk with radius 1, and take Σ_0 to be a sphere of radius $1/2$ that is not concentric with M. One obvious solution of IMCF in M is to run the IMCF in $\mathbb{R}^n$ and then restrict to M (Figure 1.4, left). We can also perturb the metric slightly outside M and run the IMCF, and then restrict to M (Figure 1.4, right). Due to the instant diffusion of parabolic equations, the two solutions become different once the hypersurfaces in the second one runs into the perturbed region, though they have the same initial data.

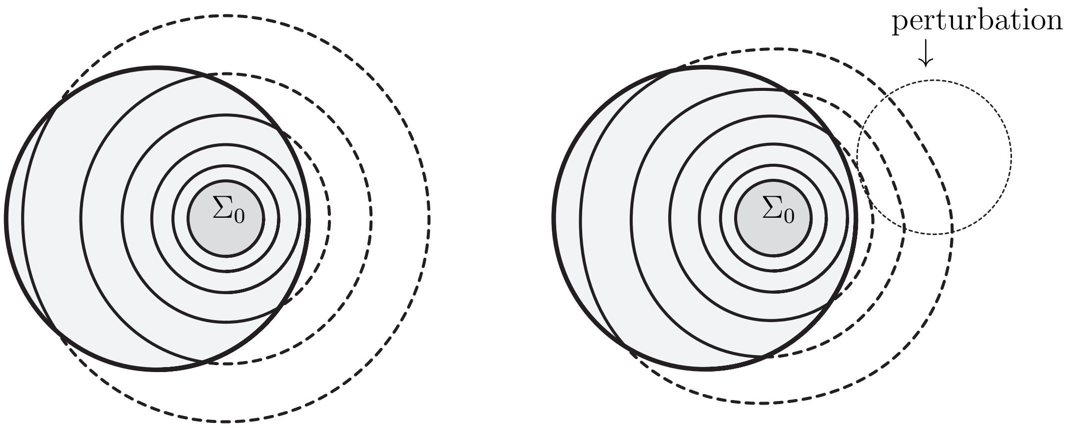

Figure 1.4: Nonuniqueness of solution when noncompact hypersurfaces appear

The resolution of the first issue is known: one shall instead consider the weak version of IMCF, which we will introduce in the next section. A resolution of the second issue is developed in this thesis: we shall consider the innermost solution, which turns out to uniquely exist; see Section 1.3.

1.2 The weak IMCF

We now introduce the weak IMCF developed in Huisken-Ilmanen [53]. Throughout this thesis, we will assume that all domains marked by Ω are connected.

Level set flow. The development in [53] starts with writing the IMCF in the level set form. This was inspired by similar approaches of Chen-Giga-Goto [28] and Evans-Spruck [40] on the mean curvature flow. Suppose $\{\Sigma_t\}$ is a solution of IMCF. Then we consider a function u such that each Σ_t is a level set of u, namely, $\Sigma_t = \{u = t\}$.

We need to transform the IMCF equation into an equation in terms of u. This is done by combining the following:

Fact 1.2.1. The flow speed of $\{\Sigma_t\}$ is equal to $1/|\nabla u|$. (An interesting observation is that infinite speed of movement corresponds to $\nabla u = 0$ – it is now not so singular!)

Fact 1.2.2. The mean curvature of $\{\Sigma_t\}$ is equal to $\operatorname{div}\big(\frac{\nabla u}{|\nabla u|}\big)$.

To see the second fact, note that the unit normal vector of Σ_t is $\frac{\nabla u}{|\nabla u|}$, and we have $\langle \nabla\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\rangle = 0$. Combining Facts 1.2.1, 1.2.2, the IMCF equation becomes

$$\operatorname{div}\Big(\frac{\nabla u}{|\nabla u|}\Big) = |\nabla u|. \tag{1.2.1}$$

In the sequel, we will call u a *smooth solution* of IMCF in a domain Ω, if $u \in C^\infty(\Omega)$ and $|\nabla u|$ is nonzero everywhere, and (1.2.1) holds in Ω.

We will also make use of subsolutions and supersolutions. A family of hypersurfaces $\{\Sigma_t\}$ is called a *subsolution* of IMCF if $\frac{\partial \Sigma_t}{\partial t} \geqslant \frac{\nu_t}{H_t}$. Transformed similarly, we will say that $u \in C^\infty(\Omega)$ is a *smooth subsolution* resp. *supersolution* of IMCF in Ω, if $|\nabla u| \neq 0$ everywhere, and

$$\operatorname{div}\Big(\frac{\nabla u}{|\nabla u|}\Big) \geqslant |\nabla u| \qquad \text{resp.} \qquad \operatorname{div}\Big(\frac{\nabla u}{|\nabla u|}\Big) \leqslant |\nabla u| \qquad \text{in } \ \Omega. \tag{1.2.2}$$

Intuitively, if a subsolution and supersolution both run from the same initial hypersurface, then the subsolution should be relatively outside (this is indeed the case, provided that the supersolution stays compact).

Transforming Examples 1.1.1, 1.1.2 to level set flows, we have

Example 1.2.3. The function $u(x) = (n-1)\log|x|$ solves (1.2.1) in $\mathbb{R}^n \setminus \{0\}$.

Example 1.2.4. In the warped product $\big(\mathbb{R}\times N, h(r)^2dr^2 + f(r)^2 g_N\big)$, the function

$$u = u(r) = (n-1)\log f(r) + C$$

solves (1.2.1) for all $C \in \mathbb{R}$.

The transition from smooth IMCF to weak IMCF is done by considering a variational version of (1.2.1), which we now introduce.

Towards weak solutions: the energy J_u. Throughout the entire thesis, we will keep the following notation: for a function u and $t \in \mathbb{R}$, we define

$$E_t(u) := \{u < t\}, \qquad E_t^+(u) := \{u \leqslant t\}. \tag{1.2.3}$$

When there is no ambiguity, we will write E_t, E_t^+ for simplicity. When u is defined on some domain Ω in the manifold, we view E_t, E_t^+ as subsets of Ω.

The weak IMCF is based on the following energy: given a function $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, a set E with locally finite perimeter in Ω, and a domain $K \Subset \Omega$, we define

$$J_u^K(E) := P(E; K) - \int_{E\cap K} |\nabla u|. \tag{1.2.4}$$

Here appear two terminologies in geometric measure theory: *sets with locally finite perimeter*, and the notation $P(E;K)$. The notation $P(E;K)$ stands for the *perimeter of E in K*. Sets with locally finite perimeters are, vaguely speaking, the roughest sets to make sense of perimeter in some way. For the definitions and properties of these notions, see Appendix A. We also recommend the textbook of Maggi [77] for a full introduction. For readers without much background or that are mainly interested in applications, below is an intuitive (but imprecise!) dictionary for temporary convenience:

- A set with finite perimeter: a set whose boundary is C^1 except for an ignorable singular set. For the rest of this list, let E be a set with locally finite perimeter.
- $P(E;K)$: viewed as the area of $\partial E \cap K$.
- $P(E)$: the total perimeter of E, viewed as the area of ∂E.
- $\partial^* E$: viewed as the smooth part of ∂E. The set $\partial E \setminus \partial^* E$ contributes zero area.
- ν_E: viewed as the outer unit normal of ∂E, which exists on $\partial^* E$.

However, since this thesis is written based on the language of sets with finite perimeter, we advise the reader to be familiar with the fundamental definitions in Appendix A.

The following *cup-cap inequality* is useful:

$$P(E \cup F; K) + P(E \cap F; K) \leqslant P(E;K) + P(F;K), \tag{1.2.5}$$

for all sets E, F with locally finite perimeter in a domain Ω, and for all domains $K \Subset \Omega$. See [77, Lemma 12.22].

We return to our main topic. The following observation lies at the heart of the theory of weak IMCF:

Fact 1.2.5. Let $u \in C^\infty(\Omega)$ be a smooth solution of IMCF in Ω. Then for each $t \in \mathbb{R}$, the set $E_t = \{u < t\}$ locally minimizes J_u in Ω in the following sense: for each set F with locally finite perimeter and each domain K, satisfying $E_t \Delta F \Subset K \Subset \Omega$, we have

$$J_u^K(E_t) \leqslant J_u^K(F). \tag{1.2.6}$$

See Figure 1.5 for a depiction of the positions of E_t, F, K. When speaking of "locally minimizing", we mean that an object minimizes some functional among all compact perturbations. The use of the precompact set K is to ensure that the energy is finite. We will usually call F a competitor set.

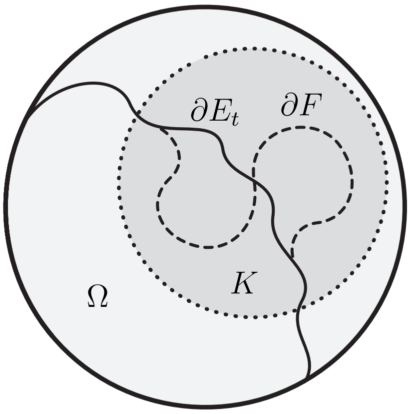

Figure 1.5: Local energy comparison

Remark 1.2.6. To verify local minimizing, we have the flexibility to choose K: for each competitor F, verifying (1.2.6) for one K implies (1.2.6) for all other K. Indeed, since we have $E_t = F$ outside K, switching to another K does not change the difference of the two sides in (1.2.6). We can always choose K at our convenience due to this reason.

Proof of Fact 1.2.5.

Let us denote by ν_G the outer unit normal of any set G. Note that $\nu_{E_t} = \frac{\nabla u}{|\nabla u|}$ since E_t is a sub-level set of u. Also, we trivially have $\nu_F \cdot \frac{\nabla u}{|\nabla u|} \leqslant 1$. Thus we directly calculate

$$\begin{aligned} J_u^K(F) - J_u^K(E_t) &= P(F;K) - P(E_t;K) + \int (\chi_{E_t} - \chi_F)|\nabla u| \\ &\geqslant \int_{\partial^* F\cap K} \nu_F \cdot \frac{\nabla u}{|\nabla u|} - \int_{\partial^* E_t \cap K} \nu_{E_t} \cdot \frac{\nabla u}{|\nabla u|} + \int (\chi_{E_t} - \chi_F)|\nabla u|. \end{aligned}$$

Through a re-combination of areas (using [77, Theorem 16.3] for rigorousness), we have

$$\begin{aligned} &\int_{\partial^* F\cap K} \nu_F \cdot \frac{\nabla u}{|\nabla u|} - \int_{\partial^* E_t \cap K} \nu_{E_t} \cdot \frac{\nabla u}{|\nabla u|} \\ &\qquad = \int_{\partial^*(F\setminus E_t)} \nu_{F\setminus E_t} \cdot \frac{\nabla u}{|\nabla u|} - \int_{\partial^*(E_t\setminus F)} \nu_{E_t\setminus F} \cdot \frac{\nabla u}{|\nabla u|}. \end{aligned}$$

Then by the divergence formula, we have

$$J_u^K(F) - J_u^K(E_t) \geqslant \int_{F\setminus E_t} \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) - \int_{E_t\setminus F} \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \int (\chi_{E_t} - \chi_F)|\nabla u| = 0,$$

making use of (1.2.1). $\square$

This proof is known as a "calibration type proof". It is helpful to connect to the following classical fact: if a smooth vector field ν satisfies $|\nu| \leqslant 1$ and $\operatorname{div}(\nu) = 0$, and a closed hypersurface Σ satisfies $\nu_\Sigma \cdot \nu = 1$, then Σ is area-minimizing in its homology class.

The weak IMCF. The definition of a weak IMCF is by making Fact 1.2.5 the definition. We say that a function u is a *weak solution* of IMCF in Ω, if

$$u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega), \text{ and each } E_t \text{ locally minimizes } J_u \text{ in } \Omega \text{ as in Fact 1.2.5.} \tag{1.2.7}$$

We reserve the notation $\mathsf{IMCF}(\Omega)$ for weak solutions: a function u is a solution of $\mathsf{IMCF}(\Omega)$ if it satisfies (1.2.7).

We also define that u is a *subsolution* (resp. *supersolution*) of $\mathsf{IMCF}(\Omega)$, if each E_t locally minimizes J_u in Ω from outside (resp. from inside). Namely, we ask $F \supset E$ (resp. $F \subset E$) in energy comparisons. Similar to Fact 1.2.5, one can show that smooth sub- or super-solutions of IMCF (see (1.2.2)) are always weak sub- or super-solutions.

A few remarks regarding the definition (1.2.7) are in order.

Remark 1.2.7. $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ denotes the space of locally Lipschitz functions in Ω. By Rademacher's theorem, locally Lipschitz functions are almost everywhere differentiable, with L^∞_{loc} derivatives. Hence, the energy (1.2.4) is well-defined for $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$.

Remark 1.2.8. It is much more commonly known that weak IMCF can be formulated using the energy

$$J_u^K(v) = \int_K |\nabla v| + v|\nabla u|. \tag{1.2.8}$$

Namely, it is commonly defined that u is a weak IMCF if $J_u^K(u) \leqslant J_u^K(v)$ for all $v \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ with $\{u \neq v\} \Subset K \Subset \Omega$. This is equivalent to our definition made above [53, Lemma 1.1]. In this thesis, we choose to develop the weak IMCF from (1.2.7), as it plays a major role in most of our results.

A less well-known (but equally important) formulation is the following calibration formulation. This first appeared in [53, Section 3]. We say that u is a *calibrated weak IMCF* if there is a measurable vector field ν, such that

$$|\nu| \leqslant 1, \qquad \nu \cdot \nabla u = |\nabla u|, \qquad \mathrm{div}(\nu) = |\nabla u|. \tag{1.2.9}$$

This formulation is equivalent to (1.2.7): every calibrated solution is a weak IMCF and every weak IMCF is calibrated. The idea of (1.2.9) is to replace the vector field $\nabla u/|\nabla u|$ in (1.2.1) by the abstract calibration vector field ν, as the former may not make sense when $\nabla u = 0$. As evidence of how this works, one may recall the calibration proof of Fact 1.2.5, and then check that the proof still makes sense if we replace each $\nabla u/|\nabla u|$ by ν and make use of (1.2.9). We refer to Section 2.3 for more details about (1.2.9).

We remark that formulations similar to (1.2.9) also appear in many works related to the 1-Laplacian; we refer to for example [46, 61, 67, 81, 88] and references therein.

Geometric behavior of weak IMCF. The following is a direct but important consequence of the definition.

Fact 1.2.9. Suppose u is a solution of $\mathsf{IMCF}(\Omega)$. Then each E_t is locally outward perimeter-minimizing in Ω.

Here, a set E is said to be *locally outward (resp. inward) perimeter-minimizing* in Ω, if for any competitor set $F \supset E$ (resp. $F \subset E$) and any domain K, satisfying $E\Delta F \Subset K \Subset \Omega$, it holds

$$P(F; K) \geqslant P(E; K).$$

For simplicity, we will often just call E locally outward minimizing in Ω.

When $E \Subset \Omega$, we will simply call E outward minimizing. In this case, the minimizing condition is much simpler: for all F with $E \subset F \Subset \Omega$, one has $P(E) \leqslant P(F)$. Here, $P(E) = P(E; M)$ stands for the *total perimeter* of E.

Proof of Fact 1.2.9. Let F and K be as stated above. By the definition of weak IMCF, we have

$$J_u^K(F) \geqslant J_u^K(E_t),$$

which implies

$$P(F;K) \geqslant P(E;K) + \int_{F\setminus E} |\nabla u| \geqslant P(E;K). \qquad \square$$

At this point, the weak and smooth IMCF start to behave differently. Indeed, recall Example 1.1.7(i), or see Figure 1.6 below, where the smooth IMCF exists until the equator Σ_T. Observe that the region enclosed by Σ_T is not outward minimizing: the surface S is a competitor. Thus, Σ_T cannot be a sub-level set of a weak IMCF. Similarly, all the surfaces Σ_s, $T' < s \leqslant T$ (where $\Sigma_{T'}$ is as in the figure), should not appear in a weak IMCF.

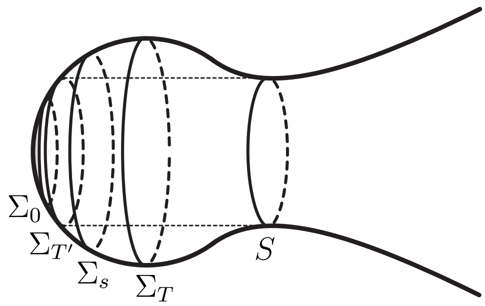

Figure 1.6: A smooth IMCF that terminates in finite time

We continue our investigation by considering the set $E_t^+ = \{u \leqslant t\}$. We have the following fact:

Fact 1.2.10. If u solves $\mathsf{IMCF}(\Omega)$, then each E_t^+ is also a local minimizer of J_u.

Proof. Note that the definition does not apply directly to E_t^+. However, since E_t^+ is a descending limit of $E_{T+\varepsilon}$, we hope to take the minimizing property of the latter and let $\varepsilon \to 0$. This proof uses what is known as the set-replacing argument, which is common in geometric measure theory. We give an outline here, and leave the analytic details in Lemma A.2.5.

Let E be an energy competitor, with $E\Delta E_t^+ \Subset K \Subset \Omega$. Choose another domain K' such that $E\Delta E_t^+ \Subset K' \Subset K$. For each $s > t$, consider the set $\widetilde{E} = (E \cap K') \cup (E_s \setminus K')$. See the left side of Figure 1.7, this choice makes sure that $\widetilde{E}\Delta E_s \Subset K$.

So we may compare $J_u^K(E_s) \leqslant J_u^K(\widetilde{E})$. This implies (see the right side of Figure 1.7)

$$A + D - \int_{E_s\cap K'} |\nabla u| \leqslant A + B + C - \int_{E\cap K'} |\nabla u|.$$

The common terms A are cancelled. Note that B lies in the region between ∂E_t^+ and ∂E_s, which should be small as $s \searrow t$ if we choose K' to be sort of transversal with ∂E_t^+. As a result,

$$D \leqslant o(1) + C + \int_{E_s\cap K'} |\nabla u| - \int_{E\cap K'} |\nabla u|.$$

Now we take $s \searrow t$. The lower semi-continuity of perimeter implies

$$\begin{aligned} P(E_t^+; K') &\leqslant \liminf_{s\to t} P(E_s; K') = \liminf_{s\to t} D \\ &\leqslant \liminf_{s\to t} \Big(o(1) + C + \int_{E_s\cap K'} |\nabla u| - \int_{E\cap K'} |\nabla u| \Big) \\ &= P(E; K') + \int_{E_t^+\cap K'} |\nabla u| - \int_{E\cap K'} |\nabla u|. \end{aligned}$$

This proves $J_u^{K'}(E_t^+) \leqslant J_u^{K'}(E)$, hence $J_u^K(E_t^+) \leqslant J_u^K(E)$ (recall Remark 1.2.6). ☐

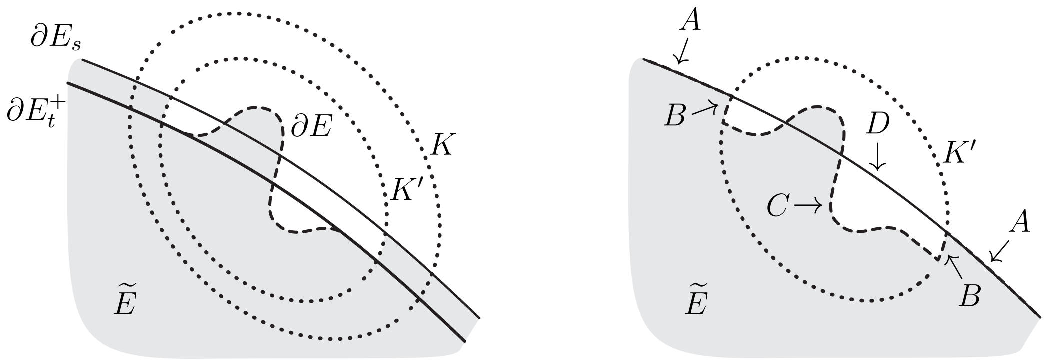

Figure 1.7: The set-replacing technique

Fact 1.2.11. Suppose u is a solution of $\mathsf{IMCF}(\Omega)$. Then each E_t^+ is strictly outward minimizing in Ω.

Here, a set E is *strictly outward (perimeter-)minimizing* in Ω, if for any competitor set $F \supset E$ and domain K satisfying $F \setminus E \Subset K \Subset \Omega$ and $|F \setminus E| > 0$, it holds

$$P(F; K) > P(E; K).$$

Proof of Fact 1.2.11. Let F, K be as stated above. By Fact 1.2.10, we have

$$P(F; K) \geqslant P(E_t^+; K) + \int_{F\setminus E_t^+} |\nabla u|.$$

Denote $G = F \setminus E_t^+$. Note that G has nonzero measure by our assumption. So if $P(F; K) = P(E_t^+; K)$ then $\nabla u \equiv 0$ in G. So u is constant in every connected component of $\overline{G}$. Since $u > t$ outside E_t^+ and $u = t$ on ∂E_t^+, it follows that $G \Subset \Omega \setminus E_t^+$. Hence $P(F; K) = P(E_t^+; K) + P(G)$. Hence $P(G) = 0$, thus G must be a union of connected components of $\Omega \setminus E_t^+$ that do not touch ∂E_t^+. But this violates our initial assumption that Ω is connected. ☐

Finally, the sets E_t, E_t^+ are related through the notion of minimizing hull. Given two sets $E \subset E' \subset \Omega$. We say that E' is the *strictly outward minimizing hull* (or "*minimizing hull*", for brevity) of E in Ω, if:

(i) E' is strictly outward minimizing in Ω,

(ii) for any another strictly outward minimizing set E'' in Ω, with $E \subset E''$, we have $E' \subset E''$ up to a set with measure zero (i.e. $|E' \setminus E''| = 0$).

One can show (see Lemma A.4.2) that if E_1, E_2 are both strictly outward minimizing, then so is $E_1 \cap E_2$. Thus the minimizing hull, if exists, is unique up to measure zero.

For many occasions in this thesis, we will be dealing with case $E, E' \Subset \Omega$. In this case, there is a convenient characterization of the minimizing hull. Given a set $E \Subset \Omega$, we say that another set $G \Subset \Omega$ is a *least area solution* outside E in Ω, if $G \supset E$, and

$$P(G) = \inf \big\{ P(F) : E \subset F \Subset \Omega \big\}.$$

We have:

Lemma 1.2.12 (identical with Theorem A.4.5)**.** *Fix an ambient domain Ω. Under the assumption $E \subset E' \Subset \Omega$, E' is the minimizing hull of E if and only if E' is the least area solution outside E with the largest volume.*

Here, it can be shown that the union of two least area solutions is again a least area solution, so the largest one is unique up to measure zero. A direct consequence of this lemma is that, if E has a precompact minimizing hull E', then $\partial E' \setminus \partial E$ is a stable minimal hypersurface.

Example 1.2.13. Consider the surface in Figure 1.8: there are three stable minimal surfaces $\Sigma_1, \Sigma_2, \Sigma_3$ with the same area. Let E be the geodesic ball such that $|\partial E| = |\Sigma_i|$. Let E' be the region enclosed by Σ_3. Then E' is the minimizing hull of E. Indeed, first note that E' is strictly outward perimeter-minimizing, but the regions enclosed by Σ_1, Σ_2 are not. Also, note that Σ_i are all least area solutions outside E, while Σ_3 is the one enclosing the largest volume.

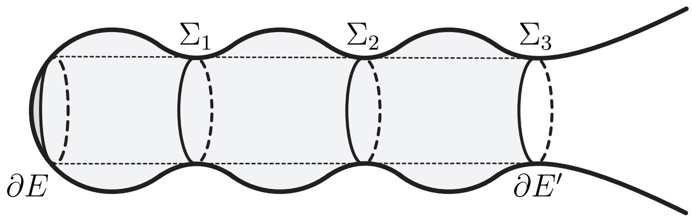

Figure 1.8: Strictly outward minimizing hull

Further note that: if one considers a surface where stable minimal surfaces repeat periodically forever, then E does not have precompact minimizing hulls.

The following lemma characterizes the relation between E_t and E_t^+:

Fact 1.2.14. Suppose u is a solution of $\mathsf{IMCF}(\Omega)$. Then each E_t^+ is the minimizing hull of E_t, provided that $E_t^+ \setminus E_t \Subset \Omega$.

Proof. By Fact 1.2.11, E_t^+ is strictly outward minimizing. It suffices to show that it is the smallest one. Choose a domain K with $E_t^+ \setminus E_t \Subset K \Subset \Omega$. Suppose $E' \supset E_t$ is another strictly outward minimizing set in Ω. We need to show that $|E_t^+ \setminus E'| = 0$.

Note that $E_t^+ \cap E'$ is also strictly outward minimizing (Lemma A.4.2). On the other hand, we can compare

$$J_u^K(E_t^+) \leqslant J_u^K(E_t^+ \cap E').$$

Since $\nabla u = 0$ in $E_t^+ \setminus E'$, this implies

$$P(E_t^+; K) \leqslant P(E_t^+ \cap E'; K).$$

By the strict minimization of $E_t^+ \cap E'$, this forces $|E_t^+ \setminus E'| = 0$. $\square$

Now we are ready to state the heuristic behavior of the weak IMCF. It is roughly a combination of two behaviors:

(i) outward movement by $1/H$,

(ii) instant jump to the minimizing hull.

And due to Fact 1.2.14, option (ii) takes priority whenever possible. This description is suitable at least as long as E_t remains precompact. When E_t becomes noncompact, due to the possible failure of Fact 1.2.14, the behavior can be more complicated.

Example 1.2.15. Figure 1.9 describes the weak IMCF in the situation of Example 1.1.7(i). The hypersurfaces ∂E_t first evolve smoothly by $1/H$ until time s, then jump, then continue evolving smoothly by $1/H$.

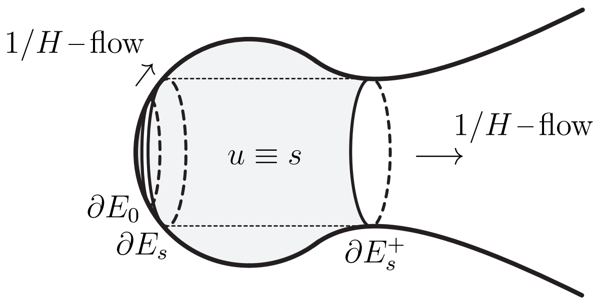

Figure 1.9: An example of weak IMCF

Example 1.2.16. For a general warped product metric $\big(\mathbb{R}\times N, dr^2+f(r)^2 g_N\big)$, where $f>0$ is not necessarily increasing, the following function is a weak solution:

$$u = u(r) = (n-1)\inf_{s\geqslant r}\log f(s).$$

Indeed, one may check that this solution is calibrated by the vector field

$$\nu = \nu(r) = \Big(\frac{\inf_{s\geqslant r} f(s)}{f(r)}\Big)^{n-1}\frac{\partial}{\partial r}.$$

Let us mention the following compactness point of view as another explanation of jumping. It is a general principle that weak objects should satisfy a certain compactness property. A limit of weak IMCFs should be again a weak IMCF. In particular, a limit of smooth IMCFs should be a weak IMCF. Now consider a family of radially symmetric

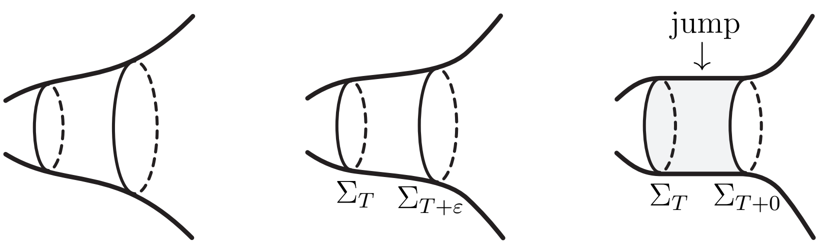

Figure 1.10: convergence to a weak IMCF

surfaces $dr^2+f_i(r)^2d\theta^2$, where each f_i is strictly increasing, but the limit $f=\lim_{i\to\infty} f_i$ is only increasing, with $f|_{1\leqslant r\leqslant 2}$ being constant. See Figure 1.10 for illustration. Due to the exponential growth of area, the time that it takes to move from $r=1$ to $r=2$ converges to zero as $i \to \infty$. In the limit, it takes zero time to move from $r = 1$ to $r = 2$, thus a jump is formed.

Further properties of the weak IMCF. Let us include some more useful facts here. We first discuss the regularity of level sets. The minimization of energy implies the following: for any competitor set F and domain K with $E_t \Delta F \Subset K \Subset \Omega$, one has

$$P(E_t; K) \leqslant P(F; K) + \sup_K |\nabla u| \cdot |E_t \Delta F|.$$

Thus, each E_t falls into the class of "almost perimeter-minimizers" defined in Section A.2. In particular, the deep regularity results in geometric measure theory show that:

Fact 1.2.17. Each ∂E_t is a $C^{1,\alpha}$ hypersurface except for a codimension 8 singular set.

We have previously discussed outward minimizing properties. Then it is natural to ask about inward minimizing. This is stated as follows:

Lemma 1.2.18 (excess inequality)**.** *Let u solve* $\mathsf{IMCF}(\Omega)$*, and $F \Subset K \Subset \Omega$. Then for all t we have*

$$P(E_t; K) \leqslant P(E_t \setminus F; K) + \big(e^{t - \inf_F(u)} - 1\big) P(F; E_t). \tag{1.2.10}$$

Proof. We present a technically simplified proof (see Lemma 2.2.1 for the full proof). See Figure 1.11: for each $\tau \leqslant t$, we denote

$$A(\tau) = P(E_\tau; F), \qquad S(\tau) = P(F; E_\tau).$$

Comparing $J_u^K(E_\tau) \leqslant J_u^K(E_\tau \setminus F)$, and cancelling the common portions, we have

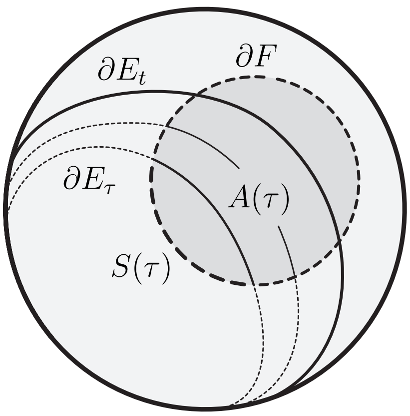

Figure 1.11: Proof of the excess inequality

$$A(\tau) \leqslant S(\tau) + \int_{E_\tau \cap F} |\nabla u| = S(\tau) + \int_{\inf_F(u)}^{\tau} A(s)\, ds \qquad \text{(by coarea formula).}$$

Then by Gronwall's inequality, this implies

$$A(t) \leqslant S(t) + \int_{\inf_F(u)}^{t} e^{t-s} S(s)\, ds.$$

Since $S(s) \leqslant P(F; E_t)$ for all $s \leqslant t$, we now have $A(t) \leqslant e^{t - \inf_F(u)} S(t)$. This is (up to some technicalities) the same as (1.2.10), by canceling the common portions in $P(E_t; K)$ and $P(E_t \setminus F; K)$. $\square$

As a consequence, if u is close to a constant, then E_t is close to being inward minimizing: by (1.2.10) we have

$$P(E_t; K) \leqslant P(E_t \setminus F; K) + \big(e^{\mathrm{osc}_K(u)} - 1\big) P(F), \qquad \forall\, F \Subset K.$$

In addition with the outer minimizing property, this actually implies that E_t is close to being perimeter-minimizing. We would later encounter a sequence of weak IMCFs that converges to a constant. Thus Lemma 1.2.18, together with the set-replacing argument, implies that a selected sequence of level sets will converge to an area-minimizing hypersurface.

The initial value problem. Next, we turn to the initial value problem of the weak IMCF. We will denote by E_0 the initial data (whose boundary is the surface that we start the flow from); it will always be a $C^{1,1}$ domain.

We say that a function u solves $\mathsf{IVP}(\Omega; E_0)$, i.e. the initial value problem in Ω starting with E_0, if the following holds:

(i) $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ and $E_0 = \{u < 0\}$,

(ii) $u|_{\Omega \setminus \overline{E_0}}$ solves $\mathsf{IMCF}(\Omega \setminus \overline{E_0})$.

A few remarks are in order. First, $E_0 = \{u < 0\}$ implies $u|_{\partial E_0} = 0$, which is consistent with the fact that ∂E_0 is the initial hypersurface. The $C^{1,1}$ regularity of E_0 guarantees that the solution is Lipschitz near ∂E_0. For those E_0 with worse regularity, the "solution" may fail to be continuous near ∂E_0 (see Example 1.1.6 and Choi-Hung [32]).

Next, note that item (ii) only states that u is a weak IMCF in $\Omega \setminus \overline{E_0}$, while it can have arbitrary values in E_0 (as long as $u|_{E_0} < 0$ and u is Lipschitz across ∂E_0). We will view two solutions u_1, u_2 as the same solution if $u_1 = u_2$ outside E_0. Let us remind the reader again that a solution of $\mathsf{IVP}(\Omega; E_0)$ is not a solution of $\mathsf{IMCF}(\Omega)$.

Proper solutions. A solution of $\mathsf{IVP}(\Omega; E_0)$ is called proper, if $E_t \Subset \Omega$ for all $t \geqslant 0$. Note that proper solutions exist only for $E_0 \Subset \Omega$. Recall the two issues of IMCF discussed in the previous section: proper solutions are exactly the ones where escaping do not happen. Proper solutions share many desirable properties, let us list a few:

Lemma 1.2.19 (jump at $t = 0$, Lemma 2.1.10)**.**
Suppose u is a proper solution of $\mathsf{IVP}(\Omega; E_0)$. Then E_0^+ is the minimizing hull of E_0.

Lemma 1.2.20 (exponential growth, Lemma 2.1.10)**.**
Suppose u is a proper solution of $\mathsf{IVP}(\Omega; E_0)$. Then $P(E_t) = e^t P(E_0^+) \leqslant e^t P(E_0)$. Equality holds if E_0 is outward minimizing in Ω.

Lemma 1.2.21 (uniqueness, Lemma 2.1.12)**.**
For a fixed E_0, there exists at most one proper solution of $\mathsf{IVP}(\Omega; E_0)$.

One fundamental question is to find effective criteria for the existence of proper solutions. The original work of Huisken-Ilmanen [53] proved the following: if there exists a single proper subsolution with a precompact initial value, then for all $E_0 \Subset M$, there exists a unique proper solution of $\mathsf{IVP}(M; E_0)$. Recall that in a subsolution of IMCF, the hypersurfaces move faster than $1/H$. In the weak setting, a subsolution of IMCF should intuitively move faster than $1/H$, and also jump further than minimizing hulls.

Example 1.2.22. See Figure 1.12 below, which is the same as in Example 1.2.15 except that the jump at time s is made further ahead. The function u described here is a weak subsolution. To prove this, one combines Lemma 2.2.4 and Remark 2.1.4(iv).

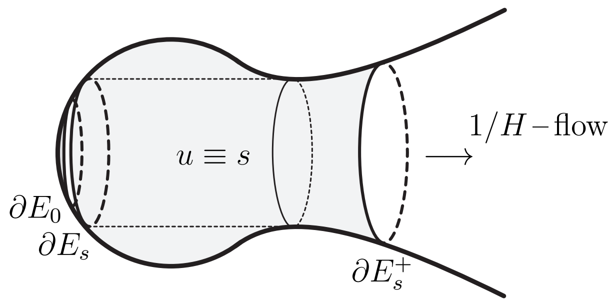

Figure 1.12: A proper weak subsolution

Analytically, the proper subsolution plays the role of a lower barrier. With this subsolution available, [53] uses the method of elliptic regularization to obtain a proper solution (see also Section 2.4). This existence theorem in [53] is applicable to manifolds with precise description at infinity, including asymptotically flat or hyperbolic manifolds. In an asymptotically flat manifold, for example, the function $(n-1-\varepsilon)\log|x|$ is a subsolution sufficiently faraway in the exterior coordinate chart, for all $\varepsilon > 0$.

We wish to investigate the relation between properness and the geometry of the underlying manifold. In this respect, the existence of a proper subsolution remains a quite unintuitive condition (since there has not been general ways of construction). Our first main theorem is a properness criterion stated in terms of the isoperimetric inequality on the ambient manifold.

Define the isoperimetric profile function

$$\mathrm{ip}(v) = \inf\Big\{P(E) : E \Subset M,\ |E| = v\Big\}, \tag{1.2.11}$$

where recall that $P(E) = P(E; M)$ is the total perimeter of E.

Theorem A. *Suppose M is complete, connected, noncompact, and the isoperimetric profile satisfies*

$$\liminf_{v\to\infty} \mathrm{ip}(v) = \infty \tag{1.2.12}$$

and

$$\int_0^{v_0} \frac{dv}{\mathrm{ip}(v)} < \infty \quad \textit{for some } v_0 > 0. \tag{1.2.13}$$

Then for any $C^{1,1}$ domain $E_0 \Subset M$, there exists a unique proper solution of $\mathsf{IVP}(M; E_0)$.

Let us remark on the conditions involved here. First, we note that (1.2.12) roughly says that the manifold is more expanding than a cylinder:

Remark 1.2.23. If (1.2.12) holds, then M has superlinear volume growth.

Indeed, if $|B(x_0, r)| \leqslant Cr$ for some constant C and for all $r > 1$, then by the coarea formula, we can find a sequence of radii $r_i \in [2^{i-1}, 2^i]$ such that $P\big(B(x_0, r_i)\big) \leqslant 2C$. This directly implies $\liminf_{v\to\infty} \mathrm{ip}(v) \leqslant 2C$.

In particular, (1.2.12) rules out examples like 1.1.8. The second condition (1.2.13) is less transparent at first glance. However, we note the following:

Remark 1.2.24. If (1.2.13) holds, then $\inf_{x\in M} |B(x, 1)| > 0$.

Indeed, fixing $x \in M$, we consider the function $V(r) = |B(x,r)|$. For almost every $r > 0$ we have

$$\frac{dV}{dr} = P\big(B(x,r)\big) \geqslant \mathrm{ip}(V(r)).$$

Since $V(r) > 0$ for all $r > 0$, and (1.2.13) holds, we can integrate this in $[0,1]$ to get

$$\int_0^{V(1)} \frac{dv}{\mathrm{ip}(v)} \geqslant 1.$$

Thus $V(1)$ has a lower bound depending on the isoperimetric profile, but not on x.

In particular, (1.2.13) rules out the appearance of ends with finite volume, which is another obstruction to properness and is not detected by (1.2.12). Here notice that if M has multiple ends, and one of the end is asymptotically cylindrical or has finite volume, then M does not admit proper IMCFs.

Let us mention several situations where the conditions (1.2.12) (1.2.13) are meet:

Remark 1.2.25.

1. For a manifold with $\mathrm{Ric} \geqslant -\lambda g$ and $\inf_{x\in M} |B(x,1)| > 0$, one can prove that

$$\liminf_{v\to 0} \mathrm{ip}(v) v^{-(n-1)/n} > 0,$$

   in particular, (1.2.13) holds. See [35] and [8, Theorem 1.3].
2. For the case of nonnegative sectional curvature and uniform lower bound on $|B(x,1)|$, it is shown in [7, 9] that (1.2.12) is equivalent to superlinear volume growth.
3. A particular case of (1.2.12) (1.2.13) is when M satisfies an Euclidean isoperimetric inequality

$$\mathrm{ip}(v) \geqslant c v^{(n-1)/n} \qquad (\forall\, v > 0). \tag{1.2.14}$$

   In [79], the existence of proper IMCF is studied under (1.2.14) and Ricci lower bounds, using p-harmonic approximations. Our theorem extends [79, Theorem 1.7].

In addition to properness, it is natural to ask further about the growth rate of a solution at infinity. When an Euclidean isoperimetric inequality (1.2.14) is present, it is conjectured that the proper solution should grow at the order of $(n-1)\log r$. This growth rate comes from the model example $u = (n-1)\log|x|$ in $\mathbb{R}^n$, and is consistent with past results in [53, 79, 89]. Here we confirm this estimate in full generality:

Theorem B. *Suppose that M satisfies the Euclidean isoperimetric inequality*

$$P(E) \geqslant c_I |E|^{\frac{n-1}{n}} \qquad \forall\, E \Subset M, \tag{1.2.15}$$

for some $c_I > 0$, then for any $C^{1,1}$ initial data $E_0 \Subset M$, the unique proper solution u of IVP$(M; E_0)$ *satisfies*

$$u(x) \geqslant (n-1)\log d(x,x_0) - C \qquad \text{in} \quad M \setminus E_0, \tag{1.2.16}$$

for some constant C.

In particular, this generalizes [6, Theorem 3.6] and the main result of [79]. Theorem B will appear in the joint note [14]. Both Theorems A, B are proved in Sections 2.5 and 2.6. We refer the reader there for ideas of proof and more details.

1.3 Innermost solutions

After the discussion on proper solutions, we shall move forward and consider the more general case. Namely, we seek a general existence theory including the case where no proper solution is expected. We keep in mind that an ideal existence theory should provide a unique solution. Let us collect some examples to keep in mind:

(i) Example 1.1.8, the asymptotically cylindrical model case. And in general, manifolds with one or multiple cylindrical ends.

(ii) Example 1.1.9, where M is an open disk, and E_0 is a sub-disk with a different center from that of M (so M is incomplete).

We start with the following interesting and crucial observation:

Fact 1.3.1. If u is a solution of $\mathsf{IVP}(\Omega; E_0)$, then so is $\min\{u, T\}$ for all $T \geqslant 0$.

See Remark 2.1.4(iv) for the proof; the reader may also try to prove this independently. This means that we can force the hypersurface to escape to infinity at any given time T. This seems strange at first glance, as forced jumping seems to produce a strict subsolution. The point here is that in a weak IMCF we only make local energy comparison, so jumps over noncompact sets are not obviously detected.

This fact suggests that decreasing the solutions leads to more non-uniqueness. We may further investigate a concrete example:

Remark 1.3.2. Recall Example 1.2.16 where we considered a warped product $dr^2 + f(r)^2 g_{\mathbb{S}^{n-1}}$. For simplicity, here let us assume $f' > 0$ everywhere. Consider $E_0 = \{r < 1\}$. Then we can show that all radial solutions of $\mathsf{IVP}(M; E_0)$ take the form

$$u(r) = \min\Big\{(n-1)\log\big[f(r)/f(1)\big], T\Big\} \qquad \text{for some } T \geqslant 0. \tag{1.3.1}$$

Namely, all of them are truncations of $(n-1)\log\big[f(r)/f(1)\big]$.

Indeed, whenever $E_t \Subset M$, we have $P(E_t) = e^t P(E_0)$ by exponential growth. The outward minimization implies that each E_t must take the form $\{r < r(t)\}$ for some $r(t) \geqslant 1$. Combining these two facts, we obtain $f(r(t)) = e^{t/(n-1)} f(r(0)) = e^{t/(n-1)} f(1)$ for all $t \geqslant 0$ such that $E_t \Subset M$. Since f is strictly increasing, this directly implies (1.3.1).

Thus, we are motivated to consider maximal solutions of IMCF. We say that u is a *maximal solution* of $\mathsf{IVP}(\Omega; E_0)$, if $u|_{\Omega\setminus E_0} \geqslant v|_{\Omega\setminus E_0}$ for any other solution v of $\mathsf{IVP}(\Omega; E_0)$. For intuitiveness, we may also call u an *innermost solution* (since a function being maximal means that its sub-level sets are innermost). We will often switch between these two terminologies, depending on our context.

The following is our main existence theorem:

Theorem C. *Let M be a (possibly incomplete) manifold, and $E_0 \subset M$ be a (possibly unbounded) $C^{1,1}$ domain. Then there exists a unique maximal solution of $\mathsf{IVP}(M; E_0)$.*

The reason that such solutions exist will be made more clear in the next section (they turn out to be a limit of IMCFs with outer obstacle). An indirect evidence is that the maximum of two solutions is a subsolution (Lemma 2.3.6). However, the major difficulty along this line is still to obtain a solution that lies above a given subsolution (the reader may compare with Huisken-Ilmanen's existence theorem, which essentially used the properness of the subsolution).

Let us briefly describe the geometric behavior of innermost solutions. For example (i) above, we saw in Remark 1.3.2 that the innermost IMCF agrees with the smooth one. Case (ii) is a bit harder to imagine. It turns out that the level sets will be tangent to ∂M upon contact. See Figure 1.13: solid lines represent the level sets in the innermost flow, while dashed lines represent the standard IMCF in $\mathbb{R}^2$ restricted to the disk, for comparison. The mean curvature diverges to infinity near ∂M (so the speed of movement becomes 0). On the other hand, it can be shown that the hypersurfaces remain $C^{1,\alpha}$ up to ∂M. Such solutions will be our focus in the next section.

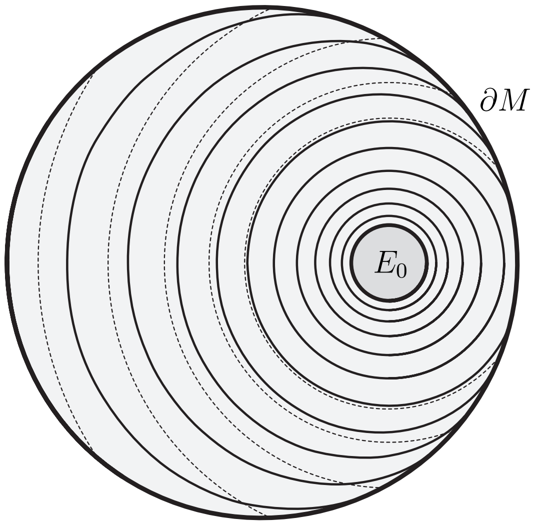

Figure 1.13: the innermost IMCF in a disk

Innermost IMCFs may generally be classified into several types, according to how the level sets diverge to infinity. Suppose $E_0 \Subset M$; we say that the innermost solution u of $\mathsf{IVP}(M; E_0)$ is:

(i) proper, if $E_t \Subset M$ for all $t > 0$;

(ii) sweeping, if there is $T \in (0, \infty)$ so that $E_t \Subset M$ for all $t < T$, and $E_T = M$.

(iii) instantly escaping, if there is $T \in (0, \infty)$ so that $E_T \Subset M$, and $u \equiv T$ in $M \setminus E_T$.

(iv) partially diverging, if there is $T \in (0, \infty)$ so that $E_T \neq M$ but $E_T \not\Subset M$.

(v) trivial, if $u \equiv 0$.

We include some examples of flows of these types. In Figures 1.14$\sim$1.17 below, dark grey regions represent the initial value E_0, and light grey regions represent jumps in the weak flow:

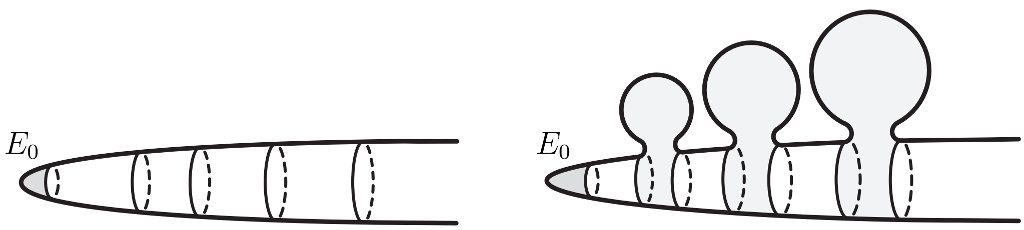

Figure 1.14: the cases of sweeping flow

Among non-proper flows, the types of sweeping and instantly escaping flows are relatively "good", since the level sets stay compact before they disappear. As shown in the

Figure 1.15: the cases of instantly escaping flow

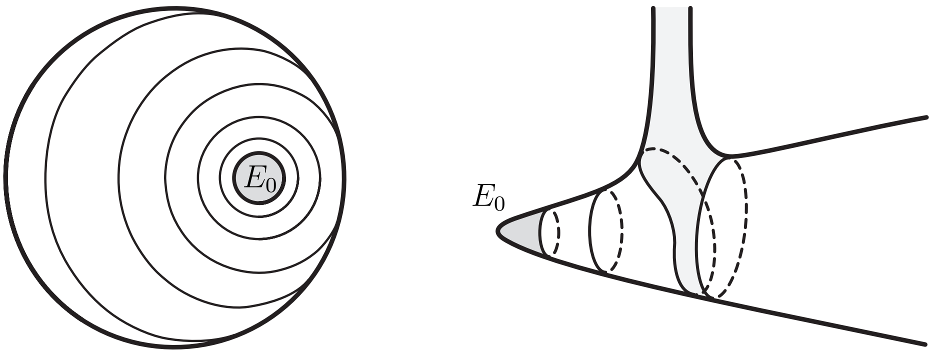

Figure 1.16: the cases of partially diverging flow

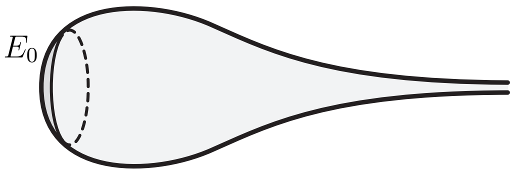

Figure 1.17: the case of trivial flow

following lemma, the *bounded geometry* condition rules out bad solution types. Furthermore, the disappearing time is exactly linked to the circumference of M at infinity.

Lemma 1.3.3 (identical with Theorem 4.3.2)**.** *Suppose M is complete, one-ended, and has bounded geometry:*

$$|\operatorname{Rm}| \leqslant \Lambda^2, \qquad \operatorname{inj} \geqslant \Lambda^{-1}.$$

Then there is a constant $A = A(\Lambda) > 0$ such that: for any $C^{1,1}$ domain $E_0 \Subset M$ with $P(E_0) \leqslant A$, the innermost solution of $\mathsf{IVP}(M; E_0)$ is either proper, sweeping, or instantly escaping. Let T be the disappearing time for the sweeping or escaping cases, and $T = +\infty$ for the proper case. Then we have

$$e^T P(E_0^+) = \inf\Big\{ \liminf_{i\to\infty} P(K_i) : K_1 \Subset K_2 \Subset \cdots \Subset M \\ \text{are smooth domains with } \bigcup K_i = M \Big\}.$$

1.4 IMCF with outer obstacle

In this section, we introduce a theory of weak IMCF inside bounded domains, which we call the IMCF with outer obstacle. There are several aspects for one to understand this object, which together would provide a comprehensive picture.

Aspect 1: as IMCF with a boundary tangency condition. Let Ω be a smooth domain. We say that $u \in C^\infty(\Omega)$ is a *smooth IMCF with outer obstacle condition* at $\partial\Omega$, if it satisfies

$$\operatorname{div}\Big(\frac{\nabla u}{|\nabla u|}\Big) = |\nabla u| \qquad \text{in } \ \Omega, \tag{1.4.1}$$

and

$$\lim_{x \to \partial\Omega} \frac{\nabla u(x)}{|\nabla u(x)|} = \nu_\Omega \qquad \text{locally uniformly,} \tag{1.4.2}$$

where ν_Ω is the outer unit normal of Ω. In terms of hypersurfaces, this is saying that the family $\{\Sigma_t\}$ evolves by $1/H$ in the interior of Ω, while each Σ_t stays tangent to $\partial\Omega$.

Recall our convention that if u is defined in a domain Ω, then E_t is viewed as a subset of Ω. Thus, its boundary contains not only $\partial E_t \cap \Omega$ but also $\partial E_t \cap \partial\Omega$. (Hence, a lot of level sets stay stationary and stack up at $\partial\Omega$.) Let us include some examples:

Example 1.4.1. Let $\Omega = B(0,1) \setminus \{0\} \subset \mathbb{R}^n \setminus \{0\} = M$. Note that $0 \notin \partial\Omega$, so Ω is a smooth domain. Then the function

$$u(x) = (n-1)\log|x|$$

satisfies (1.4.1) and (1.4.2). Note that $E_t = \Omega$ for all $t \geqslant 0$.

Example 1.4.2. Figure 1.13 depicts an IMCF with outer obstacle ∂M.

Example 1.4.3 (cycloid)**.** It is observed by Drugan-Lee-Wheeler [37] that cycloids are (the only) translating solitons of the IMCF in $\mathbb{R}^2$. With suitable positioning, let us view the cycloid γ as sliding in a strip region $\Omega \subset \mathbb{R}^2$, see Figure 1.18. There are two singularities at its endpoints, where γ is asymptotic to the graph $y = \pm C x^{3/2}$. Thus, the boundary tangency condition is satisfied. The sub-level sets of the corresponding arrival time function have $C^{1,1/2}$ boundaries (see γ_t in Figure 1.18: they are translations of γ plus two horizontal rays).

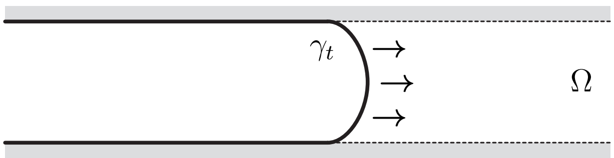

Figure 1.18: translating cycloids.

Example 1.4.4 (nephroid)**.** Let γ be a half of the nephroid, positioned in a half-plane $\Omega = \{y > 0\} \subset \mathbb{R}^2$. Then the arrival time function u associated to the family of curves

$$\gamma_t = e^{3t/4}\gamma, \qquad t \in \mathbb{R},$$

is a solution of IMCF satisfying the boundary tangency condition. Each sub-level E_t again has $C^{1,1/2}$ regularity. Note that $\lim_{x\to 0} u(x) = -\infty$ and $\lim_{x\to\infty} u(x) = +\infty$.

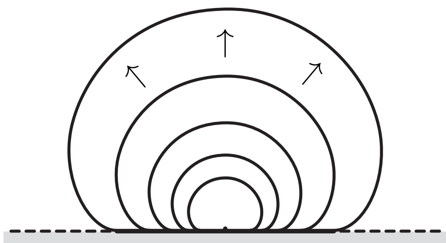

Figure 1.19: Expanding nephroids.

The nephroid soliton will appear in Chapter 3 to show the sharpness of several theorems. We shall see one instance (Theorem 1.4.13) soon in this section.

Furthermore, all the epicycloids and hypocycloids are homothetic solitons of the IMCF in suitable conic domains; see Section 3.1 for the precise computations.

Aspect 2: as the solution of an outer obstacle problem. We define that u is a weak solution of IMCF in Ω respecting the outer obstacle $\partial\Omega$, if

(i) $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$,

(ii) for each $t \in \mathbb{R}$, any competitor set $E \subset \Omega$, and any domain K satisfying $E \Delta E_t \Subset K \Subset M$, we have $\widetilde{J}_u^K(E_t) \leqslant \widetilde{J}_u^K(E)$, where

$$\widetilde{J}_u^K(E) := P(E; K) - \int_{E\cap K} |\nabla u|. \tag{1.4.3}$$

The energy $\widetilde{J}_u$ has a very similar form to J_u, the difference being that we do not require $K \Subset \Omega$. Note that each E_t is a minimizer of an outer obstacle type problem (with outer obstacle $\partial\Omega$), since E_t and its competitor E are both subsets of Ω. See Figure 1.20 for a comparison between interior energy comparison and energy comparison involving outer obstacles (the grey regions represent $E_t \Delta F$).

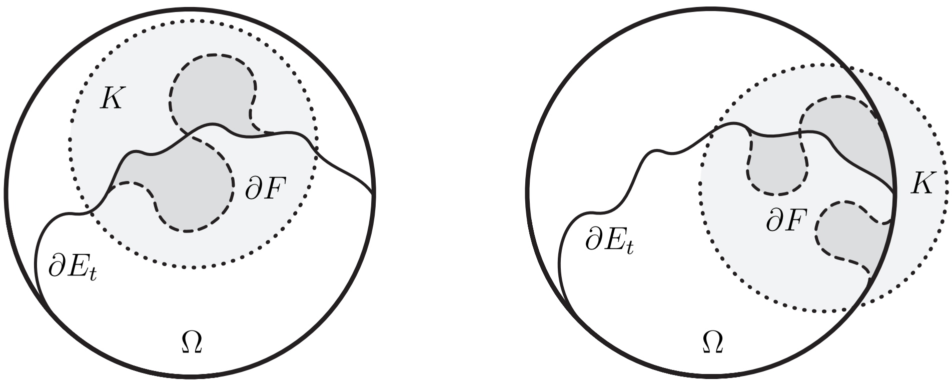

Figure 1.20: energy comparison for interior and obstacle cases

We will reserve the notation IMCF(Ω)+OBS($\partial\Omega$) to denote weak IMCFs in Ω that respect the outer obstacle $\partial\Omega$. Subsolutions and supersolutions of IMCF(Ω)+OBS($\partial\Omega$) can similarly be defined by requiring $E \subset F \subset \Omega$ or $F \subset E$ for all competitors F.

What we have made implicit in this notation is the dependence on the ambient manifold: when talking about solutions of IMCF(Ω)+OBS($\partial\Omega$), the domain Ω always sits inside a manifold M, and $\partial\Omega$ is a subset of the ambient manifold.

The outer minimizing property of level sets is similar to the interior case, except that all the comparisons are subject to the outer obstacle $\partial\Omega$. We say that a set E is *locally outward (perimeter-) minimizing* in $\overline{\Omega}$, if for any set F and domain K satisfying $E \subset F \subset \Omega$ and $F \setminus E \Subset K \Subset M$, we have

$$P(E; K) \leqslant P(F; K). \tag{1.4.4}$$

Strict outward minimizing sets, and thus minimizing hulls, can be defined in analogy with the interior case (see Subsection 3.3.2). In particular, we have:

Lemma 1.4.5 (see Theorem 3.3.12)**.** *Let u be a solution of* $\mathsf{IMCF}(\Omega)$+$\mathsf{OBS}(\partial\Omega)$. *Then*

(i) each E_t is locally outward minimizing in $\overline{\Omega}$,

(ii) each E_t^+ is strictly outward minimizing in $\overline{\Omega}$,

(iii) each E_t^+ is the minimizing hull of E_t in $\overline{\Omega}$, provided $E_t^+ \setminus E_t \Subset M$.

Below are two examples:

Example 1.4.6. See Figure 1.21: let B be a tiny ball, and we consider the IMCF in the domain $\Omega = M \setminus B$ with outer obstacle $\partial\Omega$, from the initial value E_0. In this case, the surfaces will first evolve by $1/H$ until the time T such that

$$e^T |\partial E_0| = |\partial B| + |S|.$$

Then, it has a non-trivial minimizing hull in $\overline{\Omega}$, thus will jump to the surface ∂E_T^+ as drawn in the figure. After the jump, ∂E_T^+ splits into two components $\partial B \cup S$. The component ∂B stays stationary, while the component S continues evolving by $1/H$.

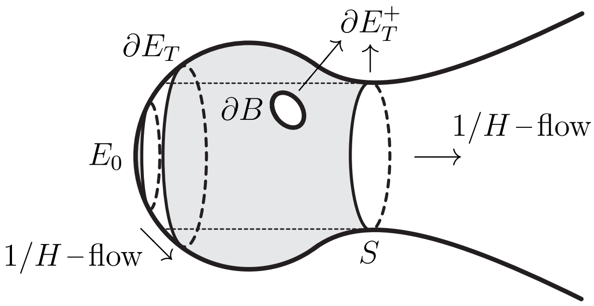

Figure 1.21: behavior of the flow in Example 1.4.6

Example 1.4.7. See Figure 1.22: the domain Ω is an ellipse with a puncture, and the initial value E_0 is the small disk marked with dark grey. The heuristic behavior of the flow is drawn in the figures by solid curves. The puncture blocks the movement, thus the curves move forward in the left and right channels around the puncture. At a certain time T, the two branches of ∂E_T merge together through a jump. See the left part of Figure 1.22: at the jump time, the lengths satisfy $A + C = B + D$.

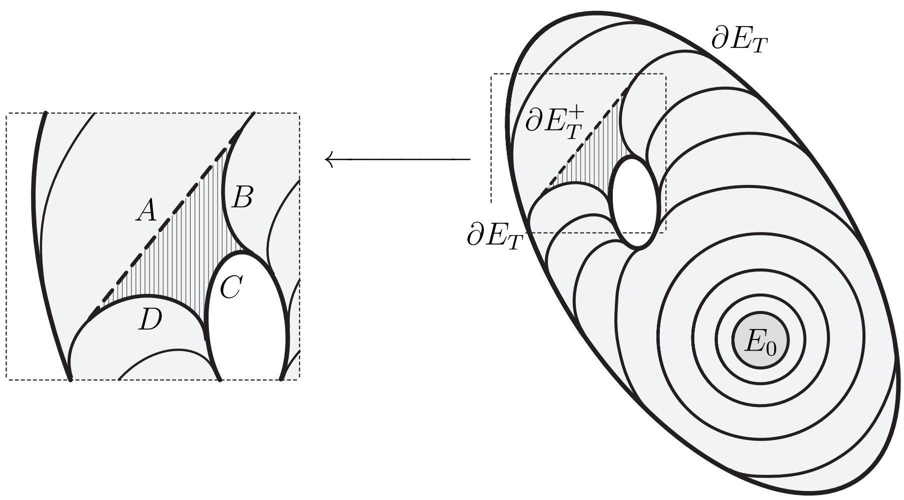

Figure 1.22: IMCF with obstacle in a punctured ellipse

It is important to notice that the area grows sub-exponentially in IMCF with outer obstacle:

$$\overline{E_t} \setminus E_s \Subset M \quad \Rightarrow \quad P(E_t) \leqslant e^{t-s} P(E_s), \qquad \forall\, s < t. \tag{1.4.5}$$

Indeed, by mutual comparison of energies $\widetilde{J}_u^K(E_t) = \widetilde{J}_u^K(E_s)$ (for any $K \Supset \overline{E_t} \setminus E_s$), we obtain

$$\begin{aligned} P(E_t) &= P(E_s) + \int_{E_t \setminus E_s} |\nabla u| \\ &= P(E_s) + \int_s^t P(E_\tau; \Omega)\, d\tau \qquad \text{(by coarea formula)} \\ &\leqslant P(E_s) + \int_s^t P(E_\tau)\, d\tau, \end{aligned}$$

which implies (1.4.5). From the derivation, we see that (1.4.5) becomes strict when ∂E_t starts to overlap nontrivially with $\partial\Omega$. An extreme case is in Example 1.4.1: there $P(E_t)$ grows exponentially for $t < 0$ but stays constant for $t \geqslant 0$.

Aspect 3: as a boundary-orthogonally calibrated IMCF. Running a calibration argument similar to Fact 1.2.5, we can show the compatibility between the energy (1.4.3) and the boundary tangency condition (1.4.2). Suppose $u \in C^\infty(\Omega)$ satisfies (1.4.1) (1.4.2). Further, assume (an ideal scenario of) sufficiently strong regularity and convergence. Let E_t be a sub-level set, and let $E \subset \Omega$ be a competitor with $E \Delta E_t \Subset M$. By the divergence theorem, we have (where ν denotes the outer unit normal of the objects):

$$\begin{aligned} P(E_t) &= \underbrace{\mathcal{H}^{n-1}(\partial E_t \cap \Omega)}_{\text{where } \nu_{E_t} = \nabla u/|\nabla u|} + \underbrace{\mathcal{H}^{n-1}(\partial E_t \cap \partial\Omega)}_{\text{where } \nu_{E_t} = \nu_\Omega = \nabla u/|\nabla u|} \\ &= \int_{\partial E_t} \frac{\nabla u}{|\nabla u|} \cdot \nu_{E_t} = \int_{\partial E} \frac{\nabla u}{|\nabla u|} \cdot \nu_E + \int_\Omega (\chi_{E_t} - \chi_E) \operatorname{div}\Big(\frac{\nabla u}{|\nabla u|}\Big) \\ &\leqslant P(E) + \int (\chi_{E_t} - \chi_E)|\nabla u|, \end{aligned} \tag{1.4.6}$$

which is exactly $\widetilde{J}_u(E_t) \leqslant \widetilde{J}_u(E)$.

To make sense of this argument for weak flows, again, we may replace the vector field $\nabla u/|\nabla u|$ by an abstract calibration. In particular, we have:

Lemma 1.4.8 (see Lemma 3.3.17)**.**
Let u be a solution of IMCF(Ω) *which is calibrated by ν. If $\nu \cdot \nu_\Omega = 1$ on $\partial\Omega$ in the sense of boundary trace, then the solution u respects the outer obstacle $\partial\Omega$ (in the sense described in Aspect 2).*

Aspect 4: as innermost (or maximal) solutions of initial value problems. Let $\Omega \Subset M$ be a smooth domain, and $E_0 \Subset \Omega$ be a $C^{1,1}$ domain. The initial value problem with obstacle is defined analogously with the interior case: we say that u is a solution of IVP($\Omega; E_0$)+OBS($\partial\Omega$), if

(i) $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ and $E_0 = \{u < 0\}$,

(ii) $u|_{\Omega \setminus \overline{E_0}}$ is a solution of IMCF($\Omega \setminus \overline{E_0}$)+OBS($\partial\Omega$) with ambient manifold $M \setminus \overline{E_0}$.

In condition (ii), note that ∂E_0 is not a part of the obstacle. The following is our main theorem regarding the existence and regularity of solutions to this initial value problem.

Theorem D. *Let $\Omega \Subset M$ be a smooth domain, and let $E_0 \Subset \Omega$ be a $C^{1,1}$ domain. Then there is a unique solution u of* $\mathsf{IVP}(\Omega; E_0)+\mathsf{OBS}(\partial\Omega)$*. Regarding u we have the following conclusions (for some $\alpha < 1$):*

(i) $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega) \cap \mathrm{BV}(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$,

(ii) u is calibrated by a vector field ν satisfying $\langle \nu, \nu_\Omega \rangle \geqslant 1 - Cd(x, \partial\Omega)^\alpha$.

(iii) ∂E_t is a $C^{1,\alpha}$ hypersurface in some small neighborhood of $\partial\Omega$, for all $t > 0$.

(iv) For any other solution v of $\mathsf{IVP}(\Omega; E_0)$*, we have $u \geqslant v$ in $\Omega \setminus E_0$.*

A refined version of Theorem D can be found in Theorem 3.6.1. Let us remark on a few aspects of this result:

Remark 1.4.9.

1. Note that item (ii) is a stronger form of Lemma 1.4.8.
2. In view of the Examples 1.4.3 in $\mathbb{R}^2$, $C^{1,\text{Hölder}}$ is the best possible regularity for the level sets. We do not know the best possible Hölder exponent.
3. Item (iv) states that the IMCF with outer obstacle coincides with the innermost solution introduced in the previous section. In fact, we will deduce Theorem C from Theorem D, by taking limit for an exhaustion of smooth bounded domains. See Section 4.1 for more details. (Thus, the actual logical order between Theorem C, D is reverse to our order of introduction.)
4. Item (i) further implies that $u \in L^\infty(\Omega)$, namely, the solution sweeps out the entire Ω within finite time. This feature strongly relies on the regularity of Ω. For instance, in a conic domain in $\mathbb{R}^2$, there exists a shrinking soliton that is unbounded above near the vertex, see Section 3.1. On the other hand, we have not used the full smoothness condition of $\partial\Omega$. From the proof of Theorem D, it follows that $\partial\Omega$ being C^3 is sufficient.

Theorem D is the most difficult result in this thesis. A sketch of proof is included in Subsection 3.6.2 (where we will need the setups in Subsection 3.6.1). Here let us explain the essential ingredient for showing maximality and $C^{1,\alpha}$ regularity.

Idea for maximality. The crucial observation is what we call an "automatic subsolution principle". The following lemma is a simpler version of it:

Fact 1.4.10. Suppose Ω is a locally Lipschitz domain. If a set E is locally outward minimizing in Ω, then it is locally outward minimizing in $\overline{\Omega}$.

Proof. Here, let us assume the case where Ω is a precompact $C^{1,1}$ domain. The general case is only of extra inessential complexity (see the proof of Lemma 2.2.3).

Suppose F is a competitor with $E \subset F \subset \Omega$. Our goal is to show that $P(E) \leqslant P(F)$, but this does not follow immediately from the lemma's condition.

We approximate F from inside. For $\varepsilon \ll 1$, consider the slightly smaller domain $\Omega_\varepsilon = \big\{x \in \Omega : d(x, \partial\Omega) > \varepsilon\big\}$. Notice that $E \cup (F \cap \Omega_\varepsilon)$ is a valid competitor with E. Hence we can compare

$$P\big(E; \Omega_{\varepsilon/2}\big) \leqslant P\big(E \cup (F \cap \Omega_\varepsilon); \Omega_{\varepsilon/2}\big).$$

By the cup-cap inequality, we have

$$P(F \cap \Omega_\varepsilon; \Omega_{\varepsilon/2}) \geqslant P\big(E \cap F \cap \Omega_\varepsilon; \Omega_{\varepsilon/2}\big) = P(E \cap \Omega_\varepsilon; \Omega_{\varepsilon/2}),$$

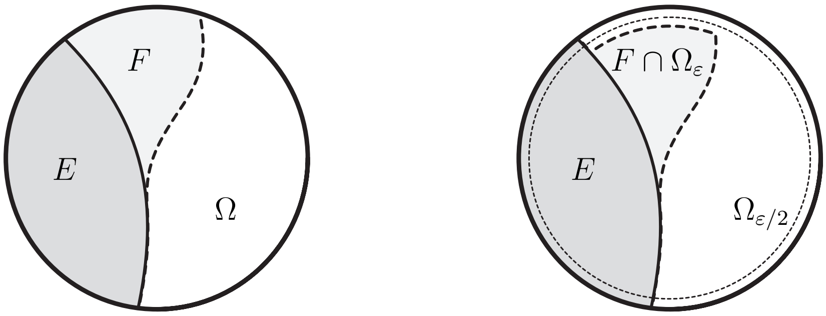

Figure 1.23: Inner approximation of sets

which trivially implies

$$P(F \cap \Omega_\varepsilon) \geqslant P(E \cap \Omega_\varepsilon).$$

Finally, we take $\varepsilon \to 0$. By the useful Lemma A.6.1, we have

$$P(F;K) = \lim_{\varepsilon \to 0} P(F \cap \Omega_\varepsilon; K), \qquad P(E;K) = \lim_{\varepsilon \to 0} P(E \cap \Omega_\varepsilon; K).$$

The result follows. □

The same fact does not hold for inward minimizing sets (this is an essential point to be noticed). There are obvious counterexamples: for any bounded domain Ω, we have that Ω is locally inward minimizing in Ω but not in $\overline{\Omega}$ (the empty set is a valid competitor).

In the same spirit with Fact 1.4.10, we have:

Lemma 1.4.11 (see Theorem 3.3.8)**.** *Let Ω be a locally Lipschitz domain. If u is a subsolution of* IMCF(Ω)*, then it is a subsolution of* IMCF(Ω)+OBS($\partial\Omega$)*.*

So it is always easy to produce subsolutions with outer obstacle but hard to produce supersolutions. With Lemma 1.4.11, the maximality in Theorem D(iv) is then implied by the following maximum principle:

Lemma 1.4.12 (see Corollary 3.3.20)**.**

Let $\Omega \Subset M$ be locally Lipschitz, and $E_0 \Subset \Omega$ be $C^{1,1}$. Suppose u, v are respectively a solution and subsolution of IVP($\Omega; E_0$)+OBS($\partial\Omega$)*. Then $u \geqslant v$ on $\Omega \setminus E_0$.*

Idea for $C^{1,\alpha}$ regularity. The key ingredient for the $C^{1,\alpha}$ regularity in Theorem D(iii) is a new parabolic estimate that we now explain. For simplicity, let us assume here that $\Omega = \{x_n < 0\} \subset \mathbb{R}^n$. Consider a solution $\{\Sigma_t\}$ of the smooth IMCF in Ω, that satisfy the boundary tangency condition

$$\lim_{x_n \to 0} \langle \nu, \partial_{x_n} \rangle = 1 \qquad \text{on each } \Sigma_t, \tag{1.4.7}$$

where ν is the outer unit normal of Σ_t. We wish to show that (1.4.7) implies the estimate

$$\langle \nu, \partial_{x_n} \rangle \geqslant 1 - C|x_n|^\gamma \qquad \text{in } \{-1/2 < x_n < 0\}, \tag{1.4.8}$$

for some constants $C > 0$, $\gamma \in (0,1)$.

Before starting this estimate, let us investigate the effect of (1.4.8). First, (1.4.8) implies that each Σ_t is tangent to $\partial\Omega$. So near $\partial\Omega$, we may express Σ_t as the graph over $\{x_n = 0\}$ of a function f with small gradient. We may do an approximate calculation

$$\langle \nu, \partial_{x_n} \rangle = \frac{1}{\sqrt{1+|\nabla f|^2}} \approx 1 - \frac{1}{2}|\nabla f|^2.$$

Hence, (1.4.8) implies an ODE inequality $|\nabla f| \leqslant C|f|^{\gamma/2}$. Solving this, we infer that near each $x \in \Sigma_t \cap \partial\Omega$, the hypersurface Σ_t is sandwiched between $\partial\Omega = \{x_n = 0\}$ and $\{x_n = -C|x'|^{1/(1-\gamma/2)}\}$. This is spiritually a $C^{1,\alpha}$ regularity statement for Σ_t at $\partial\Omega$.

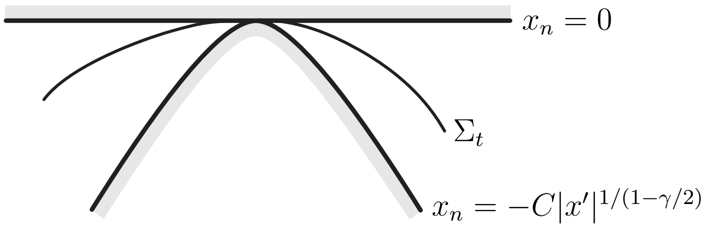

Figure 1.24: Hölder barrier from (1.4.8)

We now come back to the main estimate. It turns out that (1.4.7) alone is not enough to imply (1.4.8). We also need the following uniform condition on the normal vector:

$$\langle \nu, \partial_{x_n} \rangle > 0 \qquad \text{in } \{-1 < x_n < 0\}. \tag{1.4.9}$$

Without this condition, there can be IMCFs issuing from a pole, which do not satisfy (1.4.8). See Figure 1.25: the left case is the nephroid Example 1.4.4, while the right case is an IMCF issuing from an arbitrary interior point and respecting the obstacle $\{x_n = 0\}$. It is clear that both cases violate (1.4.8).

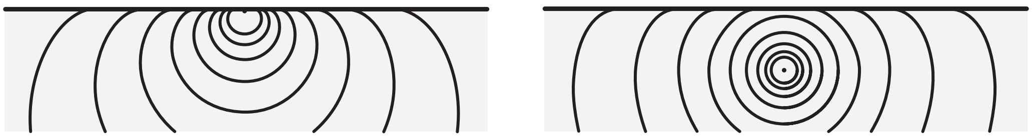

Figure 1.25: The failure of (1.4.8) without (1.4.9)

Let us now assume (1.4.7) (1.4.9) and start the estimate. We refer the reader to [31, 54] as helpful references for general parabolic estimates on the IMCF. Let $\Box = \partial_t - H^{-2}\Delta_\Sigma$ be the parabolic operator associated to the IMCF. In the region $\{-1 < x_n < 0\}$, consider the quantities

(1) $p = \langle \nu, \partial_{x_n} \rangle$,

(2) $\eta = \eta(x_n)$ a smooth positive function to be determined.

Denote $\eta' = \eta'(x_n)$ and $\eta'' = \eta''(x_n)$. We have the equations (see [31, Lemma 2.6, 2.8])

$$\Box p = \frac{|A|^2}{H^2} p, \qquad \Box \eta = 2\frac{p\eta'}{H} - \frac{\eta''}{H^2}(1-p^2). \tag{1.4.10}$$

Recall that we assumed $p > 0$ in (1.4.9). Hence, the right hand side of $\Box p$ is a strong positive term which turns out to dominate all the other terms.

Furthermore, we have the general product rule

$$\Box(XY) = X\Box Y + Y\Box X - \frac{2}{H^2}\langle\nabla_\Sigma X, \nabla_\Sigma Y\rangle, \tag{1.4.11}$$

and the useful formula

$$|\nabla_\Sigma \eta|^2 = (\eta')^2(1-p^2), \tag{1.4.12}$$

both can be directly verified.

Denote $q = 1-p$. Using (1.4.10) (1.4.11) (1.4.12) we calculate

$$\begin{aligned}\Box(q\eta) &= q\Box\eta + \eta\Box q - \frac{2}{H^2}\langle\nabla_\Sigma q, \nabla_\Sigma\eta\rangle \\ &= 2q\frac{p\eta'}{H} - \frac{\eta''}{H^2}q^2(1+p) - \frac{|A|^2}{H^2}p\eta - \frac{2}{H^2\eta}\langle\nabla_\Sigma(q\eta), \nabla_\Sigma\eta\rangle + \frac{2}{H^2}\frac{(\eta')^2}{\eta}q^2(1+p).\end{aligned}$$

Using Young's inequality, the trace inequality $|A|^2 \geqslant H^2/(n-1)$, and noting that $p > 0$, we have

$$\begin{aligned}\Box(q\eta) \leqslant& \Big[\frac{p\eta}{n-1} + \frac{q^2}{H^2}(n-1)p\frac{(\eta')^2}{\eta}\Big] \\ &- \frac{\eta''}{H^2}q^2(1+p) - \frac{p\eta}{n-1} + \frac{2}{H^2}\frac{(\eta')^2}{\eta}q^2(1+p) - \frac{2}{H^2\eta}\langle\nabla_\Sigma(q\eta), \nabla_\Sigma\eta\rangle \\ =& \frac{q^2}{H^2}\Big[-\eta''(1+p) + \frac{(\eta')^2}{\eta}(np+p+2)\Big] - \frac{2}{H^2\eta}\langle\nabla_\Sigma(q\eta), \nabla_\Sigma\eta\rangle.\end{aligned}$$

Now choose $\eta(x_n) = (\delta - x_n)^{-\gamma}$, where $0 < \delta \ll 1$ and $0 < \gamma < 1$. We obtain

$$\Box(q\eta) \leqslant \frac{q^2}{H^2}(\delta - x_n)^{-\gamma-2}\Big[-\gamma(\gamma+1)(1+p) + \gamma^2(np+p+2)\Big] - \frac{2}{H^2\eta}\langle\nabla_\Sigma(q\eta), \nabla_\Sigma\eta\rangle.$$

Choosing $\gamma = \frac{2}{n+1}$ and simplifying the expressions, we have

$$\Box(q\eta) \leqslant -\frac{2(n-1)}{(n+1)^2}\frac{q^3}{H^2}(\delta - x_n)^{-\gamma-2} - \langle\nabla_\Sigma(q\eta), X\rangle,$$

for some vector field X. This suggests from maximum principles that an estimate of the type

$$\sup_{-1<x_n\leqslant 0}(q\eta) \leqslant \max\Big\{\sup_{x_n=-1}(q\eta), \sup_{x_n\to 0^-}(q\eta)\Big\} \leqslant 4.$$

is expected. Note that $\lim_{x_n\to 0^-} q = 0$ by the tangency condition (1.4.7). Taking $\delta \to 0$, the estimate becomes

$$\langle\nu, \partial_{x_n}\rangle \geqslant 1 - C|x_n|^\gamma,$$

which is our desired result.

The above calculation is a simplified version of Section 3.5. There the full estimate is done in C^3 domains in Riemannian manifolds.

The remaining obvious question is why a uniform bound like (1.4.9) is obtainable in the context of Theorem D. This is also asking why the nephroid example 1.4.4 does not appear in Theorem D. Very roughly speaking, this is because $u \geqslant 0$ in an initial value problem, while the nephroid example is unbounded from below. This observation is linked to the following Liouville theorem:

Lemma 1.4.13 (see Theorem 3.4.1)**.** *Let* $\Omega = \{x_n < 0\} \subset \mathbb{R}^n$*, and suppose that* $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ *is a solution of* IMCF(Ω)+OBS($\partial\Omega$)*. Moreover, suppose*

$$\inf_{\Omega}(u) > -\infty \qquad \text{and} \qquad |\nabla u(x)| \leqslant \frac{C}{|x_n|} \ \text{ for a.e. } x \in \Omega. \tag{1.4.13}$$

Then u must be constant.

The way to deduce the uniform bound (1.4.9) from Lemma 1.4.13 is a blow-up argument. Very roughly speaking, if (1.4.9) does not hold, then there is a sequence of worse and worse examples, which by taking a limit leads to a solution violating Lemma 1.4.13. A precise description of this process requires much more setup. We decide to end the explanations here, and direct interested readers to Section 3.6.

Aspect 5: as an IMCF with Dirichlet boundary condition. This is related to the fact that the IMCF equation $\mathrm{div}\big(\frac{\nabla u}{|\nabla u|}\big) = |\nabla u|$ is a 1-Laplacian type equation. Here, the 1-Laplacian is defined as

$$\Delta_1(u) = \mathrm{div}\Big(\frac{\nabla u}{|\nabla u|}\Big).$$

It is the Euler-Lagrange operator for the energy $u \to \int |\nabla u|$, and is degenerate-elliptic. It is a general phenomenon that the Dirichlet problem for 1-Laplacian type equations has an outer obstacle feature. An example is the first Dirichlet eigenvalue of the 1-Laplacian, defined as

$$\lambda_1(\Delta_1) := \inf_{u \in \mathrm{Lip}_0(\Omega)} \frac{\int_\Omega |\nabla u|}{\int_\Omega |u|}. \tag{1.4.14}$$

However, the infimum in this definition is not attained by any function in $\mathrm{Lip}_0(\Omega)$. For example, taking $\Omega = B(0,1) \subset \mathbb{R}^n$, for any $u \in \mathrm{Lip}_0(\Omega)$ we have

$$\int |\nabla u| = \int |\nabla |u|| = \int_0^\infty P\big(\{|u| > t\}\big)\, dt > n \int_0^\infty \big|\{|u| > t\}\big|\, dt = n \int |u|,$$

where we used the Poincaré inequality

$$P(E) > n|E|, \qquad \forall\, E \Subset B(0,1).$$

The “minimizer” for (1.4.14) should be the constant function 1, which is a bit problematic since $1 \notin \mathrm{Lip}_0(\Omega)$. The resolution is to consider the relaxed energy

$$\lambda_1^D(\Delta_1) = \inf_{u \in \mathrm{BV}(\Omega)} \frac{\|Du\|(\Omega) + \int_{\partial\Omega} |u|}{\int_\Omega |u|}, \tag{1.4.15}$$

where the term $\int_{\partial\Omega} |u|$ is in the sense of boundary trace. In (1.4.15), we allow u to take nonzero value on $\partial\Omega$, but at the cost of the penalty term. For a minimizer u of (1.4.15), every super-level set of u is a Cheeger set, namely, a solution of the constrained minimizing problem

$$\mathrm{Ch}(\Omega) := \inf\Big\{\frac{P(E)}{|E|} : E \subset \Omega\Big\}. \tag{1.4.16}$$

See [60, 61] and references therein. In particular, the obstacle problem (1.4.16) is naturally imposed for each super-level set.

For the weak IMCF with outer obstacle, the following energy can be viewed as the relaxation of the interior energy (1.2.8)

$$\widetilde{J}_u^K(v) = \int_{\Omega\cap K} \big(|\nabla v| + v|\nabla u|\big) - \int_{\partial\Omega\cap K} v. \tag{1.4.17}$$

The formulation based on this energy turns out to be equivalent to our previous definition using $\widetilde{J}_u^K(E)$. See Subsection 3.3.3 for more details. Note that the boundary integral in (1.4.17) has negative sign. This suggests that energy minimizers tend to be maximized at $\partial\Omega$. This is consistent with the fact that solutions of IMCF with outer obstacle are maximal.

Going further in this thread of thoughts, we may also view the IMCF with outer obstacle as a "solution" with the following "properness Dirichlet condition":

$$\begin{cases} \operatorname{div}\Big(\dfrac{\nabla u}{|\nabla u|}\Big) = |\nabla u| & \text{in } \ \Omega, \\ u = +\infty & \text{on } \ \partial\Omega. \end{cases}$$

As the solution of a 1-Laplacian type equation, we do not expect u to really attain $+\infty$ on $\partial\Omega$. In turn, the boundary term $\int_{\partial\Omega} u$ in (1.4.3) is the "penalty for not achieving $+\infty$".

The above ideas may be made more clear when being connected to the topic of p-harmonic functions. Let $p > 1$, recall that the p-Laplacian is defined as

$$\Delta_p u = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big).$$

This is the Euler-Lagrange operator for the energy $u \mapsto \int |\nabla u|^p$, and is strictly elliptic.

The first Dirichlet eigenvalue of Δ_p in a bounded domain Ω is defined as

$$\lambda_1^D(\Delta_p) := \inf_{u\in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}. \tag{1.4.18}$$

It is well-known that there exists a minimizer $u \in W_0^{1,p}(\Omega)$ of this energy, and u solves the equation $\Delta_p u = -\lambda_1 |u|^{p-1}$ weakly in Ω. The eigenvalues and eigenfunctions of Δ_p and Δ_1 have an intimate relation: it can be shown, see [60], that $\lambda_1^D(\Delta_1) = \lim_{p\to 1}\lambda_1^D(\Delta_p)$. Moreover, for a sequence of Δ_p-eigenfunctions u_p normalized in some suitable sense, there is a subsequence that L^1 converges to a Δ_1-eigenfunction (i.e., a minimizer of (1.4.15)).

We are thus motivated to find the IMCF analogue of this convergence. Recall Moser's discovery [89] of the connection between p-harmonic functions and the IMCF: if v is a positive p-harmonic function, then $u := (1-p)\log v$ is a weak solution of the equation

$$\Delta_p u = |\nabla u|^p, \tag{1.4.19}$$

which is formally the IMCF equation when $p = 1$. This is made precise through the following statement [89]: if $\{v_p\}$ are positive p-harmonic functions in a domain Ω, and the convergence

$$(p-1)\log v_p \to u$$

holds in $C^0_{\mathrm{loc}}(\Omega)$ for a sequence $p \to 1$, then u is a weak solution of IMCF. This relation was further explored in [15, 65, 79] to solve the initial value problem of weak IMCF.

Now given $E_0 \Subset \Omega \Subset M$. The *p-capacitary potential* associated to Ω and E_0 is the unique solution $u \in W_0^{1,p}(\Omega)$ of the following equation:

$$\begin{cases} \Delta_p v_p = 0 & \text{in } \Omega \setminus \overline{E_0}, \\ v_p = 1 & \text{on } \partial E_0, \\ v_p = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4.20}$$

By the classical analytic results [71, 72], we have $v_p \in C^{1,\alpha}(\overline{\Omega} \setminus E_0)$. Here the exponent α and the $C^{1,\alpha}$ norm is generally not uniformly controlled as $p \to 1$.

Note that v_p is the smallest p-harmonic function in $\Omega \setminus \overline{E_0}$ with $v_p|_{\partial E_0} = 1$. Taking Moser's transformation $u_p = (p-1)\log v_p$, we see that

$$u_p|_{\partial E_0} = 0, \qquad u_p \geqslant 0, \qquad u_p|_{\partial\Omega} = +\infty,$$

and u_p is the largest solution of (1.4.19) in $\Omega \setminus \overline{E_0}$ that takes value 0 on ∂E_0. The following theorem shows that, as $p \to 1$, the functions u_p converge to the IMCF with outer obstacle. This is reasonable in the sense that the limit of maximal solutions should be maximal.

Theorem E. *Let $\Omega \Subset M$ be a smooth domain, and $E_0 \Subset \Omega$ be a $C^{1,1}$ domain. Let v_p solve* (1.4.20)*, and set $u_p = (1-p)\log v_p$. Then a subsequence of u_p converges in $C^0_{\mathrm{loc}}(\Omega \setminus E_0)$ to the unique solution of* $\mathsf{IVP}(\Omega; E_0)$+$\mathsf{OBS}(\partial\Omega)$*, as $p \to 1$.*

This theorem is a direct consequence of the proof of Theorem D and a result of Benatti-Pluda-Pozzetta [15]; see Section 3.7 for more details. The search for a potential-theoretic proof of this result may be of its own interest. Theorem E will appear in the note [14] joint with L. Benatti, L. Mari, M. Rigoli, and A. Setti.

We end this section with a remark on other boundary conditions that have appeared in the literature.

Remark 1.4.14. The IMCF with Neumann condition, or free boundary condition, has received some attention in past references. See the works of Marquart [83, 84], and a nice application due to Koerber [64]. The IMCF with Neumann condition is characterized by the energy

$$\hat{J}_u^K(E) = P(E; K \cap \Omega) - \int_{E \cap K \cap \Omega} |\nabla u|, \qquad E \subset \Omega.$$

Namely, we only include the part of perimeter in Ω. This has the effect that every ∂E_t is perpendicular to $\partial\Omega$. The initial value problem can be similarly defined. We notice that if $E_0 \Subset \Omega \Subset M$, the only solution of the initial value problem with Neumann condition is $u \equiv 0$. Indeed, we may always compare (for any $t > 0$ and $K \Supset \Omega$)

$$\hat{J}_u^K(E_t) \leqslant \hat{J}_u^K(\Omega) \quad \Rightarrow \quad P(E_t; \Omega) \leqslant P(\Omega; \Omega) - \int_{\Omega \setminus E_t} |\nabla u| = -\int_{\Omega \setminus E_t} |\nabla u|.$$

This immediately implies $u = 0$. Thus, no nontrivial theory comes out for Neumann conditions in bounded domains.

Remark 1.4.15. In parallel, the obstacle problem for mean curvature flow has been studied in some depth. See, for example, [2, 74, 87] for the smooth flow, and [45, 86, 96] for the level set flow. The flow is based on viscosity characterizations in these works, and one often obtains $C^{1,1}$ regularity for the solution, see e.g. [87, 96].

1.5 IMCF and scalar curvature

Let $\{\Sigma_t\}$ be a smooth solution of IMCF by compact surfaces in a 3-manifold. Then the following formula can be directly calculated, see for example [53, p. 395-396]:

$$\frac{d}{dt}\int_{\Sigma_t} H^2 \leqslant 4\pi\chi(\Sigma_t) - \int_{\Sigma_t} R - \frac{1}{2}\int_{\Sigma_t} H^2. \tag{1.5.1}$$

Particularly, the Euler characteristic term arises from the use of Gauss-Bonnet formula (which is the reason that we restrict to dimension three).

Now consider the weak IMCF. By our heuristic picture, a weak IMCF is a combination of $1/H$-flow and jumpings. In the $1/H$-flow periods, (1.5.1) should remain true. In a jump, the hypersurface moves from E_t to its minimizing hull E_t^+. Note that $\partial E_t^+ \setminus \partial E_t$ is a minimal surface. Hence through a jump, a part of H^2 is lost and $\int H^2$ is expected to decrease as well. This suggests that (1.5.1) may hold for the weak IMCF as well, in some weak sense. Through a nontrivial analytic argument, [53] showed that this is indeed the case: if u solves $\mathsf{IMCF}(\Omega)$, and $s < t \in \mathbb{R}$ is such that $E_t \setminus E_s \Subset \Omega$, then we have

$$\int_{\partial E_t} H^2 - \int_{\partial E_s} H^2 \leqslant \int_s^t \Big(4\pi\chi(\Sigma_\tau) - \int_{\Sigma_\tau} R - \frac{1}{2}\int_{\Sigma_\tau} H^2\Big)\,d\tau, \tag{1.5.2}$$

which is known as the weak Geroch monotonicity. This differential inequality is the key to most of the applications of IMCF to scalar curvature. It played a main role in the following works:

- Riemannian Penrose inequality for single horizon [53];
- The fact that the Yamabe constant of $\mathbb{RP}^3$ is realized by the round metric [20];
- Mass-capacity inequality for asymptotically flat manifolds [19];
- Works on the hyperbolic Penrose inequality [70, 93];
- (Anti-)isoperimetric inequalities on asymptotically flat manifolds [101], and the existence of isoperimetric sets [6] [24, Appendix K];
- The study of renormalized volume and isoperimetric sets in asymptotically hyperbolic spaces [21, 29];
- The isoperimetric Penrose inequality [12], see also [11]; a weak isoperimetric positive mass theorem [6];
- An IMCF proof of the Ricci pinching conjecture [55].

In this thesis, we prove the following two theorems as new applications:

Theorem F. *Suppose M is a closed 3-manifold such that $\pi_2(M) \neq 0$ and M is not covered by $S^2 \times S^1$. Then for any metric g on M we have*

$$\operatorname{sys}\pi_2(M,g)\cdot\min_M R_g \leqslant 24\pi\cdot\frac{2-\sqrt{2}}{4-\sqrt{2}} \qquad (\approx 5.44\pi). \tag{1.5.3}$$

Here, the π_2-systole of a manifold is defined as the minimal area of homotopically nontrivial immersed 2-spheres. For the motivation, related results and proof details of this theorem, we refer the reader to Section 5.2. Very roughly speaking, Theorem F is proved by running a proper weak IMCF on the universal cover of M.

With O. Chodosh and Y. Lai [30] we showed the following theorem:

Theorem G. *Let (M, g) be a complete, connected, contractible Riemannian 3-manifold satisfying $R \geqslant 0$ and bounded geometry:*

$$|\,\mathrm{Rm}\,| \leqslant \Lambda^2, \qquad \mathrm{inj} \geqslant \Lambda^{-1}. \tag{1.5.4}$$

Then M is diffeomorphic to $\mathbb{R}^3$.

It is also shown in [30] that handlebodies with genus $\geqslant 2$ do not admit metrics with $R \geqslant 0$ and bounded geometry. Theorem G is proved by running the innermost weak IMCF on M from a tiny geodesic ball. Again, we refer to Section 5.3 for relevant discussions.

Chapter 2

Interior weak IMCF

Section 2.1 is a technical review of the weak IMCF. In Section 2.2, we prove some useful auxiliary results which are for later use. In Section 2.3 we introduce the notion of calibrated IMCF. In Section 2.4, we summarize Huisken-Ilmanen's elliptic regularization process. Finally, in Section 2.5 and 2.6, we prove Theorems A, B respectively.

2.1 Preliminaries on the weak IMCF

In this section, we collect the precise definitions and basic properties of the weak IMCF. We closely follow the introduction chapter, as well as the materials in [53, p. 364–375]. Some proof details are provided for the reader's convenience.

2.1.1 Weak IMCF in a domain

Definition 2.1.1. Let $K \Subset \Omega \subset M$ be domains, and $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, and E be a set with locally finite perimeter in Ω. We define the energy

$$J_u^K(E) := P(E;K) - \int_{E\cap K} |\nabla u|. \tag{2.1.1}$$

We say that a set E locally minimizes J_u (resp. minimizes from outside, inside) in Ω, if for all F (resp. for all $F \supset E$, $F \subset E$) and domain K satisfying $E\Delta F \Subset K \Subset \Omega$, it holds $J_u^K(E) \leqslant J_u^K(F)$.

Definition 2.1.2 (weak IMCF)**.**
We say that $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ is a (sub-, super-) solution of $\mathsf{IMCF}(\Omega)$, if for each $t \in \mathbb{R}$, the sub-level set $E_t := \{u < t\}$ locally minimizes J_u (resp. locally minimizes from outside, inside) in Ω in the sense of Definition 2.1.1.

Solutions satisfying Definition 2.1.2 will usually be called interior weak solutions, to distinguish them from weak solutions with obstacles defined in the next chapter. When there is a need to clarify the background Riemannian metric, we will write $\mathsf{IMCF}(\Omega, g)$ to denote weak solutions of IMCF in the domain Ω with the metric g.

A few remarks and useful facts are in order:

Remark 2.1.3. A more well-known definition of the weak IMCF is the following. For functions $u, v \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ and domain $K \Subset \Omega$, define

$$J_u^K(v) := \int_K |\nabla v| + v|\nabla u|. \tag{2.1.2}$$

Then $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ is a (sub–, super–) solution of $\mathsf{IMCF}(\Omega)$ if and only if for all $v \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ (resp. for all $v \leqslant u$, $v \geqslant u$) and domain K satisfying $\{u \neq v\} \Subset K \Subset \Omega$, one has

$$J_u^K(u) \leqslant J_u^K(v).$$

The equivalence of these two definitions is proved in [53, Lemma 1.1].

Remark 2.1.4. The following properties of the weak IMCF may be useful.

(i) If u solves $\mathsf{IMCF}(\Omega)$, then u solves $\mathsf{IMCF}(\Omega')$ for any sub-domain $\Omega' \subset \Omega$. We warn that there is no converse statement: if $\Omega = \bigcup \Omega_i$ and u solves $\mathsf{IMCF}(\Omega_i)$ for each i, then u may not be a solution of $\mathsf{IMCF}(\Omega)$.

(ii) u is a solution of $\mathsf{IMCF}(\Omega)$ if and only if u is simultaneously a subsolution and supersolution of $\mathsf{IMCF}(\Omega)$. Indeed, let $t \in \mathbb{R}$ and E be a competitor set of E_t, with $E \Delta E_t \Subset K \Subset \Omega$. We may use the subsolution property to compare

$$J_u^K(E_t) \leqslant J_u^K(E \cup E_t) \quad \Rightarrow \quad P(E_t; K) \leqslant P(E \cup E_t; K) - \int_{E \setminus E_t} |\nabla u|.$$

Then, using the supersolution property, we may compare

$$J_u^K(E_t) \leqslant J_u^K(E \cap E_t) \quad \Rightarrow \quad P(E_t; K) \leqslant P(E \cap E_t; K) + \int_{E_t \setminus E} |\nabla u|.$$

Finally, the cup-cap inequality (A.1.9) states that

$$P(E \cup E_t; K) + P(E \cap E_t; K) \leqslant P(E; K) + P(E_t; K).$$

Adding these together, we obtain exactly $J_u^K(E_t) \leqslant J_u^K(E)$.

(iii) For any $K \Subset \Omega$, we have $\int_K |\nabla u| \leqslant P(K)$. This follows by choosing sufficiently negative t and comparing $J_u^{K'}(\emptyset) = J_u^{K'}(E_t) \leqslant J_u^{K'}(K)$, $\forall\, K' \Supset K$.

(iv) If u is a (sub–, super–) solution of $\mathsf{IMCF}(\Omega)$, then so is $u_T := \min\{u, T\}$ for any $T \in \mathbb{R}$. First consider the supersolution case: suppose $t \in \mathbb{R}$ and $E \subset E_t(u_T)$ is a competitor with $E_t(u_T) \setminus E \Subset K \Subset \Omega$. If $t > T$, then $E_t(u_T) = \Omega$, so we must have $E = \Omega \setminus L$ for some $L \Subset K$. Then

$$J_u^K(E) - J_u^K(E_T) = P(L) + \int_L |\nabla u| \geqslant 0.$$

If $t \leqslant T$, then $E_t(u_T) = E_t(u)$ and $u = u_T$ inside $E_t(u_T)$, so we have

$$J_{u_T}^K(E_t(u_T)) = J_u^K(E_t(u)) \leqslant J_u^K(E) = J_{u_T}^K(E).$$

Then consider the subsolution case: suppose $E \supset E_t(u_T)$ is a competitor with $E \setminus E_t(u_T) \Subset K \Subset \Omega$. If $t > T$, then we have $E_t(u_T) = E = \Omega$ and the result trivially follows. If $t \leqslant T$, then

$$\begin{aligned} J_{u_T}^K(E) - J_{u_T}^K(E_t(u_T)) &= P(E; K) - P(E_t(u); K) - \int_{E \setminus E_t(u)} |\nabla u_T| \\ &\geqslant P(E; K) - P(E_t(u); K) - \int_{E \setminus E_t(u)} |\nabla u| \\ &= J_u^K(E) - J_u^K(E_t(u)) \geqslant 0. \end{aligned}$$

(v) Exponential growth of area: if $E_t \setminus E_s \Subset \Omega$ for some $s < t \in \mathbb{R}$, then

$$P(E_t) = e^{t-s} P(E_s). \tag{2.1.3}$$

To prove this, we take any K with $E_t \setminus E_s \Subset K \Subset \Omega$. Then $J_u^K(E_t)$ must be equal to $J_u^K(E_s)$ since they are both local energy minimizers. Hence

$$P(E_t) = P(E_s) + \int_{E_t \setminus E_s} |\nabla u| = \int_s^t P(E_\tau)\, d\tau$$

by expanding the energy functional and plugging the coarea formula. Using Gronwall's inequality, (2.1.3) follows.

(vi) Constant functions are solutions of $\mathsf{IMCF}(\Omega)$. If Ω is compact (for example, a closed manifold), then constant functions are the only solutions of $\mathsf{IMCF}(\Omega)$. This follows by comparing $J_u^\Omega(E_t) \leqslant J_u^\Omega(\Omega) \Rightarrow P(E_t) = 0$ for all t.

(vii) If u solves $\mathsf{IMCF}(\Omega)$, then each E_t^+ locally minimizes J_u as well. This is proved using the standard set-replacing argument (see Lemma A.2.5).

Lemma 2.1.5 (outward minimization)**.** *Suppose u solves* $\mathsf{IMCF}(\Omega)$*. Then:*
(i) each E_t is locally outward minimizing in Ω,
(ii) each E_t^+ is strictly outward minimizing in Ω,
(iii) if $E_t^+ \setminus E_t \Subset \Omega$, then E_t^+ is the minimizing hull of E_t in Ω.

Proof. See Facts 1.2.9 ∼ 1.2.14 in the introduction. □

The measure-theoretic properties of sub-level sets are important. Suppose u solves $\mathsf{IMCF}(\Omega)$. In Lemma 2.1.5, we may compare E_t with $E_t \cup K$, for all $K \Subset \Omega$. This gives

$$P(E_t; K) \leqslant P(K), \tag{2.1.4}$$

which is a uniform mass bound for level sets. Hence, for a sequence of sub-level sets of IMCFs, we may invoke Theorem A.1.2 to extract a limit.

For each t and any competitor E with $E \Delta E_t \Subset K \Subset \Omega$, the minimization of (2.1.1) implies

$$P(E_t; K) \leqslant P(E; K) + \sup_K |\nabla u| \cdot |E \Delta E_t|. \tag{2.1.5}$$

Therefore, each E_t is an almost perimeter minimizer in Ω as defined in Section A.2. Furthermore, it holds that E_t coincides with its measure-theoretic interior (see (A.1.1)):

Lemma 2.1.6. *Suppose $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ is a solution of* $\mathsf{IMCF}(\Omega)$*. Then for every t we have $E_t = E_t^{(1)}$ and $\Omega \setminus E_t^+ = (\Omega \setminus E_t^+)^{(1)}$.*

Proof. We first show $E_t = E_t^{(1)}$. As E_t is open, it suffices to show $E_t^{(1)} \cap \partial E_t = \emptyset$. Suppose $x \in \partial E_t$. Since u is continuous, there exists a sequence of times $t_i \nearrow t$ and points $x_i \in \partial^* E_{t_i}$ with $x_i \to x$. Since $E_t = \bigcup_i E_{t_i}$, by (2.1.5) and Theorem A.2.2(iii) it follows that $x \notin E_t^{(1)}$. The other statement follows from the same argument, by approximating with $t_i \searrow t$. □

As a result, we have the following facts:

Lemma 2.1.7 (regularity of level sets)**.** *Suppose u solves $\mathsf{IMCF}(\Omega)$. Then*

(i) Each ∂E_t and ∂E_t^+ is a $C^{1,\alpha}$ ($\forall\,\alpha < 1/2$) hypersurface except for a singular set of Hausdorff dimension at most $n-8$.

(ii) For each $t \in \mathbb{R}$, we have $\partial E_s \to \partial E_t$ when $s \nearrow t$ and $\partial E_s \to \partial E_t^+$ when $s \searrow t$: the convergence holds in the local $C^{1,\beta}$ topology ($\forall\,\beta < 1/2$) when $n \leqslant 7$, and in local Hausdorff topology in all dimensions.

Proof. This follows by combining Lemma 2.1.6 and Theorems A.2.1, A.2.2. □

2.1.2 Initial value problem

Definition 2.1.8 (initial value problem; cf. [53, p. 367-368])**.**

Given a domain $\Omega \subset M$ and a $C^{1,1}$ domain $E_0 \subset \Omega$ such that $\partial E_0 \cap \partial\Omega = \emptyset$. We say that u is a (sub-, super-) solution of $\mathsf{IVP}(\Omega; E_0)$, if

$$u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega), \qquad E_0 = \{u < 0\},$$

and one of the following equivalent conditions holds.

(1) $u|_{\Omega\setminus\overline{E_0}}$ is a (sub-, super-) solution of $\mathsf{IMCF}(\Omega\setminus\overline{E_0})$.

(2) For any $t > 0$, any set $E \supset E_0$ (resp. any $E \supset E_t$, $E_0 \subset E \subset E_t$) and domain K with $E\Delta E_t \Subset K \Subset \Omega$, we have $J_u^K(E_t) \leqslant J_u^K(E)$.

Proof of equivalence.

The proof is quite technical; the reader may skip the details at first read.

(1) ⇒ (2). Let E_t, E, K be as in (2). The possible issue is that ∂E may touch ∂E_0, hence E may not be a valid competitor when trying to use (1). To resolve this issue, we use an approximation argument. We may slightly modify K so that $\mathcal{H}^{n-1}(\partial K \cap \partial E_0) = 0$. For $\varepsilon > 0$, consider the slightly larger sets

$$E_0^\varepsilon = \big\{d(\cdot, \Omega) < \varepsilon\big\}, \qquad E^\varepsilon = E \cup E_0^\varepsilon.$$

Since $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, we have $\partial E_t \cap \partial E_0 = \emptyset$. It can be checked that $E_t \Delta E^\varepsilon \Subset \Omega \setminus \overline{E_0}$, and in the case of supersolution, $E^\varepsilon \subset E_t$ for all small enough ε. Now we may compare

$$J_u^K(E_t) + \int_{K\cap E_0^{\varepsilon/2}} |\nabla u| = J_u^{K\setminus E_0^{\varepsilon/2}}(E_t) \leqslant J_u^{K\setminus E_0^{\varepsilon/2}}(E^\varepsilon) = J_u^K(E^\varepsilon) + \int_{K\cap E_0^{\varepsilon/2}} |\nabla u|,$$

which implies

$$P(E_t; K) - \int_{E_t\cap K} |\nabla u| \leqslant P(E^\varepsilon; K) - \int_{E^\varepsilon\cap K} |\nabla u|.$$

Now we take $\varepsilon \to 0$. Apply Lemma A.6.1 with "Ω" $= \Omega \setminus E_0$, "Ω_i" $= \Omega \setminus E_0^{\varepsilon_i}$ for a sequence $\varepsilon_i \to 0$, and "A" $= \Omega \setminus E$. As a result, we have $\lim_{\varepsilon\to 0} P(E^\varepsilon; K) = P(E; K)$. It is obvious that $\lim_{\varepsilon\to 0} \int_{E^\varepsilon\cap K} |\nabla u| = \int_{E\cap K} |\nabla u|$. This proves the desired conclusion.

(2) ⇒ (1). Denote $\widetilde{u} = u|_{\Omega\setminus E_0}$. Let $t \in \mathbb{R}$, and E be a competitor with $E\Delta E_t(\widetilde{u}) \Subset K \Subset \Omega \setminus \overline{E_0}$. We need to show that $J_{\widetilde{u}}^K(E_t(\widetilde{u})) \leqslant J_{\widetilde{u}}^K(E)$. If $t > 0$, then this follows verbatim from (2). Now assume $t \leqslant 0$. Then we note that $E_t(\widetilde{u}) = \emptyset$, so we are reduced to showing that

$$\int_E |\nabla u| \leqslant P(E), \qquad \forall\, E \Subset \Omega \setminus E_0.$$

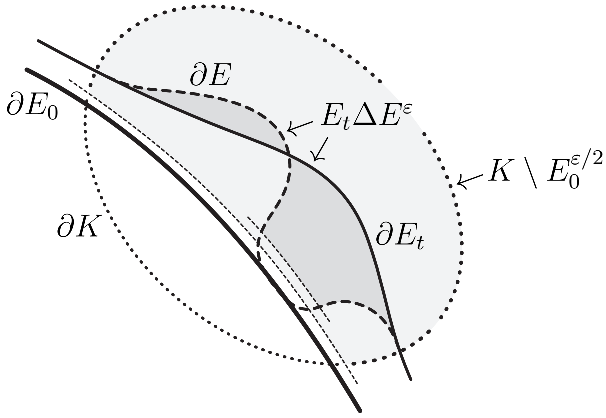

Figure 2.1: proof of equivalence, (1) ⇒ (2)

Using item (2), for all $t > 0$ we can compare

$$J_u^K(E_t) \leqslant J_u^K(E_t \cup E) \quad \Rightarrow \quad P(E_t; K) + \int_{E \setminus E_t} |\nabla u| \leqslant P(E \cup E_t; K).$$

By the cup-cap inequality (A.1.9), this implies

$$P(E \cap E_t; K) + \int_{E \setminus E_t} |\nabla u| \leqslant P(E; K).$$

Dropping the first term, letting $t \to 0$, and noting that $\int_{E \setminus E_t} |\nabla u| \to \int_{E \setminus E_0^+} |\nabla u| = \int_E |\nabla u|$, we obtain the desired conclusion. □

Similarly, we will write $\mathsf{IVP}(\Omega, g; E_0)$ when we need to clarify the background metric.

Lemma 2.1.9. *Suppose u solves $\mathsf{IVP}(\Omega; E_0)$. Then for any $t \geqslant 0$, any set $E \supset E_0$ and domain K with $E \Delta E_t \Subset K \Subset \Omega$, we have $J_u^K(E_t^+) \leqslant J_u^K(E)$.*

Proof. Note that $E_t^+ = \bigcap_{s>t} E_s$, and each E_s ($s > t \geqslant 0$) satisfies the condition in Definition 2.1.8(2). The lemma then follows from the set-replacing argument. □

Lemma 2.1.10 (minimizing hulls in IVP)**.** *Suppose u solves $\mathsf{IVP}(\Omega; E_0)$. Then:*

(i) For each $t \geqslant 0$, E_t^+ is the minimizing hull of E_t in Ω, provided $E_t^+ \setminus E_t \Subset \Omega$.
(ii) $P(E_t) = e^t P(E_0^+)$ for each $t > 0$, provided $E_t \Subset \Omega$.
(iii) $P(E_0) \geqslant P(E_0^+)$ provided $E_0^+ \Subset \Omega$. Equality holds if E_0 is outward minimizing.
(iv) $P(E_t) = P(E_t^+)$ for each $t > 0$, provided $E_t^+ \Subset \Omega$.

Note: combining items (ii)(iii)(iv), we have

$$E_t^+ \Subset \Omega \qquad \Rightarrow \qquad P(E_t^+) = P(E_t) = e^t P(E_0^+) \leqslant e^t P(E_0).$$

Proof of Lemma 2.1.10.

(i) If $t > 0$, then we have $E_t^+ \setminus E_t \Subset \Omega \setminus \overline{E_0}$, hence this is exactly Lemma 2.1.5(iii). For the case $t = 0$, recall that we have Lemma 2.1.9. So we can argue as in Fact 1.2.11 to show that E_0^+ is strictly outward minimizing in Ω, then argue in Fact 1.2.14 to show that E_0^+ is the minimizing hull of E_0.

(ii) For each $0 < s < t$, we have $E_t \setminus E_s \Subset \Omega \setminus \overline{E_0}$. By mutual comparison of energies, it holds $J_u^K(E_s) = J_u^K(E_t)$ for any $K \Supset E_t \setminus E_s$. By the coarea formula, this implies

$$P(E_t) = P(E_s) + \int_s^t P(E_\tau)\, d\tau,$$

hence implies $P(E_t) = e^{t-s}P(E_s)$. By Lemma 2.1.9, we can make the same comparison to obtain $P(E_s^+) = e^{s-\tau}P(E_\tau^+)$ for any $0 \leqslant \tau < s$. On the other hand, observe that $|E_s^+ \setminus E_s| = 0$ for almost every s. Combined together, we have $P(E_t) = e^t P(E_0^+)$.

(iii) Plug in $E = E_0$ in Lemma 2.1.9 and notice that $\nabla u = 0$ in $E_0^+ \setminus E_0$.

(iv) Notice that $E_t^+ \setminus E_t \Subset \Omega \setminus \overline{E_0}$, hence we can make mutual energy comparison $J_u(E_t) = J_u(E_t^+)$. The result follows by noticing that $\nabla u = 0$ in $E_t^+ \setminus E_t$. □

Definition 2.1.11 (properness)**.**
Let u be a solution of $\mathsf{IVP}(\Omega; E_0)$. We say that u is proper, if $E_t \Subset \Omega$ for all $t \geqslant 0$.

Note that u being proper implicitly implies $E_0 \Subset \Omega$. By Lemma 2.1.10, for a proper solution we have $P(E_t) \leqslant e^t P(E_0)$ for all $t \geqslant 0$.

2.1.3 Maximum principle and compactness

We state an important maximum principle for the weak IMCF. We call it an "interior maximum principle", to distinguish from the maximum principle for IMCF with outer obstacle (which is contained in the next chapter). In particular, item (iv) states that proper solutions are unique if exist.

Theorem 2.1.12 (interior maximum principle, [53, Theorem 2.2])**.**
(i) Let $u, v \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ be respectively a supersolution and subsolution of $\mathsf{IMCF}(\Omega)$, such that $\{u < v\} \Subset \Omega$. Then $u \geqslant v$.

(ii) Let $E_0 \Subset \Omega$ be a $C^{1,1}$ initial data, and u, v be respectively a supersolution and subsolution of $\mathsf{IVP}(\Omega; E_0)$. Then $E_t(u) \subset E_t(v)$ whenever $E_t(u) \Subset \Omega$.

(iii) Let $E_0 \Subset \Omega$, and u_1, u_2 be two solutions of $\mathsf{IVP}(\Omega; E_0)$. Then $E_t(u_1) = E_t(u_2)$ whenever $E_t(u_1) \Subset \Omega$ and $E_t(u_2) \Subset \Omega$.

(iv) Let $E_0 \Subset \Omega$. Then there exists at most one proper solution of $\mathsf{IVP}(\Omega; E_0)$.

Finally, we state the following useful compactness theorem.

Theorem 2.1.13 (interior compactness)**.** *Let $\Omega \subset M$ be a domain, and g be a fixed smooth Riemannian metric on Ω. Given the following data:*

(1) Ω_i is a sequence of domains that locally uniformly converge to Ω,

(2) g_i are smooth metrics on Ω_i, which locally uniformly converge to g,

(3) $u_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega_i)$ solves $\mathsf{IMCF}(\Omega_i, g_i)$, and $u_i \to u$ in C^0_{loc} for some u,

(4) for each $K \Subset \Omega$ we have $\sup_K |\nabla u_i|_{g_i} \leqslant C(K)$ for all sufficiently large i.

Then u solves $\mathsf{IMCF}(\Omega, g)$. Moreover, if a sequence of sets E_i locally minimizes J_{u_i} in (Ω_i, g_i), and E_i converges to a set E in L^1_{loc}, then E locally minimizes J_u in (Ω, g).

Proof. The fact that u solves $\mathsf{IMCF}(\Omega; g)$ is already proved in [53, Theorem 2.1]. The new part of this theorem is the minimizing property of E. This follows from a standard set replacing argument (see the proof of Lemma A.2.5). In this argument we need the fact that $\int_A |\nabla_g u|_g \, dV_g = \lim_{i\to\infty} \int_A |\nabla_{g_i} u_i|_{g_i} \, dV_{g_i}$ for all $A \Subset \Omega$ with finite perimeter. This is proved as follows. For any $\varphi \in \mathrm{Lip}_0(\Omega)$ with $\varphi \geqslant 0$, by (2)(3)(4) and lower semi-continuity we have

$$\int_\Omega \varphi|\nabla_g u|_g \, dV_g \leqslant \liminf_{i\to\infty} \int_\Omega \varphi|\nabla_{g_i} u_i|_{g_i} \, dV_{g_i}. \tag{2.1.6}$$

On the other hand, we have the energy comparison $J_{u_i}^K(u_i) \leqslant J_{u_i}^K\big(\varphi u + (1-\varphi)u_i\big)$, where K is chosen with $\mathrm{spt}(\varphi) \Subset K \Subset \Omega \cap \Omega_i$ for large i. Expanding this inequality we have

$$\int_\Omega \varphi|\nabla_{g_i} u_i|_{g_i} \, dV_{g_i} \leqslant \int_\Omega \varphi|\nabla_{g_i} u|_{g_i} \, dV_{g_i} + \int_\Omega |u-u_i| \left(\varphi|\nabla_{g_i} u_i|_{g_i} + |\nabla_{g_i}\varphi|_{g_i}\right) dV_{g_i}.$$

Taking $i \to \infty$, by the items (2)(3)(4) and (2.1.6), we obtain exact continuity

$$\int_\Omega \varphi|\nabla_g u|_g \, dV_g = \lim_{i\to\infty} \int_\Omega \varphi|\nabla_{g_i} u_i|_{g_i} \, dV_{g_i}. \tag{2.1.7}$$

Combined with item (4), this implies our claim. □

2.2 Useful analytic properties

We spend this section proving some useful properties of the weak IMCF.

Below is a full version of the excess inequality (Lemma 1.2.18).

Lemma 2.2.1 (excess inequality).

Suppose $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ is a supersolution of $\mathsf{IMCF}(\Omega)$*, and $t \in \mathbb{R}$. Let $F \Subset \Omega$ be a set with finite perimeter. Then for any domain K with $F \Subset K \Subset \Omega$, we have*

$$P(E_t;K) \leqslant P(E_t \setminus F;K) + \int_{\inf_F(u)}^t e^{t-s} P(F;E_s)\, ds. \tag{2.2.1}$$

In particular,

$$P(E_t;K) \leqslant P(E_t \setminus F;K) + \big(e^{t-\inf_F(u)} - 1\big) P(F;E_t). \tag{2.2.2}$$

Proof. Denote $t_0 = \inf_F(u)$. By a zero measure modification, we may assume $F = F^{(1)}$. From the energy comparison $J_u^K(E_t) \leqslant J_u^K(E_t \setminus F)$ and the coarea formula, we have

$$\begin{aligned} P(E_t;K) &\leqslant P(E_t \setminus F;K) + \int_{t_0}^t \mathcal{H}^{n-1}\big(\partial^* E_s \cap E_t \cap F\big)\, ds \\ &= P(E_t \setminus F;K) + \int_{t_0}^t \mathcal{H}^{n-1}\big(\partial^* E_s \cap F\big)\, ds \end{aligned} \tag{2.2.3}$$

Inserting the perimeter cancelation formula (A.1.14) and Lemma 2.1.6, we obtain

$$\mathcal{H}^{n-1}(\partial^* E_t \cap F) \leqslant P(F;E_t) + \int_{t_0}^t \mathcal{H}^{n-1}(\partial^* E_s \cap F)\, ds.$$

By a standard Gronwall argument, this implies

$$\mathcal{H}^{n-1}(\partial^* E_t \cap F) \leqslant P(F;E_t) + \int_{t_0}^t e^{t-s} P(F;E_s)\, ds \qquad \text{for a.e. } t. \tag{2.2.4}$$

The lemma follows by inserting (2.2.4) into (2.2.3). □

The following lemma provides regularity for interior solutions of the weak IMCF.

Lemma 2.2.2 (a priori global regularity)**.**
Let $\Omega \subset M$ be a locally Lipschitz domain, and $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$.
(i) If u is a subsolution of $\mathsf{IMCF}(\Omega)$, then for all $K \Subset M$ it holds

$$\int_{\Omega\cap K} |\nabla u| \leqslant P(\Omega \cap K) \quad \text{and} \quad P(E_t; K) \leqslant P(\Omega \cap K) \quad \forall\, t \in \mathbb{R}. \tag{2.2.5}$$

In particular, $u \in \mathrm{BV}_{\mathrm{loc}}(\overline{\Omega})$ and each E_t has locally finite perimeter in M.
(ii) If u is a supersolution of $\mathsf{IMCF}(\Omega)$, and for all $K \Subset M$ it holds $\inf_{\Omega\cap K} u \geqslant T(K)$ for some $T(K)$, then each E_t has locally finite perimeter in M. More precisely, we have

$$P(E_t; \Omega \cap K) \leqslant e^{t-T(K)} P(\Omega \cap K), \quad P(E_t; K) \leqslant 2e^{t-T(K)} P(\Omega \cap K), \tag{2.2.6}$$

for all domains $K \Subset M$. We also have

$$\int_{E_t \cap K} |\nabla u| \leqslant e^{t-T(K)} P(\Omega \cap K). \tag{2.2.7}$$

In particular, if $u \in L^\infty_{\mathrm{loc}}(\overline{\Omega})$ then $u \in \mathrm{BV}_{\mathrm{loc}}(\overline{\Omega})$.

Proof. Fix $t \in \mathbb{R}$. By Lemma A.5.2, we may find a sequence of locally Lipschitz domains $\Omega_1 \Subset \Omega_2 \Subset \cdots \Subset \Omega$, with $\bigcup \Omega_i = \Omega$, such that $\mathcal{H}^{n-1}(\partial^* E_t \cap \partial^* \Omega_i) = 0$ for each i and $|\mu_{\Omega_i}| \rightharpoonup |\mu_\Omega|$ weakly as measures. For K in either statement of the lemma, we choose another smooth precompact domain $K' \Supset K$ with $\mathcal{H}^{n-1}(\partial^*\Omega \cap \partial K') = 0$.

(i) Suppose u is a weak subsolution. For each i we have $\inf_{\Omega_{i+1}\cap K'} u \geqslant T_i$ for some $T_i \in \mathbb{R}$; thus $E_{T_i} \cap (\Omega_{i+1} \cap K') = \emptyset$. Using the subsolution property, we have

$$0 = J_u^{\Omega_{i+1}\cap K'}(E_{T_i}) \leqslant J_u^{\Omega_{i+1}\cap K'}(\Omega_i \cap K) = P(\Omega_i \cap K) - \int_{\Omega_i \cap K} |\nabla u|. \tag{2.2.8}$$

Letting $i \to \infty$ in (2.2.8) and using Lemma A.6.1 (there with the choice $A = \Omega \cap K$ and with K replaced by K'), we obtain $\int_{\Omega\cap K} |\nabla u| \leqslant P(\Omega \cap K)$. To prove the second statement of (2.2.5), we recall that E_t is locally outward minimizing in Ω, hence $P(E_t; K') \leqslant P(E_t \cup (\Omega_i \cap K); K')$ for each i. This implies $P(E_t \cap \Omega_i \cap K; K') \leqslant P(\Omega_i \cap K; K')$. Taking $i \to \infty$ and applying Lemma A.6.1 to both sides, we conclude that $P(E_t \cap K) \leqslant P(\Omega \cap K)$. In particular, it holds $P(E_t; K) \leqslant P(\Omega \cap K)$. This proves (2.2.5). To obtain $u \in \mathrm{BV}_{\mathrm{loc}}(\overline{\Omega})$ we need $u \in L^1_{\mathrm{loc}}(\overline{\Omega})$ as well, but this follows from $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ and (2.2.5) and that Ω is locally Lipschitz.

(ii) Suppose u is as in the statement. We first assume that t satisfies $\mathcal{H}^{n-1}\big(\partial^* E_t \cap \partial K\big) = 0$. From Lemma 2.2.1 we obtain

$$P(E_t; \Omega_{i+1} \cap K') \leqslant P\big(E_t \setminus (\Omega_i \cap K); \Omega_{i+1} \cap K'\big) + (e^{t-T(K)} - 1) P\big(\Omega_i \cap K; E_t\big).$$

Since $\mathcal{H}^{n-1}(\partial^* E_t \cap \partial^* K) = \mathcal{H}^{n-1}(\partial^* E_t \cap \partial^* \Omega_i) = 0$ for all i, we can use the perimeter cancellation formula (A.1.15) to deduce

$$\begin{aligned} P\big(E_t; \Omega_i \cap K\big) &\leqslant P\big(E_t; (\Omega_i \cap K)^{(1)}\big) \leqslant P\big(\Omega_i \cap K; E_t^{(1)}\big) + (e^{t-T(K)} - 1) P(\Omega_i \cap K) \\ &\leqslant e^{t-T(K)} P(\Omega_i \cap K). \end{aligned}$$

Taking $i \to \infty$, this implies the first inequality of (2.2.6). The case of all t follows by lower semi-continuity. The second inequality in (2.2.6) holds since $P(E_t; K) = P(E_t; \Omega \cap K) + \mathcal{H}^{n-1}\big(\partial^* E_t \cap \partial^* \Omega \cap K\big)$. Finally, (2.2.7) follows by the coarea formula. □

Next, we state the sutomatic subsolution principle for the weak IMCF. We also refer to Lemma 1.4.10 in the introduction, as well as Theorem 3.3.8 in the next chapter.

Lemma 2.2.3 (automatic subsolution principle).
Suppose Ω is a locally Lipschitz domain, and u is a subsolution of $\mathsf{IMCF}(\Omega)$*. Then for any $t \in \mathbb{R}$, any set E and domain K satisfying $E_t \subset E \subset \Omega$ and $E \setminus E_t \Subset K \Subset M$, we have*

$$P(E_t; K) \leqslant P(E; K) - \int_{E\setminus E_t} |\nabla u|. \tag{2.2.9}$$

Note: the finiteness of $\int_{E\setminus E_t} |\nabla u|$ is guaranteed by Lemma 2.2.2(i). We remind the reader again that on both sides of (2.2.9), the perimeters contain the portion in $\partial\Omega$.

Proof of Lemma 2.2.3.
Choose $K \Supset E \setminus E_t$ such that $\mathcal{H}^{n-1}(\partial^*\Omega \cap \partial K) = 0$ (recall our flexibility to choose K, see Remark 1.2.6). By Lemma A.5.2, we can choose Lipschitz domains $\Omega_1 \Subset \Omega_2 \Subset \cdots \Subset \Omega$, such that $\bigcup \Omega_i = \Omega$ and $|\mu_{\Omega_i}| \rightharpoonup |\mu_\Omega|$ weakly as measures. By the interior variational principle, we have

$$\begin{aligned} J_u^{\Omega_{i+1}\cap K}(E_t) &\leqslant J_u^{\Omega_{i+1}\cap K}\big(E_t \cup (E\cap\Omega_i)\big) \\ &\leqslant J_u^{\Omega_{i+1}\cap K}(E_t) + J_u^{\Omega_{i+1}\cap K}(E\cap\Omega_i) - J_u^{\Omega_{i+1}\cap K}(E_t\cap E\cap\Omega_i) \\ &= J_u^{\Omega_{i+1}\cap K}(E_t) + J_u^{K}(E\cap\Omega_i) - J_u^{K}(E_t\cap\Omega_i), \end{aligned}$$

Hence

$$P(E_t\cap\Omega_i; K) \leqslant P(E\cap\Omega_i; K) - \int_{(E\setminus E_t)\cap\Omega_i} |\nabla u|.$$

Applying Lemma A.6.1 to both E_t and E, and noticing $u \in \mathrm{BV}(K\cap\Omega)$ by Lemma 2.2.2(i), we can take $i \to \infty$ and obtain exactly (2.2.9). ☐

The following lemma addresses the merging of two subsolutions. Note that condition (3) is necessary in view of Lemma 2.1.5(i).

Lemma 2.2.4 (continuation of subsolution).
Suppose $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, and $\Omega' \subset \Omega$ is a locally Lipschitz sub-domain, such that:
(1) $u|_{\Omega'} \leqslant T$, $u|_{\partial\Omega'} \equiv T$, and $u|_{\Omega\setminus\Omega'} \geqslant T$,
(2) $u|_{\Omega'}$ is a subsolution of $\mathsf{IMCF}(\Omega')$*, and $u|_{\Omega\setminus\overline{\Omega'}}$ is a subsolution of* $\mathsf{IMCF}(\Omega\setminus\overline{\Omega'})$*,*
(3) Ω' is locally outward minimizing.
Then u is a subsolution of $\mathsf{IMCF}(\Omega)$.

Note that we require the outward minimizing of Ω'. If this does not hold, then not all subsolutions of IMCF in Ω' can be extended as a subsolution in Ω.

Proof of Lemma 2.2.4.
Suppose $t \in \mathbb{R}$ and $E \supset E_t$ is a competitor set with $E \setminus E_t \Subset K \Subset \Omega$. If $t > T$, then $E_t \supset \overline{\Omega'}$, hence $E \setminus E_t \Subset \Omega \setminus \overline{\Omega'}$. So in this case, the result follows from condition (2).
Now assume $t \leqslant T$. We have $E_t \subset \Omega'$. By Lemma 2.2.3, we have

$$P(E_t; K) \leqslant P(E\cap\Omega'; K) - \int_{(E\setminus E_t)\cap\Omega'} |\nabla u|. \tag{2.2.10}$$

For each $\varepsilon > 0$, using the subsolution property in $\Omega \setminus \overline{\Omega'}$ we have

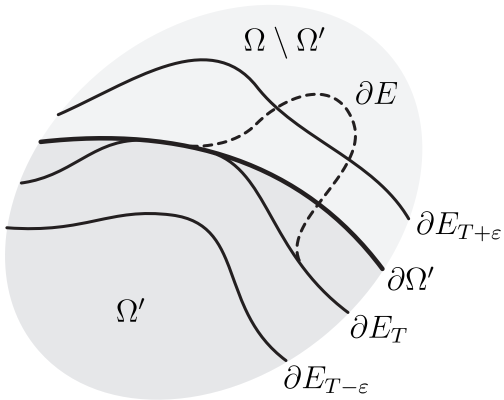

Figure 2.2: Level sets of u in Lemma 2.2.4

$$P(E_{T+\varepsilon}; K) \leqslant P(E_{T+\varepsilon} \cup E; K) - \int_{E \setminus E_{T+\varepsilon}} |\nabla u|.$$

This implies

$$P(E_{T+\varepsilon} \cap E; K) \leqslant P(E; K) - \int_{E \setminus E_{T+\varepsilon}} |\nabla u|. \tag{2.2.11}$$

By condition (3), we have

$$P(\Omega'; K) \leqslant P\big(\Omega' \cup (E_{T+\varepsilon} \cap E); K\big) \quad \Rightarrow \quad P(\Omega' \cap E; K) \leqslant P(E_{T+\varepsilon} \cap E; K). \tag{2.2.12}$$

Adding (2.2.11) (2.2.12) and taking $\varepsilon \to 0$, we obtain

$$P(\Omega' \cap E; K) \leqslant P(E; K) - \int_{E \setminus E_T^+} |\nabla u| = P(E; K) - \int_{E \setminus \Omega'} |\nabla u|. \tag{2.2.13}$$

Adding (2.2.10) and (2.2.13), we obtain exactly $J_u^K(E_t) \leqslant J_u^K(E)$. □

2.3 Calibrated solutions

As we have noticed in the introduction, the notion of calibration underlies the entire theory of weak IMCF. In this section, we give an extended discussion on this topic.

Definition 2.3.1. Let Ω be a domain in a manifold M, and $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$. We say that u is a calibrated solution of IMCF in Ω, if there is a measurable vector field ν such that:
(1) $\operatorname{ess\,sup}_\Omega |\nu| \leqslant 1$, and $\langle \nabla u, \nu \rangle = |\nabla u|$ almost everywhere,
(2) for all $\varphi \in \mathrm{Lip}_0(\Omega)$ we have

$$\int_\Omega \big(\nabla\varphi \cdot \nu + \varphi |\nabla u|\big) = 0. \tag{2.3.1}$$

Given a solution u of $\mathsf{IMCF}(\Omega)$, we say that u is calibrated by ν, if (1)(2) are satisfied for the data u, ν.

A calibrated solution is always a solution of $\mathsf{IMCF}(\Omega)$, since it satisfies Remark 2.1.3. Indeed, given $v \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ with $\{u \neq v\} \Subset M$, the inequality $J_u(u) \leqslant J_u(v)$ follows by taking $\varphi = u - v$ in (2.3.1).

Remark 2.3.2. Suppose a solution u of $\mathsf{IMCF}(\Omega)$ is calibrated by ν. Then for almost every t, we have $\nu_{E_t} = \nu$ a.e. on $\partial^* E_t \cap \Omega$.

Indeed, for a.e. t, we have that u is differentiable and $|\nabla u| > 0$ a.e. on $\partial^* E_t \cap \Omega$. Also, recall that $\partial^* E_t \cap \Omega$ is a $C^{1,\alpha}$ hypersurface. The result follows.

We prove the following compactness theorem for calibrated solutions. In item (6) below, we say that a sequence of vector fields ν_i converge to ν weakly in L^1_{loc}, if for all L^∞ vector field X with $\mathrm{supp}(X) \Subset \Omega$, we have $\int_\Omega \langle \nu, X\rangle = \lim_{i\to\infty} \int_\Omega \langle \nu_i, X\rangle$. This notion of convergence is independent of the choice of background metric.

Theorem 2.3.3 (compactness of calibrated solutions). *Let $\Omega \subset M$ be a domain, and g be a fixed smooth Riemannian metric on Ω. Given the following data:*

(1) Ω_i is a sequence of domains that locally uniformly converge to Ω,
(2) g_i are smooth metrics on Ω_i, which locally uniformly converge to g,
(3) $u_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega_i)$ solves $\mathsf{IMCF}(\Omega_i, g_i)$ and are calibrated by ν_i,
(4) for each $K \Subset \Omega$ we have $\sup_K |\nabla u_i|_{g_i} \leqslant C(K)$ for all sufficiently large i,
(5) u_i locally uniformly converges to a function u on Ω,
(6) ν_i converges to a vector field ν weakly in L^1_{loc}.

Then u is solves $\mathsf{IMCF}(\Omega, g)$ and is calibrated by ν.

Proof. From items (2)(4)(5) we have $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$. From (2)(6) and $\operatorname{ess\,sup} |v_i|_{g_i} \leqslant 1$ we have $\operatorname{ess\,sup} |v|_g \leqslant 1$. We have argued in (2.1.7) that for any $\varphi \in \mathrm{Lip}_0(\Omega)$ with $\varphi \geqslant 0$, it holds
$$\int \varphi |\nabla_g u|_g \, dV_g = \lim_{i\to\infty} \int \varphi |\nabla_{g_i} u_i|_{g_i} \, dV_{g_i}. \tag{2.3.2}$$
Clearly, this identity holds for all $\varphi \in \mathrm{Lip}_0(\Omega)$ as well.

Suppose $\varphi \in \mathrm{Lip}_0(\Omega)$. It follows from items (2)(6) that
$$\int \langle \nu, \nabla_g \varphi\rangle_g \, dV_g = \lim_{i\to\infty} \int \langle \nu_i, \nabla_{g_i} \varphi\rangle_{g_i} \, dV_{g_i}. \tag{2.3.3}$$
Recall that each pair (u_i, ν_i) satisfies
$$\int \big(\langle \nu_i, \nabla_{g_i}\varphi\rangle_{g_i} + \varphi |\nabla_{g_i} u_i|_{g_i} \big) \, dV_{g_i} = 0.$$
Taking $i \to \infty$ and combining with (2.3.2) (2.3.3), the identity (2.3.1) is verified.

Finally, we show $\langle \nu, \nabla_g u\rangle_g = |\nabla_g u|_g$ a.e.. Taking $\varphi = u_i \psi$ in the calibration condition (2.3.1) for u_i (where $\psi \in \mathrm{Lip}_0(\Omega)$ and i is sufficiently large), we obtain
$$\begin{aligned} 0 &= \int \langle \nu_i, u_i \nabla_{g_i} \psi + \psi \nabla_{g_i} u_i\rangle_{g_i} \, dV_{g_i} + \int u_i \psi |\nabla_{g_i} u_i|_{g_i} \, dV_{g_i} \\ &= \int \langle \nu_i, u_i \nabla_{g_i}\psi\rangle_{g_i} \, dV_{g_i} + \int (1 + u_i)\psi |\nabla_{g_i} u_i|_{g_i} \, dV_{g_i}. \end{aligned} \tag{2.3.4}$$
By items (2)(5)(6), the first term of (2.3.4) converges to $\int \langle \nu, u \nabla_g \psi\rangle_g \, dV_g$. By (2.3.2) and items (2)(4)(5), the second term of (2.3.4) converges to $\int (1+u)\psi |\nabla_g u|_g \, dV_g$. Therefore, it holds
$$0 = \int \langle \nu, u\nabla_g \psi\rangle_g \, dV_g + \int (1+u)\psi |\nabla_g u|_g \, dV_g.$$

On the other hand, since we have already verified (2.3.1) for the pair (u, ν), we may apply $\varphi = u\psi$ there to obtain

$$0 = \int \langle \nu, u\nabla_g \psi\rangle_g + \langle \nu, \psi \nabla_g u\rangle_g + u\psi|\nabla_g u|_g.$$

In comparison, it holds

$$\int \psi\langle \nu, \nabla_g u\rangle_g = \int \psi|\nabla_g u|_g.$$

Since ψ is arbitrary, we conclude that $\langle \nu, \nabla_g u\rangle_g = |\nabla_g u|_g$ a.e.. This proves that u is a weak solution calibrated by ν. $\square$

Now we can prove the converse that every weak IMCF is calibrated.

Lemma 2.3.4. *Every solution of* $\mathsf{IMCF}(\Omega)$ *is calibrated.*

This is a consequence of the following general theorem, whose proof uses abstract convex duality theories.

Theorem 2.3.5 (direct adaptation of [88, Theorem 3.1])**.** *Let Ω be a bounded domain, F be an L^2 vector field in Ω, and $H \in L^2(\Omega)$. Assume that*

$$\inf_{w \in W_0^{1,2}(\Omega)} \int_\Omega |\nabla w + F| + Hw > -\infty. \tag{2.3.5}$$

Then there exists a vector field ν with $||\nu||_{L^\infty} \leqslant 1$ and $\operatorname{div}(\nu) = H$ weakly. Moreover, for any minimizer w of (2.3.5), we have $\nu \cdot (\nabla w + F) = |\nabla w + F|$ a.e..

Proof of Lemma 2.3.4.
Let u be the given solution of $\mathsf{IMCF}(\Omega)$. We first produce a calibration in any subdomain $K \Subset \Omega$. Apply Theorem 2.3.5 in K, with the choice $F = \nabla u$ and $H = |\nabla u|$. In (2.3.5) we make the transform $w = v - u$. Thus the energy in (2.3.5) becomes

$$\inf \Big\{ \int_\Omega |\nabla v| + v|\nabla u| - u|\nabla u| : v \text{ is such that } v - u \in W_0^{1,2}(\Omega) \Big\}.$$

Since u is a weak solution of IMCF, it is a minimizer of this energy. Hence, (2.3.5) indeed holds, and $w = 0$ is a minimizer. Let ν be the vector field obtained from Theorem 2.3.5. Collecting the conclusions about ν, it follows that ν is a calibration of u in K.

For a general Ω, we find an exhaustion $K_1 \Subset K_2 \Subset \cdots \Subset \Omega$, $\bigcup K_i = \Omega$. For each K_i, we obtain a calibration ν_i. By the Dunford-Pettis theorem, we can extract (up to a subsequence) a limit $\nu = \lim_{i\to\infty} \nu_i$ weakly in $L^1_{\mathrm{loc}}(\Omega)$. Then applying Theorem 2.3.3, we see that ν is a calibration of u in Ω. $\square$

Lemma 2.3.6. *Suppose $u, v \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ are solutions of* $\mathsf{IMCF}(\Omega)$*. Then $w = \max\{u, v\}$ is also a subsolution of* $\mathsf{IMCF}(\Omega)$*.*

The picture is as follows: note that $E_t(\max\{u, v\}) = E_t(u) \cap E_t(v)$. Thus the speed of movement of $\max\{u, v\}$ is equal to $1/H$ on the smooth parts but is greater than $1/H$ at the corners; see Figure 2.3.

By Lemma 2.3.4, we may assume that u, v are calibrated by ν_u, ν_v respectively. Intuitively, one may consider the vector field

$$\sigma = \nu_u \chi_{\{u>v\}} + \nu_v \chi_{\{u\leqslant v\}},$$

and show that

$$\operatorname{div}\sigma \geqslant |\nabla u|\chi_{\{u>v\}} + |\nabla v|\chi_{\{u\leqslant v\}}$$

as Radon measures. Note that the inequality tends to be strict on $\{u = v\}$.

Let us naïvely argue that this works in the ideal case that u, v are smooth and 0 is a regular level set of $u - v$. Indeed, we can formally calculate

$$\begin{aligned}\operatorname{div}\sigma &= \operatorname{div}(\nu_u)\chi_{\{u>v\}} + \nu_u \cdot D\chi_{\{u>v\}} + \operatorname{div}(\nu_v)\chi_{\{u\leqslant v\}} + \nu_v \cdot D\chi_{\{u\leqslant v\}}\\ &= |\nabla u|\chi_{\{u>v\}} + \frac{\nabla u}{|\nabla u|}\cdot\frac{\nabla(u-v)}{|\nabla(u-v)|}\cdot\mathcal{H}^{n-1}\llcorner\{u=v\}\\ &\quad + |\nabla v|\chi_{u\leqslant v} - \frac{\nabla v}{|\nabla v|}\cdot\frac{\nabla(u-v)}{|\nabla(u-v)|}\cdot\mathcal{H}^{n-1}\llcorner\{u=v\},\end{aligned}$$

and one can easily check that the part concentrated on $\{u = v\}$ is nonnegative.

In general, $\{u = v\}$ need not be regular enough. To get around the possible analytic issues, in the proof we will show that $\max\{u, v + t\}$ is a subsolution for a sequence $t \to 0$, and the result will follow by taking the limit.

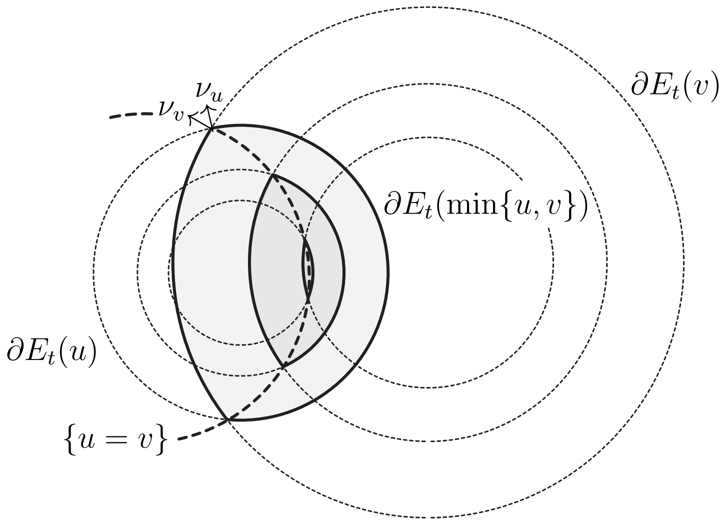

Figure 2.3: The maximum of two solutions

Proof of Lemma 2.3.6.

Let ν_u, ν_v be calibrations of u, v. For any $\varepsilon, t \in (0, 1)$, define a function $\eta_{\varepsilon,t}$ with

$$\eta_{\varepsilon,t}|_{(-\infty,t]} \equiv 0, \qquad \eta_{\varepsilon,t}|_{[t+\varepsilon,\infty)} \equiv 1, \qquad \eta_{\varepsilon,t}(x) = \varepsilon^{-1}(x-t) \ \ \forall\, x \in [t, t+\varepsilon].$$

Then set

$$\sigma_{\varepsilon,t} = \nu_u \cdot \eta_{\varepsilon,t}(u-v) + \nu_v \cdot \big(1 - \eta_{\varepsilon,t}(u-v)\big).$$

We can calculate

$$\begin{aligned}\operatorname{div}(\sigma_{\varepsilon,t}) = \eta_{\varepsilon,t}(u-v)\operatorname{div}\nu_u + \big(1-\eta_{\varepsilon,t}(u-v)\big)\operatorname{div}\nu_v&\\ + \eta'_{\varepsilon,t}(u-v)(\nu_u-\nu_v)\cdot(\nabla u - \nabla v).&\end{aligned} \tag{2.3.6}$$

Expanding the third term and using the calibration conditions, this implies

$$\operatorname{div}(\sigma_\varepsilon) \geqslant \eta_{\varepsilon,t}(u-v)|\nabla u| + \big(1-\eta_{\varepsilon,t}(u-v)\big)|\nabla v|. \tag{2.3.7}$$

On the other hand, for any $K \Subset \Omega$, we have from (2.3.6)

$$\int_K \operatorname{div}(\sigma_{\varepsilon,t}) \leqslant \int_K |\nabla u| + \int_K |\nabla v| + 2\int_K |\nabla \eta_{\varepsilon,t}(u-v)|.$$

We use the coarea formula to expand the third term:

$$\int_K |\nabla \eta_{\varepsilon,t}(u-v)| = \varepsilon^{-1}\int_t^{t+\varepsilon} P\big(\{u-v<s\};K\big)\,ds.$$

Hence, by Fatou's lemma, we have

$$\begin{aligned}\int_0^1 \liminf_{\varepsilon\to 0}\Big(\int_K |\nabla \eta_{\varepsilon,t}(u-v)|\Big)\,dt &\leqslant \liminf_{\varepsilon\to 0}\Big(\int_0^1 dt\int_K |\nabla \eta_{\varepsilon,t}(u-v)|\Big)\\ &\leqslant \liminf_{\varepsilon\to 0}\Big(\int_0^2 P\big(\{u-v<s\};K\big)\Big)\\ &\leqslant \int_K |\nabla(u-v)|.\end{aligned}$$

As a result, we have

$$\int_0^1 \liminf_{\varepsilon\to 0}\Big(\int_K \operatorname{div}(\sigma_{\varepsilon,t})\Big)\,dt \leqslant C(K).$$

Therefore, we can choose a sequence $t_i \to 0$ such that

$$\liminf_{\varepsilon\to 0}\int_K \operatorname{div}(\sigma_{\varepsilon,t_i}) \leqslant C(K,i).$$

So for each t_i, we can choose a sequence $\varepsilon\to 0$ so that

$$\sigma_{\varepsilon,t_i} \to \sigma_i := \nu_u\cdot\chi_{\{u>v+t_i\}} + \nu_v\cdot\chi_{\{u\leqslant v+t_i\}} \qquad \text{in}\quad L^1_{\mathrm{loc}},$$

and

$$\operatorname{div}(\sigma_{\varepsilon,t_i}) \rightharpoonup \mu$$

weakly for some Radon measure μ. Hence μ must be the weak divergence of σ_i.

Set $w_i = \max\{u, v+t_i\}$. Passing (2.3.7) to the limit, we obtain

$$\operatorname{div}(\sigma_i) \geqslant \chi_{\{u>v+t_i\}}|\nabla u| + \chi_{\{u\leqslant v+t_i\}}|\nabla v| = |\nabla w_i| \qquad \text{as measures.} \tag{2.3.8}$$

On the other hand, it is clear that $|\sigma_i| \leqslant 1$ and $\sigma_i\cdot\nabla w_i = |\nabla w_i|$ a.e..

Let us now show that w_i is a subsolution of $\mathsf{IMCF}(\Omega)$. By Remark 2.1.3, it suffices to show that for all $f\in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ with $f\leqslant w_i$ and $\{f<w_i\}\Subset K\Subset\Omega$, it holds $J^K_{w_i}(f) \geqslant J^K_{w_i}(w_i)$. We may directly calculate

$$\begin{aligned}J^K_{w_i}(f) - J^K_{w_i}(w_i) &= \int_K \big(|\nabla f| - |\nabla w_i|\big) + \int_K (f-w_i)|\nabla w_i|\\ &\geqslant \int_K (\nabla f - \nabla w_i)\cdot\sigma_i + \int_K (f-w_i)|\nabla w_i|\\ &= \int_K (w_i - f)\operatorname{div}(\sigma_i) + \int_K (f-w_i)|\nabla w_i| \geqslant 0.\end{aligned}$$

Finally, we show that $w=\max\{u,v\}$, as a decreasing limit of w_i, is a subsolution of $\mathsf{IMCF}(\Omega)$ as well. Let $f\leqslant w$ be a competitor with $\{f<w\}\Subset K\Subset\Omega$. Let $q\geqslant 0$ be any

Lipschitz function with $\operatorname{spt}(q) \Subset K$. Then note that $f_i := qf + (1-q)w_i \leqslant w_i$, thus is a valid competitor for w_i. Hence we can compare $J^K_{w_i}(w_i) \leqslant J^K_{w_i}(f_i)$, implying

$$\int q|\nabla w_i| + q(w_i - f)|\nabla w_i| \leqslant \int q|\nabla f| + |f - w_i|\,|\nabla q|. \tag{2.3.9}$$

First inserting $f = w$, we obtain

$$\int q|\nabla w_i| + q(w_i - w)|\nabla w_i| \leqslant \int q|\nabla w| + |w - w_i|\,|\nabla q|.$$

Taking $i \to \infty$ we note that

$$\int q|\nabla w| \geqslant \limsup_{i\to\infty} \int q|\nabla w_i| \qquad \Rightarrow \qquad \int q|\nabla w| = \lim_{i\to\infty} \int q|\nabla w_i|.$$

Replacing q by $q(w-f)$, we also have

$$\int q(w-f)|\nabla w| = \lim_{i\to\infty} \int q(w-f)|\nabla w_i| = \lim_{i\to\infty} \int q(w_i - f)|\nabla w_i|. \tag{2.3.10}$$

Now, let us choose q to be a cutoff function, such that $0 \leqslant q \leqslant 1$ and $\{f < w\} \Subset \{q = 1\}$. Taking (2.3.9) to the limit and inserting (2.3.10), we obtain

$$\int q|\nabla w| + q(w-f)|\nabla w| \leqslant \int q|\nabla f| + |f - w||\nabla q|.$$

Using the defining property of q, this directly implies

$$\int |\nabla w| + (w-f)|\nabla w| \leqslant \int |\nabla f|.$$

Hence w is a subsolution of $\mathsf{IMCF}(\Omega)$. □

2.4 Elliptic regularization

In [53], the initial value problem of weak IMCF is solved by means of elliptic regularization. For $\varepsilon > 0$, the ε-regularized IMCF equation refers to

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{\varepsilon^2 + |\nabla u|^2}}\right) = \sqrt{\varepsilon^2 + |\nabla u|^2}. \tag{2.4.1}$$

This equation is strictly elliptic, so solutions can be found by standard techniques. The strategy in [53] is to solve (2.4.1) and then pass to a limit with $\varepsilon \to 0$. It eventually led to the following existence theorem:

Theorem 2.4.1 ([53, Theorem 3.1]). *Let M be complete, connected, noncompact. Suppose there exists a function $v \in \mathrm{Lip}_{\mathrm{loc}}(M)$ such that*

(1) $E_t(v) \Subset M$ for all $t > 0$,

(2) v is a subsolution of $\mathsf{IMCF}(\{v > 0\})$.

Then for all $C^{1,1}$ domains $E_0 \Subset M$, there is a unique proper solution u of $\mathsf{IVP}(M; E_0)$. Moreover, we have the estimate

$$|\nabla u|(x) \leqslant \sup_{B(x,r)} H_+ + C(n)r^{-1}, \qquad \forall\, x \in M \setminus E_0,\ r \leqslant \sigma(x; M). \tag{2.4.2}$$

Here, the notation $\sigma(x; M)$ is defined as follows, which we shall keep using later.

Definition 2.4.2. Let Ω be a domain in M, and g be a Riemannian metric on M. For $x \in \Omega$, we define $\sigma(x; \Omega, g)$ to be the supremum of radius r such that

(1) $B_g(x, r) \Subset \Omega$, and $\mathrm{Ric}_g > -1/(100nr^2)$ in $B_g(x, r)$,

(2) the distance function $p = d(\cdot, x)^2$ is smooth and satisfies $\nabla_g^2 p < 3g$ in $B_g(x, r)$.

When there is no ambiguity, we write $\sigma(x; \Omega)$ and omit the dependence on g.

The remainder of this subsection is aimed at reviewing the basics of elliptic regularization, and recalling some analytic byproducts obtained while proving Theorem 2.4.1. The contents below will only be used in Section 3.6; the reader may skip them temporarily.

The most important geometric feature of (2.4.1) is the following: if u_ε is a solution of (2.4.1) in a domain Ω, then the function

$$U_\varepsilon(x, z) = u_\varepsilon(x) - \varepsilon z$$

is a smooth solution of IMCF in $\Omega \times \mathbb{R}$. This can be directly verified. Noticing that the family of hypersurfaces

$$\Sigma_t^\varepsilon = \mathrm{graph}\left(\varepsilon^{-1}\big(u_\varepsilon(x) - t\big)\right)$$

are level sets of U_ε, it follows that they form a downward translating soliton of IMCF.

Immediate from this observation, estimates for the smooth IMCF may be applied to solutions of the ε-regularized equation. A particular case is the interior mean curvature estimate (or gradient estimate, since $H = |\nabla u|$ in IMCF):

Theorem 2.4.3 (general smooth gradient estimate, [53, (3.6)])**.** *Let $\{\Sigma_t\}_{a \leqslant t \leqslant b}$ be a family of smooth hypersurfaces that solve the IMCF in Ω. Denote by H_t the mean curvature of Σ_t. Suppose $x \in \Sigma_t$ and $r < \sigma(x; \Omega, g)$. Define the maximal mean curvature on the parabolic boundary:*

$$H_r = \max\Big\{\sup_{\Sigma_a \cap B(x,r)} H_a,\ \sup_{s \in [a,t]}\Big(\sup_{\partial\Sigma_s \cap B(x,r)} H_s\Big)\Big\}$$

Then we have

$$H_t(x) \leqslant \max\Big\{H_r, \frac{C(n)}{r}\Big\}. \tag{2.4.3}$$

Or we can state the result as follows:

Corollary 2.4.4. *Let $u \in C^\infty(\Omega)$ solves the smooth IMCF* $\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = \nabla u$ *in Ω, with $|\nabla u|$ nonvanishing. Then*

$$|\nabla u|(x) \leqslant \frac{C(n)}{\sigma(x; \Omega, g)}, \qquad \forall\, x \in \Omega. \tag{2.4.4}$$

In particular, when $\Omega \Subset M$, we have $|\nabla u|(x) \leqslant C(\Omega) \cdot d(x, \partial\Omega)^{-1}$.

Remark 2.4.5. Modifying the argument in [53, p.382], we can show the finer estimate

$$|\nabla u|(x) \leqslant \frac{C(n)}{r} \min\Big\{1, \sqrt{\mathrm{osc}_{B(x,r)}(u)}\Big\}, \qquad \forall\, r < \sigma(x; \Omega, g). \tag{2.4.5}$$

The proof is as follows. Recall the evolution of mean curvature

$$\frac{\partial H}{\partial t} = -\Delta_\Sigma \frac{1}{H} - \big(|A|^2 + \mathrm{Ric}(\nu,\nu)\big)\frac{1}{H}.$$

Setting $\psi = 1/H$, this implies

$$\partial_t \psi = \psi^2\big[\Delta_\Sigma \psi + |A|^2\psi + \mathrm{Ric}(\nu,\nu)\psi\big] \geqslant \psi^2\Delta_\Sigma\psi - \frac{\psi^3}{r^2}. \tag{2.4.6}$$

Here, recall that $\mathrm{Ric} \geqslant -r^{-2}g$ in $B(x,r)$. Shifting u by a constant, we may assume that $\inf_{B(x,r)}(u) > 0$. Denoting $d = d(\cdot, x)$, and for some constant $c > 0$ to be chosen later, we consider the quantity

$$F = \frac{c}{r}\frac{r^2 - d^2}{t^{1/2}}.$$

Here we are implicitly switching between the hypersurface and level set forms of the flow (so the time t is actually the same as u). We calculate

$$\partial_t F = -\frac{c}{2r}\frac{r^2-d^2}{t^{3/2}} - \frac{c}{r}\frac{2d\langle \nu, \nabla d\rangle}{t^{1/2}}\psi \leqslant -\frac{F}{2t} + \frac{2c}{t^{1/2}}\psi, \tag{2.4.7}$$

and

$$\begin{aligned} \Delta_\Sigma F &= -\frac{c}{rt^{1/2}}\Delta_\Sigma d^2 = -\frac{c}{rt^{1/2}}\big[\operatorname{tr}_\Sigma \nabla^2 d^2 - H\langle \nu, \nabla d^2\rangle\big] \\ &> -\frac{c}{rt^{1/2}}\big[6n - 2\psi^{-1} d\langle \nu, \nabla d\rangle\big] > -\frac{6nc}{rt^{1/2}} - \frac{2c}{t^{1/2}\psi}. \end{aligned} \tag{2.4.8}$$

To prove (2.4.5), it suffices to assume $\operatorname{osc}_{B(x,r)}(u) \leqslant 1$ and prove that $\psi(x) > F(x)$. Once this holds, it implies $H(x) \leqslant t^{1/2}/cr$, hence implies (2.4.5) joint with Corollary 2.4.4.

Suppose that our claim is false. Then there is an exceptional point with the smallest time. So there exists $0 < t \leqslant 1$ and $y \in \Sigma_t \cap B(x,r)$, such that

$$\psi(y) = F(y), \quad \Delta_\Sigma\psi(y) \geqslant \Delta_\Sigma F(y), \quad \partial_t\psi(y) \leqslant \partial_t F(y).$$

Using (2.4.6) $\sim$ (2.4.8) to cancel all the time derivatives and Laplacians, we obtain that at y:

$$-\frac{6nc}{rt^{1/2}}\psi^2 - \frac{2c}{t^{1/2}}\psi - \frac{\psi^3}{r^2} \leqslant -\frac{\psi}{2t} + \frac{2c}{t^{1/2}}\psi.$$

Canceling a common ψ and using $\psi = F \leqslant \frac{cr}{t^{1/2}}$ to get rid of the extra ones, this implies

$$-\frac{6nc^2}{t} - \frac{2c}{t^{1/2}} - \frac{c^2}{t} \leqslant -\frac{1}{2t} + \frac{2c}{t^{1/2}}.$$

Since $t^{-1/2} \leqslant t^{-1}$, this implies

$$-6nc^2 - 2c - c^2 \leqslant -1/2 + 2c.$$

Setting $c = 1/100n$, we have the desired contradiction.

Now we return to elliptic regularization, and consider solving the regularized equation. We assume the following setups: M is complete, connected, noncompact, and $E_0 \Subset M$ is a $C^{1,1}$ domain. Moreover, v satisfies conditions (1)(2) in Theorem 2.4.1 and additionally

$$\nabla v \neq 0 \text{ everywhere in } \{v \geqslant 0\}.$$

So this makes our setup more restrictive than in Theorem 2.4.1.

For each $\varepsilon > 0$ and $L > 2$, consider the region $F_L = \{v < L\}$ (which is smooth and precompact by our assumptions), and the boundary value problem

$$\begin{cases} \operatorname{div}\Big(\dfrac{\nabla u_{\varepsilon,L}}{\sqrt{\varepsilon^2+|\nabla u_{\varepsilon,L}|^2}}\Big) = \sqrt{\varepsilon^2+|\nabla u_{\varepsilon,L}|^2} \qquad \text{on } F_L \setminus \overline{E_0}, & (2.4.9)\\ u_{\varepsilon,L} = 0 \qquad \text{on } \partial E_0, & (2.4.10)\\ u_{\varepsilon,L} = L-2 \qquad \text{on } \partial F_L. & (2.4.11)\end{cases}$$

The following existence theorem comes from [53, Lemma 3.4] and [53, Lemma 3.5]:

Theorem 2.4.6 (approximate existence)**.**

Let M be complete, and $E_0 \Subset M$ be a $C^{1,1}$ domain. Suppose $v \in C^\infty(M)$ is proper, such that $\{v < 0\} \Supset E_0$, and v is a subsolution of IMCF$(\{v > 0\})$ *with nonvanishing gradient therein. Then for each $L > 2$ there is a small $\varepsilon(L) > 0$ such that (2.4.9) ∼ (2.4.11) admits a smooth solution $u_{\varepsilon,L}$ for all $0 < \varepsilon \leqslant \varepsilon(L)$. Moreover, we have the lower bound*

$$\begin{cases} u_{\varepsilon,L} \geqslant -\varepsilon \qquad \text{in } \overline{F_L} \setminus E_0, \\ u_{\varepsilon,L} \geqslant v-2 \qquad \text{in } \overline{F_L} \setminus \{v<0\}, \end{cases} \tag{2.4.12}$$

and the gradient estimate

$$\big|\nabla u_{\varepsilon,L}(x)\big| \leqslant \max\Big\{ \sup_{B(x,r)\cap \partial E_0} H_+,\ \sup_{B(x,r)\cap\partial F_L} |\nabla u_{\varepsilon,L}|\Big\} + 2\varepsilon + \frac{C(n)}{r} \tag{2.4.13}$$

for all $x \in \overline{F_L} \setminus E_0$ and $0 < r \leqslant \sigma(x; M, g)$, where $\sigma(x; M, g)$ is as in Definition 2.4.2, and $H_+ = \max\{H_{\partial E_0}, 0\}$ is the positive part of boundary mean curvature.

See [53] for the proof. Let us briefly remark on the estimates (2.4.12) and (2.4.13): the interior estimate follows by applying Theorem 2.4.3 in $(F_L \setminus E_0) \times \mathbb{R}$. The C^0 bound (2.4.12) and the boundary gradient estimate $\sup_{\partial E_0} |\nabla u| \leqslant \sup_{\partial E_0} H_+$ comes from the use of appropriate barrier functions. Specifically, we have:

(i) For some constant $C = C(L) \gg 1$, the function

$$v_1 = \frac{\varepsilon}{C}\Big(e^{-Cd(\cdot,\partial E_0)} - 1\Big)$$

is a viscosity subsolution of (2.4.9) in $\overline{F_L} \setminus E_0$. Note that $-\varepsilon \leqslant v \leqslant 0$ in $\overline{F_l} \setminus E_0$ and $|\nabla v| = \varepsilon$ at ∂E_0.

(ii) When ε is sufficiently small (depending on L), the function

$$v_2 = \frac{L-1}{L} v - 1$$

is a strict subsolution of (2.4.9) in $\overline{F_l} \setminus \{v < 0\}$.

(iii) Choose any function f with $f|_{\partial E_0} = 0$ and $H_+ < f_\nu \leqslant H_+ + \varepsilon$ on ∂E_0. Then

$$w_1 = \frac{f}{1 - f/\delta}$$

is a strict supersolution of (2.4.9) in $\{w_1 < \infty\} \setminus E_0$, for some $\delta \ll \varepsilon$.

(iv) The constant function $w_2 \equiv L - 2$ is trivially a supersolution of (2.4.9).

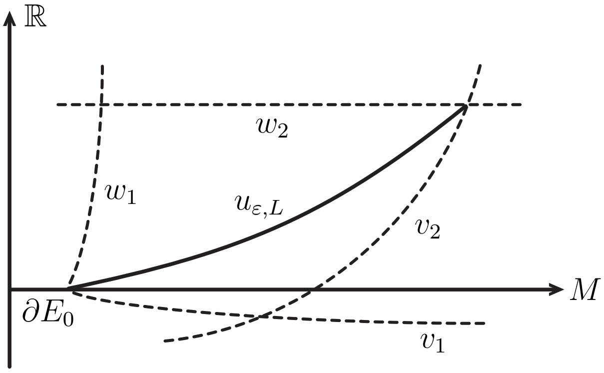

Figure 2.4: Barrier functions in elliptic regularization.

Thus $u_{\varepsilon,L}$ is bounded by these four functions; see Figure 2.4 below.

We next consider the convergence as $\varepsilon \to 0$. The next theorem states that the limit of a sequence of ε_i-regularized solutions is an actual solution of $\mathsf{IMCF}(\Omega)$. It is similar to [53, Theorem 3.1], and is proved in a broader generality for later use.

Theorem 2.4.7 (convergence to calibrated solutions)**.** *Let $\Omega \subset M$ be a domain, and g be a fixed smooth Riemannian metric on Ω. Given the following data:*

(1) Ω_i is a sequence of domains that converge locally uniformly to Ω,

(2) g_i are smooth metrics on Ω_i, which converge locally smoothly to g,

(3) $\{\varepsilon_i > 0\}$ is a sequence with $\varepsilon_i \searrow 0$, and $u_i \in C^\infty(\Omega_i)$ are solutions of the equations

$$\operatorname{div}_{g_i}\Big(\frac{\nabla_{g_i} u_i}{\sqrt{\varepsilon_i^2 + |\nabla_{g_i} u_i|_{g_i}^2}}\Big) = \sqrt{\varepsilon_i^2 + |\nabla_{g_i} u_i|_{g_i}^2}.$$

Then a subsequence of u_i converges in $C^0_{\mathrm{loc}}(\Omega)$ to a function $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, which solves $\mathsf{IMCF}(\Omega, g)$*. We have the gradient estimate*

$$|\nabla_g u(x)| \leqslant \frac{C(n)}{\sigma(x;\Omega,g)}, \qquad \forall\, x \in \Omega. \tag{2.4.14}$$

Moreover, on Ω and $\Omega \times \mathbb{R}$ respectively, the vector fields

$$\nu_i = \frac{\nabla_{g_i} u_i}{\big(\varepsilon_i^2 + |\nabla_{g_i} u_i|_{g_i}^2\big)^{1/2}} \qquad \text{and} \qquad \boldsymbol{\nu}_i = \frac{\nabla_{g_i} u_i - \varepsilon_i \partial_z}{\big(\varepsilon_i^2 + |\nabla_{g_i} u_i|_{g_i}^2\big)^{1/2}}$$

converge to some $\nu, \boldsymbol{\nu}$ in the weak topology of $L^1_{\mathrm{loc}}(\Omega)$ resp. $L^1_{\mathrm{loc}}(\Omega \times \mathbb{R})$, such that ν is the projection of $\boldsymbol{\nu}$ on the Ω factor, and ν calibrates u in the sense of Definition 2.3.1.

The proof is technical, so we postpone it to the end of this section. Note that we have two convergences:

$$u_i \xrightarrow{C^0_{\mathrm{loc}}(\Omega)} u, \qquad U_i(x,z) := u_i(x) - \varepsilon_i z \xrightarrow{C^0_{\mathrm{loc}}(\Omega\times\mathbb{R})} u(x) =: U(x,z).$$

The convergence of level sets is also interesting. Since U_i is a smooth solution of IMCF, its sub-level set is almost perimeter-minimizing (see (2.1.5)$\sim$Lemma 2.1.7). Thus, for each t we may take a limit (up to subsequence) as $\varepsilon \to 0$:

$$E_t(U_i) = \big\{z > \varepsilon_i^{-1}(u_i - t)\big\} \xrightarrow{L^1_{\mathrm{loc}}(\Omega\times\mathbb{R})} \overline{E}_t.$$

Note the following general fact: if $f_i \to f$ in C^0_{loc}, and $E_t(f_i) \to E$ in L^1_{loc}, then $E_t(f) \subset E \subset E^+_t(f)$ up to zero measure. Applying this to our case, we obtain

$$E_t(u) \times \mathbb{R} \subset \overline{E}_t \subset E^+_t(u) \times \mathbb{R} \qquad \text{up to zero measure.}$$

So for all but finitely many t, we are forced to have $\overline{E}_t = E_t(u) \times \mathbb{R} = E^+_t(u) \times \mathbb{R}$.

For those jump times t (i.e. $E^+_t \setminus E_t$ has positive measure), the limit set $\overline{E}_t$ lies between $E_t \times \mathbb{R}$ and $E^+_t \times \mathbb{R}$, and may be different from both of them. Moreover, recall from Theorem 2.1.13 that $\overline{E}_t$ is a local minimizer of J_U in $\Omega \times \mathbb{R}$. And since U is constant in $\big(E^+_t(u) \setminus \overline{E_t(u)}\big) \times \mathbb{R}$, we obtain that $\overline{E}_t$ is a local perimeter minimizer there (hence ∂E_t is a minimal surface there).

In summary, we obtain the following picture: the collection of level sets $\{E_t(U_i)\}$ converges to vertical slices $E_t(u) \times \mathbb{R}$ for all but countably many t, while it can converge to a non-vertical slice that is minimal in between $E_t(u) \times \mathbb{R}$ and $E^+_t(u) \times \mathbb{R}$.

For those exceptional times, the limit may depend on the subsequence that we choose. Moreover, by considering all the possible limits of the form

$$\lim_{i\to\infty} E_{t_i}(U_i), \qquad t_i \text{ is a sequence with } t_i \to t,$$

we should obtain a minimal foliation of $(E^+_t \setminus E_t) \times \mathbb{R}$. This foliation is further calibrated by the limit vector field $\boldsymbol{\nu}$ in Theorem 2.4.7. See Figure 2.5 for a depiction. Note a detail: a minimal leaf in $(E^+_t \setminus E_t) \times \mathbb{R}$ may touch $\partial E_t \times \mathbb{R}$, but it never touches $(\partial E^+_t \setminus \partial E_t) \times \mathbb{R}$, by the strong maximum principle.

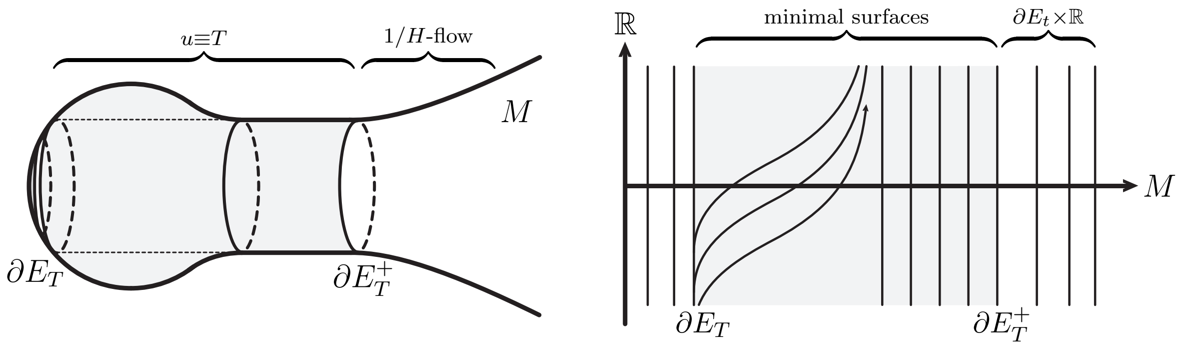

Figure 2.5: A weak flow with jump, and the limit foliation in $M \times \mathbb{R}$.

Finally, under our settings made above, we have convergence to a proper solution:

Theorem 2.4.8 (convergence to proper solution)**.** *Assume the same conditions on M, E_0, v as in Theorem 2.4.6. Given two sequences $L_i \to \infty$, $\varepsilon_i \to 0$ with $\varepsilon_i \leqslant \varepsilon(L_i)$. Then up to a subsequence, there are solutions $u_i = u_{\varepsilon_i, L_i}$ of (2.4.9)$\sim$(2.4.11) that converges in $C^0_{\mathrm{loc}}(M \setminus E_0)$ to the unique proper solution of* $\mathsf{IVP}(M; E_0)$.

Therefore, $\mathsf{IVP}(M; E_0)$ *admits a solution if we assume the conditions on M, E_0, v in Theorem 2.4.6.*

Proof. The solutions u_i are given directly by Theorem 2.4.6. We first use Theorem 2.4.7 to obtain a convergence $u_i \to u$ in $C^0_{\mathrm{loc}}(M \setminus \overline{E_0})$, so that u solves $\mathsf{IMCF}(M \setminus \overline{E_0})$. Then we notice from (2.4.13) that the gradient estimate of u_i is up to ∂E_0. Taking a further subsequence, we may assume that $u_i \to u$ in $C^0_{\mathrm{loc}}(M \setminus E_0)$. It follows immediately that $u|_{\partial E_0} = 0$. The bound (2.4.12) implies that $u \geqslant \max\{0, v-2\}$. Combining with Theorem 2.4.7, it follows that u is a proper solution of $\mathsf{IVP}(M; E_0)$. Finally, the uniqueness of u follows from the maximum principle (Theorem 2.1.12). $\square$

Proof of Theorem 2.4.7.

On $\Omega_i \times \mathbb{R}$ with the product metric $g_i + dz^2$, the functions $U_i(x,z) = u_i(x) - \varepsilon_i z$ are smooth solutions of the IMCF. By Theorem 2.4.3 and Definition 2.4.2 we have the estimate

$$|\nabla_{g_i} u_i(x)| \leqslant |\nabla_{g_i+dz^2} U_i(x,0)| \leqslant \frac{C(n+1)}{\sigma\big((x,0);\Omega_i\times\mathbb{R}, g_i+dz^2\big)} \leqslant \frac{C(n+1)}{\sigma(x;\Omega_i,g_i)}. \tag{2.4.15}$$

By condition (2) of the theorem, for all $K \Subset \Omega\times\mathbb{R}$ there exists i_0 such that

$$\inf_{x\in K}\inf_{i\geqslant i_0} \sigma(x;\Omega_i\times\mathbb{R}, g_i+dz^2) > 0.$$

By the Arzela-Ascoli theorem, there is a subsequence such that $u_i \to u$ locally uniformly for some $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$. Set $U(x,z) := u(x)$, which is clearly the C^0_{loc} limit of $U_i(x,z)$. Note that $\sigma(x;\Omega,g) \leqslant 2\sigma(x;\Omega_i,g_i)$ for sufficiently large i. Thus (2.4.15) passes to the limit and give

$$|\nabla_g u(x)| \leqslant \frac{C'(n)}{\sigma(x;\Omega,g)}.$$

Since U_i are smooth solutions, they are calibrated by the vector fields

$$\boldsymbol{\nu}_i := \frac{\nabla_{g_i+dz^2} U_i}{|\nabla_{g_i+dz^2} U_i|} = \frac{\nabla_{g_i} u_i - \varepsilon_i \partial_z}{\big(\varepsilon_i^2 + |\nabla_{g_i} u_i|^2_{g_i}\big)^{1/2}}.$$

By the Dunford-Pettis theorem and a diagonal argument, there is a subsequence such that $\boldsymbol{\nu}_i$ converges to some $\boldsymbol{\nu}$ weakly in $L^1_{\mathrm{loc}}(\Omega\times\mathbb{R})$. Now all the conditions of Theorem 2.3.3 are met, and it follows that U solves $\mathsf{IMCF}(\Omega\times\mathbb{R}; g+dz^2)$ and is calibrated by $\boldsymbol{\nu}$.

Note that $\boldsymbol{\nu}_i$ are invariant under vertical translation, and this property passes to the limit $\boldsymbol{\nu}$. Let ν be the projection of $\boldsymbol{\nu}$ on the Ω factor. It is easily seen that ν is the L^1_{loc} weak limit of ν_i.

Finally, we show that u is a weak solution in Ω calibrated by ν. Indeed, we have $|\nu| \leqslant |\boldsymbol{\nu}| \leqslant 1$ and $\nabla u(x)\cdot\nu(x) = \nabla U(x,0)\cdot\boldsymbol{\nu}(x,0) = |\nabla U(x,0)| = |\nabla u(x)|$ almost everywhere. To verify the condition (2.3.1), we fix a cutoff function $\eta \in C^\infty(\mathbb{R})$ with $\eta|_{(-\infty,-1]} \equiv 0$ and $\eta|_{[0,\infty)} \equiv 1$. For $R>0$ we set $\rho_R(z) = \eta(z)\eta(R-z)$. Now for a fixed $\phi \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, we test the calibration property of U with the function $\phi(x)\rho_R(z)$ and find

$$\begin{aligned}
0 &= \int_{\Omega\times\mathbb{R}} \big(\nabla_x\phi(x)\rho_R(z) + \phi(x)\rho_R'(z)\partial_z\big)\cdot\boldsymbol{\nu} + \phi(x)\rho_R(z)|\nabla_x u(x)|\,dx\,dz \\
&= R\int_\Omega \nabla\phi\cdot\nu + \phi|\nabla u| \\
&\quad + \int_{\Omega\times([-1,0]\times[R,R+1])} \big(\nabla_x\phi(x)\rho_R(z) + \phi(x)\rho_R'(z)\partial_z\big)\cdot\boldsymbol{\nu} + \phi(x)\rho_R(z)|\nabla_x u(x)|\,dx\,dz.
\end{aligned}$$

Taking $R\to\infty$, we have $0 = \int_\Omega \nabla\phi\cdot\nu + \phi|\nabla u|$. This verifies that ν calibrates u. □

2.5 Isoperimetry and properness

Definition 2.5.1. For a Riemannian manifold (M,g), define its isoperimetric profile by

$$\mathrm{ip}(v) = \inf\Big\{P(E) : E \Subset M \text{ has finite perimeter, } |E| = v\Big\}.$$

Then define its formal inverse by

$$\mathrm{sip}^{-1}(a) = \sup \Big\{ |E| : E \Subset M, P(E) \leqslant a \Big\}.$$

Note that $\mathrm{sip}^{-1}(a) < \infty$ when $a < \liminf_{v\to\infty} \mathrm{ip}(v)$.

The main result of this section is the following:

Theorem 2.5.2. *Given a constant $A > 0$. Let M be a connected, complete, non-compact Riemannian manifold with infinite volume, such that*

$$\liminf_{v\to\infty} \mathrm{ip}(v) > A \tag{2.5.1}$$

and

$$\int_0^{v_0} \frac{dv}{\mathrm{ip}(v)} < \infty \ \textit{for some } v_0 > 0. \tag{2.5.2}$$

Then for any $C^{1,1}$ domain $E_0 \Subset M$ with $P(E_0) < A$, there exists a solution u of $\mathsf{IVP}(M; E_0)$*, such that $E_t \Subset M$ for all $0 \leqslant t \leqslant \log\big(A/P(E_0)\big)$. If $E_0 \subset B(x_0, r_0)$ for a radius r_0, then we have $E_t \subset B(x_0, R)$, where*

$$R = r_0 + (2 + e^t) \int_0^{\mathrm{sip}^{-1}(e^t P(E_0))} \frac{dv}{\mathrm{ip}(v)} < \infty. \tag{2.5.3}$$

Moreover, we have the local gradient estimate

$$|\nabla u|(x) \leqslant \sup_{B(x,r)} H_+ + C(n) r^{-1}, \qquad \forall\, x \in M \setminus E_0,\ r \leqslant \sigma(x; M), \tag{2.5.4}$$

where as usual, H_+ denotes the positive part of the mean curvature of E_0.

Let us first note that this implies our main Theorem A.

Proof of Theorem A assuming Theorem 2.5.2.
Assume the conditions of Theorem A. For each $l \in \mathbb{N}_+$, we apply Theorem 2.5.2 with the choice $A = e^l P(E_0)$. We obtain a sequence of weak solutions u^l with initial condition E_0, such that $E_l(u^l) \Subset M$, and the quantitative diameter bound (2.5.3) holds for u^l whenever $t \leqslant l$. By Theorem 2.1.12(iii) (maximum principle), for two integers $l < l'$ we have $u^l = u^{l'}$ on $E_l(u^l)$. Therefore, the function

$$u(x) = \lim_{l\to\infty} u^l(x)$$

is defined on $\bigcup_{l\in\mathbb{N}} E_l(u^l)$ (which is M, by (2.5.4)). For each $t > 0$ we have $E_t(u) = E_t(u^{\lceil t\rceil})$, hence (2.5.3) holds for u as well. In particular, u is proper. Finally, by (2.5.4), and Theorem 2.1.13 (compactness, applied in $M \setminus \overline{E_0}$), and Definition 2.1.8, it follows that u is a solution of $\mathsf{IVP}(M; E_0)$. □

The proof of Theorem 2.5.2 consists of two components: a diameter estimate and a procedure to produce solutions. To show the ideas involved, let us first prove a result regarding the existence of precompact minimizing hulls. We refer the reader to the introduction, as well as Section A.4, for the notion of minimizing hull.

Theorem 2.5.3 (existence of precompact minimizing hull)**.**
Assume that M is complete, connected, with infinite volume, and satisfies

$$\liminf_{v\to\infty} \mathrm{ip}(v) > A \quad \text{and} \quad \int_0^{v_0} \frac{dv}{\mathrm{ip}(v)} < \infty \ \text{for some } v_0 > 0. \tag{2.5.5}$$

Then any $Q \Subset M$ with $P(Q) \leqslant A$ admits a precompact minimizing hull E in the sense of Definition A.4.5. Moreover, if $Q \subset B(x_0, r)$, then $E \subset B(x_0, R)$, where

$$R = r + 2\int_0^{\mathrm{sip}^{-1}(P(Q))} \frac{dv}{\mathrm{ip}(v)} < \infty. \tag{2.5.6}$$

We first prove a diameter estimate. In Lemma 2.5.4 below, one can view E as the imagined minimizing hull in Theorem 2.5.3.

Lemma 2.5.4. *Let M, A be as in Theorem 2.5.3. Suppose $r > 0$, and $E \Subset M$ satisfies*
(i) $P(E) \leqslant A$,
(ii) for all F with $E \cap B(x_0, r) \subset F \subset E$, it holds $P(E) \leqslant P(F)$,
then $E \subset B(x_0, R)$ up to zero measure, where

$$R = r + 2\int_0^{\mathrm{sip}^{-1}(P(E))} \frac{dv}{\mathrm{ip}(v)} < \infty. \tag{2.5.7}$$

Proof. The condition (2.5.5) implies that $\mathrm{sip}^{-1}(P(E)) \leqslant \mathrm{sip}^{-1}(A) < \infty$. Moreover, Corollary A.3.3 implies that $\mathrm{ip}(v) > 0$ for all $v > 0$. Hence $R < \infty$.

Suppose by contradiction that $\big|E \setminus B(x_0, R)\big| > 0$. With a change of measure zero, we may assume $E = E^{(1)}$. For almost every $\rho \in [r, R]$, we may define (see Remark A.6.2)

$$V(\rho) = \big|E \setminus B(x_0, \rho)\big|, \quad S(\rho) = P\big(B(x_0,\rho); E\big), \quad A(\rho) = P\big(E; M \setminus \overline{B(x_0,\rho)}\big).$$

Moreover, for almost every ρ, we have $\mathcal{H}^{n-1}\big(\partial^* E \cap \partial B(x_0, \rho)\big) = 0$.

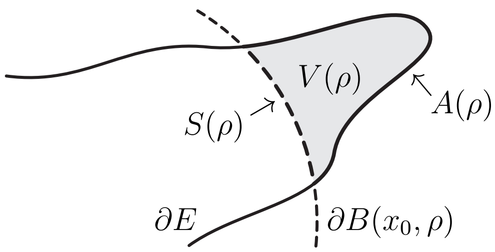

Figure 2.6: Perimeter comparison

By the coarea formula, $V(\rho)$ is absolutely continuous and satisfy

$$V'(\rho) = -S(\rho) \qquad \text{for a.e. } \rho. \tag{2.5.8}$$

By the cancellation inequality (A.1.14) and the minimization of E, we have

$$P(E) \leqslant P\big(E \cap B(x_0, \rho)\big) \ \Rightarrow \ A(\rho) \leqslant S(\rho). \tag{2.5.9}$$

By the decomposition identity (A.1.12), we have

$$S(\rho) + A(\rho) = P\big(E \setminus \overline{B(x_0,\rho)}\big) \geqslant \mathrm{ip}(V(\rho)). \tag{2.5.10}$$

Combining (2.5.8) (2.5.9) (2.5.10) we thus obtain

$$V'(\rho) \leqslant -\frac{1}{2}\,\mathrm{ip}(V(\rho)) \qquad \text{for a.e. } \rho.$$

Since $V(R) > 0$, we can integrate this inequality to obtain

$$2\int_{V(R)}^{V(r)} \frac{dv}{\mathrm{ip}(v)} > R - r.$$

On the other hand, we have $V(r) \leqslant |E| \leqslant \mathrm{sip}^{-1}(P(E))$. This contradicts (2.5.7). $\square$

Note that Lemma 2.5.4 is an a priori estimate, in the sense that it assumes the existence of the object E. To prove Theorem 2.5.3, we still need to actually find the set E. The case of minimizing hulls is simpler, and the following argument achieves our goal.

Proof of Theorem 2.5.3. Suppose $Q \subset B(x_0, r)$. Denote

$$A' = \inf\Big\{P(E) : Q \subset E \subset B(x_0, R+1)\Big\}. \tag{2.5.11}$$

By the compactness theorem and lower semi-continuity of perimeter, the least perimeter problem for A' always have a solution: there is a set E with $Q \subset E \subset B(x_0, R+1)$, such that $P(E) = A'$. Then we consider the maximal volume solution: set

$$V = \sup\big\{|F| : Q \subset F \subset B(x_0, R+1),\ P(F) = A'\big\}.$$

By compactness theorem and lower semi-continuity again, there is a set F_0 with $E \subset F_0 \subset B(x_0, R+1)$, such that $P(F_0) = A'$ and $|F_0| = V$. Note that F_0 satisfies the condition in Lemma 2.5.4, since F_0 solves the area minimization problem (2.5.11). From this we conclude that $F_0 \subset B(x_0, R)$.

We claim that F_0 is the minimizing hull of E in M, in the sense of Definition A.4.3. Suppose this is not true, so there is another precompact set $F_1 \supset Q$, with either $P(F_1) < P(F_0)$ or $|F_1| > |F_0|$. Let $R_1 \gg R$ be such that $F_1 \subset B(x_0, R_1)$. Repeating the argument above, there exists a maximal volume solution F_2 to the least area problem with inner obstacle Q and outer obstacle $B(x_0, R_1+1)$. As F_1 is a valid competitor in this minimization problem, we obtain that either $P(F_2) < P(F_1) < P(F_0)$ or $|F_2| > |F_1| > |F_0|$. On the other hand, by Lemma 2.5.4 one more time we have $F_2 \subset B(x_0, R)$, so we obtain a contradiction with the minimizing property of F_0. $\square$

2.5.1 Proof of the main properness theorem

The following lemma is our a priori diameter estimate for IMCF. In the lemma, we denote

$$\mathrm{ip}_\Omega(v) = \inf\Big\{P(E) : E \Subset \Omega,\ |E| = v\Big\}, \qquad \mathrm{sip}_\Omega^{-1}(a) = \sup\Big\{|E| : E \Subset \Omega,\ P(E) \leqslant a\Big\}.$$

Note that $\mathrm{ip}(v) = \mathrm{ip}_M(v)$, and $\mathrm{ip}_\Omega(v) \geqslant \mathrm{ip}(v)$, $\mathrm{sip}_\Omega^{-1}(a) \leqslant \mathrm{sip}^{-1}(a)$ for all $\Omega \subset M$.

Lemma 2.5.5. *Let $E_0 \Subset \Omega$ be a $C^{1,1}$ domain, and $E_0 \subset B(x_0, r_0)$ for some $x_0 \in E_0$, $r_0 > 0$. Let u solve $\mathsf{IVP}(\Omega; E_0)$, and suppose $E_t \Subset \Omega$ for some $t > 0$. Then we have $E_t \subset B(x_0, R)$, where*

$$R = r_0 + (1 + e^t)\int_0^{\mathrm{sip}_\Omega^{-1}(e^t P(E_0))} \frac{dv}{\mathrm{ip}_\Omega(v)}.$$

Proof. Suppose this is not true. Recall by Lemma 2.1.6 that $E_t = E_t^{(1)}$. Similar to the proof of Lemma 2.5.4, for almost every $\rho \in [r_0, R]$ we define

$$V(\rho) = \big|E_t \setminus B(x_0,\rho)\big|, \quad S(\rho) = P\big(B(x_0,\rho), E_t\big), \quad A(\rho) = P\big(E_t, \Omega \setminus \overline{B(x_0,\rho)}\big).$$

Choose a domain K with $E_t \Subset K \Subset \Omega$. Applying Lemma 2.2.1 with the choice $F = K \setminus B(x_0,\rho)$, we have

$$P(E_t) \leqslant P\big(E_t \setminus B(x_0,\rho)\big) + (e^t - 1)S(\rho).$$

Then by the cancellation inequality (A.1.14), we have

$$A(\rho) \leqslant e^t S(\rho) \qquad \text{for a.e. } \rho \geqslant r_0.$$

By the isoperimetric inequality, we have

$$A(\rho) + S(\rho) \geqslant \mathrm{ip}_\Omega(V(\rho)) \qquad \text{for a.e. } \rho \geqslant r_0.$$

Finally, the coarea formula gives

$$V'(\rho) = -S(\rho) \qquad \text{for a.e. } \rho \geqslant r_0.$$

Combining them together, we obtain

$$V'(\rho) \leqslant -\frac{\mathrm{ip}_\Omega(V(\rho))}{1+e^t} \qquad \text{for a.e. } \rho \geqslant r_0. \tag{2.5.12}$$

By our hypothesis, we have $V(R) > 0$. Thus we can integrate (2.5.12) to obtain

$$\frac{R - r_0}{1+e^t} \leqslant \int_0^{V(r_0)} \frac{dv}{\mathrm{ip}_\Omega(v)}.$$

Then, note that $V(r_0) \leqslant |E_t| \leqslant \mathrm{sip}_\Omega^{-1}(P(E_t)) \leqslant \mathrm{sip}_\Omega^{-1}(e^t P(E_0))$. This contradicts our choice of R, thus proves the lemma. $\square$

Having the a priori diameter estimate, the next step is to actually construct the weak solution. The main issue is that we have no existence theorem at hand except for Theorem 2.4.1. To resolve this issue, we employ a conic cutoff method, inspired by similar argument in [53, Theorem 3.1]. The idea is to truncate the manifold and attach a metric cone. Then Theorem 2.4.1 provides a proper weak solution on each modified manifold. Then we let the locus of truncation diverge to infinity, and expect to obtain the desired weak IMCF on M as a limit. Below are the details of this argument.

Assume the setups of Theorem 2.5.2. Fix a basepoint $x_0 \in E_0$. For simplicity, we denote $B(r) = B(x_0, r)$ for $r \in \mathbb{R}$. For each integer $k \in \mathbb{N}$, $k > r_0$, let W_k be a connected smooth domain with $B(k+\frac14) \subset W_k \subset B(k+\frac12)$. For each k, we construct a new manifold M_k by smoothly attaching a metric cone $\partial W_k \times [0,\infty)$ to W_k. The detailed construction is as follows. Choose $0 < \delta \leqslant \frac18$ such that 2δ is smaller than the normal injectivity radius of ∂W_k. The choice of δ depends on k, which we make implicit for brevity. Let $\eta : [0,\infty) \to [0,\infty)$ be a smooth cutoff function such that $\eta|_{[0,1/2]} \equiv 1$ and $\eta|_{[3/4,\infty)} \equiv 0$. Let $g = dr^2 + h(r,x)$ $(0 \leqslant r \leqslant \delta)$ be the metric expression in the δ-collar neighborhood of

W_k (the positive r-direction pointing outward), induced by the normal exponential map. Thus $h(0,x) = g|_{\partial W_k}$. Consider the new tensor

$$h'(r,x) = (1+e^{-1/r})\,\eta(\frac{r}{\delta})\,h(r,x) + C\big(1-\eta(\frac{r}{\delta})\big)\,\big(\frac{r}{\delta}\big)^2\,h(\delta,x).$$

By choosing the constant C sufficiently large, h' satisfies the properties

$$\begin{cases} (1)\ h'(r,x) > h(r,x) \text{ for all } 0 < r \leqslant \delta. \\ (2)\ h'(r,x) = \big(\frac{r}{\delta}\big)^2 h'(\delta,x) \text{ for all } r \geqslant \delta. \end{cases} \tag{2.5.13}$$

where $h'(r,x) > h(r,x)$ means that $\big(h'_{ij}(r,x) - h_{ij}(r,x)\big)dx^i dx^j$ is positive-definite (where $1 \leqslant i,j \leqslant n-1$). Let $M_k = W_k \cup \big(\partial W_k \times [0,\infty)\big)$, endowed with the metric g_k that coincides with g on W_k and equals to $dr^2 + h'(r,x)$ on $\partial W_k \times [0,\infty)$. Thus g_k is a smooth metric on M_k. There is a smooth map

$$\Phi : W_k \cup \big(\partial W_k \times [0,\delta)\big) \to M, \tag{2.5.14}$$

composed of the identity map on W_k and the normal exponential map on $\partial W_k \times [0,\delta)$. By item (1) in (2.5.13), Φ is 1-Lipschitz (with the source metric g_k and target metric g), and $\Phi|_{\partial W_k \times (0,\delta)}$ strictly decreases the area of any hypersurface in $\partial W_k \times (0,\delta)$.

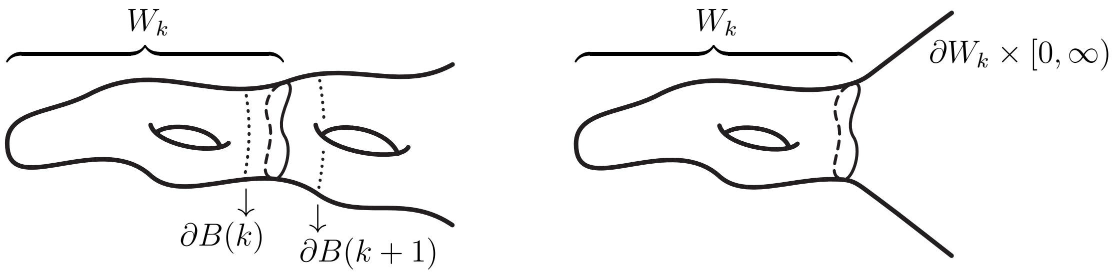

Figure 2.7: conic cutoff (left side: M, right side: M_k)

Note that $(n-1)\log(r/\delta)$ is a smooth solution to the IMCF on M_k with initial condition $W_k \cup \big(\partial W_k \times [0,\delta)\big)$. By Theorem 2.4.1, there is a proper solution u_k of $\mathsf{IVP}(M_k; E_0)$. Set

$$\overline{T} := \log\big(A/P(E_0)\big), \tag{2.5.15}$$

which is the expected maximal proper time in Theorem 2.5.2. Through a chain of lemmas below, we will show that $\min\big(u_k, \overline{T}\big)$ is the desired weak solution for sufficiently large k.

For each $k \in \mathbb{N}$, $k > r_0$, define the first escaping time of u_k as follows:

$$T_k = \sup\big\{t \geqslant 0 : E_t(u_k) \subset B(k)\big\}. \tag{2.5.16}$$

Clearly the supremum in (2.5.16) is achieved, so $E_{T_k}(u_k) \subset B(k)$.

Note the notational subtlety here. The precise statement for "$E_t(u_k) \subset B(k)$" is "$E_t(u_k) \subset W_k$ and its identical image in M is contained in $B(k)$". For brevity of statements, we keep the simplified notation for the rest of the proof. If a set $E \subset M_k$ is actually contained in W_k, then clearly $P(E)_g = P(E)_{g_k}$. Here we use $P(\cdot)_g$ to denote the perimeter with respect to a metric g.

The following lemma controls the jumping behavior of u_k, and is where the properties (2.5.13) of M_k are used. The nice behavior of u_k depends on the particular construction of M_k. For instance, if one creates a thin neck when attaching the exterior part to W_k, then u_k will quickly jump to the thin neck and ignore the geometry inside W_k.

Lemma 2.5.6. *Given $r > 0$ and $0 \leqslant t \leqslant \overline{T}$, set*

$$R = r + 2\int_0^{\mathrm{sip}^{-1}(e^t P(E_0))} \frac{dv}{\mathrm{ip}(v)}. \tag{2.5.17}$$

Then $R < \infty$ under the main conditions (2.5.1) (2.5.2). For all $k > R$ the following statement holds: if $E_t(u_k) \subset B(r)$, then $E_t^+(u_k) \subset B(R)$.

Proof. The finiteness of R follows from (2.5.1) and Corollary A.3.3.

Note that if it holds $E_t^+(u_k) \subset W_k$, then the result follows immediately from Lemma 2.5.4. So it remains to show that $E_t^+(u_k) \subset W_k$. Note that $\partial E_t^+(u_k) \setminus \partial E_t(u_k)$ is a g_k-minimal surface (rigorously speaking, the support of a g_k-stationary integral varifold) in M_k, by Lemma 2.1.10(i), Definition A.4.3 and Theorem A.4.5. Observe on the other hand that $\partial W_k \times [\delta, \infty)$ is foliated by strictly convex hypersurfaces, hence $E_t^+(u_k) \subset W_k \cup \big(\partial W_k \times (0,\delta)\big)$ by a strong maximum principle of Solomon-White [110, Theorem 4]. Suppose that $E_t^+(u_k)$ has nonempty intersection with $\partial W_k \times (0,\delta)$. Then the map Φ defined in (2.5.14) strictly decreases the area of $E_t^+(u_k)$. Denote $F_1 = \Phi(E_t^+(u_k))$, we have by Lemma 2.1.10(ii)

$$P\big(E_t(u_k)\big)_g = P\big(E_t(u_k)\big)_{g_k} \geqslant P\big(E_t^+(u_k)\big)_{g_k} > P(F_1)_g.$$

By the construction of W_k, we have $F_1 \subset B(k+1)$. Now let F_2 be any minimizer of the following double obstacle problem:

$$P(F_2)_g = \inf\big\{P(F)_g : E_t(u_k) \subset F \subset B(k+1)\big\}.$$

Therefore $P(F_2)_g \leqslant P(F_1)_g < P\big(E_t^+(u_k)\big)_{g_k}$. In particular, $P(F_2)_g \leqslant e^t P(E_0)_g \leqslant A$. Applying Lemma 2.5.4, we obtain $F_2 \subset B(R)$.

In particular, F_2 is a valid perimeter competitor for $E_t^+(u_k)$. This implies that $E_t^+(u_k)$ does not solve the least area problem outside $E_t(u_k)$ in M_k, which contradicts Lemma 2.1.10(i) and Theorem A.4.5. Hence $E_t^+(u_k) \subset W_k$. ☐

Corollary 2.5.7. *There is $k_0 \in \mathbb{N}$ such that $T_k > 0$ for all $k \geqslant k_0$.*

Proof. Apply Lemma 2.5.6 with $t = 0$, $r = r_0$. For each k we have $E_0^+(u_k) \subset B(R)$ for all $k > R$, where R is given by (2.5.17). By Lemma 2.1.7(ii) we have $E_\varepsilon(u_k) \subset B(R+1)$ for some small $\varepsilon > 0$. This proves the lemma with $k_0 = \lfloor R \rfloor + 2$. ☐

For $i = 1, 2$, define

$$R_i(t) = r_0 + (2i + e^t)\int_0^{\mathrm{sip}^{-1}(e^t|\partial E_0|)} \frac{dv}{\mathrm{ip}(v)}. \tag{2.5.18}$$

Thus R_i is finite when $t \leqslant \overline{T}$, under the conditions (2.5.1) (2.5.2).

Lemma 2.5.8. *We have $E_t(u_k) \subset B(R_1(t))$ for all $k \geqslant k_0$ and all $0 < t \leqslant \min\{T_k, \overline{T}\}$.*

Proof. Note that $E_t(u_k) \subset B(k)$ whenever $t \leqslant T_k$. Hence we can apply Lemma 2.5.5 inside $\Omega = B(k+1/8)$. The result follows by noting that $\mathrm{ip}_\Omega(v) \geqslant \mathrm{ip}(v)$, $\mathrm{sip}_\Omega^{-1}(a) \leqslant \mathrm{sip}^{-1}(a)$. ☐

Lemma 2.5.9. *There exists $k_1 \in \mathbb{N}$ such that $T_k \geqslant \overline{T}$ for all $k \geqslant k_1$. Furthermore, $E_{\overline{T}}(u_k)$ is outward minimizing in M for all $k \geqslant k_1$.*

Proof. Choose $k_1 > R_2(\overline{T})$ (in particular, $k_1 > k_0$). Suppose that $k \in \mathbb{N}$ satisfies $T_k < \overline{T}$. We will show that $k < k_1$, which proves the first statement. Applying Lemma 2.5.8, we obtain $E_{T_k}(u_k) \subset B(R_1(T_k))$. If further $R_2(T_k) < k$, then Lemma 2.5.6 applies to yield $E^+_{T_k}(u_k) \subset B(R_2(T_k) - 1)$. Therefore $E_{T_k+\varepsilon}(u_k) \subset B(R_2(T_k)) \subset B(k)$ for some small $\varepsilon > 0$, by Lemma 2.1.7(ii). This contradicts the maximality of T_k, hence $R_2(T_k) \geqslant k$. We have

$$k \leqslant R_2(T_k) < R_2(\overline{T}) < k_1,$$

and the first statement follows.

For each $k \geqslant k_1$, Lemma 2.5.8 implies $E_{\overline{T}}(u_k) \subset B(R_1(\overline{T}))$. By Theorem 2.5.3, $E_{\overline{T}}(u_k)$ admits a strictly outward minimizing hull that is contained in $B(R_2(\overline{T}) - 1)$. Thus we note that: if $E_{\overline{T}}(u_k)$ is outward minimizing in $B(R_2(\overline{T})-1)$, then it is outward minimizing in M. The former must hold, since u is a weak solution in $B(R_2(\overline{T})) \subset B(k)$. This proves the second statement. $\square$

Proof of Theorem 2.5.2.

Given all the settings described above, we choose k_1 as in Lemma 2.5.9 and consider the function

$$u(x) = \begin{cases} \min(u_{k_1}(x), \overline{T}) & (x \in B(k_1)), \\ \overline{T} & (x \notin B(k_1)). \end{cases}$$

Since $E_{\overline{T}}(u_{k_1}) \subset B(R_1(\overline{T})) \Subset B(k_1)$ by Lemma 2.5.8, u is a continuous function. The quantitative bound (2.5.3) for $0 \leqslant t \leqslant \overline{T}$ is inherited from Lemma 2.5.8. The gradient estimate (2.5.4) comes from Theorem 2.4.1. It remains to show that u solves $\mathsf{IMCF}(M \setminus \overline{E_0})$.[1] Given $0 < t \leqslant \overline{T}$ and a competitor set E such that $E \Delta E_t(u) \subset K \Subset M \setminus E_0$. By the outward minimizing property in Lemma 2.5.9, we have

$$J^K_u(E \cup E_{\overline{T}}(u)) \geqslant J^K_u(E_{\overline{T}}(u)).$$

Hence

$$J^K_u(E) \geqslant J^K_u(E \cap E_{\overline{T}}(u)) + J^K_u(E \cup E_{\overline{T}}(u)) - J^K_u(E_{\overline{T}}(u)) \geqslant J^K_u(E \cap E_{\overline{T}}(u)).$$

Since u is a weak solution in $B(k_1)$, we have

$$J^K_u(E \cap E_{\overline{T}}(u)) \geqslant J^K_u(E_t(u)).$$

It follows that $J^K_u(E) \geqslant J^K_u(E_t(u))$, hence u is a weak solution. $\square$

2.5.2 On Huisken-Ilmanen's existence theorem

For the reader's convenience, we include a proof of Theorem 2.4.1. We refer to Section 2.4 for the context. Recall from Theorem 2.4.8 that we already proved the case where v is smooth with nonvanishing gradient in $\{v \geqslant 0\}$. Using the conic cutoff trick, we can now prove the full case of Theorem 2.4.1. Assume the conditions there: we are given a proper function $v \in \mathrm{Lip}_{\mathrm{loc}}(M)$ which is a subsolution of $\mathsf{IMCF}(\{v > 0\})$.

For each $k > 0$, let W_k be a connected smooth domain such that $\{v < k\} \Subset W_k$. Then we perform the conic cutoff construction, see above Figure 2.7, to obtain a new manifold $M_k = W_k \cup \big(\partial W_k \times [0, \infty]\big)$. Let r be the radial factor on $\partial W_k \times [0, \infty)$. As we have

[1] This does not follow directly from Remark 2.1.4(iv), since the ambient manifold has changed.

noticed near Figure 2.7, the function $(n-1)\log(r/\delta)$ is a smooth proper IMCF on M_k. This function fulfills the conditions of Theorem 2.4.8, so we get a proper solution u_k of $\mathsf{IVP}(M_k; E_0)$. On the other hand, the function

$$v_k = \begin{cases} \min\{v,k\} & \text{in } W_k \\ k & \text{in } \partial W_k \times [0,\infty) \end{cases}$$

is a subsolution of $\mathsf{IVP}(M_k; E_0)$. This can be verified by noticing that $E_k(v_k)$ is outward minimizing in M_k. By the standard maximum principle, we have $u_k \geqslant v_k$.

Finally, letting $k \to \infty$, the standard compactness argument yields a solution u of $\mathsf{IVP}(M; E_0)$. Passing $u_k \geqslant v_k$ to the limit, we have $u \geqslant v$ hence u is proper. The gradient estimate comes from passing (2.4.13) to the limit. This proves Theorem 2.4.1.

2.6 Euclidean growth estimate

The aim of this section is to prove our main Theorem B. Recall that we assume an Euclidean isoperimetric inequality

$$P(E) \geqslant c_I |E|^{\frac{n-1}{n}}, \qquad \forall\, E \Subset M. \tag{2.6.1}$$

To prove Theorem B, we first show that for any $x_0 \in M$, there exists an IMCF that "starts from the point x_0", and satisfies the growth estimate $u \geqslant (n-1)\log d(\cdot, x_0) - C$. Following G. Huisken, we call such a solution an IMCF core.

Definition 2.6.1. Given a point $x_0 \in M$ and a function $u : M \setminus \{x_0\} \to \mathbb{R}$. We say that u is an IMCF core with pole x_0, if the following conditions hold:

(1) $u \in \mathrm{Lip}_{\mathrm{loc}}\big(M \setminus \{x_0\}\big)$, and $\lim_{x\to x_0} u(x) = -\infty$,

(2) u solves $\mathsf{IMCF}(M \setminus \{x_0\})$.

We say that such a u is proper, if $E_t(u) \Subset M$ for all $t \in \mathbb{R}$.

The following growth estimate, and its relation with Theorem B, are inspired by similar arguments in Mari-Rigoli-Setti [79]:

Theorem 2.6.2. *Suppose M satisfies (2.6.1). Then for any $x_0 \in M$, there exists a proper IMCF core u with pole x_0, with the asymptotic*

$$\lim_{x\to x_0} \big|u(x) - (n-1)\log d(x,x_0)\big| \to 0 \tag{2.6.2}$$

and the growth estimate

$$u(x) \geqslant (n-1)\log d(x,x_0) - C(n,c_I) \qquad \textit{on} \;\; M \setminus \{x_0\}. \tag{2.6.3}$$

Theorem B is a straightforward consequence of this result:

Proof of Theorem B assuming Theorem 2.6.2.

Fix $x_0 \in E_0$. Let v be the IMCF core with pole x_0, as given by Theorem 2.6.2. Since v is proper, there exists $T \in \mathbb{R}$ so that $E_0 \Subset E_T(v)$. Let u be the proper solution of $\mathsf{IVP}(M; E_0)$. By the maximum principle (Theorem 2.1.12(iii)), it follows that $u \geqslant v - T \geqslant (n-1)\log d(\cdot, x_0) - (n-1)\log C_1 - T$ on $M \setminus E_0$. □

The idea of proving Theorem 2.6.2 is the following: for a radius $r \ll 1$, consider the proper IMCF $\widetilde{u}_r$ with initial value $E_0 = B(x_0, r)$, then normalize $u_r = \widetilde{u}_r + (n-1)\log r$. If a uniform lower bound for u_r is obtained, then the (well-defined) limit $u = \lim_{r\to 0} u_r$ would be our desired object.

The main issue here is that the estimate in Theorem 2.5.2 is too weak for this purpose. Indeed, inserting $\mathrm{ip}(v) \geqslant c_I v^{(n-1)/n}$ into (2.5.3) and simplifying the expressions, we have

$$E_t(\widetilde{u}_r) \subset B\big(x_0, r + C(n, c_I)(2+e^t)e^{t/(n-1)}r\big). \tag{2.6.4}$$

This only implies

$$u_r \geqslant \frac{n-1}{n}\log d(\cdot, x_0) + \frac{(n-1)^2}{n}\log r - C(n, r_I).$$

In particular, the leading $\frac{n-1}{n}$ is weaker than optimal, and more seriously, the $\log r$ term blows up to $-\infty$ when $r \to 0$, which prevents us from getting anything useful.

The essential technical reason is the $(2+e^t)$ term in (2.6.4). Tracing back in the proof of (2.6.4), this term comes from Lemma 2.5.5. More precisely, it comes from applying Lemma 2.5.5 from the initial time to a time that is roughly $(n-1)\log(1/r)$. This long time interval is the reason that makes the estimate particularly weak. The key to refining the estimate is to use Lemma 2.5.5 (or its proof) only in a bounded time interval. The proof of Theorem 2.6.2 involves a soft blow-up argument, whose meaning will be manifest in the proof below. We will need the following technical lemma:

Lemma 2.6.3. *Suppose M satisfies (2.6.1). Then for each $x_0 \in M$, there exists a sufficiently small radius r_0, so that $B(x_0, r)$ is outward minimizing in M for all $r \leqslant r_0$.*

Proof. Suppose $r \leqslant r_0$. By Theorem 2.5.3, $B(x_0, r)$ admits a precompact minimizing hull E in M. Note the following two properties:

(i) if $B(x_0, r)$ is not outer area-minimizing, then $E \supsetneq B(x_0, r)$;

(ii) $\partial E \setminus \partial B(x_0, r)$ is a minimal surface (said rigorously, the support of a stationary integral varifold).

Inserting our main condition (2.6.1) into the diameter estimate (2.5.6), we obtain $E \subset B(x_0, R)$ with

$$R = r + 2nc_I^{-\frac{n}{n-1}} P\big(B(x_0, r)\big)^{\frac{1}{n-1}}.$$

Note that $R \to 0$ when $r \to 0$. So we may choose $r_0 \ll 1$ so that all the geodesic balls $B(x, \rho)$, $\rho \leqslant R$, are strictly mean convex. In view of fact (ii) above, the maximum principle forces us to have $E = B(x_0, r)$. The lemma then follows by fact (i) stated above. □

Proof of Theorem 2.6.2.

Let $r_0 < 10^{-n}$ satisfy Lemma 2.6.3. Further decreasing r_0, we may assume that

$$P\big(B(x_0, r)\big) \leqslant 2\omega_{n-1} r^{n-1}, \qquad \forall\, r \leqslant r_0, \tag{2.6.5}$$

the mean curvature H_r of $B(x_0, r)$ satisfies

$$\big|H_r - (n-1)r^{-1}\big| < 1, \qquad \forall\, r \leqslant r_0. \tag{2.6.6}$$

For each $r \leqslant r_0$, let $\widetilde{u}_r$ be the proper solution of $\mathsf{IVP}(M; B(x_0, r))$ given by Theorem A, and then set

$$u_r = \widetilde{u}_r + (n-1)\log r.$$

We first establish a precise asymptotic of u_r near x_0. Denote $d = d(\cdot, x_0)$, and set the functions

$$\underline{u} = (n-1)\log d - d, \qquad \overline{u} = (n-1)\log d + d + \frac{1}{1 - d/r_0}.$$

Due to (2.6.6), it is easy to verify

$$\operatorname{div}\Big(\frac{\nabla \underline{u}}{|\nabla \underline{u}|}\Big) > |\nabla \underline{u}| \qquad \text{and} \qquad \operatorname{div}\Big(\frac{\nabla \overline{u}}{|\nabla \overline{u}|}\Big) < |\nabla \overline{u}| \qquad \text{in } \ B(x_0, r_0) \setminus \{x_0\},$$

namely, $\underline{u}, \overline{u}$ are subsolution and supersolution of IMCF in $B(x_0, r_0) \setminus \{x_0\}$.

Claim 1. $u_r \geqslant \underline{u}$ in $B(x_0, r_0) \setminus B(x_0, r)$ and $u_r \geqslant (n-1)\log r_0 - r_0$ on $M \setminus B(x_0, r_0)$.

Proof. Extend $\underline{u}$ by the constant value $(n-1)\log r_0 - r_0$ on $M \setminus B(x_0, r_0)$. Combining Lemma 2.6.3 and 2.2.4 (extension of subsolutions, with the choice $\Omega = B(x_0, r_0) \setminus \{x_0\}$ there), $\underline{u}$ is a subsolution of $\mathsf{IMCF}(M \setminus \{x_0\})$. Moreover, we have $u_r > \underline{u}$ in $\partial B(x_0, r)$. Thus by the maximum principle (Theorem 2.1.12(i)), we have $u_r \geqslant \underline{u}$ on $M \setminus B(x_0, r)$. The claim follows. □

Claim 2. $u_r \leqslant \overline{u}$ in $B(x_0, r_0) \setminus B(x_0, r)$.

Proof. Apply Theorem 2.1.12(i) in $B(x_0, r_0)$, noticing that $u_r < \overline{u}$ in $\partial B(x_0, r)$. □

Next, we prove the key growth estimate for u_r. For each $r \leqslant r_0$ and $t \geqslant (n-1)\log r$, define

$$D(r,t) := e^{-\frac{t}{n-1}} \sup_{E_t(u_r)} \big(d(\cdot, x_0)\big).$$

This quantity roughly measures the diameter of $E_t(u_r)$ compared to its area.

Claim 3. $D(r,t) < 4$ whenever $t \leqslant (n-1)\log(r_0/2)$.

Proof. By Claim 1, we have

$$u|_{M\setminus B(x_0,r_0)} \geqslant (n-1)\log r_0 - r_0 > t,$$

thus $E_t(u_r) \Subset B(x_0, r_0)$. Using Claim 1 again, we have

$$E_t(u_r) \subset \big\{(n-1)\log d - d < t\big\} \subset \big\{(n-1)\log d - 1 < t\big\} = \big\{d < e^{\frac{t+1}{n-1}}\big\}.$$

The claim follows. □

Define the constant

$$C_1 = 1 + c_I^{-\frac{n}{n-1}} n (2\omega_{n-1})^{\frac{1}{n-1}} \frac{1 + e^{n-1}}{1 - e^{-1}}. \tag{2.6.7}$$

The following claim is the core of the proof, which involves the soft blow-up argument.

Claim 4. $D(r,t) \leqslant C_1$ for all $r < r_0/10$ and $t \geqslant (n-1)\log r$.

Proof. Fix $r < r_0/10$. By Claim 3, we already have $D(r,t) < C_1$ for all $t \in \big[(n-1)\log r, (n-1)\log(r_0/2)\big]$. Suppose that Claim 4 is not true. Then there exists a time $T > (n-1)\log(r_0/2)$ so that

$$D(r,T) > C_1 \qquad \text{and} \qquad D(r, T-n+1) \leqslant C_1, \tag{2.6.8}$$

where we notice that $T - n + 1 > (n-1)\log r$. By the definition of $D(r,t)$, (2.6.8) means that

$$E_{T-n+1}(u_r) \subset B\big(x_0, C_1 e^{T/(n-1)-1}\big) \tag{2.6.9}$$

while
$$\big|E_T(u_r)\setminus B\big(x_0,C_1e^{T/(n-1)}\big)\big|>0. \tag{2.6.10}$$

We now argue similarly as in Lemma 2.5.5. For a.e. $\rho\in\big[C_1e^{T/(n-1)-1},C_1e^{T/(n-1)}\big]$, define the quantities
$$A(\rho)=P\big(E_T(u_r);M\setminus\overline{B(x_0,\rho)}\big),\quad S(\rho)=P\big(B(x_0,\rho);E_T(u_r)\big)$$
and
$$V(\rho)=\big|E_T(u_r)\setminus B(x_0,\rho)\big|.$$
Using the excess inequality (Lemma 2.2.1), we have
$$\begin{aligned}P\big(E_T(u_r)\big)\leqslant{}&P\big(E_T(u_r)\cap B(x_0,\rho)\big)\\&+\Big[\exp\big(T-\inf_{E_T(u_r)\setminus B(x_0,\rho)}(u_r)\big)-1\Big]P\big(B(x_0,\rho);E_T(u_r)\big).\end{aligned}$$
By (2.6.9), we have $\inf_{E_T(u_r)\setminus B(x_0,\rho)}(u_r)\geqslant T-n+1$. So for almost every ρ it holds
$$A(\rho)\leqslant e^{n-1}S(\rho). \tag{2.6.11}$$
Next, the isoperimetric inequality and coarea formula provide
$$A(\rho)+S(\rho)\geqslant c_IV(\rho)^{\frac{n-1}{n}},\qquad S(\rho)=-\frac{d}{d\rho}V(\rho), \tag{2.6.12}$$
for almost every ρ. Combining (2.6.11) (2.6.12), we have the differential inequality
$$\frac{d}{d\rho}V(\rho)\leqslant-\frac{c_I}{1+e^{n-1}}V(\rho)^{\frac{n-1}{n}}. \tag{2.6.13}$$
Integrating in $\rho\in\big[C_1e^{T/(n-1)-1},C_1e^{T/(n-1)}\big]$, and using (2.6.10), we have
$$V\big(C_1e^{T/(n-1)-1}\big)^{\frac1n}\geqslant C_1e^{T/(n-1)}\cdot\frac{c_I(1-e^{-1})}{n(1+e^{n-1})}. \tag{2.6.14}$$
On the other hand, by the isoperimetric inequality we have
$$V\big(C_1e^{T/(n-1)-1}\big)\leqslant|E_T(u_r)|\leqslant c_I^{-\frac{n}{n-1}}P\big(E_T(u_r)\big)^{\frac{n}{n-1}}, \tag{2.6.15}$$
and by (2.6.5) and the exponential growth of area (Lemma 2.1.10), we have
$$P\big(E_T(u_r)\big)\leqslant e^{T-(n-1)\log r}P\big(B(x_0,r)\big)\leqslant e^Tr^{1-n}\cdot2\omega_{n-1}r^{n-1}=2\omega_{n-1}e^T. \tag{2.6.16}$$
Combining (2.6.14) (2.6.15) (2.6.16), we obtain
$$\frac{c_I(1-e^{-1})}{n(1+e^{n-1})}\cdot C_1e^{\frac{T}{n-1}}\leqslant c_I^{-\frac{1}{n-1}}(2\omega_{n-1})^{\frac{1}{n-1}}e^{\frac{T}{n-1}},$$
contradicting our choice of C_1 in (2.6.7) (note that $e^{T/(n-1)}$ are cancelled). $\square$

Suppose $r<r_0/10$. Note that Claim 4 is directly equivalent to
$$E_t(u_r)\subset B\big(x_0,C_1e^{t/(n-1)}\big),\qquad\forall\,t\geqslant(n-1)\log r,$$

which means that

$$u_r \geqslant (n-1)\log\left[C_1^{-1}d(\cdot,x_0)\right] \qquad \text{on} \;\; M\setminus B(x_0,r). \tag{2.6.17}$$

We are ready to take the limit $r\to 0$. Recall the local gradient estimate from Theorem 2.5.2, which implies

$$|\nabla u_r|(x) \leqslant \frac{C(n)}{\min\left\{d(x,x_0),\sigma(x;M)\right\}}, \qquad \forall\, r\leqslant d(x,x_0)/2, \tag{2.6.18}$$

where $\sigma(x;M)$ is a quantity that depends only on the local geometry near x. Combining (2.6.18) and Claim 1,2, we can use the Arzela-Ascoli theorem to extract a limit (for some subsequence)

$$u = \lim_{r\to 0} u_r \qquad \text{in} \;\; C^0_{\mathrm{loc}}\big(M\setminus\{x_0\}\big).$$

It follows from Claim 1,2 that

$$\big|u-(n-1)\log d\big| < 2d \qquad \text{on} \;\; B(x_0,r_0/2)\setminus\{x_0\}, \tag{2.6.19}$$

and it follows from (2.6.17) that

$$u \geqslant (n-1)\log d(\cdot,x_0) - (n-1)\log C_1 \qquad \text{on} \;\; M\setminus\{x_0\}. \tag{2.6.20}$$

Finally, by compactness (Theorem 2.1.13), u solves $\mathsf{IMCF}(M\setminus\{x_0\})$. Now Theorem 2.6.2 has been proved. $\square$

Chapter 3

IMCF with outer obstacle

This chapter is devoted to the study of IMCF with outer obstacle. The reader may first read the introduction chapter to obtain a preliminary understanding of this object. We shall directly enter the technical contents.

Section 3.1 contains some computations for the smooth IMCF in $\mathbb{R}^2$. Section 3.2 contains some technical lemmas. In Section 3.3 we present the precise definition of IMCF with outer obstacle, and prove its fundamental properties. In Sections 3.4 and 3.5 respectively, we prove Liouville theorems on the half space, and derive some parabolic evolution inequalities. These results aid the proof of the main existence theorem. Finally, in Section 3.6, we prove the main Theorem D.

3.1 Some smooth calculations in $\mathbb{R}^2$

In this section, we carry out some specific computations for the smooth IMCF in $\mathbb{R}^2$. In particular, we are interested in the smooth IMCF with outer obstacle in conic domains.

The following setups are used. Suppose $\gamma \subset \mathbb{R}^2$ is a smooth curve with nonvanishing curvature, with unit speed parametrization s. Let $\tau := d\gamma/ds$ be its unit tangential vector, and θ be a continuous angle function so that $\tau = (\cos\theta, \sin\theta)$. Let $\nu := (\sin\theta, -\cos\theta)$ be the normal vector, and $\kappa := d\theta/ds$ be the curvature. Then note that

$$\frac{d\nu}{ds} = \kappa\tau, \qquad \frac{d\tau}{ds} = -\kappa\nu.$$

As we have assumed that $\kappa \neq 0$ everywhere, the angle θ is also a valid parametrization of γ, with $\frac{d}{d\theta} = \frac{1}{\kappa}\frac{d}{ds}$. Set the support function $h = x \cdot \nu$, where x is the position vector. Note that

$$\frac{dh}{d\theta} = \kappa^{-1}\big(\tau \cdot \nu + x \cdot \kappa\tau\big) = x \cdot \tau,$$

and

$$\frac{d}{d\theta}(x \cdot \tau) = \kappa^{-1}\big(\tau \cdot \tau - \kappa x \cdot \nu\big) = \kappa^{-1} - h.$$

Thus we obtain the useful general formulas

$$x = h\nu + \frac{dh}{d\theta}\tau, \tag{3.1.1}$$

and

$$\frac{d^2h}{d\theta^2} = \kappa^{-1} - h. \tag{3.1.2}$$

Here, note that $\partial_{\theta\theta}h + h$ is always positive for strictly convex curves.

Now suppose that a family of curves $\{\gamma_t\}$ evolves under the IMCF, while each γ_t is parametrized by the angle θ. Note that the curve evolves by

$$\frac{\partial\gamma}{\partial t} = \kappa^{-1}\nu + \varphi\tau$$

for some function φ on γ_t. The tangential factor does not affect us computing

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial t}(x\cdot\nu) = \frac{\partial x}{\partial t}\cdot\nu = \kappa^{-1}. \tag{3.1.3}$$

Joint with (3.1.2), we obtain the evolution of support function

$$\frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial\theta^2} + h. \tag{3.1.4}$$

Conversely, if a function $h(t,\theta)$ solves (3.1.4) in a square domain $(t_1,t_2)\times(\theta_1,\theta_2)$, then the resulting curves γ_t given by (3.1.1) is a solution of IMCF. These computations are already well-known, see [34, 107].

If we are given a closed convex curve $\gamma_0\subset\mathbb{R}^2$ whose support function is h_0, then the IMCF $\{\gamma_t\}$ starting with γ_0 is given by the support function

$$h_t(\theta) = e^t\int_{\mathbb{S}^1} K_t(\theta,\alpha)h_0(\alpha)\,d\alpha, \tag{3.1.5}$$

where K_t is the heat kernel on S^1. It is easily seen that when $t\to\infty$, the rescaled curve $e^{-t}\gamma_t$ smoothly converges to a round circle of radius $\frac{1}{2\pi}\int_{\mathbb{S}^1} h_0(\alpha)\,d\alpha$.

Remark 3.1.1. The general formula (3.1.5) can even be used to solve the IMCF from degenerate data. For example, the segment from $(-1,0)$ to $(1,0)$ has support function

$$h_0(\theta) = |\cos\theta|.$$

There is no issue with plugging in this function into (3.1.5). The resulting functions $\{h_t\}$ are the support functions of a smooth IMCF for all $t>0$, with convergence to round circles as $t\to\infty$. The more interesting aspect is when $t\to 0$. We notice that:

(i) γ_t converges to the segment from $(-1,0)$ to $(1,0)$ in the Hausdorff topology. This easily follows from the fact that $h_t\searrow h_0$ in $C^0(\mathbb{S}^1)$.

(ii) $d\big(\gamma_t(\pi/2),(0,0)\big) = \frac{2}{\sqrt{\pi}}\sqrt{t}+o(\sqrt{t})$. Indeed, we have

$$\begin{aligned} d\big(\gamma_t(\pi/2),(0,0)\big) &= h_t(\pi/2)\\ &= \frac{e^t}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-x^2/4t}|\sin x|\,dx\\ &= \frac{e^t}{\sqrt{\pi t}}\Big[\int_0^{\pi/2}+\int_{\pi/2}^{\infty}\Big]e^{-x^2/4t}|\sin x|\,dx =: \mathrm{I}+\mathrm{II}. \end{aligned}$$

Note that

$$\mathrm{I} = \frac{e^t}{\sqrt{\pi t}}\int_0^{\pi/2} e^{-x^2/4t}\big(x\pm Cx^3\big)\,dx = \frac{2}{\sqrt{\pi}}\sqrt{t}\big(1+o(1)\big), \tag{3.1.6}$$

where $\pm Cx^3$ is an expression bounded by Cx^3 in $[-\pi/2, \pi/2]$. Then note that

$$0 \leqslant \mathrm{II} \leqslant \frac{e^t}{\sqrt{4\pi t}} \int_{\mathbb{R}\setminus[-\pi/2,\pi/2]} e^{-x^2/4t}\, dx \leqslant Ce^{-\pi^2/16t}. \tag{3.1.7}$$

The result follows. We remark that the $\sqrt{t}$ rate of movement has been observed in all dimensions, see [54, Theorem 1.1], [31, Theorem A.5], and the comments in [54, p.434].

(iii) $d\big(\gamma_t(0), (1,0)\big) \leqslant 1 + Ce^{-\pi^2/16t}$. Indeed, we have

$$\begin{aligned} 1 + d\big(\gamma_t(0), (1,0)\big) = h_t(0) &= \frac{e^t}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-x^2/4t} |\cos x|\, dx \\ &= \frac{e^t}{\sqrt{\pi t}} \Big[\int_0^{\pi/2} + \int_{\pi/2}^{\infty} \Big] e^{-x^2/4t} |\cos x|\, dx =: \mathrm{I} + \mathrm{II}\,. \end{aligned}$$

We have $\mathrm{II} \leqslant Ce^{-\pi^2/16t}$ similarly as (3.1.7). It is well-known that

$$\int_0^\infty e^{-x^2/4t} \cos x\, dx = e^{-t}\sqrt{\pi t},$$

hence

$$\mathrm{I} = 1 - \frac{e^t}{\sqrt{\pi t}} \int_{\pi/2}^{\infty} e^{-x^2/4t} \cos x\, dx \leqslant 1 + \frac{e^t}{\sqrt{\pi t}} \int_{\pi/2}^{\infty} e^{-x^2/4t}\, dx \leqslant 1 + Ce^{-\pi^2/16t},$$

The result follows. See Figure 3.1 for a depiction of the small time behavior.

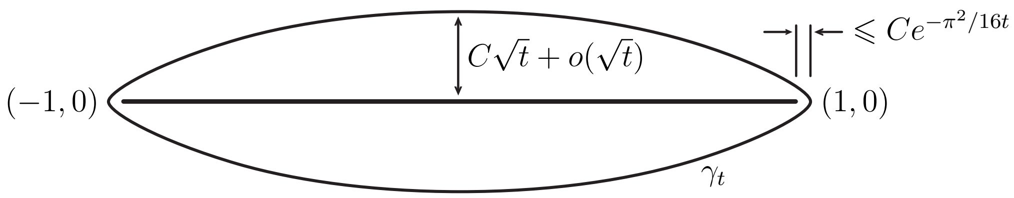

Figure 3.1: The IMCF in $\mathbb{R}^2$ starting from a segment

In general, starting with the boundary of a convex domain $E_0 \Subset \mathbb{R}^2$ there always exists a smooth IMCF $\{\gamma_t\}_{t>0}$ such that:

(i) $\gamma_t \to \partial E_0$ in the Hausdorff sense as $t \to 0$;
(ii) if $x_0 \in \partial E_0$ is a regular point, i.e. ∂E_0 is a smooth curve with nonvanishing curvature near x_0, then $\gamma_t \to \partial E_0$ smoothly near x_0;
(iii) if $x_0 \in \partial E_0$ is a flat point, then $d(x_0, \gamma_t) = C\sqrt{t}\big(1 + o(1)\big)$ for some constant C;
(iv) if $x_0 \in \partial E_0$ is a corner point, then $d(x_0, \gamma_t) \leqslant C_1 e^{-C_2/t}$ for some $C_1, C_2 > 0$.

Note that (iv) is unique to dimension 2. In higher dimensions, the IMCF starting with a convex hypersurface will stick at cone vertices, due to the presence of nontrivial cone solutions (see Example 1.1.5).

3.1.1 Solitons

We summarize the classification of translating and homothetic solitons of IMCF in $\mathbb{R}^2$, following the previous works [25, 26, 37, 62, 63]. See also [26] for more complicated solitons with simultaneous scaling and rotating behaviors.

In the above mentioned works, it was noticed that planar solitons of IMCF are usually incomplete. However, due to the nature of their endpoint singularities, the soliton curves contact tangentially with the boundary of the region that they sweep out. Therefore, the solutions that they generate would respect the boundary obstacle. This observation provides us the first class of nontrivial explicit examples. The setups in the previous subsection are assumed.

Translating soliton. A curve γ is called a *translating soliton* of IMCF, if for some fixed vector $w \in \mathbb{R}^2$, the family of curves $\gamma_t = \gamma + tw$ solves the IMCF. Since the evolution speed of this family is $\nu \cdot w$, we obtain the soliton equation

$$\kappa^{-1} = \nu \cdot w. \tag{3.1.8}$$

Up to a scaling and rigid motion, let us assume $w = \partial_x$, thus $\kappa^{-1} = \nu \cdot \partial_x = \sin\theta$. Inserting this into (3.1.2) and adding a boundary condition $h(0) = \frac{dh}{d\theta}(0) = 0$, we obtain

$$h(\theta) = -\frac{1}{2}\theta\cos\theta + \frac{1}{2}\sin\theta.$$

Then inserting into (3.1.1), we obtain

$$\gamma(\theta) = \frac{1}{4}\big(1 - \cos 2\theta, 2\theta - \sin 2\theta\big), \qquad \theta \in (0, \pi). \tag{3.1.9}$$

This is exactly the cycloid equation.

The curve γ is embedded into the strip region $\Omega = \mathbb{R} \times (0, \pi/2)$, and its endpoints $\gamma(0)$, $\gamma(\pi)$ lie on $\partial\Omega$. By Taylor expansion, near $\gamma(0)$ the curve is the graph of

$$y = \frac{2\sqrt{2}}{3}x^{3/2} + O(x^{5/2}), \qquad x > 0. \tag{3.1.10}$$

A similar result holds near $\gamma(\pi)$ by symmetry. Thus, the boundary tangency condition is met (at a Hölder modulus). Let u be the level set function. The boundary of $E_t(u)$ consists of a translation of γ and two horizontal radial lines, hence is a $C^{1,1/2}$ curve.

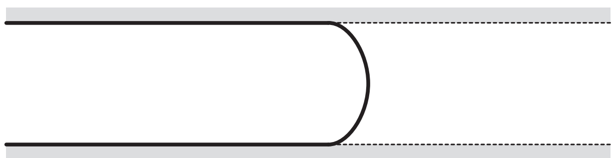

Figure 3.2: translating cycloids in a strip region.

Homothetic soliton. A curve γ is called a *homothetic soliton* of IMCF, if for some constant $c \in \mathbb{R}$, the family of curves $\gamma_t = e^{ct}\gamma$ solves the IMCF. Since the evolution speed for this family at $t = 0$ is $cx \cdot \nu = ch$, the equation for homothetic solitons reads

$$\kappa^{-1} = ch. \tag{3.1.11}$$

Thus by (3.1.2), this implies

$$\frac{d^2h}{d\theta^2} = (c-1)h. \tag{3.1.12}$$

Let us assume $c < 1$ since this is the case of relevance. Solving (3.1.12), we obtain up to a normalization $h(\theta) = \sin\big(\sqrt{1-c}\,\theta\big)$. Since the curve γ becomes singular when $h(\theta) = 0$, due to (3.1.11), we restrict the parameter to a single period $\theta \in (0, \pi/\sqrt{1-c})$. Then we insert the expression into (3.1.1) and obtain the curve $\gamma(\theta)$ in complex form:

$$\gamma(\theta) = -\sin\left(\sqrt{1-c}\,\theta\right) ie^{i\theta} + \sqrt{1-c}\cos\left(\sqrt{1-c}\,\theta\right) e^{i\theta}. \tag{3.1.13}$$

Setting $\mu = 1 - \sqrt{1-c}$ and $k = \frac{1+\sqrt{1-c}}{1-\sqrt{1-c}}$, we can rewrite (3.1.13) as

$$\gamma(\theta) = \frac{1}{2}\mu\left(ke^{i\mu\theta} - e^{ik\mu\theta}\right). \tag{3.1.14}$$

Comparing with the classical equations [68, Section 6.3, 6.5], we find that (3.1.14) describes a hypocycloid ($c < 0$) or epicycloid ($c > 0$). This was previously observed in [25, 37]. Notice that $h(0) = h\left(\pi/\sqrt{1-c}\right) = 0$, so γ is tangent to the radial line connecting the origin and its endpoints.

By Taylor expanding (3.1.14), we have

$$\gamma(\theta) = \frac{1}{2}\mu(k-1) + \frac{1}{4}\mu^3(k^2-k)\theta^2 + \frac{i}{12}\mu^4(k^3-k)\theta^3 + O(\theta^4).$$

Note that the quadratic term is real and the cubic term is imaginary. This shows again that γ is asymptotic to $y = Cx^{3/2}$ near the endpoint $\theta = 0$.

Example 3.1.2.

(i) Assume $c < 0$, and set $T = \pi/\sqrt{1-c}$. From (3.1.13) we have $\gamma(0) = \sqrt{1-c}$ and $\gamma(T) = \sqrt{1-c}\,e^{i\pi/\sqrt{1-c}-i\pi}$. Hence γ is supported in a planar cone domain Ω with angle $\left(1 - \frac{1}{\sqrt{1-c}}\right)\pi$. By the above discussion, the family $\{e^{ct}\gamma\}$ solves the IMCF and satisfies the boundary tangency condition at $\partial\Omega$. Taking u to be the level set function for the family $\gamma_t = e^{ct}\gamma$, we notice that $u \to +\infty$ at the origin.

(ii) Assume $c > 0$. Similarly as above, the family of curves $\{e^{ct}\gamma\}$ solve the IMCF in a cone domain Ω with angle $\left(\frac{1}{\sqrt{1-c}} - 1\right)\pi$, and satisfies the boundary tangency condition. This time, the correponding level set u satisfies $\lim_{x\to 0} u(x) = -\infty$. Notice that γ is embedded in $\mathbb{R}^2$ only when $c \leqslant 8/9$. For the case $c > 8/9$, we may view γ as embedded in the universal cover of $\mathbb{R}^2 \setminus \{0\}$.

(iii) In the special case $c = \frac{3}{4}$, the supporting domain Ω becomes a half plane. In this case, we obtain expanding nephroid as introduced in Example 1.4.4.

(iv) Let u be the expanding soliton as in (ii). Then the restriction of u to $\Omega := \{u < 0\}$ is a non-constant IMCF inside a bounded domain.

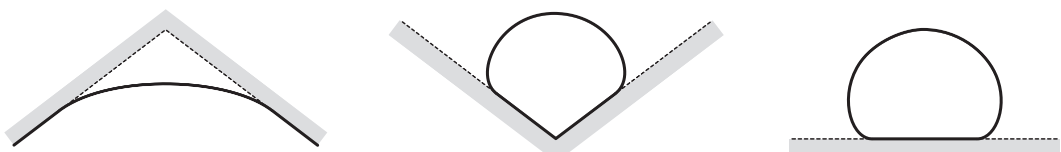

Figure 3.3: shrinking hypocycloid, expanding epicycloid and nephroid.

3.1.2 IMCF in conic domains

For $0 < \alpha \leqslant \pi$, consider the conical domain

$$\Omega = \left\{re^{i\theta} : r > 0,\ 0 < \theta < \alpha\right\} \subset \mathbb{R}^2.$$

Suppose $S_0 \subset \overline{\Omega}$ is a compact convex set with $0 \in \partial S_0$. Let us consider the IMCF that starts from ∂S_0 in Ω, and satisfies the boundary tangency condition at $\partial\Omega$ (namely, the evolving curves must be tangent to $\partial\Omega$). Similar to Remark 3.1.1, we allow S_0 to be non-strictly convex, non-smooth, or have empty interior. We even allow ∂S_0 to not satisfy the boundary tangency condition at $\partial\Omega$. See Figure 3.4: an extreme case is where S_0 is a segment in the middle of Ω.

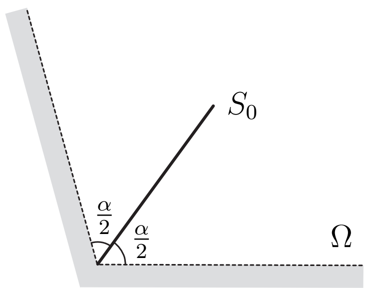

Figure 3.4: The case where S_0 is a segment

We seek a family of C^1 curves $\{\gamma_t\}_{t>0}$, such that:
(i) $\gamma_t \to \partial S_0$ in the Hausdorff sense as $t \to 0$,
(ii) the interior of γ_t is smooth and solves the IMCF for all $t > 0$,
(iii) γ_t is tangent to $\partial\Omega$ at its endpoints.
The resulting solution will be an IMCF with "initial value S_0" and outer obstacle $\partial\Omega$. See Figure 3.5 below for a depiction of such a solution.

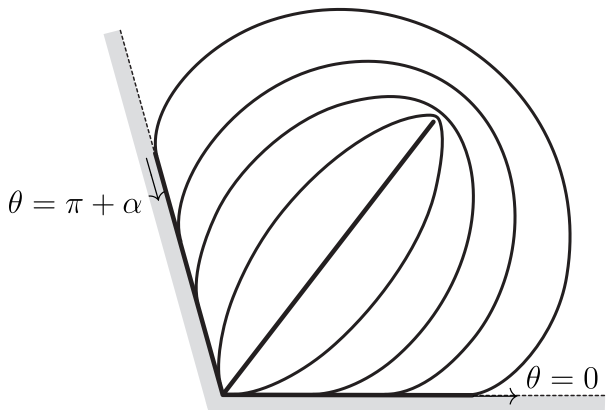

Figure 3.5: The IMCF issuing from S_0.

To explicitly compute the solution, we solve the general equation (3.1.4) with Dirichlet boundary conditions. We fix the domain of angles

$$I = [0, \pi + \alpha].$$

For $\theta \in I$, define the support function h_0 as usual:

$$h_0(\theta) = \max\left\{x \cdot (\cos\theta, \sin\theta), x \in S_0\right\}.$$

Note that h_0 is Lipschitz in $[0,\pi+\alpha]$, with $h_0 \geqslant 0$ and $h_0(0) = h_0(\pi+\alpha) = 0$. Furthermore, it is well-known that $\partial_{\theta\theta}h_0 + h_0 \geqslant 0$ in the viscosity sense. In the case of Figure 3.4, we have $h_0|_{[0,\alpha/2]} = h_0|_{[\pi+\alpha/2,\pi+\alpha]} = 0$ and $h_0|_{[\alpha/2,\pi+\alpha/2]} = \sin(\theta - \alpha/2)$.

Then we consider the system

$$\begin{cases} \partial_t h = \partial_{\theta\theta}h + h & \text{in } [0,\pi+\alpha] \times (0,\infty), \\ h(0,\theta) = h_0(\theta), & \forall\, \theta \in [0,\pi+\alpha], \\ h(t,0) = h(t,\pi+\alpha) = 0, & \forall\, t \geqslant 0. \end{cases}$$

This has the solution

$$h_t(\theta) = e^t \int_0^{\pi+\alpha} \widetilde{K}_t(\theta,\lambda) h_0(\lambda)\, d\lambda, \tag{3.1.15}$$

where $\widetilde{K}_t$ is the Dirichlet heat kernel in $[0,\pi+\alpha]$, given by

$$\widetilde{K}_t(\theta,\lambda) = \frac{2}{\pi+\alpha} \sum_{k\geqslant 1} \exp\Big[-\frac{k^2\pi^2 t}{(\pi+\alpha)^2}\Big] \sin\Big(\frac{k\pi\theta}{\pi+\alpha}\Big) \sin\Big(\frac{k\pi\lambda}{\pi+\alpha}\Big). \tag{3.1.16}$$

Then, the curve γ_t can be recovered from h_t using the formula $\gamma_t = h_t \cdot \nu + \partial_\theta h_t \cdot \tau$, $0 \leqslant \theta \leqslant \pi+\alpha$, see (3.1.1). Extending h_t to $[0,2\pi]$ with zero value on $[\pi+\alpha,2\pi]$, the resulting support function corresponds to a convex domain which we call E_t. Note that $E_t \subset \Omega$ and $\gamma_t \subset \partial E_t$. We note the following facts about γ_t.

(i) The interior of γ_t is smooth for all $t > 0$. By the strong maximum principle, we have $\partial_{\theta\theta}h_t + h_t > 0$ for all $t > 0$, $\theta \in (0,\pi+\alpha)$. Then we may calculate

$$\frac{\partial\gamma_t}{\partial\theta} = \Big[\frac{\partial h_t}{\partial\theta}\nu + h_t\tau\Big] + \Big[\frac{\partial^2 h_t}{\partial\theta^2}\tau - \frac{\partial h_t}{\partial\theta}\nu\Big] = \big(h_t + \partial_{\theta\theta}h_t\big)\tau,$$

hence the angle θ is a non-degenerate parametrization, showing that γ_t is a smooth curve.

(ii) γ_t is tangent to $\partial\Omega$ at its endpoints. This follows from $h_t(0) = h_t(\pi+\alpha) = 0$.

(iii) Convergence as $t \to 0$. Note that h_0 is a continuous function on $[0,\pi+\alpha]$, with $h_0(0) = h_0(\pi+\alpha) = 0$. Hence $h_t \to h_0$ in $C^0[0,\pi+\alpha]$. This implies $E_t \to S_0$ in the Hausdorff sense as $t \to 0$.

(iv) Convergence as $t \to \infty$. Dropping the higher order term in (3.1.16), we have

$$\lim_{t\to\infty} \exp\Big[\Big(\frac{\pi^2}{(\pi+\alpha)^2} - 1\Big)t\Big] h_t(\theta) = \sin\Big(\frac{\pi\theta}{\pi+\alpha}\Big) \cdot \underbrace{\frac{2}{\pi+\alpha}\int_0^{\pi+\alpha} \sin\Big(\frac{\pi\lambda}{\pi+\alpha}\Big) h_0(\lambda)\, d\lambda}_{\text{constant}},$$

where the convergence holds in $C^\infty\big([0,\pi+\alpha]\big)$. Hence, a suitable rescaling of γ_t converges to the curve whose support function is

$$h_\infty(\theta) = \sin\Big(\frac{\pi\theta}{\pi+\alpha}\Big), \qquad 0 \leqslant \theta \leqslant \pi+\alpha.$$

Denoting $\beta = \pi/(\pi+\alpha)$, the general formula (3.1.1) gives

$$\gamma_\infty(\theta) = -i\sin(\beta\theta)e^{i\theta} + \beta\cos(\beta\theta)e^{i\theta}.$$

This aligns with (3.1.13), thus γ_∞ is exactly an epicycloid supported in Ω. So the conclusion is that $\exp\big[\big(\frac{\pi^2}{(\pi+\alpha)^2} - 1\big)t\big]\gamma_t$ converges to a scaling of the epicycloid γ_∞.

3.2 Further auxiliary results

From now on, we work with the weak IMCF. This section contains a few useful technical lemmas.

3.2.1 Weak solutions with weights

This is a brief technical section, where we introduce the notion of IMCF with a weight function. Given a function $\psi \in C^\infty(\Omega)$, the corresponding weighted IMCF refers to the following equation:

$$\operatorname{div}\Big(e^\psi \frac{\nabla u}{|\nabla u|}\Big) = e^\psi |\nabla u|. \tag{3.2.1}$$

In the smooth regime, this corresponds to the curvature flow

$$\frac{\partial \Sigma_t}{\partial t} = \frac{\nu}{H + \partial\psi/\partial\nu}.$$

We call u a subsolution (resp. supersolution) of (3.2.1), if the equality sign is replaced by "$\geqslant$" (resp. "$\leqslant$").

Lemma 3.2.1. *Regarding a function $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, the following are equivalent:*

(1) for all $v \in \mathrm{Lip}_{loc}(\Omega)$ and every domain K with $\{u \neq v\} \Subset K \Subset \Omega$, we have

$$\int_K e^\psi\big(|\nabla u| + u|\nabla u|\big) \leqslant \int_K e^\psi\big(|\nabla v| + v|\nabla u|\big). \tag{3.2.2}$$

(2) Setting $\widetilde\Omega = \Omega \times S^1$ with the warped product metric $\widetilde g = g + e^{2\psi(x)}dz^2$, the function $\widetilde u(x,z) = u(x)$ is a solution of $\mathsf{IMCF}(\widetilde\Omega, \widetilde g)$.

(3) u is a solution of $\mathsf{IMCF}(\Omega; g')$, where $g' = e^{\frac{2\psi}{n-1}}g$.

We call u a weak solution of (3.2.1), if any of the above conditions holds.

The characterization (3) allows the standard theory of IMCF to be extended to the weighted case. In particular, if u weakly solves (3.2.1), then each E_t locally minimizes

$$E \mapsto \int_{\partial^* E} e^\psi \, d\mathcal{H}^{n-1} - \int_E e^\psi |\nabla u| \tag{3.2.3}$$

in the same sense with Definitions 2.1.1, 2.1.2. The notion of weak sub- and supersolutions can be defined similarly. The maximum principle (Theorem 2.1.12) holds for weighted solutions as well. When $\psi = 0$ in a certain region, u reduces to a usual weak solution of IMCF.

In the context of mean curvature flow, similar warping and conformal transformations are used to relate mean curvature flow to minimal surfaces, see [56, 82, 102].

Proof of Lemma 3.2.1.

(1) $\Rightarrow$ (2). Given a function $\widetilde v(x,t)$ and domain $\widetilde K$ with $\{\widetilde u \neq \widetilde v\} \Subset \widetilde K \Subset \Omega \times S^1$. By enlarging $\widetilde K$, we may assume that $\widetilde K = K \times S^1$ for some $K \Subset \Omega$. We need to show

$$\int_{\widetilde K} \big(|\widetilde\nabla \widetilde u| + \widetilde u|\widetilde\nabla\widetilde u|\big)\, dV_{\widetilde g} \leqslant \int_{\widetilde K} \Big(|\widetilde\nabla \widetilde v| + \widetilde v|\widetilde\nabla\widetilde u|\Big)\, dV_{\widetilde g}, \tag{3.2.4}$$

where $|\widetilde{\nabla}\cdot|$ denotes the norm of gradient with respect to $\widetilde{g}$. Using the facts $dV_{\widetilde{g}}(x,z) = e^{\psi(x)}dV_g(x)dz$, $\widetilde{u}(x,z) = u(x)$ and $|\nabla\widetilde{v}(x,z)| \geqslant |\nabla_x\widetilde{v}(x,z)|$, it is sufficient to show

$$2\pi\int_K e^{\psi}\big(|\nabla u| + u|\nabla u|\big)dV_g \leqslant \int_{K\times S^1} e^{\psi(x)}\Big(|\nabla_x\widetilde{v}(x,z)| + \widetilde{v}(x,z)|\widetilde{\nabla}\widetilde{u}(x)|\Big)\,dV_g(x)dz.$$

However, this follows from Fubini's theorem and (3.2.2).

(2) ⇒ (1): It is now our hypothesis that (3.2.4) holds for any competitor $\widetilde{v}$. Choosing $\widetilde{v}(x,t) = v(x)$ and $\widetilde{K} = K\times S^1$, we find that (3.2.4) implies item (1).

(1) ⇔ (3): Note that $dV_{g'} = e^{\frac{n\psi}{n-1}}dV_g$ and $|\nabla_{g'}f| = e^{-\frac{\psi}{n-1}}|\nabla_g f|$ for all f. Therefore, u being a solution of $\mathsf{IMCF}(\Omega, g')$ is equivalent to

$$\int_K \big(e^{-\frac{\psi}{n-1}}|\nabla_g u| + ue^{-\frac{\psi}{n-1}}|\nabla_g u|\big)e^{\frac{n\psi}{n-1}}dV_g \leqslant \int_K \big(e^{-\frac{\psi}{n-1}}|\nabla_g v| + ve^{-\frac{\psi}{n-1}}|\nabla_g u|\big)e^{\frac{n\psi}{n-1}}dV_g$$

whenever $\{u\neq v\}\Subset K\Subset\Omega$. This inequality is identical with (3.2.2). □

Remark 3.2.2. When a weighted IMCF is considered, with the interpretation of Lemma 3.2.1(3), the corresponding elliptic regularization takes the form

$$\operatorname{div}\left(e^{\psi}\frac{\nabla u}{\sqrt{\varepsilon^2 e^{2\psi/(n-1)} + |\nabla u|^2}}\right) = e^{\psi}\sqrt{\varepsilon^2 e^{2\psi/(n-1)} + |\nabla u|^2}. \tag{3.2.5}$$

To derive this, we note that the regularized equation writes

$$\operatorname{div}_{g'}\left(\frac{\nabla_{g'}u}{\sqrt{\varepsilon^2 + |\nabla_{g'}u|^2}}\right) = \sqrt{\varepsilon^2 + |\nabla_{g'}u|^2}, \qquad \text{where } \; g' = e^{\frac{2\psi}{n-1}}g.$$

Since $\nabla_{g'}u = e^{-2\psi/(n-1)}\nabla_g u$ and $|\nabla_{g'}u|^2 = e^{-2\psi/(n-1)}|\nabla_g u|^2$, this is equivalent to

$$\operatorname{div}_{g'}\left(e^{-\psi/(n-1)}\frac{\nabla u}{\sqrt{\varepsilon^2 e^{2\psi/(n-1)} + |\nabla u|^2}}\right) = e^{-\psi/(n-1)}\sqrt{\varepsilon^2 e^{2\psi/(n-1)} + |\nabla u|^2}.$$

Since $\operatorname{div}_{g'}(X) = \operatorname{div}_g X + \frac{n}{n-1}\langle X, \nabla_g\psi\rangle_g$ for all X, from this we obtain (3.2.5).

3.2.2 A higher integrability lemma

The following lemma is used in Subsection 3.3.3, but may be of its own interest:

Lemma 3.2.3 (higher integrability)**.** *Suppose $\Omega\subset M$ is locally Lipschitz, and $u\in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ is a subsolution of $\mathsf{IMCF}(\Omega)$. Then*

$$\int_{\Omega\cap K}|u|^p\,|\nabla u| < \infty \qquad \forall\, K\Subset M,\ p\geqslant 1. \tag{3.2.6}$$

In fact, the proof of Lemma 3.2.3 yields the stronger result:

Lemma 3.2.4. *Assume the same conditions as in Lemma 3.2.3. Then for all $K\Subset M$, there is a constant c so that*

$$\int_{\Omega\cap K} e^{c|u|} < \infty. \tag{3.2.7}$$

Proof of Lemma 3.2.3.

By Lemma A.5.1, we may assume without loss of generality that $\Omega \Subset K \Subset M$. Then by Lemma A.5.3, $\partial\Omega$ admits a Lipschitz collar neighborhood. Let $N \subset \overline{\Omega}$ and $\Phi : N \to \partial\Omega$ be the neighborhood and Lipschitz retraction map obtained there.

Since $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, we have $\sup_{\Omega \setminus N} |u| < T$ for some T. By Theorem 3.3.8, u is a subsolution of $\mathsf{IMCF}(\Omega)$+$\mathsf{OBS}(\partial\Omega)$. Comparing $\widetilde{J}_u^K(E_t(u)) \leqslant \widetilde{J}_u^K(\Omega)$, by the coarea formula we obtain

$$P(E_t(u)) + \int_t^\infty P\big(E_s(u);\Omega\big)\, ds \leqslant P(\Omega). \tag{3.2.8}$$

For $t \geqslant T$ we have $E_t(u) \Supset \Omega \setminus N$, thus Φ maps $\partial^* E_t(u)$ surjectively to $\partial^*\Omega$ up to a $\mathcal{H}^{n-1}$-null set. By the area formula we have

$$P(\Omega) \leqslant \mathcal{H}^{n-1}\big(\partial^* E_t(u) \cap \partial^*\Omega\big) + \mathrm{Lip}(\Phi)^{n-1} P\big(E_t(u);\Omega\big). \tag{3.2.9}$$

Cancelling the common portion of perimeters in (3.2.8) and (3.2.9), we obtain

$$\int_t^\infty P(E_s(u);\Omega)\, ds \leqslant \big(\mathrm{Lip}(\Phi)^{n-1} - 1\big) P\big(E_t(u);\Omega\big), \qquad \forall\, t \geqslant T.$$

Hence $P\big(E_t(u);\Omega'\big)$ exponentially decays, and $\int_{u \geqslant T} u^p |\nabla u| < \infty$ by the coarea formula.

For $t \leqslant -T$, we compare $\widetilde{J}_u(E_s(u)) \leqslant \widetilde{J}_u(E_t(u))$ and take $s \to -\infty$, to find

$$\int_{-\infty}^t P\big(E_s(u);\Omega\big)\, ds \leqslant P(E_t(u)).$$

Since $E_t(u) \Subset N$, projecting via Φ we find that

$$P\big(E_t(u)\big) \leqslant \big(\mathrm{Lip}(\Phi)^{n-1} + 1\big) P\big(E_t(u);\Omega\big).$$

The combined inequality implies that $P(E_t(u);\Omega)$ decays exponentially when $t \to -\infty$. By the coarea formula, we have $\int_{u \leqslant -T} (-u)^p |\nabla u| < \infty$. Finally, we have $\int_{|u| \leqslant T} |u|^p |\nabla u| \leqslant 2T^p P(\Omega)$ by Lemma 2.2.2(i), and thus (3.2.6) is proved. □

Proof of Lemma 3.2.4.

By Lemma A.5.1, we may enlarge K and assume that $\Omega \cap K$ is a Lipschitz domain. From the above proof, we infer that there is are constants $c, C > 0$ independent of t, so that

$$P\big(E_t(u);\Omega\big) \leqslant Ce^{-c|t|}, \qquad \forall\, t \in \mathbb{R}.$$

By the coarea formula, this implies

$$\int_{\Omega \cap K} \big|\nabla e^{c'|u|}\big| = c' \int_{\Omega \cap K} e^{c'|u|} |\nabla u| < \infty$$

for all $c' < c$. Then by the Poincare inequality, we have

$$\int_{\Omega \times \Omega} \Big| e^{c'|u(x)|} - e^{c'|u(y)|} \Big|\, dx\, dy < \infty.$$

Thus for some $x \in \Omega$ we have $\int_\Omega \big| e^{c'|u|} - e^{c'|u(x)|} \big| < \infty$. This implies the desired result. □

3.3 Definitions and basic properties

In this section, we set up the general framework for IMCF with outer obstacle.

3.3.1 Formulation using sub-level sets

We use the following variational principle as the starting point of the theory. Recall our convention that all the manifolds are assumed to be smooth, connected, oriented, without boundary, and all domains Ω are assumed to be connected.

Definition 3.3.1. Let $\Omega \subset M$ be a locally Lipschitz domain. For a function $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, a domain $K \Subset M$, and a set $E \subset \Omega$, we define the energy

$$\widetilde{J}_u^K(E) := P(E;K) - \int_{E\cap K} |\nabla u| \tag{3.3.1}$$

whenever the two terms are not both infinite.

We say that a set $E \subset \Omega$ locally minimizes $\widetilde{J}_u$ (resp. minimizes from inside, outside), if for any $F \subset \Omega$ (resp. for any $F \subset \Omega$ with $F \subset E$, $F \supset E$) and any domain K that satisfy $E\Delta F \Subset K \Subset M$, we have

$$\widetilde{J}_u^K(E) \leqslant \widetilde{J}_u^K(F) \tag{3.3.2}$$

whenever both energies are defined.

Definition 3.3.2 (outer obstacle I)**.**
Let Ω be a locally Lipschitz domain in M. We say that u is a (sub-, super-) solution of $\mathsf{IMCF}(\Omega)$+$\mathsf{OBS}(\partial\Omega)$, if $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, and $E_t := \{u < t\}$ locally minimizes $\widetilde{J}_u$ (resp. minimizes from outside, inside) for each $t \in \mathbb{R}$.

Notice that:
(1) A solution of $\mathsf{IMCF}(\Omega)$+$\mathsf{OBS}(\partial\Omega)$ is clearly a solution of $\mathsf{IMCF}(\Omega)$.
(2) If $M = \Omega$, then $\mathsf{IMCF}(\Omega)$+$\mathsf{OBS}(\partial\Omega)$ is equivalent to $\mathsf{IMCF}(\Omega)$.
Given an interior solution u (i.e. u solves $\mathsf{IMCF}(\Omega)$), we say that u *respects the obstacle* $\partial\Omega$ if u actually solves $\mathsf{IMCF}(\Omega)$+$\mathsf{OBS}(\partial\Omega)$. When we need to clarify the background metric, we will write $\mathsf{IMCF}(\Omega, g)$+$\mathsf{OBS}(\partial\Omega)$.

Remark 3.3.3 (integrability in the definition of $\widetilde{J}_u$)**.**
The energy (3.3.1) is not a priori defined, since u is only guaranteed interior regularity. However, if u is a subsolution of $\mathsf{IMCF}(\Omega)$, then $\widetilde{J}_u^K(E_t)$ is always defined and is finite, due to Lemma 2.2.2(i). By the same lemma, $\widetilde{J}_u^K(E)$ is defined and finite provided E has locally finite perimeter (which will always be the case in specific energy comparisons). Hence, there is no integrability issue for interior subsolutions.

The case of supersolution is more complicated. If u is a supersolution of $\mathsf{IMCF}(\Omega)$ with additionally

$$\inf_{\Omega\cap K} u > -\infty \quad \text{for all} \quad K \Subset M, \tag{3.3.3}$$

then $\widetilde{J}_u^K(E_t)$ is always defined and is finite, by Lemma 2.2.2(ii). Since any competitor set E is contained in E_t, it follows that $\int_{E\cap K} |\nabla u| \leqslant \int_{E_t\cap K} |\nabla u|$, thus $\widetilde{J}_u^K(E)$ is also defined with finite value. On the other hand, see Remark 3.3.4 below that $\widetilde{J}_u = -\infty$ may occur for general supersolutions.

Remark 3.3.4 (a supersolution not belonging to $\mathrm{BV}_{\mathrm{loc}}(\overline{\Omega})$).

Consider $\Omega = \{(x,y) : -1 < x < 1\} \subset \mathbb{R}^2$ and the function $u(x,y) = \tan(\pi x/2)$. We argue that u is a supersolution of $\mathsf{IMCF}(\Omega)$+$\mathsf{OBS}(\partial\Omega)$. First, it is easy to see that u is a smooth supersolution of $\mathsf{IMCF}(\Omega)$. Let $F \subset E_t$ be a competitor with $E_t \setminus F \Subset \mathbb{R}^2$. One of the following two cases must occur for F.

(1) $E_t \Delta F \Subset \Omega$. In this case the energy comparison is entirely interior.

(2) $\overline{E_t \Delta F}$ has nonempty intersection with $\{x = -1\}$. In this case $\widetilde{J}_u^K(E_t) = -\infty$ for all $K \Supset E_t \Delta F$, thus $\widetilde{J}_u^K(E_t) \leqslant \widetilde{J}_u^K(F)$ trivially holds.

Therefore, u is a supersolution satisfying Definition 3.3.2.

On the other hand, for any increasing function $f \in C^\infty(-1,1)$ with $\lim_{x\to -1} f(x) > -\infty$, the interior supersolution $u(x,y) = f(x)$ does not respect the obstacle $\partial\Omega$. Informally speaking, such function is only a supersolution of $\mathsf{IMCF}(\Omega)$+$\mathsf{OBS}(\{x = 1\})$.

The following remark includes several useful observations regarding the definitions.

Remark 3.3.5 (properties and relations).

(i) Note the differences between Definition 3.3.1, 3.3.2 and the interior formulation: (a) the comparison set E must be contained in Ω, but the difference set $E \Delta E_t$ is allowed to touch $\partial\Omega$ (this characterizes an outer obstacle problem); (b) the boundary portion of perimeter $\partial^* E \cap \partial\Omega$ is contained in the energy $\widetilde{J}_u$. If we remove the boundary portion in item (b), then we obtain the energy

$$\hat{J}^K_{0;u}(E) = P\big(E; \Omega \cap K\big) - \int_{E\cap\Omega\cap K} |\nabla u|,$$

which describes the weak IMCF with free boundary [84]. For a constant $\theta \in (-1,1)$, we may vary the energy to be

$$\hat{J}^K_{\theta;u}(E) = P\big(E; \Omega \cap K\big) + \theta \cdot \mathcal{H}^{n-1}\big(\partial^* E \cap \partial^*\Omega \cap K\big) - \int_{E\cap\Omega\cap K} |\nabla u|.$$

This describes weak solutions with capillary boundary condition, corresponding to the flow such that each hypersurface keeps the contact angle $\arccos(\theta)$ with $\partial\Omega$. No existence result is known for the weak IMCF with capillary conditions, except for the free boundary case $\theta = 0$ [64, 84] and the obstacle case $\theta = 1$ considered by us.

(ii) The following inequality is useful:

$$\widetilde{J}_u^K(E \cap F) + \widetilde{J}_u^K(E \cup F) \leqslant \widetilde{J}_u^K(E) + \widetilde{J}_u^K(F), \tag{3.3.4}$$

for all $E, F \subset \Omega$ with finite perimeter in K and $u \in \mathrm{BV}(\Omega \cap K)$. It follows that u is a solution of $\mathsf{IMCF}(\Omega)$+$\mathsf{OBS}(\partial\Omega)$ if it is both a subsolution and supersolution.

(iii) It is natural to wonder how the area grows in the IMCF with outer obstacle. Let u be a solution of $\mathsf{IMCF}(\Omega)$+$\mathsf{OBS}(\partial\Omega)$. If $E_t \Subset M$ for each t, then the energy comparison forces $\widetilde{J}_u^K(E_s) = \widetilde{J}_u^K(E_t)$ for all $s < t$. This implies

$$P(E_s) = P(E_t) - \int_s^t P(E_r; \Omega)\, dr \geqslant P(E_t) - \int_s^t P(E_r)\, dr. \tag{3.3.5}$$

In particular, we have sub-exponential growth of area

$$P(E_t) \leqslant e^{t-s} P(E_s), \qquad \forall\, t > s, \tag{3.3.6}$$

with strict inequality when $\mathcal{H}^{n-1}\big(\partial^* E_s \cap \partial^*\Omega\big) \neq 0$. In Example 3.1.2(ii) of expanding epicycloids, for instance, it holds $P(E_t) = e^{c(t-s)}P(E_s)$ where $c \in (0,1)$ is the expanding rate defined there. From (3.3.5) and using $\mathcal{H}^{n-1}\big(\partial^* E_s \cap \partial\Omega\big) \leqslant \mathcal{H}^{n-1}\big(\partial^* E_t \cap \partial\Omega\big)$, we have

$$P(E_s;\Omega) \geqslant P(E_t;\Omega) - \int_s^t P(E_r;\Omega)\,dr,$$

which implies

$$P(E_t;\Omega) \leqslant e^{t-s}P(E_s;\Omega), \qquad \forall\, t > s. \tag{3.3.7}$$

(iv) For u a solution of $\mathsf{IMCF}(\Omega)$+$\mathsf{OBS}(\partial\Omega)$, the set E_t^+ also locally minimizes $\widetilde{J}_u$ for each $t \in \mathbb{R}$. This follows from $E_t^+ = \bigcap_{s>t} E_s$ and the set-replacing argument.

(v) When $\Omega \Subset M$, the constant function on Ω is *not* a solution of $\mathsf{IMCF}(\Omega)$+$\mathsf{OBS}(\partial\Omega)$. Indeed, setting $u \equiv c$ we find that $\widetilde{J}_u^K(E_{c+1}(u)) = P(\Omega) > 0 = \widetilde{J}_u^K(\emptyset)$, for any $K \Supset \Omega$. For a noncompact Ω, constant functions are solutions of $\mathsf{IMCF}(\Omega)$+$\mathsf{OBS}(\partial\Omega)$ if and only if Ω is locally inward perimeter-minimizing (we leave this to the reader as an exercise). An interesting example for this is $M = \mathbb{H}^2 \times \mathbb{R}$ and $\Omega = \{-\pi/2 < z < \pi/2\}$, where z is the coordinate in the $\mathbb{R}$-direction. This set is inward-minimizing since it is calibrated by the vector field $\nu = \sin(z)\partial_z - \cos(z)\tanh(r/2)\partial_r$.

(vi) When $\Omega \Subset M$, there exists no solution u of $\mathsf{IMCF}(\Omega)$+$\mathsf{OBS}(\partial\Omega)$ such that $\inf_\Omega(u) > -\infty$. Indeed, if $\inf_\Omega(u) = -T > -\infty$, then by (3.3.6) we have

$$P(E_t) \leqslant e^{t+T+1}P(E_{-T-1}) = 0$$

for all t, which is impossible. On the other hand, the nephroid Example 3.1.2(iv) shows that there do exist solutions that are unbounded below. For these solutions u, it holds $P(\Omega) = \int_\Omega |\nabla u|$ and $P(E_t) = \int_{E_t} |\nabla u|$ for all t. In general, all solutions of $\mathsf{IMCF}(\Omega)$+$\mathsf{OBS}(\partial\Omega)$ for bounded Ω must have poles at $\partial\Omega$. The reader may thought of these solutions as Piosson kernels.

(vii) If u is a subsolution of $\mathsf{IMCF}(\Omega)$+$\mathsf{OBS}(\partial\Omega)$, then so is $\min\{u, T\}$ for all $T \in \mathbb{R}$. On the other hand, this truncation property does not hold for supersolutions.

Definition 3.3.6 (initial value problem)**.**
Let M be a manifold without boundary, and Ω be a locally Lipschitz domain in M, and $E_0 \subset \Omega$ be a $C^{1,1}$ domain with $\partial E_0 \cap \partial\Omega = \emptyset$. A function $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ is called a (sub-, super-) solution of $\mathsf{IVP}(\Omega; E_0)$+$\mathsf{OBS}(\partial\Omega)$ in M, if
(i) $E_0 = \{u < 0\}$,
(ii) $u|_{\Omega\setminus\overline{E_0}}$ is a (sub-, super-) solution of $\mathsf{IMCF}(\Omega\setminus\overline{E_0})$+$\mathsf{OBS}(\partial\Omega)$.
We say that two solutions u_1, u_2 are equivalent, if $u_1 = u_2$ on $\Omega \setminus E_0$.

Similar to the interior case, we have an equivalent formulation stated below. The proof is is almost the same as in Definition 2.1.8, thus we choose to omit it here.

Theorem 3.3.7 (initial value problem II)**.**
Let M, Ω, E_0 be as in Definition 3.3.6, and suppose $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$. Then u is a solution of $\mathsf{IVP}(\Omega; E_0)$+$\mathsf{OBS}(\partial\Omega)$ *if and only if*
(i) $E_0 = \{u < 0\}$,
(ii) for any $t > 0$, any set E with $E_0 \subset E \subset \Omega$, and any domain K with $E\Delta E_t \Subset K \Subset M$, we have $\widetilde{J}_u^K(E_t(u)) \leqslant \widetilde{J}_u^K(E)$.

The following observation is important in relating interior solutions to solutions with outer obstacles.

Theorem 3.3.8 (automatic subsolution).

Suppose Ω is a locally Lipschitz domain, and u is a subsolution of $\mathsf{IMCF}(\Omega)$. *Then u is a subsolution of* $\mathsf{IMCF}(\Omega)$+$\mathsf{OBS}(\partial\Omega)$.

Proof. This is just a re-statement of Lemma 2.2.3. □

We end this subsection with the following connectedness lemma.

Lemma 3.3.9. *Suppose $E_0 \Subset \Omega \Subset M$ and E_0 is connected, and u is a solution of* $\mathsf{IVP}(\Omega; E_0)$+$\mathsf{OBS}(\partial\Omega)$. *Then $\overline{E_t(u)}$ is connected for all $t > 0$.*

Proof. Otherwise, suppose S is a connected component of $\overline{E_t(u)}$ that does not intersect $\overline{E_0}$. Thus there is a domain U with $S \Subset U \Subset M \setminus \overline{E_0}$ and $\overline{E_t(u)} \cap U = S$. Denote $\widetilde{u} = u|_{\Omega\setminus\overline{E_0}}$ and $T = \inf_U(\widetilde{u}) \geqslant 0$. For each $s < t$, note that $E_s(\widetilde{u}) \cap U \Subset U$, hence we may compare the energies $\widetilde{J}^U_{\widetilde{u}}(E_s(\widetilde{u})) \leqslant \widetilde{J}^U_{\widetilde{u}}\big(E_s(\widetilde{u}) \setminus U\big) = \widetilde{J}^U_{\widetilde{u}}(\emptyset) = 0$ and obtain

$$P\big(E_s(\widetilde{u}); U\big) \leqslant \int_{E_s(\widetilde{u})\cap U} |\nabla u| = \int_T^s P\big(E_{s'}(\widetilde{u}); U\big)\, ds'.$$

Then by Gronwall's inequality $P\big(E_s(\widetilde{u}); U\big) = 0$ for all $s < t$, which is a contradiction. □

3.3.2 Outward minimizing properties

For a weak solution u respecting an outer obstacle, the sub-level sets of u satisfy certain outward minimizing properties subject to the outer obstacle $\partial\Omega$. When $M = \Omega$, the conclusions here reduce to the interior case.

Definition 3.3.10. Given a locally Lipschitz domain Ω. We say that a set $E \subset \Omega$ is locally outward minimizing in $\overline{\Omega}$, if for any competitor F and domain K satisfying $E \subset F \subset \Omega$ and $F \setminus E \Subset K \Subset M$, we have

$$P(E; K) \leqslant P(F; K). \tag{3.3.8}$$

We say that E is strictly locally outward-minimizing, if (3.3.8) is a strict inequality whenever $|F \setminus E| > 0$.

Definition 3.3.11. Given two sets E, E' with $E \subset E' \subset \Omega$. We say that E' is the minimizing hull of E in $\overline{\Omega}$, provided that:

(i) E' is strictly outward minimizing in $\overline{\Omega}$,

(ii) if E'' is another strictly outward minimizing set in $\overline{\Omega}$, with $E \subset E'' \subset \Omega$, then we have $E' \subset E''$ up to a set with zero measure.

Similar to the interior case (see Section A.4), it can be verified that any set $E \subset \Omega$ has at most one minimizing hull, up to modifications of zero measure. The proof of the following result is the same as in Facts 1.2.9 ∼ 1.2.14.

Theorem 3.3.12 (minimizing properties).

Let u be a solution of $\mathsf{IMCF}(\Omega)$+$\mathsf{OBS}(\partial\Omega)$. *Then for all $t \in \mathbb{R}$ it holds*

(i) E_t is locally outward minimizing in $\overline{\Omega}$,

(ii) E_t^+ is strictly locally outward minimizing in $\overline{\Omega}$,

(iii) E_t^+ is the minimizing hull of E_t in $\overline{\Omega}$, provided $E_t^+ \setminus E_t \Subset M$.

For solutions of initial value problems, the initial time $t = 0$ is not included. Hence we additionally state

Theorem 3.3.13. *Let M, Ω, E_0 be as in Definition 3.3.6, and suppose that u is a solution of* $\mathsf{IVP}(\Omega; E_0)$+$\mathsf{OBS}(\partial\Omega)$. *Then for each $t \geqslant 0$, E_t^+ is the minimizing hull of E_t in $\overline{\Omega}$, provided that $E_t^+ \setminus E_t \Subset M$.*

Proof. The case $t > 0$ is already covered by Theorem 3.3.12, since $\partial E_t \cap \partial E_0 = \emptyset$. Taking Theorem 3.3.7(ii) and approximating $t \to 0^+$, it follows that $\widetilde{J}_u^K(E_0^+) \leqslant \widetilde{J}_u^K(E)$ whenever $E_0 \subset E \subset \Omega$ and $E \Delta E_0^+ \Subset K \Subset M$. Then the same argument as in Theorem 3.3.12 shows that E_0^+ is the strictly minimizing hull of E_0. $\square$

In initial value problems with outer obstacle, the area of level sets follow the rule of sub-exponential growth. Note that it may hold $P(E_t) < e^t P(E_0)$ even if E_0 is outward minimizing in $\overline{\Omega}$; see Remark 3.3.5(iii).

Corollary 3.3.14. *Let M, Ω, E_0 be as in Definition 3.3.6, and suppose that u is a solution of* $\mathsf{IVP}(\Omega; E_0)$+$\mathsf{OBS}(\partial\Omega)$. *Then for any $0 \leqslant s < t$ with $E_t \Subset M$, we have*

$$P(E_t) \leqslant e^{t-s} P(E_s).$$

Proof. The fact $E_t \Subset M$ implies $E_0 \Subset M$. In particular, it implies $E_0 \Subset \Omega$ as we have assumed $\partial E_0 \cap \partial\Omega = \emptyset$ in Definition 3.3.6. Since $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, for every $0 < s < t$ we have $E_t \setminus E_s \Subset M \setminus \overline{E_0}$. Hence the energy comarison forces $\widetilde{J}_u^K(E_s) = \widetilde{J}_u^K(E_t)$ for any $K \Supset \Omega$. Arguing as in Remark 3.3.5(iii), this implies $P(E_t) \leqslant e^{t-s} P(E_s)$.

Then we consider the case $s = 0$. By Theorem 3.3.7 and the standard approximation argument, each E_s^+ ($s \geqslant 0$) has the same minimizing property as described in Theorem 3.3.7(ii). By mutual energy comparison, this implies $P(E_s^+) \leqslant e^s P(E_0^+)$ by arguing in the same manner. For almost every $s \leqslant t$ we have $E_s = E_s^+$, hence we obtain $P(E_t) \leqslant e^t P(E_0^+)$. Finally, by Theorem 3.3.13 we have $P(E_0^+) \leqslant P(E_0)$. $\square$

3.3.3 Formulation using 1-Dirichlet energy

Similar to the interior case, there is another variational principle in terms of a 1-Dirichlet type energy. The presence of an obstacle results in a boundary term:

$$\widetilde{J}_u(v) = \int_\Omega \big(|\nabla v| + v|\nabla u|\big) - \int_{\partial^*\Omega} v^\partial \, d\mathcal{H}^{n-1}. \tag{3.3.9}$$

where we use v^∂ to denote the BV boundary trace of v. The choice of sign in the boundary term becomes manifest in the proof of the equivalence theorem.

We show the equivalence between the main Definition 3.3.2 and the definition using this energy. This is in analogy with Remark 2.1.3. The proofs in this section are mostly technical; the reader is recommended to to skip them.

Definition 3.3.15. Let $\Omega \subset M$ be a locally Lipschitz domain. For a function $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, a domain $K \Subset M$ and another function $v \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, set the energy

$$\widetilde{J}_u^K(v) = \begin{cases} \displaystyle\int_{\Omega\cap K} \big(|\nabla v| + v|\nabla u|\big) - \int_{\partial^*\Omega\cap K} v^\partial \, d\mathcal{H}^{n-1} & (\text{if } v \in BV(\Omega \cap K), \\ & \quad v|\nabla u| \in L^1(\Omega \cap K)), \\ \text{undefined} \quad (\text{otherwise}). \end{cases}$$

We say that u locally minimizes $\widetilde{J}_u$ (resp. minimizes from below, from above), if for all $v \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ (resp. for all $v \leqslant u$, $v \geqslant u$) and any domain K satisfying $\{u \neq v\} \Subset K \Subset M$, we have

$$\widetilde{J}_u^K(u) \leqslant \widetilde{J}_u^K(v) \tag{3.3.10}$$

whenever both sides are defined.

At this point, it is important to recall Lemma 2.2.2(i) and 3.2.3, where we have shown that $u \in \mathrm{BV}(\Omega \cap K)$ and $u|\nabla u| \in L^1(\Omega \cap K)$ for all $K \Subset M$ whenever u is an interior subsolution of $\mathsf{IMCF}(\Omega)$, and thus, the energy $\widetilde{J}_u^K(u)$ is always defined.

Theorem 3.3.16 (outer obstacle II)**.**
Let $\Omega \subset M$ be a locally Lipschitz domain, and $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$. Then u is a solution (resp. subsolution) of $\mathsf{IMCF}(\Omega)$+$\mathsf{OBS}(\partial\Omega)$ if and only if u locally minimizes $\widetilde{J}_u$ (resp. locally minimizes $\widetilde{J}_u$ from below). If we further assume

$$u \in \mathrm{BV}_{\mathrm{loc}}(\overline{\Omega}), \qquad |u|\,|\nabla u| \in L^1_{\mathrm{loc}}(\overline{\Omega}), \tag{3.3.11}$$

then u is a supersolution if and only if u locally minimizes $\widetilde{J}_u$ from above.

Proof. The proof is a direct generalization of the interior case [53, Lemma 1.1].

(1) Suppose u is a weak (sub-, super-) solution of $\mathsf{IMCF}(\Omega)$+$\mathsf{OBS}(\partial\Omega)$. By Lemma 2.2.2 and 3.2.3 or by our assumption (3.3.11), we have $u \in \mathrm{BV}_{\mathrm{loc}}(\overline{\Omega})$ and $|u|\,|\nabla u| \in L^1_{\mathrm{loc}}(\overline{\Omega})$ in either case. Let $v \in \mathrm{Lip}_{\mathrm{loc}}(\overline{\Omega})$ be a competitor (with additionally $v \leqslant u$ or $v \geqslant u$ for the case of subsolution or supersolution), with $\{u \neq v\} \Subset K \Subset M$ and $\widetilde{J}_u^K(v)$ is defined. For $b \in \mathbb{R}$, we set $v_b = \min\{v, b\}$ and truncate the integrals (we suppress the boundary measure $d\mathcal{H}^{n-1}$):

$$\int_{\Omega\cap K} |\nabla v| = \int_{\{v>b\}\cap K} |\nabla v| + \int_{\Omega\cap K} |\nabla v_b|, \tag{3.3.12}$$

$$\int_{\partial^*\Omega\cap K} v^\partial = \int_{\partial^*\Omega\cap\{v^\partial>b\}\cap K} (v-b)^\partial + b\mathcal{H}^{n-1}\big(\partial^*\Omega \cap K\big) - \int_{\partial^*\Omega\cap K} (b-v_b)^\partial, \tag{3.3.13}$$

$$\int_{\Omega\cap K} v|\nabla u| = \int_{\{v>b\}\cap K} (v-b)|\nabla u| + b\int_{\Omega\cap K} |\nabla u| - \int_{\Omega\cap K} (b-v_b)|\nabla u|. \tag{3.3.14}$$

The first term on the right hand sides converge to 0 as $b \to +\infty$, by our integrability assumption for v. Then by Cavalieri's formula and the coarea formula for BV functions (see [39, Section 5.4, 5.5]), we have

$$\int_{\Omega\cap K} \Big(|\nabla v_b| - (b-v_b)|\nabla u|\Big) + \int_{\partial^*\Omega\cap K} (b-v_b)^\partial = \int_{-\infty}^{b} \widetilde{J}_u^K(E_t(v))\,dt.$$

As a result,

$$\widetilde{J}_u^K(v) = o(1) + \int_{-\infty}^{b} \widetilde{J}_u^K(E_t(v))\,dt + b\int_{\Omega\cap K} |\nabla u| - b\mathcal{H}^{n-1}\big(\partial^*\Omega \cap K\big). \tag{3.3.15}$$

Decomposing $\widetilde{J}_u^K(u)$ in the same manner, applying Definition 3.3.2, cancelling the common terms in (3.3.15) and finally taking $b \to \infty$, we find that $\widetilde{J}_u^K(u) \leqslant \widetilde{J}_u^K(v)$.

(2) Suppose u satisfies (3.3.11) and locally minimizes $\widetilde{J}_u$ from above; let us verify the supersolution condition: $\widetilde{J}_u^K(E_t(u)) \leqslant \widetilde{J}_u^K(E)$ for all $E \subset E_t$ with $E_t \setminus E \Subset K \Subset M$.

The proof here differs only slightly from [53, Lemma 1.1]. By selecting a $\widetilde{J}_u^K$–minimizer among all the sets F with $E \subset F \subset E_t$, we may assume that $\widetilde{J}_u^K(E) \leqslant \widetilde{J}_u^K(E')$ for all $E \subset E' \subset E_t$. Next, we define the function

$$v = \begin{cases} t & \text{on } E_t \setminus E, \\ u & \text{elsewhere.} \end{cases}$$

It is easy to see that $u \leqslant v \leqslant \max\{u, t\}$ and $\{u \neq v\} \Subset K$. Writing $v = u + (t-u)\chi_{E_t \setminus E}$, we find $v \in \mathrm{BV}_{\mathrm{loc}}(\overline{\Omega})$ and $|v|\,|\nabla u| \in L^1_{\mathrm{loc}}(\overline{\Omega})$ by (3.3.11). Note that $E_s(v) = E_s(u) \cap E$ for all $s \leqslant t$, and therefore we have

$$\widetilde{J}_u^K(E_s(v)) \leqslant \widetilde{J}_u^K(E_s(u)) + \widetilde{J}_u^K(E) - \widetilde{J}_u^K(E_s(v) \cup E) \leqslant \widetilde{J}_u^K(E_s(u)) \tag{3.3.16}$$

By the standard approximation results (see [4, Theorem 3.9 and 3.88]), there exist non-negative functions $w_i \in C^\infty(\Omega) \cap \mathrm{BV}(\Omega)$ supported in K, such that

$$w_i \xrightarrow{L^1} (v-u), \qquad \|Dw_i\|(\Omega) \to \|D(v-u)\|(\Omega), \qquad \int_{\partial^*\Omega} w_i^\partial \to \int_{\partial^*\Omega} (v^\partial - u^\partial).$$

Since $|\nabla u| \in L^\infty_{\mathrm{loc}}(\Omega)$, then with a slight modification of [4, Theorem 3.9] we can achieve

$$\int_\Omega w_i |\nabla u| \to \int_\Omega (v-u)|\nabla u|.$$

Utilizing the fact $\widetilde{J}_u^K(u) \leqslant \widetilde{J}_u^K(u + w_i)$ from our hypotheses, then taking $i \to \infty$, we find that $\widetilde{J}_u^K(u) \leqslant \widetilde{J}_u^K(v)$ in the BV sense. Truncating the integrals at an arbitrary $b > t$ as in (3.3.12) ∼ (3.3.14), and noting that $u = v \in \mathrm{Lip}_{\mathrm{loc}}(\{u > t\})$, we find that

$$\int_{-\infty}^t \widetilde{J}_u^K(E_s(u))\, ds \leqslant \int_{-\infty}^t \widetilde{J}_u^K(E_s(v))\, ds. \tag{3.3.17}$$

The combination (3.3.16) (3.3.17) implies $\widetilde{J}_u^K(E_s(v)) = \widetilde{J}_u^K(E_s(u))$ for a.e. $s < t$. This implies $\widetilde{J}_u^K(E_s(u) \cup E) \leqslant \widetilde{J}_u^K(E)$, and taking $s \nearrow t$ implies $\widetilde{J}_u^K(E_t(u)) \leqslant \widetilde{J}_u^K(E)$.

This argument verifies the supersolution case. For subsolutions we argue in the same manner, where the integrability condition for u comes from Lemma 2.2.2 and 3.2.3. The case of solution follows by combining the supersolution and subsolution case. □

3.3.4 Boundary orthogonality of calibration

Recall from Subsection 2.3 the notion of a calibrated solution: there exists a measurable vector field ν satisfying $|\nu| \leqslant 1$ and $\nu \cdot \nabla u = |\nabla u|$ almost everywhere, and $\mathrm{div}(\nu) = |\nabla u|$ weakly in Ω. Moreover, recall by Lemma 2.2.2(i) that $|\nabla u| \in L^1_{\mathrm{loc}}(\overline{\Omega})$. Here we show that if $\langle \nu, \nu_\Omega \rangle = 1$ on $\partial\Omega$ in the trace sense, then u solves $\mathsf{IMCF}(\Omega)$+$\mathsf{OBS}(\partial\Omega)$.

Let us first introduce the notion of boundary trace. Suppose Ω is locally Lipschitz. Define the space

$$X = \big\{\nu \in L^\infty(T\Omega) : \mathrm{div}(\nu) \in L^1_{\mathrm{loc}}(\overline{\Omega})\big\}.$$

There is a well-defined operator $\nu \mapsto [\nu \cdot \nu_\Omega]$ from X to $L^\infty(\partial\Omega, \mathcal{H}^{n-1})$, called the *normal trace*, with the following properties:

(i) for all $\nu \in X$ we have $\|[\nu \cdot \nu_\Omega]\|_\infty \leqslant \|\nu\|_\infty$;

(ii) for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$ we have

$$[\nu \cdot \nu_\Omega](x) = -\frac{n}{|B^{n-1}|} \lim_{r\to 0} \frac{1}{r^n} \int_{\Omega\cap B(x,r)} \langle \nu, \nabla d_x \rangle, \tag{3.3.18}$$

where $d_x = d(\cdot, x)$ is the distance function from x. In particular, if Ω is a C^1 domain and $\nu \in C^1(\overline{\Omega})$, then $[\nu \cdot \nu_\Omega] = \nu \cdot \nu_\Omega$ in the classical sense;

(iii) the following divergence formula holds: for $\varphi \in \mathrm{BV}(\Omega) \cap \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ with $\mathrm{spt}(\varphi) \Subset M$ and $\varphi\,\mathrm{div}(\nu) \in L^1(\Omega)$, we have

$$\int_\Omega \varphi\,\mathrm{div}(\nu) + \langle \nu, \nabla\varphi \rangle = \int_{\partial\Omega} [\nu \cdot \nu_\Omega] \varphi^\partial \, d\mathcal{H}^{n-1}. \tag{3.3.19}$$

Properties (i)(iii) can be found (as the Riemannian and local version) in [10, Section 1]. Property (ii) is proved in [100, Theorem 4.4]. The formula (3.3.19) is proved in [10, Theorem 1.9] assuming $\varphi \in \mathrm{BV}(\Omega) \cap \mathrm{Lip}_{\mathrm{loc}}(\Omega) \cap L^\infty$, but what stated here follows by a truncation argument. The divergence formula in fact holds in a much broader sense: it suffices to assume that $\mathrm{div}(\nu)$ is a Radon measure and $\varphi \in \mathrm{BV}(\Omega)$ has bounded support. In this case, the second term in (3.3.19) needs to be replaced with an abstract pairing. We refer the reader to [10, 27] for more details; the current formulation is enough for our purpose.

Lemma 3.3.17. *Suppose $\Omega \subset M$ is a locally Lipschitz, and u solves* $\mathsf{IMCF}(\Omega)$ *and is calibrated by ν. If $[\nu \cdot \nu_\Omega] = 1$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$, then u solves* $\mathsf{IMCF}(\Omega)$+$\mathsf{OBS}(\partial\Omega)$.

Combining Lemma 3.3.17, (3.3.18), and an elementary computation, we obtain the following convenient criterion:

Corollary 3.3.18. *Let Ω, u, ν be as in Lemma 3.3.17. If for $\mathcal{H}^{n-1}$-almost every $x \in \partial^*\Omega$, and in some geodesic normal coordinate near x, we have*

$$\lim_{r\to 0} \operatorname{ess\,sup} \Big\{ \big|\nu(y) - \nu_\Omega(x)\big| : y \in \Omega \cap B(x,r) \Big\} = 0, \tag{3.3.20}$$

then u solves $\mathsf{IMCF}(\Omega)$+$\mathsf{OBS}(\partial\Omega)$.

Proof of Lemma 3.3.17. Since $\mathrm{div}(\nu) = |\nabla u|$, and $u \in \mathrm{BV}_{\mathrm{loc}}(\overline{\Omega})$ due to Lemma 2.2.2, we indeed have $\nu \in X$. Let $v \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ be a competitor, such that $\{u \neq v\} \Subset K \Subset M$ and $\widetilde{J}_u^K(v)$ is defined. Applying (3.3.19) with the function $\varphi = v - u$, we obtain

$$\begin{aligned}\int_{\partial\Omega} (v^\partial - u^\partial)\, d\mathcal{H}^{n-1} = \int_{\partial\Omega} [\nu \cdot \nu_\Omega] \varphi^\partial \, d\mathcal{H}^{n-1} &= \int_\Omega (v-u)|\nabla u| + \int_\Omega \nu \cdot (\nabla v - \nabla u) \\ &\leqslant \int_\Omega (v-u)|\nabla u| + \int_\Omega \big(|\nabla v| - |\nabla u|\big),\end{aligned}$$

since $\nu \cdot \nabla u = |\nabla u|$ a.e.. This exactly implies $\widetilde{J}_u^K(u) \leqslant \widetilde{J}_u^K(v)$. $\square$

3.3.5 A weak maximum principle

We extend the interior maximum principle (Theorem 2.1.12) to a version with the presence of obstacle.

Theorem 3.3.19 (weak maximum principle).

Given $\Omega \subset M$ a locally Lipschitz domain. Let $u, v \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ be respectively a solution and subsolution of IMCF(Ω)+OBS($\partial\Omega$). *If $\{u < v\} \Subset M$, and*

$$\inf_{\Omega\cap K} u > -\infty \qquad \text{for some } K \text{ with } \{u<v\} \Subset K \Subset M, \tag{3.3.21}$$

then $u \geqslant v$ in Ω.

In complement with Theorem 3.3.8, this has the following consequence: the solution of IVP($\Omega; E_0$)+OBS($\partial\Omega$), if it exists, is maximal among all solutions of IVP($\Omega; E_0$):

Corollary 3.3.20 (maximality).

Given a Lipschitz domain $\Omega \Subset M$ and $C^{1,1}$ domains $E_0 \subset E_0' \Subset \Omega$. Suppose

(1) $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ is a solution of IVP($\Omega; E_0$)+OBS($\partial\Omega$),

(2) $v \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ is a subsolution of IVP($\Omega; E_0'$).

Then $u \geqslant v$ in $\Omega \setminus E_0'$. In particular, the weak solution of IVP($\Omega; E_0$)+OBS($\partial\Omega$) *is unique up to equivalence (if it exists).*

Proof. For any $\varepsilon > 0$, the functions $u + \varepsilon$ and v are respectively a solution (by Definition 3.3.6 and restriction) and subsolution (by Theorem 3.3.8) of IMCF($\Omega\setminus\overline{E_0'}$)+OBS($\partial\Omega$). Moreover, we have $\{v > u + \varepsilon\} \Subset M \setminus \overline{E_0'}$. Then we have $u \geqslant v$ in $\Omega \setminus E_0'$, by Theorem 3.3.19 and by taking $\varepsilon \to 0$. Finally, the uniqueness follows from maximality. ☐

Let us come back to Theorem 3.3.19. It is important to note that the lower bound (3.3.21) is not removable. To see this, recall the nephroid soliton in Example 1.4.4. Let u be the corresponding level set function, and let us restrict to $\Omega = \{u < 0\} \Subset \mathbb{R}^2$. Now u is a solution of IMCF(Ω)+OBS($\partial\Omega$) which is unbounded from below due to the presense of the pole. Setting $v = u + 1$, we have $\{u < v\} \Subset \mathbb{R}^2$ but obviously $u \ngeqslant v$.

The proof here is a minor adaptation of [53, Theorem 2.2]. We assume u to be an exact solution here, but it can be checked that the proof works when u is a supersolution with enough integrability $u \in BV(\Omega \cap K)$, $u|\nabla u| \in L^1(\Omega \cap K)$. These assumptions can actually be removed, by invoking the original Definition 3.3.2 (hence avoiding Theorem 3.3.16). By comparing the energies $\widetilde{J}_u(E_t(u_\varepsilon)) \leqslant \widetilde{J}_u(E_t(u_\varepsilon) \cap E_t(v))$ and $\widetilde{J}_v(E_t(v)) \leqslant \widetilde{J}_v(E_t(v) \cup E_t(u_\varepsilon))$, and integrating over t, we can recover the imtermediate inequality (3.3.24). The other inequality (3.3.25) can similarly be obtained by comparing among various sub-level sets. For technical simplicity, we do not pursue proving the case with the widest possible generality.

Proof of Theorem 3.3.19.

Adding a common constant to u, v, we may assume that $u|_K \geqslant 0$. Replacing v with $\min\{v, T\}$ (which is still a weak subsolution of IMCF(Ω)+OBS($\partial\Omega$) by Remark 2.1.4(iv) and Theorem 3.3.8), and taking $T \to +\infty$, we may assume that $v \leqslant T$ for some $T > 0$. Thus we are reduced to the case $u \geqslant 0$, $v \leqslant T$. For a constant $\varepsilon > 0$ we set $u_\varepsilon = \frac{u}{1-\varepsilon}$. Then it is sufficient to show that $u_\varepsilon \geqslant v$. Suppose that this fails to hold. Then we may increase u by an appropriate constant, and assume that $u_\varepsilon > v - \varepsilon$ but $\{u_\varepsilon < v\} \neq \emptyset$.

From Theorem 3.3.16, it is not hard to see that u_ε satisfies a strict minimizing condition

$$\int \left(|\nabla u_\varepsilon| - |\nabla w|\right) - \int_{\partial^*\Omega} (u_\varepsilon^\partial - w^\partial) \leqslant (1-\varepsilon) \int (w - u_\varepsilon)|\nabla u_\varepsilon|$$

for all competitors $w \geqslant u_\varepsilon$ such that $\{w \neq u_\varepsilon\} \Subset M$ and $\widetilde{J}_u^K(w)$ is defined. Setting $w = \max\{u_\varepsilon, v\}$ (which is a valid competitor), we obtain

$$\int_{\{v>u_\varepsilon\}} \big(|\nabla u_\varepsilon| - |\nabla v|\big) - \int_{\partial^*\Omega\cap\{v^\partial>u_\varepsilon^\partial\}} (u_\varepsilon^\partial - v^\partial) \leqslant (1-\varepsilon)\int_{\{v>u_\varepsilon\}} (v-u_\varepsilon)|\nabla u_\varepsilon|. \quad (3.3.22)$$

Testing the subsolution property of v with the competitor $\min\{u_\varepsilon, v\}$, we find that

$$\int_{\{u_\varepsilon<v\}} \big(|\nabla v| - |\nabla u_\varepsilon|\big) - \int_{\partial^*\Omega\cap\{u_\varepsilon^\partial<v^\partial\}} (v^\partial - u_\varepsilon^\partial) \leqslant \int_{\{u_\varepsilon<v\}} (u_\varepsilon - v)|\nabla v|. \quad (3.3.23)$$

Adding (3.3.22) (3.3.23) and canceling the common terms, we obtain

$$\int_{\{v>u_\varepsilon\}} (v-u_\varepsilon)|\nabla v| \leqslant (1-\varepsilon)\int_{\{v>u_\varepsilon\}} (v-u_\varepsilon)|\nabla u_\varepsilon|. \quad (3.3.24)$$

Next, for $s \geqslant 0$ we compare $\widetilde{J}_{u_\varepsilon}^K(u_\varepsilon) \leqslant \widetilde{J}_{u_\varepsilon}^K(\max\{u_\varepsilon, v-s\})$ to obtain

$$\int_{\{v-s>u_\varepsilon\}} \big(|\nabla u_\varepsilon| - |\nabla v|\big) - \int_{\partial^*\Omega\cap\{v^\partial-s>u_\varepsilon^\partial\}} (u_\varepsilon^\partial - v^\partial + s) \leqslant \int_{\{v-s>u_\varepsilon\}} (v-s-u_\varepsilon)|\nabla u_\varepsilon|.$$

The boundary term has the favorable sign and thus can be discarded. Integrating this inequality over $0 \leqslant s < \infty$ (note that the integrand is nonzero only within a bounded interval of s), by Fubini's theorem we obtain

$$0 \leqslant \int_{\{v>u_\varepsilon\}} (v-u_\varepsilon)\big(|\nabla v| - |\nabla u_\varepsilon|\big) + \frac{1}{2}(v-u_\varepsilon)^2|\nabla u_\varepsilon|. \quad (3.3.25)$$

Combining (3.3.24) (3.3.25) we have

$$\varepsilon\int_{\{v>u_\varepsilon\}} (v-u_\varepsilon)|\nabla u_\varepsilon| \leqslant \frac{1}{2}\int_{\{v>u_\varepsilon\}} (v-u_\varepsilon)^2|\nabla u_\varepsilon|. \quad (3.3.26)$$

Recall our reduction hypothesis at the beginning, that $u_\varepsilon > v - \varepsilon$ and $\{v > u - \varepsilon\} \neq \emptyset$. Then (3.3.26) implies that u_ε, hence u as well, is constant on $\{v > u_\varepsilon\}$. Then (3.3.24) implies in turn that v is constant on $\{v > u_\varepsilon\}$. Then the only possibility is that $\{v > u_\varepsilon\}$ is a precompact connected component of Ω. However, this contradicts the supersolution property of u, see Remark 3.3.5(v). Thus this completes the proof. □

3.4 Liouville theorems on the half space

The aim of this section is to establish Theorem 3.4.1 and 3.4.3. They are Liouville theorems in the half space, for weak solutions that respect the boundary obstacle and respectively a "soft obstacle" respectively. These results are applied in the main existence theorem (see Section 6) in showing that there are only trivial limits in the blow-up procedures.

We adopt the following notations: for a point $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, we denote $x' = (x_1, \cdots, x_{n-1})$ and write $x = (x', x_n)$. Moreover, set $e_n = (0, \cdots, 0, 1)$.

Theorem 3.4.1. *Let $\Omega = \{x_n < 0\} \subset \mathbb{R}^n$, and suppose that $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ is a solution of* $\mathsf{IMCF}(\Omega)$+$\mathsf{OBS}(\partial\Omega)$. *Moreover, suppose*

$$\inf_{\Omega}(u) > -\infty \qquad \text{and} \qquad |\nabla u(x)| \leqslant \frac{C}{|x_n|} \text{ for a.e. } x \in \Omega. \tag{3.4.1}$$

Then u must be constant.

The condition $\inf_\Omega(u) > -\infty$ cannot be removed, due to the Poisson kernel type Example 1.4.4. In the proof we need the following direct extension of Lemma 2.2.1.

Lemma 3.4.2. *Suppose Ω is a locally Lipschitz domain, and $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ is a supersolution of* $\mathsf{IMCF}(\Omega)$+$\mathsf{OBS}(\partial\Omega)$, *such that $\inf_\Omega(u) = t_0 > -\infty$. Suppose F has finite perimeter, and K is a domain, such that $F \Subset K \Subset M$. Then we have*

$$P(E_t; K) \leqslant P(E_t \setminus F; K) + \int_{t_0}^{t} e^{t-s} P(F; E_s)\, ds. \tag{3.4.2}$$

Proof. By Lemma 2.2.2(ii), both $\widetilde{J}_u^K(E_t)$ and $\widetilde{J}_u^K(E_t \setminus F)$ are finite. From the fact $\widetilde{J}_u^K(E_t) \leqslant \widetilde{J}_u^K(E_t \setminus F)$ and the coarea formula, we see that

$$\begin{aligned} P(E_t; K) &\leqslant P(E_t \setminus F; K) + \int_{t_0}^{t} \mathcal{H}^{n-1}\big(\partial^* E_s \cap E_t \cap F \cap \Omega\big)\, ds \\ &\leqslant P(E_t \setminus F; K) + \int_{t_0}^{t} \mathcal{H}^{n-1}\big(\partial^* E_s \cap F\big)\, ds. \end{aligned}$$

The remaining proof is the same as in Lemma 2.2.1. □

Proof of Theorem 3.4.1.
Shifting u by a constant, we may assume $\inf_\Omega(u) = 0$. Thus we are aimed at showing $u \equiv 0$. We separate the proof into two cases according to whether the level sets become flat when $t \searrow 0$. If they do become flat (Case (i)), then we use a barrier argument. If they are uniformly non-flat (Case (ii)), then we use a blow-up argument to obtain a contradiction.

Case (i): suppose there exist sequences $t_i \searrow 0$ and $\theta_i \searrow 0$, such that

$$\operatorname*{ess\,inf}_{\partial^* E_{t_i}} \langle \nu_{E_{t_i}}, e_n \rangle \geqslant \cos\theta_i. \tag{3.4.3}$$

For this case we use a barrier argument. Suppose that u is not identically zero, so there exists $x_0 \in \Omega$ with $u(x_0) > 0$. We may remove finitely many terms in the sequence and assume $u(x_0) > t_i$ for all i. By continuity and $\inf_\Omega(u) = 0$, for each i there exists a point $y_i = y_i' + y_{i,n} e_n$ ($y_{i,n} < 0$) that lies on $\partial^* E_{t_i}$. By Lemma A.6.3, (3.4.3), and Lemma 2.1.6, we have

$$E_{t_i} \supset \Big\{(x', x_n) : x_n < y_{i,n} - \big|x' - y_i'\big| \tan\theta_i\Big\}. \tag{3.4.4}$$

For a constant $N > 1$, consider the family of sets

$$F_t = B\big(y_i' + N y_{i,n} e_n, e^{\frac{t}{n-1}}(N-1)|y_{i,n}|\cos\theta_i\big),$$

whose boundary smoothly solves the IMCF. By the interior maximum principle and the fact that $F_0 \subset E_{t_i}$, we have

$$F_t \subset E_{t+t_i} \qquad \text{whenever } F_t \Subset \Omega.$$

The condition $F_t \Subset \Omega$ holds precisely when $t < T := (n-1)\log\big(\frac{N}{(N-1)\cos\theta_i}\big)$. Taking $t \nearrow T$, we obtain

$$E_{t_i+T} \supset B\big(y_i' + Ny_{i,n}e_n, N|y_{i,n}|\big).$$

Finally, taking $N \to \infty$ and then $i \to \infty$, we find that $E_t \supset \{x_n < 0\}$ for any $t > 0$. This implies $u \equiv 0$.

Case (ii): suppose the hypothesis of case (i) fails. This implies that we can find a constant $\theta_0 < 1$, a sequence of times $t_i \searrow 0$, and a sequence of points $y_i = (y_i', y_{i,n}) \in \partial^* E_{t_i}$, with $\langle \nu_{E_{t_i}}(y_i), e_n\rangle \leqslant \cos\theta_0$. The points y_i are necessarily contained in the interior of Ω (otherwise $\nu_{E_{t_i}}(y_i) = \nu_\Omega = e_n$). Consider the rescaled sets

$$F_i = \frac{1}{|y_{i,n}|}(E_{t_i} - y_i') \subset \Omega,$$

where it clearly holds $-e_n \in \partial^* F_i$ and $\langle \nu_{F_i}(-e_n), e_n\rangle \leqslant \cos\theta_0$. We investigate the minimizing properties of F_i: by Theorem 3.3.12, each F_i is locally outward minimizing in $\overline{\Omega}$, hence locally outward minimizing in $\mathbb{R}^n$, by direct verification. On the other hand, by Lemma 3.4.2 and the scale invariance of (3.4.2), we obtain the inward-minimizing property

$$P(F_i; K) \leqslant P(F_i \setminus G; K) + (e^{t_i} - 1)P(G) \qquad \forall\, G \Subset K \Subset \mathbb{R}^n.$$

Taking $i \to \infty$, a subsequence of F_i converges locally to some $F_\infty \subset \Omega$. Taking limit of the minimizing properties stated above, and by a standard set replacing argument, it follows that F_∞ is locally perimeter-minimizing in $\mathbb{R}^n$.

The gradient estimate in (3.4.1) and energy comparison implies the following: for all $x \in \Omega$ and any competitor set E with $E\Delta E_t \Subset B(x, |x_n|/2)$, we have

$$P\big(E_t; B(x, \frac{|x_n|}{2})\big) \geqslant P\big(E; B(x, \frac{|x_n|}{2})\big) + \frac{C}{|x_n|}\big|E\Delta E_t\big|.$$

Rescaling this, we see that F_i satisfies the condition

$$P\big(F_i; B(-e_n, \frac{1}{2})\big) \geqslant P\big(F; B(-e_n, \frac{1}{2})\big) + C\big|F\Delta F_i\big|,$$

for all F with $F\Delta F_i \Subset B(-e_n, 1/2)$. Thus F_i are uniform almost perimeter-minimizers in $B(-e_n, 1/2)$, in the sense stated in Section A.2. Then we apply Theorem A.2.2 to obtain $-e_n \in \operatorname{spt}|\mu_{F_\infty}|$. Recalling from above that F_∞ is locally perimeter-minimizing in $\mathbb{R}^n$, we then apply Lemma A.6.4 to obtain $F_\infty = \big\{x_n < -1\big\}$. Finally, by Theorem A.2.2(iii) we have the convergence of normal vectors $\nu_{F_i}(-e_n) \to \nu_{F_\infty}(-e_n) = e_n$, contradicting our hypothesis. This proves the theorem. □

We also establish the following approximate Liouville theorem, where the obstacle $\{x_n = 0\}$ is replaced by a "soft obstacle", represented by the weight function $\psi(x_n)$ unbounded near $x_n = 1$. See Section 3.2 for the definition of weighted weak solutions.

Theorem 3.4.3. *Fix a smooth function $\psi : (-\infty, 1) \to [0, \infty)$ that satisfies*
- $\psi|_{(-\infty,0]} \equiv 0$,
- $\psi > 0$, $\psi' > 0$, $\psi'' > 0$, $\psi''' > 0$ *on* $(0, 1)$,
- $\lim_{x\to 1} \psi(x) = +\infty$.

Let $\Omega = \{x_n < 1\} \subset \mathbb{R}^n$, and $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ be a weak solution of the weighted IMCF

$$\operatorname{div}\Big(e^{\psi(x_n)}\frac{\nabla u}{|\nabla u|}\Big) = e^{\psi(x_n)}|\nabla u| \tag{3.4.5}$$

in Ω. Moreover, assume there is a constant C so that
(1) $u \geqslant \psi(x_n) - C$ in Ω (in particular, it holds $\inf_\Omega(u) > -\infty$),
(2) $|\nabla u(x)| \leqslant \frac{C}{|x_n|}$ for a.e. $x \in \{x_n < 0\}$.
Then $u(x', x_n) = \psi(x_n) - C'$ for some other constant C'.

Let us remark that condition (1) is not removable. Consider the weight function $\psi(x) = (n-1)\log\frac{1}{1-x}$ for $x > 0$ and $\psi(x) = 0$ for $x \leqslant 0$ (the non-smoothness of ψ does not really matter). By Lemma 3.2.1, the weighted IMCF is equivalent to the usual weak IMCF in the metric $g' = e^{2\psi(x_n)/(n-1)}g$, which is hyperbolic in $\{0 < x_n < 1\}$. Without condition (1), the following solution is a counterexample: the solution starts from initial the horoball $E_0 = \{x_n < 0\}$, then instantly jumps to the half-space $E_0^+ = \{x_n < 1 - \sqrt{1 - |x'|^2}\}$, and then continue to evolve by $\Sigma_t = \big\{x : d(x, E_0^+) = \operatorname{arccosh} e^{t/(n-1)}\big\}$. See Figure 3.6.

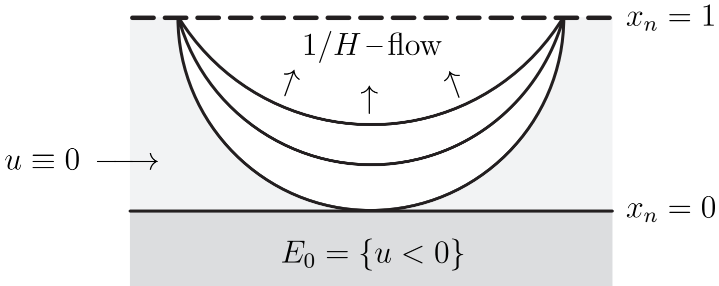

Figure 3.6: Counterexample of Theorem 3.4.3 without condition (1).

Proof of Theorem 3.4.3.
Shifting u by a constant, we may assume $\inf_\Omega(u) = 0$. We are aimed at proving $u(x', x_n) = \psi(x_n)$. The proof consists of three steps: we first show $u \geqslant \psi(x_n)$ using outer barriers, and next show that $u = 0$ on $\{x_n \leqslant 0\}$ by arguing similarly as in Theorem 3.4.1, and finally prove $u \leqslant \psi(x_n)$ using inner barriers.

Step 1. We show that $u \geqslant \psi(x_n)$. The following fact can be directly verified: if Σ_t are graphs of functions $f = f(\cdot, t)$ over $\mathbb{R}^{n-1}$, then Σ_t is a solution (resp. subsolution, supersolution) of the smooth weighted IMCF $\frac{\partial \Sigma_t}{\partial t} = \frac{\nu}{H + \partial\psi/\partial\nu}$ if and only if

$$\frac{\partial f}{\partial t}\Big(-\operatorname{div}\frac{\nabla f}{\sqrt{1+|\nabla f|^2}} + \frac{\psi'(f)}{\sqrt{1+|\nabla f|^2}}\Big) = \ (\text{resp. } \geqslant, \leqslant)\ \sqrt{1+|\nabla f|^2}. \tag{3.4.6}$$

For $\mu > 0$ and $R \geqslant 4$, consider the function

$$\underline{f}(x', t) = (1+\mu)\psi^{-1}(t) + \mu + R - \sqrt{R^2 - |x'|^2}, \tag{3.4.7}$$

where ψ^{-1} is the inverse of ψ. Note that $\underline{f}$ is smooth in the region $\{(x', t) \in \mathbb{R}^{n-1}\times(0,\infty) : \underline{f}(x', t) < 1\}$. We claim that there exists R_0 depending on ψ and μ, such that $\underline{f}$ is a subsolution of (3.4.6) whenever $R \geqslant R_0$, $t > 0$ and $\underline{f} < 1$. To verify this, we compute

$$\frac{\partial \underline{f}}{\partial t} = \frac{1+\mu}{\psi'(\psi^{-1}(t))} \qquad \text{and} \qquad \operatorname{div}\frac{\nabla \underline{f}}{\sqrt{1+|\nabla \underline{f}|^2}} = \frac{n-1}{R}. \tag{3.4.8}$$

To evaluate the second term in (3.4.6), we note that $\underline{f} < 1$ implies $R - \sqrt{R^2 - |x'|^2} < 1$, which further implies $|x'|^2 \leqslant 2R - 1$. Therefore, by direct estimation

$$1 + |\nabla \underline{f}|^2 = \frac{R^2}{R^2 - |x'|^2} \leqslant \frac{R^2}{(R-1)^2} \leqslant 1 + \frac{4}{R} \qquad \text{whenever } \underline{f} < 1, \tag{3.4.9}$$

where we used $R \geqslant 4$. Moreover, using the convexity of ψ and ψ' we have

$$\begin{aligned} \psi'(\underline{f}) &\geqslant \psi'\big(\psi^{-1}(t) + \frac{\mu}{2}\big) + \psi''\big(\psi^{-1}(t) + \frac{\mu}{2}\big) \cdot \big(\underline{f} - \psi^{-1}(t) - \frac{\mu}{2}\big) \\ &\geqslant \psi'(\psi^{-1}(t)) + \psi''(\frac{\mu}{2}) \cdot \frac{\mu}{2}. \end{aligned} \tag{3.4.10}$$

Combining (3.4.8) (3.4.9) (3.4.10), we obtain

$$\begin{aligned} &\frac{\partial \underline{f}}{\partial t}\Big(-\operatorname{div} \frac{\nabla \underline{f}}{\sqrt{1 + |\nabla \underline{f}|^2}} + \frac{\psi'(\underline{f})}{\sqrt{1 + |\nabla \underline{f}|^2}}\Big) \\ &\geqslant \frac{1+\mu}{\psi'(\psi^{-1}(t))} \cdot \Big(-\frac{n-1}{R} + \frac{\psi'(\psi^{-1}(t)) + (\mu/2)\psi''(\mu/2)}{\sqrt{1+4/R}}\Big). \end{aligned}$$

As a result, if we choose R_0 sufficiently large so that

$$\frac{\mu}{2}\psi''(\frac{\mu}{2}) > \sqrt{1 + \frac{4}{R}} \cdot \frac{n-1}{R} \quad \text{ and } \quad \frac{1+\mu}{\sqrt{1+4/R}} > \sqrt{1 + \frac{4}{R}} \quad \text{ for all } R \geqslant R_0,$$

then combined with (3.4.9), it follows that $\underline{f}$ is a strict subsolution of (3.4.6) whenever $t > 0$ and $\underline{f} < 1$. Switching to the level set description, we find that the function

$$\underline{u}(x', x_n) = \psi\Big(\frac{1}{1+\mu}\big(x_n - \mu - R + \sqrt{R - |x'|^2}\big)\Big),$$

defined such that $\{\underline{u} = t\} = \operatorname{graph}\big(f(\cdot, t)\big)$, satisfies the subsolution condition

$$\operatorname{div}\Big(e^{\psi}\frac{\nabla \underline{u}}{|\nabla \underline{u}|}\Big) > e^{\psi}|\nabla \underline{u}|$$

in the region $\Omega' = \big\{\mu + R - \sqrt{R^2 - |x'|^2} < x_n < 1\big\} \subset \Omega$.

For $\varepsilon > 0$, we apply the interior maximum principle (Theorem 2.1.12) to the functions $u' = u + \varepsilon$ and $\underline{u}$. By our hypothesis we have $u \geqslant \max\{0, \psi(x_n) - C\}$. Hence

$$u'(x) < \underline{u}(x) \quad \Rightarrow \quad \psi(x_n) - C < \psi\big(\frac{x_n}{1+\mu}\big) \quad \Rightarrow \quad x_n < C' \qquad \text{for all } x \in \Omega',$$

where $C' < 1$ is a constant depending only on C, μ, ψ. On the other hand, $u'(x) < \underline{u}(x)$ implies $\underline{u}(x) > \varepsilon$, therefore the closure of $\{u' < \underline{u}\}$ does not intersect $\{x_n = \mu + R - \sqrt{R^2 - |x'|^2}\}$. As a result, we have $\{u' < \underline{u}\} \Subset \Omega'$. By Lemma 3.2.1(3) and Theorem 2.1.12, we obtain $u' \geqslant \underline{u}$. Taking $\varepsilon \to 0$, $R \to \infty$ and then $\mu \to 0$, we obtain $u \geqslant \psi(x_n)$.

Step 2. we show that $u = 0$ on $\{x_n \leqslant 0\}$, using similar methods as in Theorem 3.4.1. Due to the presence of weight function, the discussion here is more complicated. Recall that we have assumed $\inf(u) = 0$ and showed that $E_t \subset \{x_n < \psi^{-1}(t)\}$ for all $t > 0$.

Case 2(i): suppose for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\partial^* E_t \subset \{x \geqslant -\varepsilon\}$ for all $0 < t \leqslant \delta$. Thus for each $t \in (0, \delta]$ there are only two possibilities: either $E_t \supset \{x_n < -\varepsilon\}$, or $E_t \subset \{-\varepsilon < x_n < \psi^{-1}(t)\}$. We claim that latter case never occurs. Once this is proved, we have $\{u \leqslant 0\} \supset \{x_n < -\varepsilon\}$, which by taking $\varepsilon \to 0$ proves the goal of Step 2. It remains to prove our claim. Suppose we have $E_t \subset \{-\varepsilon < x < \psi^{-1}(t)\}$ for some ε, δ and $t \in (0, \delta]$. Let $\pi : (x', x_n) \mapsto x'$ be the projection map. Note that $P\big(E_t; A \times \mathbb{R}\big) \geqslant \mathcal{L}^{n-1}(\pi(E_t) \cap A)$ for any open set $A \subset \mathbb{R}^{n-1}$. Lemma 2.2.1 is easily generalized to the weighted case, which implies that

$$\int_{\partial^* E_t \cap K} e^{\psi}\, d\mathcal{H}^{n-1} \leqslant \int_{\partial^*(E_t \setminus F) \cap K} e^{\psi}\, d\mathcal{H}^{n-1} + (e^t - 1) \int_{\partial^* F \cap E_t} e^{\psi}\, d\mathcal{H}^{n-1}, \tag{3.4.11}$$

whenever $F \Subset K \Subset \Omega$. If $\mathcal{H}^{n-1}(\partial^* E_t \cap \partial^* F) = 0$ and $F = F^{(1)}$, then (3.4.11) is reduced to

$$\int_{\partial^* E_t \cap F} e^{\psi}\, d\mathcal{H}^{n-1} \leqslant e^t \int_{\partial^* F \cap E_t} e^{\psi}\, d\mathcal{H}^{n-1}. \tag{3.4.12}$$

Since $e^{\psi} \geqslant 1$ everywhere and $e^{\psi} \leqslant e^t$ in $\overline{E_t}$, we have

$$P(E_t; F) \leqslant e^{2t} P(F; E_t). \tag{3.4.13}$$

Now for a radius $R > 0$ we choose $F = B'(0, R) \times (-1/2, 1/2)$, where B' denotes open balls in $\mathbb{R}^{n-1}$. For almost every R, (3.4.13) implies

$$\begin{aligned} \mathcal{L}^{n-1}(\pi(E_t) \cap B'(0, R)) &\leqslant P\big(E_t; B'(0, R) \times \mathbb{R}\big) = P(E_t; F) \leqslant e^{2t} P(F; E_t) \\ &\leqslant e^{2t}\big(\varepsilon + \psi^{-1}(t)\big)\mathcal{H}^{n-2}(\partial B'(0, R) \cap \pi(E_t)). \end{aligned}$$

Denoting $f(R) = \mathcal{L}^{n-1}(\pi(E_t) \cap B'(0, R))$, thus $f(R) \leqslant e^{2t}\big(\varepsilon + \psi^{-1}(t)\big) f'(R)$. Since $\inf_\Omega(u) = 0$, we have $f(R) > 0$ for some R, therefore $f(R)$ is exponentially growing as $R \to \infty$. This is impossible. From the setups made above, this proves the desired result for Case 2(i).

For the rest of Step 2, we may assume the contrary of Case 2(i): that there is a fixed $\varepsilon > 0$ and a sequence of times $t_i \to 0$, such that $\partial^* E_{t_i} \cap \{x_n < -\varepsilon\} \neq \emptyset$ for all i.

Case 2(ii): assume that there are angles $\theta_i \to 0$, such that for each i it holds

$$\operatorname*{ess\,inf}_{\partial^* E_{t_i} \cap \{x_n < -\varepsilon\}} \langle \nu_{E_{t_i}}, e_n \rangle \geqslant \cos\theta_i.$$

Then arguing similarly as in Case (i) of Theorem 3.4.1, we have $E_t \supset \{x_n < 0\}$ for all $t > 0$. This proves the desired result for Step 2.

Case 2(iii): suppose the hypothesis of Case 2(ii) does not hold. Then passing to a subsequence, there is a constant $\theta_0 > 0$ and points $y_i = (y_i', y_{i,n}) \in \partial^* E_{t_i} \cap \{x_n < -\varepsilon\}$, such that $\langle \nu_{E_{t_i}}(y_i), e_n \rangle \leqslant \cos\theta_0$. We argue that this is impossible. For convenience, we denote $E_i = E_{t_i}$. Recall from step 1 that $E_i \subset \{x_n < \psi^{-1}(t_i)\}$. We start with investigating the minimizing properties of E_i.

1. E_i are local outward minimizers of the weighted perimeter $E \mapsto \int_{\partial^* E} e^{\psi}\, d\mathcal{H}^{n-1}$ in $\{x_n < 1\}$, by Lemma 3.2.1(iii). This immediately implies that E_i are local outward minimizers of $E \mapsto \int_{\partial^* E} e^{\min\{\psi, t_i\}}\, d\mathcal{H}^{n-1}$ in $\big\{x_n < \psi^{-1}(t_i)\big\}$. Then by direct verification, E_i locally outward minimizes the same functional $E \mapsto \int_{\partial^* E} e^{\min\{\psi, t_i\}}\, d\mathcal{H}^{n-1}$ in $\mathbb{R}^n$.

2. The inward minimizing effect of E_i is given by (3.4.11). Since all the integrals in (3.4.11) occur inside the set $\{\psi \leqslant t_i\}$, it makes no difference to replace each ψ by $\min\{\psi, t_i\}$.

Combining the inward and outward minimizing properties, we conclude that

$$\int_{\partial^* E_i \cap K} e^{\min\{\psi,t_i\}}\, d\mathcal{H}^{n-1} \leqslant \int_{\partial^* F \cap K} e^{\min\{\psi,t_i\}}\, d\mathcal{H}^{n-1} + (e^{t_i}-1)\int_{\partial^*(E_i\setminus F)} e^{\min\{\psi,t_i\}}\, d\mathcal{H}^{n-1}$$

whenever $F\Delta E_i \Subset K \Subset \mathbb{R}^n$. This directly implies

$$P(E_i;K) \leqslant e^{t_i}P(F;K) + e^{t_i}(e^{t_i}-1)P(E_i\setminus F).$$

Next, consider the rescaled sets $F_i = (E_i - y_i')/|y_{i,n}|$. It follows that $F_i \subset \{x_n < 1/\varepsilon\}$, $-e_n \in \partial^* F_i$, and $\langle \nu_{F_i}(-e_n), e_n\rangle \leqslant \cos\theta_0$, and F_i satisfy the minimizing property

$$P(F_i;K) \leqslant e^{t_i}P(F;K) + e^{t_i}(e^{t_i}-1)P(F_i\setminus F) \tag{3.4.14}$$

whenever $F\Delta F_i \Subset K \Subset \mathbb{R}^n$. Moreover, since $\psi_i = 0$ in $\{x_n<0\}$, and by the gradient estimate $|\nabla u| \leqslant C/|x_n|$, we have the uniform almost-minimizing property

$$P\big(F_i; B(-e_n,\tfrac{1}{2})\big) \leqslant P\big(F; B(-e_n,\tfrac{1}{2})\big) + C|F\Delta F_i| \tag{3.4.15}$$

whenever $F\Delta F_i \Subset B(-e_n,\frac{1}{2})$.

We are in a position to pass to the limit: there is a subsequence of F_i that converge to a set F_∞ in L^1_{loc}. By (3.4.15) and Theorem A.2.2(ii) we have $-e_n \in \partial^* F_\infty$. By (3.4.14) and the fact $t_i \to 0$ and the standard set replacing argument, the limit F_∞ locally minimizes the perimeter in $\mathbb{R}^n$. By Lemma A.6.4 and the fact $F_\infty \subset \{x_n < 1/\varepsilon\}$, we have $F_\infty = \{x_n < -1\}$. Finally, by Theorem A.2.2(iii) we have convergence of normal vectors $\nu_{F_i}(-e_n) \to \nu_{F_\infty}(-e_n) = e_n$, contradicting our hypothesis. This completes step 2.

Step 3. We have shown that $u \geqslant \psi(x_n)$ and $u=0$ on $\{x_n<0\}$. In this step we use inner barriers to show that $u \leqslant \psi(x_n)$, which completes the proof of the theorem. For constants $\mu \ll 1$, $R \gg 1$ and for $t>0$, we consider the function

$$\overline{f}(x',t) = (1-\mu)\int_0^t \frac{ds}{\psi'(\psi^{-1}(s)) + (n-1)/R} - R + \sqrt{R^2 - |x'|^2}.$$

We claim that $\overline{f}$ is a supersolution of (3.4.6) whenever $t>0$ and $\overline{f}>0$. Note that $\overline{f} \leqslant (1-\mu)\int_0^t \frac{ds}{\psi'(\psi^{-1}(s))} < (1-\mu)\psi^{-1}(t)$. In particular, $\overline{f}<1$ always holds. To verify the supersolution property, according to (3.4.6) we compute

$$\frac{\partial \overline{f}}{\partial t} = \frac{1-\mu}{\psi'(\psi^{-1}(t)) + (n-1)/R} \qquad \text{and} \qquad \operatorname{div}\Big(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\Big) = -\frac{n-1}{R}.$$

By the convexity of ψ, we have $\psi'(\overline{f}) \leqslant \psi'(\psi^{-1}(t))$. Therefore,

$$\begin{aligned}
&\frac{\partial \overline{f}}{\partial t}\Big(-\operatorname{div}\frac{\nabla \overline{f}}{\sqrt{1+|\nabla \overline{f}|^2}} + \frac{\psi'(\overline{f})}{\sqrt{1+|\nabla \overline{f}|^2}}\Big)\\
\leqslant\, &\frac{1-\mu}{\psi'(\psi^{-1}(t)) + (n-1)/R}\cdot\Big(\frac{n-1}{R} + \psi'(\psi^{-1}(t))\Big)\\
=\, &1-\mu < \sqrt{1+|\nabla \overline{f}|^2}.
\end{aligned}$$

Thus $\overline{f}$ is a supersolution of (3.4.6).

Consider the unique positive function $\overline{u}$ on $\{0 < x_n < 1\}$, such that $E_t(\overline{u}) = \big\{(x', x_n) : 0 < x_n < \overline{f}(x', t)\big\}$ for all $t > 0$. Then $\overline{u}$ is a smooth supersolution of IMCF in the region $\Omega' := \{0 < x_n < 1, \overline{u} < \infty\}$. For $\varepsilon > 0$ we wish to compare u with the function $\overline{u} + \varepsilon$. Assembling the facts: $u = 0$ on $\{x_n = 0\}$, and $\{\overline{u} < \infty\} \subset \big\{x_n < 1-\mu-R+\sqrt{R^2 - |x'|^2}\big\}$, and $u \in \mathrm{Lip}_{\mathrm{loc}}(\{x_n < 1\})$, we conclude that $\{u > \overline{u} + \varepsilon\} \Subset \Omega'$. By Lemma 3.2.1 (3) and Theorem 2.1.12, we obtain $u \leqslant \overline{u} + \varepsilon$. Taking $\varepsilon \to 0$ and then $R \to \infty$ and then $\mu \to 0$, we eventually obtain $u \leqslant \psi(x_n)$. □

3.5 Parabolic estimates near smooth obstacles

In this section, we compute several parabolic evolution equations for the smooth IMCF near a smooth obstacle. The main results are Lemma 3.5.4 and 3.5.5.

The aim is to show the following: if Σ_t evolves under the IMCF in a smooth domain Ω, such that (1) ν_{Σ_t} is approximately equal to ν_Ω on $\partial\Omega$ (where ν denote the outer unit normal of the corresponding objects), and (2) in some neighborhood of $\partial\Omega$ we have $\langle \nu_{\Sigma_t}, \partial_r\rangle \geqslant \frac{1}{2}$ (where ∂_r is an extention of ν_Ω, see below), then it holds $\langle \nu_{\Sigma_t}, \partial_r\rangle \geqslant 1 - Cr^\gamma - o(1)$ in some smaller neighborhood of $\partial\Omega$. This estimate enters the proofs in Section 3.6 in showing the $C^{1,\alpha}$ regularity of level sets and boundary regularity of blow-up limits.

The following notations are used consistently in the present and the next sections. Suppose Ω is a smooth domain in a Riemannian manifold M. Define the signed distance function

$$r(x) := \begin{cases} -d(x, \partial\Omega) & x \in \Omega, \\ d(x, \partial\Omega) & x \notin \Omega, \end{cases}$$

For $\delta \in \mathbb{R}$, we denote $\Omega_\delta := \big\{x \in M : r(x) < \delta\big\}$. We use r_0 to denote a radius such that $r(x)$ is smooth in $\Omega \setminus \overline{\Omega_{-r_0}}$. In Lemma 3.5.1$\sim$3.5.3, the existence of such a radius is implicitly assumed. In $\Omega \setminus \overline{\Omega_{-r_0}}$ we define the outer radial vector field $\partial_r := \nabla r$.

For a hypersurface $\Sigma \subset \Omega \setminus \overline{\Omega_{-r_0}}$, we use ν, A, H to denote the unit normal vector, second fundamental form and mean curvature. We use $|\cdot|_\Sigma, \nabla_\Sigma, \Delta_\Sigma$ to denote the (Hilbert-Schmidt) norm, gradient and Laplacian on Σ_t, and use $|\cdot|, \nabla, \Delta$ for objects with respect to the ambient manifold M. Also, denote

$$p := \langle \nu, \partial_r\rangle.$$

The following algebraic fact is useful: $|\nu - p\partial_r| = |\partial_r - p\nu| = |\nabla_\Sigma r| = \sqrt{1-p^2}$.

When there is a family of hypersurfaces $\{\Sigma_t\}$, we often omit the dependence on t in the above notations. When Σ_t evolves under the IMCF, we denote $\Box = \partial_t - \frac{1}{H^2}\Delta_\Sigma$ the associated heat operator.

We start with evaluating $\Delta_\Sigma p$. The main formula (3.5.1) is arranged such that each term vanishes when $\Sigma = \partial\Omega_{-r}$ for some r, and the terms with unfavorable sign become small when M and $\partial\Omega$ are close to being flat.

Lemma 3.5.1. *Let Σ be a smooth hypersurface in $\Omega \setminus \overline{\Omega_{-r_0}}$, such that $p > 0$ on Σ. Then we have*

$$\begin{aligned} \Delta_\Sigma p \leqslant\ & \langle \nabla_\Sigma H, \partial_r\rangle - p\big|A - p^{-1}\nabla^2 r\big|^2_\Sigma \\ & + |\nabla^2 r|^2 p^{-1}(1-p^2) + \big(|\mathrm{Ric}| + n|\nabla^2\partial_r|\big)\sqrt{1-p^2}. \end{aligned} \tag{3.5.1}$$

(In the second term, the expression $\nabla^2 r$ means the restriction of $\nabla^2_M r$ to Σ.)

Proof. Near a given point $x \in \Sigma$, we let $\{e_i\}_{1 \leqslant i \leqslant n-1}$ be a local orthonormal frame on Σ, such that $\nabla^\Sigma_{e_i} e_j = 0$ at x. We compute

$$\begin{aligned}
\Delta_\Sigma p &= \sum_i e_i e_i \langle \nu, \partial_r \rangle = \sum_i e_i \Big[\langle A(e_i), \partial_r \rangle + \langle \nu, \nabla_{e_i} \partial_r \rangle \Big] \\
&= \sum_i \Big[\langle \nabla_{e_i}(A(e_i)), \partial_r \rangle + 2 \langle A(e_i), \nabla_{e_i} \partial_r \rangle + \langle \nu, \nabla_{e_i} \nabla_{e_i} \partial_r \rangle \Big]. \qquad (3.5.2)
\end{aligned}$$

Here $A(e_i)$ is understood as a tangent vector field of Σ. First observe that

$$2 \sum_i \langle A(e_i), \nabla_{e_i} \partial_r \rangle = 2 \sum_i \nabla^2 r(A(e_i), e_i) = 2 \langle A, \nabla^2 r \rangle_\Sigma. \qquad (3.5.3)$$

Next, the first term in (3.5.2) is calculated as

$$\begin{aligned}
\sum_i \langle \nabla_{e_i} A(e_i), \partial_r \rangle &= \sum_i \Big\langle (\nabla^\Sigma_{e_i} A)(e_i) + A(\nabla^\Sigma_{e_i} e_i) - A(e_i, A(e_i)) \nu, \, \partial_r \Big\rangle \\
&= \langle \operatorname{div}_\Sigma A - |A|^2_\Sigma \, \nu, \partial_r \rangle \\
&= \langle \operatorname{div}_\Sigma A, \partial_r - p\nu \rangle - p|A|^2 \\
&\leqslant \langle \nabla_\Sigma H, \partial_r \rangle - p|A|^2_\Sigma + |\operatorname{Ric}| \sqrt{1 - p^2}, \qquad (3.5.4)
\end{aligned}$$

where we use the traced Codazzi equation and note that $\operatorname{div}_\Sigma A$, $\nabla_\Sigma H$ are tangent to Σ. It remains to simplify the last term in (3.5.2). We calculate

$$\begin{aligned}
\sum_i \langle \nu, \nabla_{e_i} \nabla_{e_i} \partial_r \rangle &= \sum_i \Big[\langle \nu - p\partial_r, \nabla_{e_i} \nabla_{e_i} \partial_r \rangle + p \langle \partial_r, \nabla_{e_i} \nabla_{e_i} \partial_r \rangle \Big] \\
&\leqslant n |\nabla^2 \partial_r| \sqrt{1 - p^2} - p \sum_i \big| \nabla_{e_i} \partial_r \big|^2. \qquad (3.5.5)
\end{aligned}$$

Continuing to evaluate the second term in (3.5.5):

$$\sum_i \big| \nabla_{e_i} \partial_r \big|^2 = \sum_{ij} \langle \nabla_{e_i} \partial_r, e_j \rangle^2 + \sum_i \langle \nabla_{e_i} \partial_r, \nu \rangle^2 \geqslant \sum_{ij} \langle \nabla_{e_i} \partial_r, e_j \rangle^2 = |\nabla^2 r|^2_\Sigma. \qquad (3.5.6)$$

Inserting (3.5.3) $\sim$ (3.5.6) into (3.5.2), we obtain

$$\begin{aligned}
\Delta_\Sigma p \leqslant \langle \nabla_\Sigma H, \partial_r \rangle - p|A|^2_\Sigma + 2 \langle A, \nabla^2 r \rangle_\Sigma - p |\nabla^2 r|^2_\Sigma \\
+ |\operatorname{Ric}| \sqrt{1 - p^2} + n |\nabla^2 \partial_r| \sqrt{1 - p^2}.
\end{aligned}$$

This implies (3.5.1) by completing the square and noting that $|\nabla^2 r|_\Sigma \leqslant |\nabla^2 r|$. □

Lemma 3.5.2. *Suppose $\{\Sigma_t\}$ evolves under the IMCF in $\Omega \setminus \overline{\Omega_{-r_0}}$, such that $p > 0$ holds everywhere. Then we have*

$$\Box p \geqslant \frac{|A - p^{-1} \nabla^2 r|^2_\Sigma}{2H^2} p - 2n \frac{|\nabla^2 r|^2}{H^2 p} (1 - p^2) - \frac{|\operatorname{Ric}| + n |\nabla^2 \partial_r|}{H^2} \sqrt{1 - p^2}. \qquad (3.5.7)$$

Proof. We compute

$$\frac{\partial p}{\partial t} = \partial_t \langle \nu, \partial_r \rangle = \langle H^{-2} \nabla_\Sigma H, \partial_r \rangle + \langle \nu, \nabla_{H^{-1}\nu} \partial_r \rangle.$$

For the second term, we use the fact $\nabla^2 r(\partial_r, X) = 0$ for all X to evaluate

$$\langle \nu, \nabla_{H^{-1}\nu}\partial_r\rangle = \frac{1}{H}\nabla^2 r(\nu,\nu) = \frac{1}{H}\nabla^2 r(\nu - p\partial_r, \nu - p\partial_r) \geqslant -\frac{1}{H}|\nabla^2 r|(1-p^2).$$

Combined with (3.5.1), we obtain

$$\begin{aligned}\Box p \geqslant \frac{|A - p^{-1}\nabla^2 r|_\Sigma^2}{H^2}p - \frac{1}{H}|\nabla^2 r|(1-p^2) \\ - \frac{|\nabla^2 r|^2}{H^2 p}(1-p^2) - \frac{|\operatorname{Ric}| + n|\nabla^2\partial_r|}{H^2}\sqrt{1-p^2}.\end{aligned} \tag{3.5.8}$$

We bound the second term in this expression as follows:

$$\begin{aligned}\frac{1}{H}|\nabla^2 r|(1-p^2) &= \frac{H - p^{-1}\operatorname{tr}_\Sigma \nabla^2 r + p^{-1}\operatorname{tr}_\Sigma \nabla^2 r}{H^2}|\nabla^2 r|(1-p^2) \\ &\leqslant \sqrt{n-1}\frac{|A - p^{-1}\nabla^2 r|_\Sigma + p^{-1}|\nabla^2 r|_\Sigma}{H^2}|\nabla^2 r|(1-p^2).\end{aligned}$$

Using Young's inequality, we continue the estimate:

$$\frac{1}{H}|\nabla^2 r|(1-p^2) \leqslant \Big[\frac{|A - p^{-1}\nabla^2 r|_\Sigma^2}{2H^2}p + \frac{n}{2H^2 p}|\nabla^2 r|^2(1-p^2)^2\Big] + n\frac{|\nabla^2 r|^2}{H^2 p}(1-p^2).$$

The result follows by combining this into (3.5.8) and noting that $0 < p \leqslant 1$. $\square$

Lemma 3.5.3. *Assume the same conditions as in Lemma 3.5.2. Let $\eta = \eta(r)$ be a smooth radial function, and denote $\eta' = d\eta/dr$, $\eta'' = d^2\eta/dr^2$. Then we have*

$$\Box\eta \leqslant \frac{2p\eta'}{H^2}\big(H - p^{-1}\operatorname{tr}_\Sigma \nabla^2 r\big) - \frac{\eta''}{H^2}(1-p^2) + \frac{n|\eta'|}{H^2}|\nabla^2 r|. \tag{3.5.9}$$

and

$$\Box\eta \leqslant \frac{2p\eta'}{H} - \frac{\eta''}{H^2}(1-p^2) + \frac{n|\eta'|}{H^2}|\nabla^2 r|. \tag{3.5.10}$$

Proof. This is obtained by combining

$$\frac{\partial\eta}{\partial t} = \frac{\langle\nabla\eta,\nu\rangle}{H} = \frac{p\eta'}{H} = \frac{p\eta'}{H^2}\big(H - p^{-1}\operatorname{tr}_\Sigma \nabla^2 r + p^{-1}\operatorname{tr}_\Sigma \nabla^2 r\big)$$

with

$$\Delta_\Sigma\eta = \eta'\Delta_\Sigma r + \eta''|\nabla_\Sigma r|^2 = \eta'(\operatorname{tr}_\Sigma \nabla^2 r - pH) + \eta''(1-p^2),$$

and noting that $\operatorname{tr}_\Sigma \nabla^2 r \leqslant \sqrt{n-1}|\nabla^2 r|_\Sigma \leqslant n|\nabla^2 r|$. $\square$

Lemma 3.5.4. *Suppose Ω is a smooth domain, and $r_0 > 0$ is a radius such that $r(x)$ is smooth in $\Omega \setminus \overline{\Omega_{-r_0}}$. Assume that*

$$|\operatorname{Ric}| \leqslant r_0^{-2}, \qquad |\nabla^2 r| \leqslant r_0^{-1}, \qquad |\nabla^2\partial_r| \leqslant r_0^{-2} \tag{3.5.11}$$

hold inside $\Omega \setminus \overline{\Omega_{-r_0}}$. Let $\big\{\Sigma_t \subset \Omega \setminus \overline{\Omega_{-r_0}}\big\}$ be a smooth family of hypersurfaces evolving under the IMCF. Assume additionally that $p \geqslant \frac{1}{2}$ on all Σ_t. Then there exist constants $\gamma \in (0, 1/2)$ and $C_1, C_2 > 0$ depending only on n, such that the following holds: if we set

$$\eta(r) = \big(b - r/r_0\big)^{-\gamma}, \qquad F = (1-p)\eta, \tag{3.5.12}$$

with any choice $b \in (0,1]$, then we have the evolution inequality

$$(1-p)^{\frac{2}{\gamma}-1}H^2\Box F + \langle \nabla_\Sigma F, X\rangle \leqslant \frac{C_1}{r_0^2}\big(-F^{\frac{\gamma+2}{\gamma}} + C_2\big) \tag{3.5.13}$$

on Σ_t, where X is a vertain smooth vector field.

Proof. Note that $r<0$ inside Ω, hence $\eta>0$ and increasing when approaching $\partial\Omega$. Combining Lemma 3.5.2 and equation (3.5.9) in Lemma 3.5.3, then using (3.5.11) with $\frac{1}{2}\leqslant p\leqslant 1$ to simplify the resulting expressions, we have

$$\begin{aligned}
\Box F &= -\eta\Box p + (1-p)\Box\eta - \frac{2}{H^2}\langle\nabla_\Sigma(1-p), \nabla_\Sigma\eta\rangle\\
&\leqslant -\frac{|A-p^{-1}\nabla^2 r|_\Sigma^2}{2H^2}\eta p + 2n\frac{|\nabla^2 r|^2}{H^2 p}\eta(1-p^2) + \frac{|\operatorname{Ric}| + n|\nabla^2\partial_r|}{H^2}\eta\sqrt{1-p^2}\\
&\quad + \frac{2p\eta'}{H^2}(1-p)\big(H - p^{-1}\operatorname{tr}_\Sigma\nabla^2 r\big) - \frac{\eta''}{H^2}(1-p)(1-p^2) + n\frac{|\eta'|}{H^2}|\nabla^2 r|(1-p)\\
&\quad - \frac{2}{H^2\eta}\langle\nabla_\Sigma F, \nabla_\Sigma\eta\rangle + \frac{2}{H^2\eta}(1-p)|\nabla_\Sigma\eta|^2\\
&\leqslant -\frac{|A-p^{-1}\nabla^2 r|_\Sigma^2}{2H^2}\eta p + \frac{8n}{H^2 r_0^2}\eta(1-p) + \frac{4n\eta}{H^2 r_0^2}\sqrt{1-p}\\
&\quad + \frac{2p\eta'}{H^2}(1-p)\big(H - p^{-1}\operatorname{tr}_\Sigma\nabla^2 r\big) - \frac{\eta''}{H^2}(1-p)^2(1+p) + \frac{n|\eta'|}{H^2 r_0}(1-p)\\
&\quad - \frac{2}{H^2\eta}\langle\nabla_\Sigma F, \nabla_\Sigma\eta\rangle + \frac{2}{H^2}\frac{(\eta')^2}{\eta}(1-p)^2(1+p).
\end{aligned}$$

We use Young's inequality to estimate the fourth term:

$$\begin{aligned}
\frac{2p\eta'}{H^2}(1-p)\big(H - p^{-1}\operatorname{tr}_\Sigma\nabla^2 r\big) &\leqslant \frac{(H-p^{-1}\operatorname{tr}_\Sigma\nabla^2 r)^2}{2(n-1)H^2}\eta p + 8(n-1)\frac{(\eta')^2}{H^2\eta}p(1-p)^2\\
&\leqslant \frac{|A-p^{-1}\nabla^2 r|_\Sigma^2}{2H^2}\eta p + 8n\frac{(\eta')^2}{H^2\eta}p(1-p)^2.
\end{aligned}$$

Further, we calculate

$$\eta' = \frac{\gamma}{r_0}\eta^{\frac{\gamma+1}{\gamma}}, \qquad \eta'' = \frac{\gamma(\gamma+1)}{r_0^2}\eta^{\frac{\gamma+2}{\gamma}}.$$

Inserting these to the main estimate, we obtain

$$\begin{aligned}
H^2\Box F \leqslant \frac{8n}{r_0^2}\eta(1-p) &+ \frac{4n\eta}{r_0^2}\sqrt{1-p} + 8n\frac{\gamma^2}{r_0^2}\eta^{\frac{\gamma+2}{\gamma}}p(1-p)^2\\
&- \frac{\gamma(\gamma+1)}{r_0^2}\eta^{\frac{\gamma+2}{\gamma}}(1-p)^2(1+p) + \frac{n\gamma}{r_0^2}\eta^{\frac{\gamma+1}{\gamma}}(1-p)\\
&- \frac{2}{\eta}\langle\nabla_\Sigma F, \nabla_\Sigma\eta\rangle + \frac{2\gamma^2}{r_0^2}\eta^{\frac{\gamma+2}{\gamma}}(1-p)^2(1+p).
\end{aligned}$$

Multiplying both sides by $(1-p)^{\frac{2}{\gamma}-1}$, and using the facts $0<p\leqslant 1$, $1-p\leqslant\sqrt{1-p}$, we obtain

$$\begin{aligned}
(1-p)^{\frac{2}{\gamma}-1}H^2\Box F \leqslant \frac{12n}{r_0^2}\eta(1-p)^{\frac{2}{\gamma}-\frac{1}{2}} &+ \frac{n\gamma}{r_0^2}\eta^{\frac{\gamma+1}{\gamma}}(1-p)^{\frac{2}{\gamma}} - \langle\nabla_\Sigma F, X\rangle\\
&+ \frac{1}{r_0^2}\Big[8n\gamma^2 + 4\gamma^2 - \gamma(\gamma+1)\Big]\eta^{\frac{\gamma+2}{\gamma}}(1-p)^{\frac{\gamma+2}{\gamma}},
\end{aligned} \tag{3.5.14}$$

where X is a certain smooth vector field. Choosing γ sufficiently small (depending only on n), we can realize

$$8n\gamma^2 + 4\gamma^2 - \gamma(\gamma+1) < -3\gamma^2. \tag{3.5.15}$$

Using Hölder's inequality, we have

$$12n\eta(1-p)^{\frac{2}{\gamma}-\frac{1}{2}} \leqslant \gamma^2\eta^{\frac{\gamma+2}{\gamma}}(1-p)^{\frac{\gamma+2}{\gamma}} + C(n)\eta^{-\frac{(4-3\gamma)(2+\gamma)}{3\gamma^2}} \tag{3.5.16}$$

and

$$n\gamma\eta^{\frac{\gamma+1}{\gamma}}(1-p)^{\frac{2}{\gamma}} \leqslant \gamma^2\eta^{\frac{\gamma+2}{\gamma}}(1-p)^{\frac{\gamma+2}{\gamma}} + C(n)\eta^{-\frac{(1-\gamma)(2+\gamma)}{\gamma^2}} \tag{3.5.17}$$

Notice that $\eta \geqslant 2^{-\gamma}$ in $\Omega \setminus \overline{\Omega_{-r_0}}$, which bounds the last terms uniformly by constants. Inserting (3.5.15) $\sim$ (3.5.17) into (3.5.14), we finally obtain

$$(1-p)^{\frac{2}{\gamma}-1}H^2\Box F + \langle\nabla_\Sigma F, X\rangle \leqslant \frac{1}{r_0^2}\Big(-\gamma^2F^{\frac{\gamma+2}{\gamma}} + C(n)\Big). \qquad \Box$$

Lemma 3.5.5. *Suppose $b, \gamma \in (0, 1/2)$ are constants, Ω is a smooth domain, and $r_0 > 0$ is a radius such that $r(x)$ is smooth in $\Omega \setminus \overline{\Omega_{-r_0}}$. Assume that*

$$|\operatorname{Ric}| \leqslant r_0^{-2}, \qquad |\nabla^2 r| \leqslant r_0^{-1}, \qquad |\nabla^2\partial_r| \leqslant r_0^{-2} \tag{3.5.18}$$

holds inside $\Omega\setminus\Omega_{-r_0}$. Suppose $\big\{\Sigma_t \subset \Omega\setminus\overline{\Omega_{-r_0}}\big\}$ evolves by the IMCF. Additionally, assume $p \geqslant \frac{1}{2}$ and $1-p \leqslant C_3(b-r/r_0)^\gamma$ on each Σ_t. Set

$$\eta(r) = \big(b - r/r_0\big)^{1-\gamma} - (2b)^{1-\gamma}, \qquad G = r_0H\eta(r). \tag{3.5.19}$$

Then in $\Omega_{-br_0} \setminus \overline{\Omega_{-r_0}}$ we have the evolution inequality

$$\Box G + \langle\nabla_\Sigma G, X\rangle \leqslant -\frac{G}{n} + \frac{4n}{G} + C_4\big(b - r/r_0\big)^{-\gamma}\Big(\frac{C_5}{G} - 1\Big), \tag{3.5.20}$$

where X is a certain smooth vector field, and the constants C_4, C_5 depend on n, C_3.

Proof. Recall the evolution equation of the mean curvature:

$$\Box H = -\frac{2}{H^3}|\nabla_\Sigma H|^2 - \Big(|A|^2 + \operatorname{Ric}(\nu,\nu)\Big)\frac{1}{H}.$$

Combining (3.5.10) and (3.5.18) we have

$$\begin{aligned}
r_0^{-1}\Box G &= \eta\Box H + H\Box\eta - \frac{2}{H^2}\langle\nabla_\Sigma H, \nabla_\Sigma\eta\rangle \\
&\leqslant -\frac{2\eta}{H^3}|\nabla_\Sigma H|^2 - \frac{\eta H}{n-1} + \frac{\eta}{r_0^2H} + 2p\eta' - \frac{\eta''}{H}(1-p^2) + \frac{n|\eta'|}{r_0H} \\
&\quad - \frac{2}{H^3}\langle\nabla_\Sigma H, \nabla_\Sigma(H\eta) - \eta\nabla_\Sigma H\rangle \\
&= -\frac{G}{(n-1)r_0} + \frac{\eta^2}{r_0G} + 2p\eta' - r_0\frac{\eta\eta''}{G}(1-p^2) + \frac{n\eta|\eta'|}{G} - \langle\nabla_\Sigma G, X\rangle.
\end{aligned}$$

Note that

$$\eta' = -(1-\gamma)r_0^{-1}(b-r/r_0)^{-\gamma}, \qquad \eta'' = -\gamma(1-\gamma)r_0^{-2}(b-r/r_0)^{-1-\gamma}.$$

These imply that $|\eta\eta'| \leqslant 2r_0^{-1}$ in $\Omega_{-br_0} \setminus \overline{\Omega_{-r_0}}$, since $\gamma < 1/2$. In addition, we have $\eta \leqslant 2$ and $1-p^2 \leqslant 2(1-p)$. Hence

$$\Box G \leqslant -\frac{G}{n} + \frac{4+2n}{G} - 2p(1-\gamma)(b-r/r_0)^{-\gamma} + 2\gamma(1-\gamma)\frac{1-p}{G}(b-r/r_0)^{-2\gamma}.$$

Inserting our assumption $1-p \leqslant C_3(b-r/r_0)^{\gamma}$, we obtain

$$\Box G \leqslant -\frac{G}{n} + \frac{4n}{G} + (b-r/r_0)^{-\gamma}\Big[-2p(1-\gamma) + \frac{C}{G}\Big] - \langle \nabla_\Sigma G, X\rangle.$$

Since $p \geqslant \frac{1}{2}, \gamma \leqslant \frac{1}{2} \Rightarrow -2p(1-\gamma) \leqslant -\frac{1}{2}$, the result follows. $\Box$

3.6 Initial value problems in smooth domains

In this section we prove the following main existence and regularity theorem, which implies the main Theorem D.

Theorem 3.6.1. *Let $\Omega \subset M$ be a precompact domain with smooth boundary, and $E_0 \Subset \Omega$ be a $C^{1,1}$ domain. Then there exists a solution u of* $\mathsf{IVP}(\Omega; E_0)+\mathsf{OBS}(\partial\Omega)$*, unique up to equivalence. There exists $\gamma \in (0,1)$ depending on n, such that the following holds.*

(i) We have $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega) \cap \mathrm{BV}(\Omega) \cap C^{0,\gamma}(\overline{\Omega})$. More precisely, it holds

$$|\nabla u(x)| \leqslant \sup_{\partial E_0 \cap B(x,r)} H_+ + \frac{C(n)}{r}, \qquad x \in \Omega \setminus E_0, \quad r \leqslant \sigma(x;\Omega,g), \tag{3.6.1}$$

where $\sigma(x;\Omega,g)$ is as in Definition 2.4.2, and

$$|\nabla u(x)| \leqslant Cd(x,\partial\Omega)^{\gamma-1}, \qquad x \in \Omega \setminus \overline{E_0}, \tag{3.6.2}$$

for some constant $C > 0$.

(ii) The solution u is calibrated in $\Omega \setminus \overline{E_0}$ by a vector field ν, which satisfies

$$\langle \nu, \partial_r\rangle(x) \geqslant 1 - Cd(x,\partial\Omega)^{\gamma} \tag{3.6.3}$$

in some small neighborhood of $\partial\Omega$. Here $C > 0$ is a constant, and $\partial_r := -\nabla d(\cdot,\partial\Omega)$ is the outpointing unit vector perpendicular to $\partial\Omega$.

(iii) Each level set ∂E_t is a $C^{1,\gamma/2}$ hypersurface in some small neighborhood of $\partial\Omega$.

(iv) If $v \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ is some other solution of $\mathsf{IVP}(\Omega; E_0)$*, then $u \geqslant v$ in $\Omega \setminus E_0$.*

The proof is to do an approximation, where we replace the obstacle $\partial\Omega$ by a "soft obstacle" represented by a weight function ψ_δ. This function is defined such that $\psi_\delta|_\Omega = 0$ and $\psi_\delta(x) \to +\infty$ when $d(x,\Omega) \to \delta$. It follows that

$$\lim_{\delta\to 0} \psi_\delta(x) = \begin{cases} 0 & (x \in \Omega), \\ \infty & (x \notin \Omega), \end{cases}$$

and we recover a hard obstacle when $\delta \to 0$. Consider the weighted IMCF

$$\operatorname{div}\Big(e^{\psi_\delta}\frac{\nabla u_\delta}{|\nabla u_\delta|}\Big) = e^{\psi_\delta}|\nabla u_\delta| \tag{3.6.4}$$

in the δ-neighborhood of Ω. The rapid growth of ψ_δ ensures the existence of a solution u_δ such that $\lim_{d(x,\Omega)\to\delta} u_\delta(x) = +\infty$. Then we take the limit (of a subsequqence) $u = \lim_{\delta\to 0} u_\delta$. Note that (3.6.4) is the usual IMCF inside Ω, thus u is a solution of $\mathsf{IVP}(\Omega; E_0)$ by the standard compactness theorem. The level sets of u_δ are bent drastically by ψ_δ when going outside Ω, and when $\delta \to 0$, a large amount of level sets pile up at $\partial\Omega$. We will try to show that the limit solution u satisfies the terms of Theorem 3.6.1.

In reality, there is another layer of complexity in this argument: the parabolic estimates in Section 3.5 only work for smooth solutions. Therefore, we will not directly solve (3.6.4), but instead solve the elliptic regularization of it. See Section 2.4 for a general introduction. In particular, due to the geometric meaning of elliptic regularized equation, we may apply the results of Section 3.5 to (the spacetime graph of) the regularized solutions.

The remainder of this section is organized as follows. In Subsection 3.6.1, we set up some definitions and preliminary estimates, then we make precise the above approximation scheme. In the end of Subsection 3.6.1, we summarize the remining tasks needed to conclude Theorem 3.6.1. In Subsection 3.6.2 we outline the strategy to prove these (highly nontrivial) tasks. In particular, we will explain how elliptic regularization, blow-up techniques and parabolic estimates are combined into the proofs. Finally, we prove these tasks in Subsection 3.6.3.

3.6.1 Setups, notations, and the approximating scheme

We make the following setups and constructions.

The signed distance $r(x)$ and regular radius r_I.

Fix Ω, E_0 as in Theorem 3.6.1. Let $r(x)$ be the signed distance function to $\partial\Omega$, taking negative values in Ω and positive values in $M \setminus \Omega$. For $\delta \in \mathbb{R}$, we set $\Omega_\delta = \big\{x \in M : r(x) < \delta\big\}$. Thus $\Omega_\delta \subset \Omega$ when $\delta < 0$, and $\Omega_\delta \supset \Omega$ when $\delta > 0$. Let r_I be a sufficiently small radius, so that the following holds:

(1) $d(\partial E_0, \partial\Omega) > 3r_I$,

(2) $r(x)$ is smooth in $\Omega_{3r_I} \setminus \overline{\Omega_{-3r_I}}$

Define the radial vector field $\partial_r := \nabla r$, which is smooth in the same region. We further decrease r_I such that the following holds:

(3) in $\Omega_{3r_I} \setminus \overline{\Omega_{-3r_I}}$ we have the small curvature condition

$$|\operatorname{Ric}| \leqslant \frac{1}{100n^2 r_I^2}, \qquad |\nabla\partial_r| \leqslant \frac{1}{100n^2 r_I}, \qquad |\nabla^2\partial_r| \leqslant \frac{1}{100n^2 r_I^2}, \tag{3.6.5}$$

(4) for all $x \in \Omega \setminus \Omega_{-r_I}$ and $r, s \leqslant r_I$, we have $\mathcal{H}^{n-1}\big(B(x,r) \cap \partial\Omega_{-s}\big) \leqslant 2|B^{n-1}|r^{n-1}$,

(5) for all $x \in \Omega \setminus \Omega_{-r_I}$ we have $\sigma(x; \Omega, g) \geqslant \frac{1}{2}|r(x)|$ (see Definition 2.4.2).

The weight functions ψ_δ.

We fix a function $\psi_0 : (-\infty, 1) \to \mathbb{R}_+$ such that:

(1) the conditions of Theorem 3.4.3 are satisfied (in particular, ψ_0 is strictly increasing and convex in $(0,1)$, and $\psi_0|_{[-\infty,0]} \equiv 0$, $\lim_{x\to 1}\psi_0(x) = +\infty$),

(2) $\psi_0(\frac{1}{2}) > \frac{1}{2}$, and $\psi_0'(x) > 1$ for all $x \in [\frac{1}{2}, 1)$, and $\psi_0(x) > \psi_0(\frac{1}{2}) + 3$ for all $x \in [\frac{3}{4}, 1)$.

(3) $\psi_0(x) \geqslant (n-1)\log\frac{1}{1-x}$ for all $x \in [\frac{1}{2}, 1)$.

For each $\delta \in (0, r_I)$, we define the (smooth) weight function

$$\psi_\delta(x) = \begin{cases} 0, & x \in \Omega, \\ \psi_0\big(\delta^{-1} r(x)\big), & x \in \Omega_\delta \setminus \Omega. \end{cases}$$

Note that $\lim_{x\to\partial\Omega_\delta}\psi_\delta(x) = +\infty$.

The approximating equations.

For $\delta > 0$, we consider the weighted IMCF (see Subsection 3.2.1)

$$\operatorname{div}\Big(e^{\psi_\delta}\frac{\nabla u}{|\nabla u|}\Big) = e^{\psi_\delta}|\nabla u|, \tag{3.6.6}$$

and for $\varepsilon > 0$ consider its elliptic regularization (see Remark 3.2.2):

$$\operatorname{div}\Big(e^{\psi_\delta}\frac{\nabla u}{\sqrt{\varepsilon^2 e^{2\psi_\delta/(n-1)} + |\nabla u|^2}}\Big) = e^{\psi_\delta}\sqrt{\varepsilon^2 e^{2\psi_\delta/(n-1)} + |\nabla u|^2}. \tag{3.6.7}$$

The following lemma ensures the presence of good enough barriers.

Lemma 3.6.2 (barrier functions)**.** *Suppose $\delta \in (0, r_I)$. Then the function*

$$\underline{u}_\delta(x) = \psi_\delta(x) - \psi_0\big(\frac{1}{2}\big) - \Big(\frac{r(x)}{\delta} - \frac{1}{2}\Big) \tag{3.6.8}$$

is negative in $\Omega_{\delta/2}\setminus\Omega$ and positive in $\Omega\setminus\overline{\Omega_{\delta/2}}$. Moreover, $\underline{u}_\delta$ is a strict subsolution of (3.6.6) with nonvanishing gradient in $\Omega_\delta\setminus\Omega_{\delta/2}$. The function

$$\overline{u}_\delta(x) = \psi_\delta(x) + \frac{r(x)}{r_I} \tag{3.6.9}$$

is a strict supersolution of (3.6.6) in the region $\Omega_\delta\setminus\Omega_{-r_I}$.

Proof. The claim on the sign of $\underline{u}_\delta$ follows from $\psi_0(1/2) > 1/2$ and the strict convexity of ψ_0. In $\Omega_\delta\setminus\Omega_{\delta/2}$ we have

$$|\nabla\underline{u}_\delta| = \Big|\frac{\partial\psi_\delta}{\partial r} - \frac{1}{\delta}\Big| = \frac{1}{\delta}\big|\psi_0'(r(x)) - 1\big| = \frac{1}{\delta}\Big(\psi_0'(\delta^{-1}r(x)) - 1\Big) = \frac{\partial\psi_\delta}{\partial r} - \frac{1}{\delta}.$$

The third equality is because $\psi_0' > 1$ on $[1/2, 1)$. The same facts imply that $|\nabla\underline{u}_\delta| \neq 0$ everywhere in $\Omega_\delta\setminus\Omega_{\delta/2}$. On the other hand, we calculate

$$\operatorname{div}\Big(e^{\psi_\delta}\frac{\nabla\underline{u}_\delta}{|\nabla\underline{u}_\delta|}\Big) = \operatorname{div}\big(e^{\psi_\delta}\partial_r\big) = e^{\psi_\delta}\Big(\frac{\partial\psi_\delta}{\partial r} + \operatorname{div}\partial_r\Big) > e^{\psi_\delta}\Big(\frac{\partial\psi_\delta}{\partial r} - \frac{1}{100r_I}\Big),$$

where the last inequality comes from (3.6.5). Hence $\underline{u}_\delta$ is a strict subsolution on $\Omega_\delta\setminus\Omega_{\delta/2}$.

Next, inside $\Omega_\delta\setminus\Omega_{-r_I}$ we calculate

$$\operatorname{div}\Big(e^{\psi_\delta}\frac{\nabla\overline{u}_\delta}{|\nabla\overline{u}_\delta|}\Big) \leqslant e^{\psi_\delta}\Big(\frac{\partial\psi_\delta}{\partial r} + \frac{1}{100r_I}\Big) < e^{\psi_\delta}\Big(\frac{\partial\psi_\delta}{\partial r} + \frac{1}{r_I}\Big) = |\nabla\overline{u}_\delta|,$$

confirming the supersolution property. □

Now the following lemma provides elliptic regularized solutions. Note that the system (3.6.10) ∼ (3.6.10) is nothing else but (2.4.9) ∼ (2.4.11). This is seen from Remark 3.2.2 and the fact that $\Omega_{\lambda\delta}$ is a sub-level set of $\underline{u}_\delta$.

Lemma 3.6.3 (approximating solutions)**.** *For each $\delta < r_I$ and $\lambda \in (3/4, 1)$, there exists $\varepsilon(\delta, \lambda) > 0$ such that the boundary value problem*

$$\begin{cases} \operatorname{div}\Big(e^{\psi_\delta} \dfrac{\nabla u_{\varepsilon,\delta,\lambda}}{\sqrt{\varepsilon^2 e^{2\psi_\delta/(n-1)} + |\nabla u_{\varepsilon,\delta,\lambda}|^2}}\Big) = e^{\psi_\delta}\sqrt{\varepsilon^2 e^{2\psi_\delta/(n-1)} + |\nabla u_{\varepsilon,\delta,\lambda}|^2} & (3.6.10) \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{in } \Omega_{\lambda\delta} \setminus \overline{E_0}, \\ u_{\varepsilon,\delta,\lambda} = 0 \qquad \textit{on } \partial E_0, & (3.6.11) \\ u_{\varepsilon,\delta,\lambda} = \underline{u}_\delta - 2 \qquad \textit{on } \partial\Omega_{\lambda\delta} & (3.6.12) \end{cases}$$

admits a solution $u_{\varepsilon,\delta,\lambda} \in C^\infty\big(\overline{\Omega_{\lambda\delta}} \setminus E_0\big)$ for all $0 < \varepsilon \leqslant \varepsilon(\delta, \lambda)$. We have the C^0 bounds

$$\max\big\{-\varepsilon, \psi_\delta(x) - C\big\} \leqslant u_{\varepsilon,\delta,\lambda}(x) \leqslant \psi_\delta(x) + C \qquad \forall x \in \Omega_{\lambda\delta} \setminus E_0, \tag{3.6.13}$$

for some constant $C > 0$ independent of $\varepsilon, \delta, \lambda$. In particular, $-\varepsilon \leqslant u_{\varepsilon,\delta,\lambda} \leqslant C$ in $\overline{\Omega} \setminus E_0$.

We also have the gradient estimate

$$|\nabla u_{\varepsilon,\delta,\lambda}(x)| \leqslant \sup_{\partial E_0 \cap B(x,r)} H_+ + 2\varepsilon + \frac{C(n)}{r} \tag{3.6.14}$$

for all $x \in \Omega \setminus E_0$ and $0 < r \leqslant \sigma(x; \Omega, g)$, where $H_+ = \max\big\{H_{\partial E_0}, 0\big\}$.

Proof. In Ω_δ we consider the conformally transformed metric $g' = e^{2\psi_\delta/(n-1)}g$. The fact $\psi_0 \geqslant (n-1)\log\frac{1}{1-x}$ in $[1/2, 1)$ ensures that g' is a complete metric in Ω_δ. Let $\underline{u}_\delta$ be as in (3.6.8). For convenience, we modify $\underline{u}_\delta$ inside Ω, so that it is smooth with negative value there (this does not affect any argument below). Then $\underline{u}_\delta$ is smooth and proper in Ω_δ, with $\{\underline{u}_\delta < 0\} = \Omega_{\delta/2}$. By Lemma 3.6.2 and Lemma 3.2.1(3), $\underline{u}_\delta$ is a smooth subsolution of IMCF in the region $\big(\Omega_\delta \setminus \overline{\Omega_{\delta/2}}, g'\big)$, with nonvanishing gradient there. Finally, note that $\Omega_{\lambda\delta}$ is a sub-level set of $\underline{u}_\delta$ (namely, $\Omega_{\lambda\delta} = \{\underline{u}_\delta < L\}$ for some $L > 2$). Thus we may invoke Theorem 2.4.6 on (Ω_δ, g') to obtain that: there is $\varepsilon(\delta, \lambda) > 0$ such that the regularized equation (2.4.9) ∼ (2.4.11) admits a solution for all $\varepsilon \leqslant \varepsilon(\delta, \lambda)$. But under the present settings, the regularized equation is exactly (3.6.10) ∼ (3.6.12), by Remark 3.2.2. This shows the existence of the solution $u_{\varepsilon,\delta,\lambda}$.

To obtain (3.6.14), we note that the gradient estimate in Theorem 2.4.6 states that

$$\big|\nabla_{g'} u_{\varepsilon,\delta,\lambda}(x)\big| \leqslant \max\Big\{\sup_{B_{g'}(x,r)\cap\partial E_0} H_+, \sup_{B_{g'}(x,r)\cap\partial\Omega_{\lambda\delta}} |\nabla_{g'} u_{\varepsilon,\delta,\lambda}|_{g'}\Big\} + 2\varepsilon + \frac{C(n)}{r}$$

for all $x \in \overline{\Omega_{\lambda\delta}} \setminus E_0$ and $r \leqslant \sigma(x; \Omega_\delta, g')$. Since $g' = g$ inside Ω, and $\sigma(x; \Omega_\delta, g') \geqslant \sigma(x; \Omega, g)$ for all $x \in \Omega$, and $B_{g'}(x,r) \cap \partial\Omega_{\lambda\delta} = \emptyset$ when $r \leqslant \sigma(x; \Omega, g)$, this gradient estimate directly implies (3.6.14). Next, the lower bound in (3.6.13) follows from (2.4.12) and (3.6.8). The upper bound is derived as follows: by (3.6.14) we have

$$\sup\big\{u_{\varepsilon,\delta,\lambda}(x) : x \in \overline{\Omega_{-r_I}}\big\} \leqslant C, \tag{3.6.15}$$

where $C > 0$ is independent of $\varepsilon, \delta, \lambda$. We compare $u_{\varepsilon,\delta,\lambda}$ with $\overline{u}_\delta + C + 1$ inside $\overline{\Omega_{\lambda\delta}} \setminus \Omega_{-r_I}$, where $\overline{u}_\delta$ is as in (3.6.9). On $\partial\Omega_{-r_I}$ we have

$$u_{\varepsilon,\delta,\lambda} \leqslant C = \overline{u}_\delta + C + 1.$$

On $\partial\Omega_{\lambda\delta}$ we have

$$u_{\varepsilon,\delta,\lambda} = \underline{u}_\delta - 2 < \overline{u}_\delta < \overline{u}_\delta + C + 1.$$

Since $\overline{u}_\delta$ is a strict supersolution of (3.6.6) in the compact set $\overline{\Omega_{\lambda\delta}} \setminus \Omega_{-r_I}$, by continuity, we may further decrease $\varepsilon(\delta,\lambda)$ so that $\overline{u}_\delta$ is a strict supersolution of (3.6.10) in the same region. By the maximum principle, we obtain $u_{\varepsilon,\delta,\lambda} \leqslant \overline{u}_\delta + C + 1$ in $\overline{\Omega_{\lambda\delta}} \setminus \Omega_{-r_I}$. This implies the desired upper bound along with (3.6.15). $\square$

Convergence to an interior solution.

Lemma 3.6.4. *For any sequences $\delta_i \to 0$, $\lambda_i \to 1$, $\varepsilon_i \to 0$ with $\varepsilon_i \leqslant \varepsilon(\delta_i,\lambda_i)$, there exists a sequence of solutions $u_i := u_{\varepsilon_i,\delta_i,\lambda_i}$ of (3.6.10) $\sim$ (3.6.12). Moreover, a subsequence of u_i converges in $C^0_{\mathrm{loc}}(\Omega\backslash E_0)$ to a function $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega\backslash E_0)$ as $i \to \infty$, and u solves* $\mathsf{IVP}(\Omega; E_0)$ *and is calibrated by some vector field ν in $\Omega \setminus \overline{E_0}$.*

Proof. The approximate solutions u_i are directly given by Lemma 3.6.3. With an application of Theorem 2.4.7 to the data $\Omega_i = \Omega \setminus \overline{E_0}$, $g_i = g$ (note that $\psi_{\delta_i} \equiv 0$ in Ω, so the weighted equation is reduced to the ordinary IMCF), we find a subsequence of u_i that converges in $C^0_{\mathrm{loc}}(\Omega \setminus \overline{E_0})$ to a calibrated solution $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega \setminus \overline{E_0})$. Additionally, due to (3.6.14), the functions u_i are uniformly Lipschitz up to ∂E_0. Therefore, the uniform convergence and Lipschitz regularity of u holds up to ∂E_0. Then by (3.6.13) (3.6.11), we have $u \geqslant 0$ in $\Omega \setminus E_0$ and $u|_{\partial E_0} = 0$. Extending u with negative values in E_0, we obtain a solution of $\mathsf{IVP}(\Omega; E_0)$. $\square$

The remaining tasks.

Take any sequence $\delta_i \to 0$, $\lambda_i \to 1$ and $\varepsilon_i \leqslant \varepsilon(\delta_i,\lambda_i)$ with $\varepsilon_i \to 0$. Let u be the limit solution given by Lemma 3.6.4, and ν be the corresponding calibration. Note that ν is not an arbitrary calibration, but one that comes from the use of Theorem 2.4.7 (later we will make use of the statements in Theorem 2.4.7). We would prove Theorem 3.6.1 if we show that u respects the boundary obstacle and satisfies conditions (i) $\sim$ (iv) there.

We note that since Theorem 3.6.1 contains a unique statement, it in turn implies that the limit function u in Lemma 3.6.4 is independent of the choice of $\delta_i, \lambda_i, \varepsilon_i$.

For the clarity of proofs, we split the remaining part of Theorem 3.6.1 as follows:

Proposition 3.6.5. *Let u, ν be the solution and calibration obtained in Lemma 3.6.4. Then there is a radius $r_0 \in (0, r_I)$ and dimensional constants $\gamma = \gamma(n) \in (0,1)$, $C = C(n) > 0$, such that*

$$1 - \langle \nu, \partial_r \rangle \leqslant C r_0^{-\gamma} |r|^\gamma \tag{3.6.16}$$

and

$$|\nabla u| \leqslant C r_0^{-\gamma} |r|^{\gamma-1} \tag{3.6.17}$$

hold inside $\Omega \setminus \overline{\Omega_{-r_0}}$. In particular, u solves $\mathsf{IMCF}(\Omega \setminus \overline{E_0})$+$\mathsf{OBS}(\partial\Omega)$.

Proposition 3.6.6. *u can be extend continuously to $\partial\Omega$, and we have $u \in C^{0,\gamma}(\overline{\Omega}\backslash\Omega_{-r_0/2})$.*

Proposition 3.6.7. *There exists a sufficiently small radius $r_1 \in (0, r_0)$ such that: for all $t > 0$, the set $\partial E_t \setminus \overline{\Omega_{-r_1}}$ is a $C^{1,\gamma/2}$ hypersurface.*

Proof of Theorem 3.6.1 assuming these propositions.

It is stated in Proposition 3.6.5 that u respects the obstacle $\partial\Omega$. The condition $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega) \cap BV(\Omega) \cap C^{0,\gamma}(\overline{\Omega})$ follows by interior regularity and Lemma 2.2.2 and Poposition 3.6.6. The gradient bound (3.6.1) follows by taking limit of (3.6.14). The boundary regularity (3.6.2) follows from (3.6.17) inside $\Omega \setminus \Omega_{-r_0}$, and follows from the interior regularity inside Ω_{-r_0}. The bound on calibration follows from (3.6.16). Finally, the regularity of level sets follows from Proposition 3.6.7, and the maximality follows from Corollary 3.3.20. □

3.6.2 The spacetime foliation; strategies of the proof

The approximation process described above, which essentially invokes the elliptic regularization in Theorem 2.4.7, provides an additional set of data. For each i, the family of hypersurfaces

$$\Sigma_t^i := \text{graph}\left(\varepsilon_i^{-1}(u_i - t)\right)$$

form a downward translating soliton of the IMCF in the product domain

$$\left(\Omega_{\lambda_i\delta_i} \times \mathbb{R},\ \exp(\frac{2\psi_{\delta_i}}{n-1})g + dz^2\right),$$

due to Remark 3.2.2 and the geometric meaning of elliptic regularization. Equivalently, the function $U_i(x,z) = u_i(x) - \varepsilon_i z$ solves the smooth IMCF

$$\operatorname{div}\left(\frac{\nabla_{\boldsymbol{g}} U_i}{|\nabla_{\boldsymbol{g}} U_i|}\right) = |\nabla_{\boldsymbol{g}} U_i|$$

in the same region. The proof of Proposition 3.6.5 ∼ 3.6.7 is done by obtaining the corresponding estimates on Σ_t^i, and then passing to the limit.

To distinguish from objects on M, we will use bold symbols to denote objects on $M \times \mathbb{R}$. We denote by $\boldsymbol{r}(x,z) = r(x)$ the signed distance to $\partial\Omega \times \mathbb{R}$. Note that $\boldsymbol{r}$ is smooth in $(\Omega_{3r_I} \setminus \overline{\Omega_{-3r_I}}) \times \mathbb{R}$. Then denote the product metric $\boldsymbol{g} = g + dz^2$ and radial vector field $\boldsymbol{\partial_r} = \nabla_{\boldsymbol{g}} \boldsymbol{r}$. In the region $(\overline{\Omega} \setminus \overline{E_0}) \times \mathbb{R}$, let

$$\boldsymbol{\nu}_i := \frac{\nabla_{\boldsymbol{g}} U_i}{|\nabla_{\boldsymbol{g}} U_i|_{\boldsymbol{g}}} = \frac{\nabla u_i - \varepsilon_i \partial_z}{\sqrt{\varepsilon_i + |\nabla u_i|^2}} \tag{3.6.18}$$

be the downward unit normal vector field of the foliation Σ_t^i (we remind that $\psi_{\delta_i} = 0$ in $\overline{\Omega}$, so the weighted IMCF is reduced to the usual one). According to Theorem 2.4.7, the vector fields $\boldsymbol{\nu}_i$ converge to some vector field $\boldsymbol{\nu}$ weakly in $L^1_{\text{loc}}(\Omega \setminus \overline{E_0})$, and the projection of $\boldsymbol{\nu}$ onto the Ω factor is the calibration ν given in Lemma 3.6.4.

Propositions 3.6.5 ∼ 3.6.7 are proved via the following steps:

1. We show that when $i \to \infty$, we have $\inf_{\partial\Omega\times\mathbb{R}} \langle \boldsymbol{\nu}_i, \boldsymbol{\partial_r} \rangle_{\boldsymbol{g}} \to 1$ (Lemma 3.6.8). The proof uses a blow-up argument: if there is a sequence of exceptional points $x_i \in \partial\Omega \times \mathbb{R}$, then we may rescale the solutions u_i near x_i and obtain an exceptional blow-up limit. The limit function will be a weak solution of

$$\operatorname{div}\left(e^{\psi_0(x_n)} \frac{\nabla u}{|\nabla u|}\right) = e^{\psi_0(x_n)} |\nabla u|.$$

However, Theorem 3.4.3 implies that this limit must have the form $u = \psi(x_n) - C$. Then, standard geometric measure theory is used to show that the space-time graphs Σ_t^i converges to a hyperplane at x_i in C^1 sense, contradicting our hypotheses for bad points.

2. Let r_i be the largest radius so that $\langle \boldsymbol{\nu}_i, \boldsymbol{\partial_r} \rangle_{\boldsymbol{g}} \geqslant 1/2$ holds inside $(\Omega \setminus \Omega_{-r_i}) \times \mathbb{R}$. We show that r_i is uniformly bounded below (Lemma 3.6.9). If this is false, then we find a subsequence of radii $r_i \to 0$ and exceptional points $x_i \in \partial\Omega_{-r_i} \times \mathbb{R}$, with $\langle \boldsymbol{\nu}_i, \boldsymbol{\partial_r} \rangle_{\boldsymbol{g}}(x_i, 0) =$

1/2. Using the result of step 1 and the parabolic estimate in Lemma 3.5.4, we will obtain the bounds

$$1-\langle \boldsymbol{\nu}_i,\boldsymbol{\partial_r}\rangle_{\boldsymbol{g}}\leqslant C\big(o(1)+|r|/r_i\big)^{\gamma} \tag{3.6.19}$$

for some uniform constants C,γ. Then consider a blow-up sequence centered at x_i. The limit is an exceptional weak solution u' on $\{x_n<0\}\subset\mathbb{R}^{n+1}$. The bound (3.6.19) passes to the limit and implies that u' respects the obstacle $\{x_n=0\}$, due to Corollary 3.3.18. This eventually contradicts Theorem 3.4.1, by looking at the convergence of the space-time graphs Σ_t^i (this is more involved compared to the previous step).

3. Combining the above two steps, we eventually obtain (3.6.16) through another application of parabolic estimate (Lemma 3.5.4). The boundary gradient estimate (3.6.17) and Hölder regularity follows by applying Lemma 3.5.5. Finally, Proposition 3.6.7 follows by an ε-regularity theorem for almost perimeter-minimizers (in the sense of (3.6.42) below).

3.6.3 Proof of Propositions 3.6.5 ∼ 3.6.7

We assume all the setups and notations made in subsection 3.6.1 and 3.6.2.

Lemma 3.6.8. *For any $\theta<1$, there exists $i_0\in\mathbb{N}$ such that for all $i\geqslant i_0$ we have*

$$\inf_{\partial\Omega\times\mathbb{R}}\langle\boldsymbol{\nu}_i,\boldsymbol{\partial_r}\rangle_{\boldsymbol{g}}\geqslant\theta.$$

Proof. Suppose that the conclusion does not hold. Then passing to a subsequence (which we do not relabel), we can find a constant $\theta_0<1$ and a sequence of points $q_i=(x_i,z_i)\in\partial\Omega\times\mathbb{R}$, such that $\langle\boldsymbol{\nu}_i,\boldsymbol{\partial_r}\rangle_{\boldsymbol{g}}(x_i,z_i)\leqslant\theta_0$. Since $\boldsymbol{\nu}_i$ is invariant under vertical translation, we may assume $z_i=0$. Passing to a further subsequence, we may assume $x_i\to x_0\in\partial\Omega$.

Consider the domains

$$\Omega_i:=B_g(x_i,\sqrt{\delta_i})\cap\Omega_{\lambda_i\delta_i},\qquad \boldsymbol{\Omega}_i:=\Omega_i\times(-1,1)$$

with the rescaled and conformally transformed metrics

$$h_i:=\delta_i^{-2}e^{2\psi_{\delta_i}/(n-1)}g,\qquad \boldsymbol{h}_i:=h_i+\delta_i^{-2}dz^2.$$

Set the normalized function $u_i'(x):=u_i(x)-u_i(x_i)$. By the equation (3.6.10), Remark 3.2.2 and the scaling invariance of IMCF, the functions $U_i'(x,z):=u_i'(x)-\varepsilon_i z$ are smooth solutions of the IMCF in $(\boldsymbol{\Omega}_i,\boldsymbol{h}_i)$.

To properly state the blow-up process, we define some suitable coordinate maps. In each tangent space $T_{x_i}M$, fix an orthonormal frame $\{e_i\}$ such that $e_1,\cdots,e_{n-1}$ are tangential to $\partial\Omega$, and $e_n=\partial_r$. With respect to this frame, the g-exponential map at x_i (denoted by $\exp_{x_i}$) is a diffeomorphism from a small ball in $\mathbb{R}^n$ to its image. Define the maps

$$\Phi_i:x\mapsto\exp_{x_i}(\delta_i x),\qquad \boldsymbol{\Phi}_i:(x,z)\mapsto\big(\exp_{x_i}(\delta_i x),\delta_i z\big),$$

and set the preimages[1] $\widetilde{\Omega}_i:=\Phi_i^{-1}(\Omega_i)\subset\mathbb{R}^n$, $\widetilde{\boldsymbol{\Omega}}_i:=\boldsymbol{\Phi}_i^{-1}(\boldsymbol{\Omega}_i)=\widetilde{\Omega}_i\times(-\delta_i^{-1},\delta_i^{-1})$. Also, set the pullbacks $\widetilde{h}_i:=\Phi_i^*h_i$, $\widetilde{\boldsymbol{h}}_i:=\boldsymbol{\Phi}_i^*\boldsymbol{h}_i=\widetilde{h}_i+dz^2$, and $\widetilde{u}_i:=u_i'\circ\Phi_i$, $\widetilde{U}_i':=U_i'\circ\boldsymbol{\Phi}_i$. Note that $\widetilde{U}_i'(0)=U_i'(x_i,0)=0$ and $\widetilde{U}_i(x,z)=\widetilde{u}_i'(x)-\varepsilon_i\delta_i z$. By diffeomorphism invariance,

[1] The rule for setting up the notations: boldface → objects in $M\times\mathbb{R}$; tilde → objects in the pulled-back spaces; prime → normalized functions (so that the value at x_i is zero).

$\widetilde{U}_i'$ are smooth solutions of IMCF in $(\widetilde{\boldsymbol{\Omega}}_i, \widetilde{\boldsymbol{h}}_i)$. Therefore, $\widetilde{u}_i$ solves the $(\varepsilon_i\delta_i)$-regularized equation

$$\operatorname{div}\left(\frac{\nabla_{\widetilde{h}_i}\widetilde{u}_i}{\left(\varepsilon_i^2\delta_i^2+|\nabla_{\widetilde{h}_i}\widetilde{u}_i|^2\right)^{1/2}}\right)=\sqrt{\varepsilon_i^2\delta_i^2+|\nabla_{\widetilde{h}_i}\widetilde{u}_i|^2}\qquad \text{in } (\widetilde{\Omega}_i,\widetilde{h}_i).$$

Next, we take limit of the rescaled objects. Note that $\widetilde{\Omega}_i \to \{x_n < 1\} \subset \mathbb{R}^n$ locally, since $\delta_i \to 0$ and $\lambda_i \to 1$. Also, $\widetilde{h}_i \to \widetilde{h} := e^{2\psi_0(x_n)/(n-1)}(dx_1^2+\cdots+dx_n^2)$ locally smoothly. Apply Theorem 2.4.7 with the data $\widetilde{\Omega}_i$, $\widetilde{h}_i$ and $\widetilde{u}_i$. As a result, there is a subsequence such that $\widetilde{u}_i'$ converges in C^0_{loc} to a function $\widetilde{u}' \in \mathrm{Lip}_{\mathrm{loc}}(\{x_n<1\})$, and $\widetilde{u}'$ solves $\mathsf{IMCF}(\{x_n<1\},\widetilde{h})$. By Lemma 3.2.1(3), $\widetilde{u}'$ is a weak solution of the weighted IMCF

$$\operatorname{div}\left(e^{\psi_0(x_n)}\frac{\nabla_0\widetilde{u}'}{|\nabla_0\widetilde{u}'|}\right)=e^{\psi_0(x_n)}|\nabla_0\widetilde{u}'|,$$

where ∇_0 denotes the Euclidean gradient. The gradient estimate (2.4.14) implies

$$|\nabla_0\widetilde{u}'(x)|=|\nabla_{\widetilde{h}}\widetilde{u}'(x)|\leqslant\frac{C(n)}{\sigma\big(x;\{x_n<1\},\widetilde{h}\big)}\leqslant\frac{C(n)}{|x_n|},\qquad \forall\, x\in\{x_n<0\}.$$

Furthermore, the C^0 bounds in Theorem 3.6.3 gives (recall $u_i'=u_i-u_i(x_i)$)

$$u_i'\geqslant\psi_{\delta_i}-2C\quad\Rightarrow\quad \widetilde{u}_i'(x)\geqslant\psi_{\delta_i}\big(\exp_{x_i}(\delta_i x)\big)-2C=\psi_0(x_n)+o(1)-2C.$$

This passes to the limit and gives the bound $\widetilde{u}'(x)\geqslant\psi_0(x_n)-2C$. Now all the assumptions of Theorem 3.4.3 are met, and we obtain $\widetilde{u}'(x)=\psi_0(x_n)-C'$ for some other C'.

Set $\widetilde{U}'(x,z):=\widetilde{u}'(x)=\psi_0(x_n)-C'$, which is clearly the C^0_{loc} limit of $\widetilde{U}_i'$. The sub-level sets $E_0(\widetilde{U}_i')$ have locally uniformly bounded perimeter, by Lemma 2.2.2(i), Thus passing to a further sequence, we may assume $E_0(\widetilde{U}_i')\to E$ in L^1_{loc}, for some set $E\subset\big\{(x,z):x_n<1\big\}$. Since $\widetilde{U}_i'\to\widetilde{U}'$ in C^0_{loc}, we have

$$E_0(\widetilde{U}')\subset E\subset E_0^+(\widetilde{U}')\tag{3.6.20}$$

up to zero measure. Next, by Corollary 2.4.4 and the nice convergence of $\widetilde{\boldsymbol{h}}_i$, the gradients $|\nabla_{\widetilde{\boldsymbol{h}}_i}\widetilde{U}_i'|$ are uniformly bounded in $B_{\mathbb{R}^{n+1}}(0,1/2)$, for all large i. Then by (2.1.5), $E_0(\widetilde{U}_i')$ are uniform almost perimeter minimizers in $\big(B_0(0,1/2),\widetilde{\boldsymbol{h}}_i\big)$. By Theorem A.2.2(ii) and the fact $0\in\partial E_0(\widetilde{U}_i')$, we have $0\in\operatorname{spt}|\mu_E|$. Combined with (3.6.20) and the expression $\widetilde{U}'=\psi(x_n)-C$, we see that there are only two possibilities:

(1) $C'>0$ and $E=\{x_n<0\}$, or

(2) $C'=0$, $\widetilde{U}'(x,z)=\psi_0(x_n)$, and $E\subset\{x_n<0\}$.

Suppose the second case holds. By Theorem 2.1.13, the limit set E locally minimizes the energy $J_{\widetilde{U}'}$ in $\big\{(x,z):x_n<1\big\}$. As $\widetilde{U}'\equiv 0$ in $\big\{(x,z):x_n<0\big\}$, this implies that E is locally inward-minimizing. At the same time, this also implies that E is locally outward minimizing in $\{(x,z):x_n\leqslant 0\}$. By direct verification, this implies that E is locally outward perimeter-minimizing in $\mathbb{R}^{n+1}$. As a result, E is locally perimeter-minimizing in $\mathbb{R}^{n+1}$. By Lemma A.6.4 and the facts $E\subset\{x_n<0\}$, $0\in\operatorname{spt}(|\mu_E|)$, we have $E=\{x_n<0\}$.

So in either case we conclude that $E=\{x_n<0\}$. Then Theorem A.2.2(iii) implies

$$\nu_{E_0(\widetilde{U}_i')}(0)\to\nu_E(0)=\partial_{x_n},$$

where $\nu_{E_0(\widetilde{U}_i')}$ is the outer unit normal of $E_0(\widetilde{U}_i')$ with respect to $\widetilde{\boldsymbol{h}}_i$. On the other hand, we may evaluate by unraveling the pullbacks

$$\big\langle \nu_{E_0(\widetilde{U}_i')}, \partial_{x_n} \big\rangle_{\widetilde{\boldsymbol{h}}_i}(0) = \langle \boldsymbol{\nu}_i, \boldsymbol{\partial_r} \rangle_{\boldsymbol{g}}(x_i, 0).$$

This contradicts our initial assumption $\langle \boldsymbol{\nu}_i, \boldsymbol{\partial_r} \rangle_{\boldsymbol{g}}(x_i, 0) \leqslant \theta_0$, thus proves the lemma. □

Lemma 3.6.9. *There exists $r_0 < r_I$ and $i_0 \in \mathbb{N}$, such that for all $i \geqslant i_0$ we have*

$$\inf_{(\Omega \setminus \Omega_{-r_0}) \times \mathbb{R}} \langle \boldsymbol{\nu}_i, \boldsymbol{\partial_r} \rangle_{\boldsymbol{g}} \geqslant \frac{1}{2}.$$

Proof. By Lemma 3.6.8, we have $\inf_{\partial\Omega \times \mathbb{R}} \langle \boldsymbol{\nu}_i, \boldsymbol{\partial_r} \rangle_{\boldsymbol{g}} \geqslant 3/4$ for all sufficiently large i. Suppose that the lemma is false. Then passing to a subsequence, we may find radii $r_i \in (0, r_I)$, $r_i \to 0$, and a sequence of points $q_i = (x_i, z_i) \in \partial\Omega_{-r_i} \times \mathbb{R}$, such that

$$\langle \boldsymbol{\nu}_i, \boldsymbol{\partial_r} \rangle_{\boldsymbol{g}}(x_i, z_i) \leqslant \frac{1}{2}. \tag{3.6.21}$$

By translation invariance, we may assume $z_i = 0$. By re-selecting each x_i to have the smallest distance to $\partial\Omega$ (and decreasing r_i correspondingly), we can further assume that

$$\langle \boldsymbol{\nu}_i, \boldsymbol{\partial_r} \rangle_{\boldsymbol{g}} \geqslant \frac{1}{2} \quad \text{inside} \quad (\Omega \setminus \Omega_{-r_i}) \times \mathbb{R}. \tag{3.6.22}$$

Finally, passing to a further subsequence, we may assume $x_i \to x_0 \in \partial\Omega$. By Lemma 3.6.8, there exist numbers $\theta_i \to 1$ such that

$$\inf_{\partial\Omega \times \mathbb{R}} \langle \boldsymbol{\nu}_i, \boldsymbol{\partial_r} \rangle_{\boldsymbol{g}} \geqslant \theta_i. \tag{3.6.23}$$

Step 1. We establish a uniform bound on $\boldsymbol{\nu}_i$ by parabolic estimates.

Recall that $\Sigma_t^i = \operatorname{graph}\big(\varepsilon_i^{-1}(u_i - t)\big)$ forms a downward translating soliton of the usual IMCF in $(\overline{\Omega} \setminus \overline{E_0}) \times \mathbb{R}$. We apply Lemma 3.5.4 to the flow Σ_t^i, with Ω replaced by $\Omega \times \mathbb{R}$ and with the choice $r_0 = r_i$. The small curvature condition (3.5.11) is implied by (3.6.5), and the condition $p \geqslant \frac{1}{2}$ there comes from (3.6.22). Choosing the parameter $b_i = (1 - \theta_i)^{1/\gamma}$ there, it follows that the quantity $F = \big(1 - \langle \boldsymbol{\nu}_i, \boldsymbol{\partial_r} \rangle_{\boldsymbol{g}}\big)(b_i - r/r_i)^{-\gamma}$ satisfies

$$\big(1 - \langle \boldsymbol{\nu}_i, \boldsymbol{\partial_r} \rangle_{\boldsymbol{g}}\big)^{\frac{2}{\gamma} - 1} H^2 \square F + \langle \nabla_\Sigma F, X \rangle \leqslant \frac{C_1}{r_i^2}\big(- F^{\frac{\gamma+2}{\gamma}} + C_2\big)$$

inside $(\Omega \setminus \overline{\Omega_{-r_i}}) \times \mathbb{R}$, where $\square = \partial_t - H^{-2}\Delta_\Sigma$ is the heat operator associated to the IMCF, and the subscripts Σ are shorthands for Σ_t^i. Since Σ_t^i evolves as translating soliton in the z-direction, we have $\partial_t F = \langle \nabla_\Sigma F, Y \rangle$ for some smooth vector field Y. Hence

$$-\big(1 - \langle \boldsymbol{\nu}_i, \boldsymbol{\partial_r} \rangle_{\boldsymbol{g}}\big)^{\frac{2}{\gamma} - 1} \Delta_\Sigma F + \langle \nabla_\Sigma F, Z \rangle \leqslant \frac{C_1}{r_i^2}\big(- F^{\frac{\gamma+2}{\gamma}} + C_2\big),$$

for some $\gamma \in (0, 1/2)$ and $C_1, C_2 > 0$ depending only on n. Since Σ_t^i is graphical, the intersection $\Sigma_t^i \cap \big((\overline{\Omega} \setminus \Omega_{-r_I}) \times \mathbb{R}\big)$ is compact. By the maximum principle and invariance in the z-direction, we obtain

$$\begin{aligned} \max_{(\overline{\Omega} \setminus \Omega_{-r_i}) \times \mathbb{R}} (F) &\leqslant \max\Big\{ \max_{\partial\Omega \times \mathbb{R}} (F),\ \max_{\partial\Omega_{-r_i} \times \mathbb{R}} (F),\ C_2^{\frac{\gamma}{\gamma+2}} \Big\} \\ &\leqslant \max\Big\{ 1,\ \frac{1}{2},\ C_2^{\frac{\gamma}{\gamma+2}} \Big\}. \end{aligned}$$

From this we have

$$1 - \langle \boldsymbol{\nu}_i, \boldsymbol{\partial_r} \rangle_{\boldsymbol{g}} \leqslant C_3 (b_i - \boldsymbol{r}/r_i)^\gamma \qquad \text{in} \ \ (\overline{\Omega} \setminus \Omega_{-r_i}) \times \mathbb{R}. \tag{3.6.24}$$

Step 2. We establish an inward minimizing property of the level sets of U_i, where recall $U_i(x,z) = u_i(x) - \varepsilon_i z$. Denote $\boldsymbol{\Omega} = \Omega \times \mathbb{R}$. Fix $t \in \mathbb{R}$, and consider $E = E_t(U_i) \cap \boldsymbol{\Omega}$. Suppose $F \Subset K \Subset (M \setminus \overline{E_0}) \times \mathbb{R}$. Using the boundary condition (3.6.23), the fact $\nu_E = \boldsymbol{\nu}_i$ in $\boldsymbol{\Omega}$, and the smooth IMCF equation $\operatorname{div}(\boldsymbol{\nu}_i) = |\nabla_{\boldsymbol{g}} U_i|$, we evaluate by the divergence theorem as follows. In the expressions, we write $|\cdot| = \mathcal{H}^{n-1}(\cdot)$.

$$\begin{aligned}
&P\big(E; \boldsymbol{\Omega} \cap K\big) - P\big(E \backslash F; \boldsymbol{\Omega} \cap K\big) + \theta_i \Big(\big|\partial^* E \cap \partial \boldsymbol{\Omega} \cap K\big| - \big|\partial^* (E \backslash F) \cap \partial \boldsymbol{\Omega} \cap K\big| \Big) \\
&\leqslant \int_{\partial^* E \cap \boldsymbol{\Omega} \cap K} \langle \nu_E, \boldsymbol{\nu}_i \rangle_{\boldsymbol{g}} - \int_{\partial^* (E \backslash F) \cap \boldsymbol{\Omega} \cap K} \langle \nu_{E \backslash F}, \boldsymbol{\nu}_i \rangle_{\boldsymbol{g}} + \int_{\partial \boldsymbol{\Omega} \cap K} \big(\chi_{\partial^* E} - \chi_{\partial^* (E \backslash F)} \big) \langle \nu_{\boldsymbol{\Omega}}, \boldsymbol{\nu}_i \rangle_{\boldsymbol{g}} \\
&= \int_{E \cap F} |\nabla_{\boldsymbol{g}} U_i|_{\boldsymbol{g}}.
\end{aligned}$$

Next, by the coarea formula we have

$$\int_{E \cap F} |\nabla_{\boldsymbol{g}} U_i|_{\boldsymbol{g}} = \int_{\inf_{F \cap \boldsymbol{\Omega}}(U_i)}^t \big| \partial^* E_s(U_i) \cap \boldsymbol{\Omega} \cap F \big| \, ds.$$

By a perimeter decomposition and Gronwall argument, as in Lemma 2.2.1, these imply

$$\begin{aligned}
P\big(E; \boldsymbol{\Omega} \cap K\big) + \theta_i \big|\partial^* E \cap \partial \boldsymbol{\Omega} \cap K\big| \leqslant P\big(E \backslash F; \boldsymbol{\Omega} \cap K\big) + \theta_i \big|\partial^* (E \backslash F) \cap \partial \boldsymbol{\Omega} \cap K\big| \\
+ \int_{\inf_{F \cap \boldsymbol{\Omega}}(U_i)}^t e^{t-s} P\big(F; E_s(U_i) \cap \boldsymbol{\Omega}\big) \, ds.
\end{aligned}$$

Notice that $P(E;K) = P(E; \boldsymbol{\Omega} \cap K) + |\partial^* E \cap \partial \boldsymbol{\Omega} \cap K|$, since $E \subset \boldsymbol{\Omega}$. So the above inequality clearly implies (where we have plugged in the definition of E):

$$\theta_i P\big(E_t(U_i) \cap \boldsymbol{\Omega}; K\big) \leqslant P\big((E_t(U_i) \cap \boldsymbol{\Omega}) \setminus F; K\big) + \big(e^{t - \inf_{F \cap \boldsymbol{\Omega}}(U_i)} - 1\big) P(F). \tag{3.6.25}$$

We remark that, by the spirit of Theorem 3.3.8, outward-minimizing properties are sort of automatic. On the other hand, the inward-minimizing property is closely related to the boundary condition. This explains why (3.6.23) is invoked in this step.

Step 3. We are ready for the blow-up argument. Let $y_i \in \partial\Omega$ be the unique point with the smallest distance from x_i, thus $d(x_i, y_i) = r_i$. Consider the domains

$$\Omega_i := B_g(y_i, \sqrt{r_i}) \cap \Omega, \qquad \boldsymbol{\Omega}_i := \Omega_i \times (-1, 1),$$

with the metrics

$$h_i := r_i^{-2} g, \qquad \boldsymbol{h}_i := r_i^{-2} \big(g + dz^2\big).$$

Define $u_i'(x) := u(x) - u(x_i)$, $U_i'(x,z) := u_i'(x) - \varepsilon z$. Near y_i we set up a geodesic normal coordinate in the same way as in Lemma 3.6.8. Let $\exp_{y_i}$ be the induced exponential map; thus we have $\exp_{y_i}(-r_i e_n) = x_i$. Consider the scaled coordinate maps

$$\Phi_i : x \mapsto \exp_{y_i}(r_i x), \qquad \boldsymbol{\Phi}_i : (x,z) \mapsto (\exp_{y_i}(r_i x), r_i z).$$

Denote $\widetilde{\Omega}_i := \Phi_i^{-1}(\Omega_i) \subset \mathbb{R}^n$ and $\widetilde{\boldsymbol{\Omega}}_i := \boldsymbol{\Phi}_i^{-1}(\boldsymbol{\Omega}_i) = \Omega_i \times (-r_i^{-1}, r_i^{-1}) \subset \mathbb{R}^{n+1}$, equipped with the metrics and functions

$$\begin{aligned} \widetilde{h}_i &= \Phi_i^* h_i, & \widetilde{\boldsymbol{h}}_i &:= \boldsymbol{\Phi}_i^* \boldsymbol{h}_i = \widetilde{h}_i + dz^2, \\ \widetilde{u}_i' &:= u_i' \circ \Phi_i, & \widetilde{U}_i' &:= U_i' \circ \boldsymbol{\Phi}_i. \end{aligned}$$

So $\widetilde{\Omega}_i$ is the intersection of $B_{\mathbb{R}^n}(0, 1/\sqrt{r_i})$ with a set that locally approximates $\{x_n < 0\}$ as $i \to \infty$. Also, notice that $\Phi_i^{-1}(\partial\Omega_{-r_i})$ locally approaches $\{x_n = -1\}$ when $i \to \infty$. Let us view $\widetilde{u}_i'$, $\widetilde{U}_i'$ as functions defined only on $\widetilde{\Omega}_i$, $\widetilde{\boldsymbol{\Omega}}_i$ (thus we are discarding the values outside these domains).

As usual, denote $e_n = (0, \cdots, 0, 1) \in \mathbb{R}^n$. Further denote $\boldsymbol{e}_n = (e_n, 0) \in \mathbb{R}^{n+1}$. Note the following facts: $x_i = \Phi_i(-e_n)$, and $\widetilde{U}_i'(x, z) = \widetilde{u}_i'(x) - \varepsilon_i r_i z$, so $\widetilde{U}_i'(-\boldsymbol{e}_n) = 0$. Also, $\widetilde{U}_i'$ solves the smooth IMCF in $(\widetilde{\boldsymbol{\Omega}}_i, \widetilde{\boldsymbol{h}}_i)$, by scaling and diffeomorphism invariance.

We have $\widetilde{\boldsymbol{\Omega}}_i \to \{(x, z) : x_n < 0\} \subset \mathbb{R}^{n+1}$ and $\widetilde{\boldsymbol{h}}_i \to \mathbf{euc} := dx_1^2 + \cdots + dx_{n+1}^2$ locally smoothly. By Corollary 2.4.4, we have the gradient estimate

$$\big|\nabla_{\widetilde{\boldsymbol{h}}_i} \widetilde{U}_i'\big|_{\widetilde{\boldsymbol{h}}_i}(x, z) \leqslant \frac{C(n)}{|x_n|} + o(1), \tag{3.6.26}$$

where the $o(1)$ term is uniform in all compact sets and goes to zero as $i \to \infty$.

By Theorem 2.4.7 (applied with the data $\widetilde{\boldsymbol{\Omega}}_i$, $\widetilde{\boldsymbol{h}}_i$, $\widetilde{U}_i'$), up to a subsequence, we have $\widetilde{U}_i' \to \widetilde{U}'$ in C^0_{loc} for some $\widetilde{U}' \in \mathrm{Lip}_{\mathrm{loc}}(\{(x, z) : x_n < 0\})$, and the calibrations $\widetilde{\boldsymbol{\nu}}_i := \nabla_{\widetilde{\boldsymbol{h}}_i} \widetilde{U}_i' / |\nabla_{\widetilde{\boldsymbol{h}}_i} \widetilde{U}_i'|$ converges to some $\widetilde{\boldsymbol{\nu}}$ in the weak L^1_{loc} topology. Finally, $\widetilde{\boldsymbol{\nu}}$ calibrates $\widetilde{U}'$ as a solution of $\mathsf{IMCF}(\{x_n < 0\}; \mathbf{euc})$.

Taking limit of (3.6.26) and the C^0 bounds in Theorem 3.6.3, we have

$$|\nabla_{\widetilde{\boldsymbol{h}}_0} \widetilde{U}'| \leqslant \frac{C(n)}{|x_n|}, \qquad \widetilde{U}(x, z) \geqslant -C. \tag{3.6.27}$$

For $(x, z) \in \widetilde{\boldsymbol{\Omega}}_i$ with $-1 \leqslant x_n < 0$, we may calculate by unraveling the pullbacks and using the asymptotics of the exponential map:

$$\big\langle \widetilde{\boldsymbol{\nu}}_i, \partial_{x_n} \big\rangle_{\widetilde{\boldsymbol{h}}_i}(x, z) = \langle \boldsymbol{\nu}_i, \boldsymbol{\partial_r} \rangle_{\boldsymbol{g}}(\Phi_i(x), r_i z) + o(1), \tag{3.6.28}$$

where the $o(1)$ term locally uniformly converges to zero as $\delta \to 0$. By (3.6.24) this implies

$$\big\langle \widetilde{\boldsymbol{\nu}}_i, \partial_{x_n} \big\rangle_{\widetilde{\boldsymbol{h}}_i}(x, z) \geqslant 1 - C_3\big(b_i - r_i^{-1} r(\Phi_i(x))\big)^\gamma - o(1). \tag{3.6.29}$$

Note that $\lim_{i\to\infty} r_i^{-1} r(\Phi_i(x)) = x_n$ and $\lim_{i\to\infty} b_i = 0$. Thus taking the limit $i \to \infty$, we obtain

$$\big\langle \widetilde{\boldsymbol{\nu}}, \partial_{x_n} \big\rangle_{\mathbf{euc}}(x, z) \geqslant 1 - C_3|x_n|^\gamma \qquad \text{a.e. when} \quad -1 < x_n < 0. \tag{3.6.30}$$

Then by Corollary 3.3.18, $\widetilde{U}'$ actually solves of $\mathsf{IMCF}(\{x_n < 0\}, \mathbf{euc})$+$\mathsf{OBS}(\{x_n = 0\})$. Then applying Theorem 3.4.1 with (3.6.27) with $\widetilde{U}'(-\boldsymbol{e}_n) = 0$, it follows that $\widetilde{U}' \equiv 0$.

Step 4. We argue similarly as in Lemma 3.6.8, to obtain a contradiction from the level sets. Recall $-\boldsymbol{e}_n \in \partial E_0(\widetilde{U}_i')$. Applying Lemma 2.2.2(i) to $\widetilde{\boldsymbol{\Omega}}_i \subset \mathbb{R}^{n+1}$, we have the uniform bound (recall that we are treating $E_0(\widetilde{U}_i')$ as subsets of $\widetilde{\boldsymbol{\Omega}}_i$)

$$P\big(E_0(\widetilde{U}_i'); K\big) \leqslant P\big(\widetilde{\boldsymbol{\Omega}}_i \cap K\big), \qquad \forall K \Subset \mathbb{R}^{n+1}.$$

Thus, a subsequence of $E_0(\widetilde{U}_i')$ converges to some $E \subset \big\{(x,z) : x_n < 0\big\}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$.

The gradient bound (3.6.26) and (2.1.5) implies the uniform almost minimizing effect

$$P\big(E_0(\widetilde{U}_i'); B(-\boldsymbol{e}_n, \frac{1}{2})\big) \leqslant P(F; B(-\boldsymbol{e}_n, \frac{1}{2})) + C\big|E_0(\widetilde{U}_i')\Delta F\big|, \tag{3.6.31}$$

whenever $E_0(\widetilde{U}_i')\Delta F \Subset B(-\boldsymbol{e}_n, 1/2)$. Then by Theorem A.2.2(ii), we obtain $(-e_n, 0) \in \operatorname{spt}|\mu_E|$. By Theorem 2.1.13, E is a local minimizer of $J_{\widetilde{U}'}$ in $\big\{(x,z) : x_n < 0\big\}$, and since $\widetilde{U}'$ is shown to be constant, E is actually a local perimeter minimizer in $\big\{(x,z) : x_n < 0\big\}$ (with respect to the Euclidean metric). At this point, the minimizing is unknown to hold up to the boundary. However, one may show that E is locally outward-minimizing in $\mathbb{R}^{n+1}$, by the automatic subsolution principle (see Fact 1.4.10, Theorem 3.3.8).

If we can show that E is locally inward perimeter-minimizing in $\mathbb{R}^{n+1}$, then Lemma A.6.4 implies $E = \{(x,z) : x_n < -1\}$, and then Theorem A.2.2(iii) implies the convergence of normal vectors

$$\widetilde{\boldsymbol{\nu}}_i(-\boldsymbol{e}_n) = \nu_{E_0(\widetilde{U}_i')}(-\boldsymbol{e}_n) \to \partial_{x_n}.$$

This would contradict (3.6.21) due to (3.6.28), thus proving the lemma. It remains to show the (non-trivial) inward-minimizing of E.

For each fixed R, we denote $Q_R = \{|x'| \leqslant R, |x_n| \leqslant R, |z| \leqslant R\} \subset \mathbb{R}^{n+1}$. Let us prove the following almost constancy conclusion: for each $R \geqslant 2$ we have

$$\lim_{i\to\infty} \min\Big\{\widetilde{U}_i'(x,z) : (x,z) \in Q_R \cap \widetilde{\boldsymbol{\Omega}}_i\Big\} = 0. \tag{3.6.32}$$

Recall the direction bound (3.6.22) and the definition $\boldsymbol{\nu}_i = \nabla_{\boldsymbol{g}} U_i'/|\nabla_{\boldsymbol{g}} U_i'|_{\boldsymbol{g}}$. Pulling back to $\widetilde{\boldsymbol{\Omega}}_i$, these imply that

$$\frac{\partial \widetilde{U}_i'}{\partial x_n} \geqslant 0 \quad \text{in} \quad \Big\{-1/2 \leqslant x_n < 0, |x'| \leqslant R, |z| \leqslant R\Big\} \cap \widetilde{\boldsymbol{\Omega}}_i.$$

for sufficiently large i. Therefore, the minimum in (3.6.32) is attained at a point with $-R \leqslant x_n \leqslant -1/2$, for each large i. Since $\widetilde{U}_i' \to \widetilde{U}' \equiv 0$ uniformly on the compact set $\big\{|x'| \leqslant R, -R \leqslant x_n \leqslant -1/2, |z| \leqslant R\big\}$, our claim (3.6.32) immediately follows.

Now we pull back (3.6.25) via $\boldsymbol{\Phi}_i$. In this process we notice that: U_i and U_i' differs only by additive constants, so (3.6.25) holds for $\widetilde{U}_i'$ as well. Also, the inequality (3.6.25) is invariant under scaling. Moreover, notice a subtlety in notations: in our setting, $E_0(\widetilde{U}_i')$ is the pull back of $E_0(U_i') \cap \boldsymbol{\Omega}$ (i.e. the intersection with $\boldsymbol{\Omega}$ is already done via $\boldsymbol{\Phi}_i$). So for any $F \Subset Q_R \Subset B_{\mathbb{R}^{n+1}}(0, 1/\sqrt{r_i})$, we have

$$\theta_i P\big(E_0(\widetilde{U}_i'); Q_R\big) \leqslant P\big(E_0(\widetilde{U}_i') \setminus F; Q_R\big) + \big(e^{-\inf_{F\cap\widetilde{\boldsymbol{\Omega}}_i}(\widetilde{U}_i')} - 1\big)P(F),$$

where the perimeters are with respsect to the rescaled and pulled-back metric $\widetilde{\boldsymbol{h}}_i$. Taking $i \to \infty$, using (3.6.32), $\theta_i \to 1$, and the convergence $\widetilde{\boldsymbol{h}}_i \to \mathbf{euc}$ in $C^0_{\mathrm{loc}}(\mathbb{R}^{n+1})$, we obtain by the standard set replacing argument

$$P(E; Q_R) \leqslant P(E \setminus F; Q_R),$$

where the perimeters are with respect to the Euclidean metric. This proves the inward minimizing of E in $\mathbb{R}^{n+1}$, thus completes the proof. $\square$

Proof of Proposition 3.6.5.

Combining the results of Lemma 3.6.8 and 3.6.9, by taking a subsequence we can assume the following: there is a radius $r_0 < r_I$ such that

$$\inf_{\partial\Omega\times\mathbb{R}} \langle \boldsymbol{\nu}_i, \boldsymbol{\partial_r}\rangle_{\boldsymbol{g}} \geqslant 1 - \frac{1}{i}, \qquad \inf_{(\Omega\setminus\Omega_{-r_0})\times\mathbb{R}} \langle \boldsymbol{\nu}_i, \boldsymbol{\partial_r}\rangle_{\boldsymbol{g}} \geqslant \frac{1}{2}, \qquad \text{for all } i. \tag{3.6.33}$$

Apply Lemma 3.5.4 with the domain $\Omega\times\mathbb{R}$ in place of Ω, and with the smooth solutions $\Sigma^i_t = \operatorname{graph}\big(\varepsilon_i^{-1}(u_i - t)\big)$ and the parameter $b = i^{-1/\gamma}$. The small curvature and $p \geqslant \frac{1}{2}$ condition there are satisfied by (3.6.5) (3.6.33). Arguing verbatim as in Step 1 of Lemma 3.6.9, we obtain

$$1 - \langle \boldsymbol{\nu}_i, \boldsymbol{\partial_r}\rangle_{\boldsymbol{g}} \leqslant C_3\big(i^{-1/\gamma} - r/r_0\big)^{\gamma} \qquad \text{in } (\Omega\setminus\overline{\Omega_{-r_0}})\times\mathbb{R}. \tag{3.6.34}$$

We next apply Lemma 3.5.5 in the region $(\Omega\setminus\overline{\Omega_{-r_0}})\times\mathbb{R}$. Consider the quantity

$$G = r_0\Big((i^{-1/\gamma} - r/r_0)^{1-\gamma} - (2i^{-1/\gamma})^{1-\gamma}\Big)H,$$

where H is the mean curvature of Σ^i_t. The conditions in Lemma 3.5.5 are satisfied by (3.6.5) (3.6.33) (3.6.34) respectively. By the result (3.5.20) and the fact that Σ^i_t forms a translating soliton, we obtain the inequality

$$-\frac{1}{H^2}\Delta_\Sigma G + \langle \nabla_\Sigma G, Y\rangle \leqslant -\frac{G}{n} + \frac{4n}{G} + C_4\big(i^{-1/\gamma} - r/r_0\big)^{-\gamma}\big(\frac{C_5}{G} - 1\big)$$

in $(\Omega\setminus\overline{\Omega_{-r_0}})\times\mathbb{R}$, for some $C_4, C_5 > 0$ independent of i. By the maximum principle, this implies

$$\begin{aligned}\max_{(\overline{\Omega_{-br_0}}\setminus\Omega_{-r_0})\times\mathbb{R}}(G) &\leqslant \max\Big\{\max_{\partial\Omega_{-r_0}\times\mathbb{R}}(G),\ \max_{\partial\Omega_{-br_0}\times\mathbb{R}}(G), 2n + C_5\Big\}\\ &\leqslant \max\Big\{C(n), 0, 2n + C_5\Big\}.\end{aligned} \tag{3.6.35}$$

Here, the control of G on $\partial\Omega_{-r_0}\times\mathbb{R}$ follows by Corollary 2.4.4 and the assumptions for r_I:

$$G(x) \leqslant r_0\cdot 2\cdot H(x) \leqslant \frac{2C(n+1)r_0}{\sigma(x;\Omega\times\mathbb{R},\boldsymbol{g})} \leqslant \frac{2C(n+1)r_0}{\sigma(x;\Omega,g)} \leqslant C, \quad x\in\partial\Omega_{-r_0}\times\mathbb{R}.$$

So (3.6.35) implies

$$\sqrt{\varepsilon_i^2 + |\nabla u_i^2|} = H \leqslant \frac{C(n)r_0^{-1}}{(i^{-1/\gamma} - r/r_0)^{1-\gamma} - (2i^{-1/\gamma})^{1-\gamma}}. \tag{3.6.36}$$

As $i\to\infty$, we pass (3.6.34) to the limit and project onto the Ω factor, and obtain

$$1 - \langle \nu, \partial_r\rangle \leqslant C_3\big(|r|/r_0\big)^{\gamma} \qquad \text{a.e. in } \ \Omega\setminus\Omega_{-r_0}, \tag{3.6.37}$$

see below (3.6.18) for the convergence to ν. Passing (3.6.36) to the limit, we obtain

$$|\nabla u| \leqslant C(n)r_0^{-\gamma}|r|^{\gamma-1} \qquad \text{in } (\Omega\setminus\overline{\Omega_{-r_0}})\times\mathbb{R}.$$

Finally, by Corollary 3.3.18 and (3.6.37), the solution u respects the obstacle $\partial\Omega$. □

Proof of Proposition 3.6.6.

Let g_0 be the product metric on $\partial\Omega \times (0, r_0)$, and $\Phi : \partial\Omega \times (0, r_0) \to \Omega \setminus \overline{\Omega_{-r_0}}$ be the normal exponential map. By (3.6.5), we have $\frac{1}{2}g_0 \leqslant \Phi^* g \leqslant 2g_0$. By (3.6.17), (pulling back via Φ,) we may view $u = u(y, r)$ as a function on $\partial\Omega \times (0, r_0)$ with $|\nabla_{g_0} u| \leqslant C r_0^{-\gamma} r^{\gamma-1}$.

For $0 < r_1, r_2 \leqslant r_0/2$, from the gradient bound we have

$$\big|u(y, r_1) - u(y, r_2)\big| \leqslant C\gamma^{-1} r_0^{-\gamma} |r_1^\gamma - r_2^\gamma| \leqslant C\gamma^{-1} r_0^{-\gamma} |r_1 - r_2|^\gamma. \tag{3.6.38}$$

Note that this implies $u \in L^\infty\big(\partial\Omega \times (0, r_0/2)\big)$. Next, for $r \leqslant r_0/2$ and $d_{\partial\Omega}(y_1, y_2) \leqslant r$, we have

$$\big|u(y_1, r) - u(y_2, r)\big| \leqslant C r_0^{-\gamma} r^{\gamma-1} d_{\partial\Omega}(y_1, y_2) \leqslant C r_0^{-\gamma} d_{\partial\Omega}(y_1, y_2)^\gamma. \tag{3.6.39}$$

Finally, suppose $r \leqslant r_0/2$ and $d_{\partial\Omega}(y_1, y_2) \geqslant r$. If $d_{\partial\Omega}(y_1, y_2) \geqslant r_0/2$, then we have

$$\big|u(y_1, r) - u(y_2, r)\big| \leqslant 2\|u\|_{L^\infty(\partial\Omega \times (0, r_0/2))} \cdot (r_0/2)^{-\gamma} \cdot d_{\partial\Omega}(y_1, y_2)^\gamma. \tag{3.6.40}$$

If $r \leqslant d_{\partial\Omega}(y_1, y_2) \leqslant r_0/2$, then we set $s = d_{\partial\Omega}(y_1, y_2)$ and estimate

$$\begin{aligned}
\big|u(y_1, r) - u(y_2, r)\big| &\leqslant \big|u(y_1, r) - u(y_1, s)\big| + \big|u(y_2, r) - u(y_2, s)\big| + \big|u(y_1, s) - u(y_2, s)\big| \\
&\leqslant 2C\gamma^{-1} r_0^{-\gamma} |s - r|^\gamma + C r_0^{-\gamma} s^{\gamma-1} d_{\partial\Omega}(y_1, y_2) \\
&\leqslant (2C\gamma^{-1} + C) r_0^{-\gamma} d_{\partial\Omega}(y_1, y_2)^\gamma.
\end{aligned}$$

Combined with (3.6.38) $\sim$ (3.6.40), it follows that u can be extend to a $C^{0,\gamma}$ function on $\partial\Omega \times [0, r_0/2]$. Finally, by the smallness of r_0, there is a constant C such that $d_{\partial\Omega \times [0, r_0/2]}(x, y) \leqslant C d_g(\Phi(x), \Phi(y))$. This shows that u extends to a $C^{0,\gamma}$ function on $\overline{\Omega} \setminus \Omega_{-r_0/2}$. $\square$

Proof of Proposition 3.6.7.

For any $x \in \Omega \setminus \Omega_{-r_0/3}$ and $r < r_0/3$, by (3.6.17) and the choice of r_I we have

$$\begin{aligned}
\int_{B(x,r)\cap\Omega} |\nabla u| &\leqslant \int_{\max\{0, |r(x)| - r\}}^{|r(x)| + r} \mathcal{H}^{n-1}\big(B(x, r) \cap \Omega_{-s}\big) C r_0^{-\gamma} s^{\gamma-1}\, ds \\
&\leqslant C' r_0^{-\gamma} r^{n-1+\gamma}.
\end{aligned} \tag{3.6.41}$$

It is already known that u respects the obstacle $\partial\Omega$. For all $t > 0$ and each competitor set $E \subset M$ satisfying $E \Delta E_t \Subset B(x, r)$, we compare the energy $\widetilde{J}_u^{B(x,r)}(E_t) \leqslant \widetilde{J}_u^{B(x,r)}(E \cap \Omega)$ and obtain (note that $E_t \Delta (E \cap \Omega) \Subset B(x, r)$ since $E_t \subset \Omega$)

$$\begin{aligned}
P\big(E_t; B(x, r)\big) &\leqslant P\big(E \cap \Omega; B(x, r)\big) + \int_{E_t \Delta (E\cap\Omega)} |\nabla u| \\
&\leqslant P\big(E; B(x, r)\big) + P\big(\Omega; B(x, r)\big) - P\big(E \cup \Omega; B(x, r)\big) + C r_0^{-\gamma} r^{n-1+\gamma}.
\end{aligned}$$

It is directly verifiable, using the C^2 smoothness of $\partial\Omega$ (see [105, Section 1.6]), that

$$P\big(\Omega; B(x, r)\big) \leqslant P\big(E \cup \Omega; B(x, r)\big) + C r_0^{-2} r^{n+1}.$$

Thus we obtain the almost-minimizing condition

$$P\big(E_t; B(x, r)\big) \leqslant P\big(E; B(x, r)\big) + C r_0^{-\gamma} r^{n-1+\gamma}. \tag{3.6.42}$$

Next, combining (3.6.16) and Remark 2.3.2, we obtain that almost every E_t satisfies

$$1 - \langle \nu_{E_t}, \partial_r \rangle \leqslant C_3 r_0^{-\gamma} |r|^\gamma. \tag{3.6.43}$$

In addition, on $\partial^* E_t \cap \partial^* \Omega$ it is clear that $\nu_{E_t} = \partial_r$. Since ∂_r is smooth, this has the following consequence: for any $l > 0$ there exists a sufficiently small radius $r(l)$, such that for all $x \in \overline{\Omega} \setminus \Omega_{-r(l)}$, $s \leqslant r(l)$, the set $E_t \cap B(x, s)$ is representable as the sub-graph of a l-Lipschitz function in some geodesic normal coordinate near x. Conbining (3.6.42) and this small slope condition, the classical small excess regularity theorem (see [104] and [3, 105]) implies the following: there is a sufficiently small $r_1 < r_0$, such that almost every $E_t \setminus \overline{\Omega_{-r_1}}$ is a $C^{1,\gamma/2}$ hypersurface. By the Arzela-Ascoli theorem, the same conclusion holds for all $t > 0$. This proves the desired result. □

3.7 p-Harmonic approximation

Suppose $p > 1$. The p-Laplacian is defined as

$$\Delta_p v = \operatorname{div}\big(|\nabla v|^{p-2} \nabla v\big).$$

A function $v \in W^{1,p}(\Omega)$ is called p-harmonic if $\Delta_p v = 0$ weakly, namely, if

$$\int |\nabla v|^{p-2} \langle \nabla v, \nabla \varphi \rangle = 0, \qquad \forall\, \varphi \in C_0^1(\Omega).$$

It is a classical anlytical fact that p-harmonic functions are locally $C^{1,\alpha}$ [71]. The relation between IMCF and p-harmonic functions was initiated from the work of Moser [89], and was further investigated by Kotschwar-Ni [65], Mari-Rigoli-Setti [79] (see also [14]). The following fundamental relation was shown:

Lemma 3.7.1 ([65, 89]). *Suppose $p_i \searrow 1$, and $v_{p_i} \in W^{1,p_i}_{\mathrm{loc}}(\Omega)$ are positive p_i-harmonic functions in a domain Ω, such that*

$$\lim_{i \to \infty} (1 - p_i) \log v_{p_i} = u \quad \text{in} \quad C^0_{\mathrm{loc}}(\Omega).$$

Then u is a solution of $\mathsf{IMCF}(\Omega)$.

When an initial value is involved, we have convergence up to ∂E_0:

Lemma 3.7.2 ([65, p.13]). *Suppose $E_0 \Subset \Omega$ is a $C^{1,1}$ domain, $p_i \searrow 1$, and v_{p_i} satisfy*

$$\begin{cases} \Delta_{p_i} v_{p_i} = 0, \ 0 < v_{p_i} \leqslant 1 & \text{in } \Omega \setminus E_0, \\ v_{p_i} = 1 & \text{on } \partial E_0. \end{cases}$$

Then a subsequence of $(1 - p_i) \log v_{p_i}$ converges in $C^0_{\mathrm{loc}}(\Omega \setminus E_0)$ to a solution of $\mathsf{IVP}(\Omega; E_0)$.

Lemma 3.7.2 is obtained by combining Lemma 3.7.1 with interior and boundary gradient estimates in [65, Theorem 1.1 and (1.7)].

The goal of this section is to prove the following theorem, which is to be included in [14]. We also refer to the introduction chapter for more backgrounds.

Theorem 3.7.3 (=Theorem E).
Let $\Omega \Subset M$ be a smooth domain, and $E_0 \Subset \Omega$ be a $C^{1,1}$ domain. Let $v_p \in W^{1,p}(\Omega \setminus E_0)$ solve the boundary value problem

$$\begin{cases} \Delta_p v_p = 0 & \text{in } \Omega \setminus \overline{E_0}, \\ v_p = 1 & \text{on } \partial E_0, \\ v_p = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.7.1}$$

and set $u_p = (1-p)\log v_p$. Then a subsequence of u_p converges in $C^0_{\mathrm{loc}}(\Omega \setminus E_0)$ to the unique solution of $\mathsf{IVP}(\Omega; E_0)$+$\mathsf{OBS}(\partial\Omega)$*, as $p \to 1$.*

The proof makes essential use of the following observation due to Benatti-Pluda-Pozzetta. The author thanks Luca Benatti for pointing out this result.

Theorem 3.7.4 ([15, Theorem 2.8]).
Let M be complete, noncompact, and $E_0 \Subset M$ be a $C^{1,1}$ domain. Suppose that there exists a proper solution w of $\mathsf{IVP}(M; E_0)$*. Then for all domain Ω with $E_0 \Subset \Omega \Subset M$, there is a family of positive p-harmonic functions v_p, such that*

$$v_p|_{\partial E_0} = 1, \qquad \lim_{p\to 1}(1-p)\log v_p = w \quad \text{in} \quad C^0(\overline{\Omega \setminus E_0}).$$

Proof of Theorem 3.7.3.
The convergence (up to a subsequence) $\widetilde{u} = \lim_{p\to 1} u_p$ follows from Lemma 3.7.2. There we also know that $\widetilde{u}$ solves $\mathsf{IVP}(\Omega; E_0)$. Next, let u the solution of $\mathsf{IVP}(\Omega; E_0)$+$\mathsf{OBS}(\partial\Omega)$ given by Theorem 3.6.1. Our goal is to show that $\widetilde{u} = u$. By Theorem 3.6.1(iv), namely the maximality of u, it suffices to show that $\widetilde{u} \geqslant u$ in $\Omega \setminus E_0$.

Let us briefly recall the construction of u. We assume the setups made in Subsection 3.6.1. Fix a sequence $\delta_i \to 0$. Recall that we conidered the slightly larger domain Ω_{δ_i} and the weight function ψ_{δ_i}. By Lemma 3.6.2 and keeping in mind that weighted IMCF is equivalent to the usual IMCF under a conformal change, the domain

$$(\Omega_{\delta_i}, g_i'), \qquad \text{where } g_i' = e^{2\psi_{\delta_i}/(n-1)} g$$

admits a proper subsolution of IMCF (which is the function $\underline{u}_{\delta_i}$ in Lemma 3.6.2). Therefore, there is a proper solution w_i of $\mathsf{IVP}(\Omega_{\delta_i}, g_i'; E_0)$, by Theorem 2.4.1. For any $\lambda_i \in (0,1)$ and sufficiently small ε_i, Lemma 3.6.3 yields a solution $u_i = u_{\varepsilon_i,\delta_i,\lambda_i}$ of the regularized equation (3.6.10) ∼ (3.6.12). Recall that (3.6.10) ∼ (3.6.12) is the same as (2.4.9) ∼ (2.4.11) with the domain "F_L" $= \Omega_{\lambda_i\delta_i}$ and the metric given by g_i'. By the convergence statement in Theorem 2.4.8, for each individual i we may choose λ_i sufficiently close to 1 and then ε_i sufficiently small (depending on i, δ_i, λ_i), so that

$$\|u_i - w_i\|_{C^0(\Omega\setminus E_0)} \leqslant i^{-1}. \tag{3.7.2}$$

By Theorem 3.7.4 above, for each i there exists p_i close enough to 1, such that there is a function $f_i \in \mathrm{Lip}(\overline{\Omega} \setminus E_0)$ with

$$f_i|_{\partial E_0} = 0, \qquad e^{\frac{f_i}{1-p_i}} \text{ being } p_i\text{-harmonic}, \qquad \|f_i - w_i\|_{C^0(\Omega\setminus E_0)} \leqslant i^{-1}. \tag{3.7.3}$$

Note that the function v_p in (3.7.1) is the minimal p-harmonic function in Ω with $v_p|_{\partial E_0} = 1$. Hence $v_p \leqslant e^{-f_i/(p-1)}$, which implies

$$u_{p_i} \geqslant f_i \qquad \text{in } \ \Omega \setminus E_0. \tag{3.7.4}$$

Combining (3.7.2) (3.7.3) (3.7.4), we obtain

$$u_{p_i} \geqslant u_i - 2i^{-1} \qquad \text{in } \ \Omega \setminus E_0.$$

Taking $i \to \infty$, it follows that $\widetilde{u} \geqslant u$ in $\Omega \setminus E_0$. $\square$

Chapter 4

Innermost solutions

Let $E_0 \subset M$ be a $C^{1,1}$ initial value. Recall that u is a *maximal* or *innermost solution* of $\mathsf{IVP}(M; E_0)$, if $u \geqslant v$ for any other solution v of $\mathsf{IVP}(M; E_0)$. In this section, we prove the existence and basic properties of such a solution, including Theorem C and Lemma 1.3.3. Section 4.3 is based on joint work with O. Chodosh and Y. Lai [30].

4.1 Existence and properties

The following is a restatement of Theorem C.

Theorem 4.1.1 (existence).
Let (M, g) be a (possibly incomplete) smooth Riemannian manifold, and $E_0 \subset M$ be a (possibly unbounded) $C^{1,1}$ domain. Then there exists, up to equivalence, a unique maximal solution of $\mathsf{IVP}(M; E_0)$.

Proof. Find a sequence of precompact $C^{1,1}$ domains E_0^i, such that $E_0^1 \subset E_0^2 \subset \cdots \subset E_0$, and $\bigcup_{i \geqslant 1} E_0^i = E_0$, and that for all $K \Subset M$, the sets $E_0^i \cap K$ eventually stablize (i.e. $E_0^i \cap K = E_0^{i+1} \cap K = E_0^{i+2} \cap K = \cdots$ for large enough i). Next, find a sequence of smooth precompact domains $\Omega_i \Supset E_0^{i+1}$ such that $\Omega_1 \Subset \Omega_2 \Subset \cdots \Subset M$ and $\bigcup_{i \geqslant 1} \Omega_i = M$. Let u_i be the (unique) solution of $\mathsf{IVP}(\Omega_i; E_0^i)$+$\mathsf{OBS}(\partial\Omega_i)$, given by Theorem 3.6.1.

Applying Corollary 3.3.20 to each Ω_i, we have $u_1 \geqslant u_2 \geqslant u_3 \geqslant \cdots$ outside E_0. By the interior gradient estimates (3.6.1) and Arzela-Ascoli theorem, some subsequence of u_i converge in C^0_{loc} to a function $u \in \mathrm{Lip}_{\mathrm{loc}}(M \setminus E_0)$. By Theorem 2.3.3, u is a calibrated weak solution in $M \setminus \overline{E_0}$. Since (3.6.1) provides a uniform gradient estimate up to ∂E_0, the resulting function u is Lipschitz up to ∂E_0, and satisfies $u \geqslant 0$, $u|_{\partial E_0} = 0$. Extending u by negative values inside E_0, we obtain a solution of $\mathsf{IVP}(M; E_0)$.

It remains to show that u is maximal. Suppose $v \in \mathrm{Lip}_{\mathrm{loc}}(M)$ is another solution of $\mathsf{IVP}(M; E_0)$. For each i we find functions $v_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega_i)$, such that $v_i = v$ on $\Omega_i \setminus E_0$, and $v_i = 0$ on $E_0 \setminus E_0^i$, and $v_i < 0$ on E_0^i. By Definition 2.1.2 it can be verified that v_i is a subsolution of $\mathsf{IVP}(\Omega_i; E_0^i)$. Then by Corollary 3.3.20, we obtain $v_i \leqslant u_i$ on $\Omega_i \setminus E_0^i$, hence $v \leqslant u_i$ on $\Omega_i \setminus E_0$. As u is the descending limit of u_i, it follows that $v \leqslant u$ on $M \setminus E_0$. $\square$

For our future convenience, let us summarize the following construction lemma. Its proof can be extracted from above by taking $E_0^1 = E_0^2 = \cdots = E_0$.

Lemma 4.1.2. *Suppose $E_0 \Subset \Omega_1 \Subset \Omega_2 \Subset \cdots \Subset M$, where E_0 is a $C^{1,1}$ domain, and each Ω_i is a smooth domain, and $\bigcup \Omega_i = M$. Let $u_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega_i)$ be the solution of*

$\mathsf{IVP}(\Omega_i, E_0)$+$\mathsf{OBS}(\partial\Omega_i)$ *given by Theorem 3.6.1. Then we have* $u_1 \geqslant u_2 \geqslant u_3 \geqslant \cdots$, *and* $u_i \to u$ *in* $C^0_{\mathrm{loc}}(M \setminus E_0)$ *where* u *is the maximal solution of* $\mathsf{IVP}(M; E_0)$.

The following theorem summarizes the useful properties of maximal solutions.

Theorem 4.1.3 (properties of maximal solutions)**.**
Let M, g, E_0 *be as in Theorem 4.1.1, and* u *be the maximal solution of* $\mathsf{IVP}(M; E_0)$ *given there. Then the following hold.*

(i) We have the interior gradient estimate

$$|\nabla u|(x) \leqslant \sup_{\partial E_0 \cap B(x,r)} H_+ + \frac{C(n)}{r}, \qquad x \in M \setminus E_0, \quad r \leqslant \sigma(x; M, g),$$

where H_+ *denotes the positive part of the mean curvature of* ∂E_0*, and* $\sigma(x; M, g)$ *is as in Definition 2.4.2.*

(ii) If E_0 *is connected, then* E_t *is connected for all* $t > 0$*. Moreover,* $M \setminus E_t$ *does not have compact connected components.*

(iii) If $E_0 \Subset M$*, then we have* $P(E_t) \leqslant e^t P(E_0)$*.*

Proof. By the proof of Theorem 4.1.1, the maximal solution u arises as a limit of the solutions u_i of $\mathsf{IVP}(\Omega_i; E_0^i)$+$\mathsf{OBS}(\partial\Omega_i)$, where E_0^i and Ω_i are precompact exhaustions of E_0 and Ω as defined in Theorem 4.1.1. Moreover, u does not depend on the choice of these exhaustions (since it is a unique object).

Given this setup, item (i) follows by passing Theorem 3.6.1(i) to the limit. Item (iii) follows by passing Corollary 3.3.14 to the limit and using the lower semi-continuity of perimeter. It remains to prove (ii). Given that E_0 is connected, we claim that there is a sequence of *connected* precompact $C^{1,1}$ domains $E_0^1 \subset E_0^2 \subset \cdots \subset E_0$ with $\bigcup E_0^i = E_0$. Once this is proved, it follows by Lemma 3.3.9 that each $\overline{E_t(u_i)}$ is connected. This then implies that $E_t(u_i)$ is connected as well: otherwise, since $u_i \in C^0(\overline{\Omega_i})$ by Theorem 3.6.1(i), there will be a slightly smaller t' so that $\overline{E_{t'}(u_i)}$ is disconnected, contradiction. Finally, as $E_t(u) = \bigcup_{i \geqslant 1} E_t(u_i)$, it follows that $E_t(u)$ is connected.

The connected exhaustion $\{E_0^i\}$ is constructed as follows. Fix a basepoint $x_0 \in E_0$. We start with an arbitrary precompact $C^{1,1}$ exhaustion $\overline{E}_0^1 \subset \overline{E}_0^2 \subset \cdots \subset E_0$, then let E_0^i be the connected component of $\overline{E}_0^i$ containing x_0. It follows that $\{E_0^i\}$ is also an exhaustion: for any $x \in E_0$, there is a path $\gamma \Subset E_0$ joining x and x_0. We have $\gamma \Subset \overline{E_0^i}$ for all large i, hence $\gamma \Subset E_0^i$ as well, hence $x \in E_0^i$. □

When Ω is a non-compact domain, the fact that u solves $\mathsf{IVP}(\Omega; E_0)$+$\mathsf{OBS}(\partial\Omega)$ does not imply that it is the maximal solution of $\mathsf{IVP}(\Omega; E_0)$. Intuitively, the maximal solution respects not only the obstacle $\partial\Omega$, but also the "invisible obstacle" at infinity. Note that item (iii) may be a strict inequality, whereas we always have equality for proper solutions.

We mention the following examples of maximal solutions.

1. In Choi-Daskalopoulos [31], one considers the (smooth) IMCF with with a convex, non-compact, $C^{1,1}$ initial domain $E_0 \subset \mathbb{R}^n$. A compact approximation argument is employed to obtain a solution: one finds bounded convex domains $E_0^1 \subset E_0^2 \subset \cdots$ with $\bigcup_{i \geqslant 1} E_0^i = E_0$, and then take u_i to be the proper (hence maximal) solution of $\mathsf{IVP}(\mathbb{R}^n; E_0^i)$, and finally take the descending limit $u = \lim_{i \to \infty} u_i$. Arguing similarly as above, it follows that u is the maximal solution of $\mathsf{IVP}(\mathbb{R}^n; E_0)$.

2. In the setting of Remark 1.3.2, the function

$$u_0(r) = (n-1) \log \big[f(r)/f(r_0) \big]$$

is the maximal solution of $\mathsf{IVP}(\Omega; \{r < r_0\})$.

3. Consider E_0 a half-space in the hyperbolic space, and let u be the maximal solution of $\mathsf{IVP}(\mathbb{H}^n; E_0)$. By uniqueness, the function $u|_{\mathbb{H}^n \setminus E_0}$ must be invariant under any isometry that preserves E_0. Therefore, the level sets of u are equi-distance sets from E_0. This reduces the flow to a one-dimension problem: the maximal solution must be

$$u = (n-1) \log \cosh d(x, E_0).$$

Remark 4.1.4. We note that the connectedness of level sets (Theorem 4.1.3(ii)) only holds for maximal solutions, but not general solutions. A concrete counterexample is as follows. Let $E_0, E' \subset \mathbb{H}^n$ be two disjoint half-spaces that are sufficiently far separated. Denote $l_1 = \operatorname{arccosh}\big(e^{1/(n-1)}\big)$ and $l_2 = \operatorname{arccosh}\big(e^{2/(n-1)}\big)$. Set $u \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{H}^n)$ such that

(1) $u|_{E_0} < 0$, and $u|_{N(E_0, l_2)} = (n-1) \log \cosh d(\cdot, E_0)$;
(2) $u|_{E'} \equiv 1$, and $u|_{N(E', l_1)} = 1 + (n-1) \log \cosh d(\cdot, E')$;
(3) $u|_{\mathbb{H}^n \setminus (N(E_0, l_2) \cup N(E', l_1))} \equiv 2$.

It follows that u solves $\mathsf{IVP}(\mathbb{H}^n; E_0)$, but $E_2(u)$ is not connected.

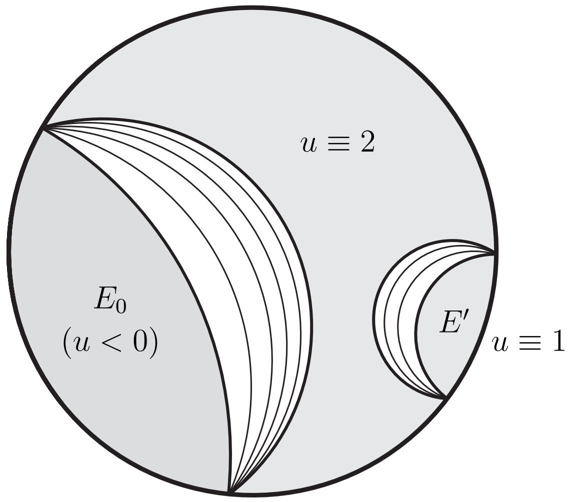

Figure 4.1: The example in Remark 4.1.4.

4.2 Isoperimetry and properness revisited

In this section, we give a second proof of Theorem A by replacing the conic cutoff argument in Section 2.5 with the IMCF with outer obstacle. This proof is conceptually cleaner (but not actually simpler since it relies on Theorem 3.6.1).

Recall the notion of isoperimetric profile $\mathrm{ip}(v)$ and its formal inverse $\mathrm{sip}^{-1}(a)$ from Definition 2.5.1, as well as the following condition

$$\liminf_{v \to \infty} \mathrm{ip}(v) = \infty \quad \text{and} \quad \int_0^{v_0} \frac{dv}{\mathrm{ip}(v)} < \infty \ \text{ for some } v_0 > 0, \tag{4.2.1}$$

which we assumed in Theorem A. As a consequence of these conditions, we have noticed by Corollary A.3.3 that the quantity

$$\int_0^{\mathrm{sip}^{-1}(A)} \frac{dv}{\mathrm{ip}(v)}$$

is finite for all $A > 0$.

A second proof of Theorem A.

Fix an exhaustion $E_0 \Subset \Omega_1 \Subset \Omega_2 \Subset \cdots$, with Ω_i smooth and $\bigcup_{i\geqslant 1} \Omega_i = M$. Let u_i be the solution of $\mathsf{IVP}(\Omega_i; E_0)$+$\mathsf{OBS}(\partial\Omega_i)$ given by Theorem 3.6.1. Fix a basepoint $x_0 \in E_0$, and denote $B(r) = B(x_0, r)$. For each $i \in \mathbb{N}$, $t > 0$, and for almost every $r > \operatorname{diam} E_0$ (so that $\partial B(r)$ is rectifiable), we apply Lemma 3.4.2 (the excess inequality with obstacle) with $F = \Omega_{i+1} \setminus B(r)$ to find

$$P\big(E_t(u_i)\big) \leqslant P\big(E_t(u_i) \cap B(r)\big) + (e^t - 1)P\big(B(r); E_t(u_i)\big).$$

The remining argument is similar to that of Lemma 2.5.5: using the decomposition identities (A.1.15), this further implies

$$\mathcal{H}^{n-1}\big(E_t(u_i); B(r)^{(0)}\big) \leqslant e^t P\big(B(r); E_t\big) \qquad \text{for a.e. } r > \operatorname{diam} E_0. \tag{4.2.2}$$

Consider $V(r) := |E_t(u_i) \setminus B(r)|$, so $V'(r) = -P\big(B(r); E_t(u_i)\big)$ for a.e. r. We also have

$$\mathcal{H}^{n-1}\big(E_t(u_i); B(r)^{(0)}\big) + P\big(B(r); E_t\big) = P\big(E_t(u_i) \setminus B(r)\big) \geqslant \operatorname{ip}(V(r)) \tag{4.2.3}$$

for a.e. r. Combining (4.2.2) (4.2.3), we obtain

$$V'(r) \leqslant -\frac{\operatorname{ip}(V(r))}{e^t + 1} \qquad \text{for a.e. } r > \operatorname{diam} E_0. \tag{4.2.4}$$

In addition, $V(\operatorname{diam} E_0) \leqslant |E_t(u_i)| \leqslant \operatorname{sip}^{-1}\big(P(E_t(u_i))\big) \leqslant \operatorname{sip}^{-1}\big(e^t P(E_0)\big)$, where we used the sub-exponential growth of area (Corollary 3.3.14). Solving (4.2.4) yields

$$V(r) \equiv 0 \quad \text{for all} \quad r > r_*(t) := \operatorname{diam} E_0 + (1 + e^t) \int_0^{\operatorname{sip}^{-1}(e^t P(E_0))} \frac{dv}{\operatorname{ip}(v)}.$$

This implies $E_t(u_i) \subset B(r_*(t))$ for all $t > 0$, which is a bound independent of i. Now we take the descending limit $u = \lim_{i\to\infty} u_i$. Arguing as in Theorem 4.1.1, such limit exists and is the maximal solution of $\mathsf{IVP}(M; E_0)$. The bounds $E_t(u_i) \subset B(r_*(t))$ pass to the limit and gives $E_t(u) \subset B(r_*(t)) \Subset M$. Hence u is proper. □

4.3 The effect of bounded geometry

Suppose M is complete, noncompact and connected. We say that M has *bounded geometry*, if it holds

$$|\operatorname{Rm}| \leqslant \Lambda^2, \qquad \operatorname{inj} \geqslant \Lambda^{-1} \tag{BG}$$

for some constant $\Lambda > 0$. The bounded geometry condition has several strong implications on the geometry of M:

(i) The volume of $B(x, 1)$ is bounded below independently of x. This follows from the classical comparison geometry.

(ii) We have a uniform Neumann isoperimetric inequality

$$\min\Big\{\big|B(x,1) \cap E\big|, \big|B(x,1) \setminus E\big|\Big\} \leqslant C(n,\Lambda) P\big(E; B(x,1)\big)^{\frac{n}{n-1}}, \tag{4.3.1}$$

for all $x \in M$ and all sets E with locally finite perimeter. This is a consequence of the abstract Hajłasz-Koskela theorem; see [95, Theorem 7.1.13].

(iii) The regular radius $\sigma(x;M)$ in Definition 2.4.2 is bounded below in terms of Λ.
(iv) The C^0 norm of the metric near any $x\in M$ (see the discussions around (A.2.2)) is bounded above in terms of Λ.

On manifolds with bounded geometry, the maximal IMCF shows much better behavior. In particular, the level set always stay compact before it disappears at infinity. We define the following solution types:

Definition 4.3.1. Let u be a solution of $\mathsf{IVP}(M;E_0)$. We say that:
(i) u is *sweeping*, if $T:=\sup(u)\in(0,\infty)$, and $E_t\Subset M$ for all $t<T$, and $E_T=M$.
(ii) u is *instantly escaping*, if $T:=\sup(u)\in(0,\infty)$, and $E_T\Subset M$, and $u\equiv T$ in $M\setminus E_T$. In both cases, we call T the escape time of u.

See Figures 1.14 and 1.15 for examples of sweeping and instantly escaping flows. Our goal in this section is to prove the following:

Theorem 4.3.2 (= Lemma 1.3.3)**.**
Suppose M is complete, noncompact, one-ended, and satisfies (BG)*. Then there is a constant $A=A(\Lambda)>0$ such that: for any $C^{1,1}$ domain $E_0\Subset M$ with $P(E_0)\leqslant A$, the maximal solution u of $\mathsf{IVP}(M;E_0)$ is either proper, sweeping, or instantly escaping.*
Let T be as in Definition 4.3.1, or $T=+\infty$ for the proper case. Then

$$\begin{aligned} e^T P(E_0^+) = \inf\Big\{ \liminf_{i\to\infty} P(F_i) : F_1\Subset F_2\Subset\cdots\Subset M,\\ F_i \text{ has finite perimeter}, \bigcup F_i = M\Big\}. \end{aligned} \tag{4.3.2}$$

In (4.3.2), note that $E_0^+\Subset M$ by the conclusion on the solution type of u. Hence, E_0^+ is the minimizing hull of E_0 in M. The right hand side of (4.3.2) is different from that in Lemma 1.3.3. However, those two expressions are equal since sets with finite perimeter are smoothable (Theorem A.1.3). All the three conditions in Theorem 4.3.2, namely completeness, one-endedness and bounded geometry, are not removable: see Figure 1.16 and 1.17 for examples.

Theorem 4.3.2 is the consequence of a chain of intermediate results, as follows.

Lemma 4.3.3 (regularity and density bounds)**.**
Let M satisfy (BG)*, and u be a solution of $\mathsf{IMCF}(\Omega)$ for some domain $\Omega\subset M$. Moreover, assume $|\nabla u|\leqslant L$ in Ω. Then for all $t\in\mathbb{R}$ the following hold:*
(i) for all $x\in\partial E_t$ and $r\leqslant\min\big\{\Lambda^{-1},\, d(x,\partial\Omega)/2\big\}$, we have the density bound

$$P\big(E_t;B(x,r)\big)\geqslant c(n,L,\Lambda)r^{n-1};$$

(ii) if $n\leqslant 7$, and M further satisfies $|\nabla^k\operatorname{Rm}|\leqslant\Lambda'$, $\forall\,1\leqslant k\leqslant 5$, then $\partial E_t(u)$ is a $C^{1,\alpha}$ surface. For each $x\in\partial E_t$, we have $\|E_t\|_{C^{1,\alpha}(x)}\leqslant C\big(n,\alpha,L,\Lambda,\Lambda',d(x,\partial\Omega)\big)$. See the discussions around (A.2.4) *for the precise definition of the $C^{1,\alpha}$ norm.*

Proof. The gradient bound implies the almost perimeter-minimizing condition

$$P(E_t\cap K)\leqslant P(F\cap K)+L|E_t\Delta F|,\quad \text{for all } F \text{ with } E_t\Delta F\Subset K\Subset\Omega.$$

Then item (ii) follows directly from Theorem A.2.3.

For item (i), we set $R=\min\big\{\Lambda^{-1},\,d(x,\partial\Omega)/2\big\}$. By Theorem A.2.4, there is a constant $c=c(n,L,\Lambda)$ so that $P\big(E_t;B(x,r)\big)\geqslant c(n)r^{n-1}$ for all $r\leqslant cR$. But in the range $cR\leqslant r\leqslant R$, we simply estimate

$$P\big(E_t;B(x,r)\big)\geqslant P\big(E_t;B(x,cR)\big)\geqslant c(n)(cR)^{n-1}\geqslant\big(c(n)c^{n-1}\big)r^{n-1}. \qquad\square$$

Lemma 4.3.4 (compactness and diameter bound).
Let M be complete, connected, one-ended, satisfying (BG), and $E_0 \Subset M$ be connected. Let u be the maximal solution of $\mathsf{IVP}(M; E_0)$. *Suppose $E_t \neq M$ for $t > 0$. Then:*
(i) it actually holds $E_t \Subset M$,
(ii) if ∂E_t is connected, then

$$\operatorname{diam}(\partial E_t) \leqslant C\big(n, \Lambda, t, P(E_0), \operatorname{diam}(\partial E_0)\big). \tag{4.3.3}$$

The lemma's condition does not imply the connectedness of ∂E_t, see Figure 4.2.

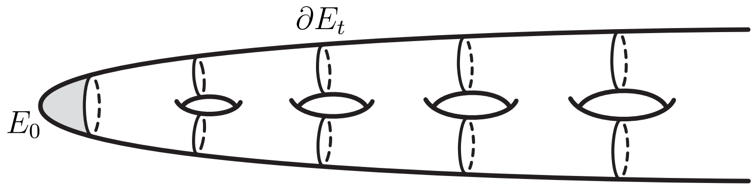

Figure 4.2: A maximal solution with disconnected ∂E_t

Proof of Lemma 4.3.4.
Let us first prove that $\partial E_t \Subset M$. Otherwise, we can find infinitely many disjoint balls $B(x_i, 1/2\Lambda)$ with $x \in \partial E_t$, $d(x_i, E_0) \geqslant 1/\Lambda$. Recall by Theore 4.1.3(i) that

$$|\nabla u| \leqslant C(\Lambda) \quad \text{in} \quad \big\{d(\cdot, E_0) \geqslant 1/\Lambda\big\}.$$

Thus, Lemma 4.3.3(i) provides a uniform lower bound $P\big(E_t; B(x_i, 1/2\Lambda)\big) \geqslant c(\Lambda)$. This contradicts the fact $P(E_t) \leqslant e^t P(E_0)$ from Theorem 4.1.3(i).

Next we show that $E_t \Subset M$. By Theorem 4.1.3(ii) and our condition, we know that E_t is connected and $M \setminus E_t$ is noncompact and nonempty. Since M is one-ended and $\partial E_t \Subset M$, we may find a domain K such that $\partial E_t \Subset K \Subset M$, and $M \setminus K$ is connected and noncompact. Since E_t is connected and $\partial E_t \cap (M \setminus K) = \emptyset$, we have either $E_t \cap (M \setminus K) = \emptyset$ or $E_t \supset M \setminus K$. The first case implies $E_t \subset K \Subset M$. The second case implies $M \setminus E_t \subset K$, which is impossible.

Now we assume that ∂E_t is connected. Setting

$$N = \Big\lfloor \Lambda\big(\operatorname{diam}(\partial E_t)/2 - \operatorname{diam}(\partial E_0) - 1/\Lambda\big) \Big\rfloor,$$

we can find N points $x_i \in \partial E_t$ such that $d(x_i, E_0) \geqslant 1/\Lambda$, and the balls $B(x_i, 1/2\Lambda)$ are pairwise disjoint. This implies (4.3.3) by the same area counting argument as above. $\square$

In the following lemma, recall Definition 2.5.1 for the notion of isoperimetric profile.

Lemma 4.3.5. *Assume M is complete, connected, noncompact, and satisfies (BG). Then*

$$\operatorname{ip}(v) \geqslant C(\Lambda)^{-1} \min\big\{1, v^{(n-1)/n}\big\}, \qquad \forall\, v > 0. \tag{4.3.4}$$

Proof. Note that (BG) implies a uniform lower bound on the volume of balls: there exists $V = V(\Lambda) > 0$ such that

$$|B(x, 1)| \geqslant V, \qquad \forall\, x \in M.$$

This further implies $|M| = \infty$. Suppose $E \Subset M$. We divide into two cases.

Case 1: $|E \cap B(x,1)| \geqslant V/2$ for some $x \in M$. By continuity, we can find another point $x' \in M$ such that $|E \cap B(x',1)| = V/2$. Then by the isoperimetric inequality (4.3.1), we obtain $P(E) \geqslant P\big(E; B(x',1)\big) \geqslant C(\Lambda)^{-1}$ for this case.

Case 2: $|E \cap B(x,1)| < V/2$ for all $x \in M$. Then (4.3.1) implies $|E \cap B(x,1)| \leqslant C(\Lambda) P\big(E; B(x,1)\big)^{n/(n-1)}$ for all $x \in M$. By volume doubling, we may find finitely many balls $\{B(x_i,1)\}_{i=1}^m$ that covers $\overline{\Omega}$, whose covering multiplicity is bounded by $C(\Lambda)$. Thus

$$\begin{aligned}|E| \leqslant \sum_{i=1}^m |E \cap B(x_i,1)| &\leqslant C(\Lambda) \sum_{i=1}^m P\big(E; B(x_i,1)\big)^{\frac{n}{n-1}} \\ &\leqslant C(\Lambda) \Big(\sum_{i=1}^m P\big(E; B(x_i,1)\big)\Big)^{\frac{n}{n-1}} \leqslant C(\Lambda) P(E)^{\frac{n}{n-1}}.\end{aligned}$$

Then (4.3.4) follows by combining the two cases. □

We are in a position to prove the main result.

Proof of Theorem 4.3.2.

Combining Lemma 4.3.5 and Theorem 2.5.2, there exists $A = A(n,\Lambda) > 0$ so that the following holds: for all E_0 with $P(E_0) \leqslant A$, there exists a solution u' of $\mathsf{IVP}(M; E_0)$ with $E_t(u') \Subset M$ for some $t > 0$. Let u be the maximal solution of $\mathsf{IVP}(M; E_0)$, we have $E_t(u) \subset E_t(u') \Subset M$ for the same t.

Let $T' := \sup\big\{t \geqslant 0 : E_t \Subset M\big\}$, thus $T' > 0$. If $T' = \infty$ then u is proper. Now assume $T' < \infty$. By Lemma 4.3.4(i), we must have $E_t = M$ for all $t > T'$. Therefore

$$T' = \sup(u).$$

Finally, consider $E_{T'}$: if $E_{T'} \Subset M$ then u is instantly escaping by Definition 4.3.1. Otherwise, we must have $E_{T'} = M$ by Lemma 4.3.4(i) again. Hence $M = \bigcup_{t<T'} E_t$ with $E_t \Subset M$ for each $t < T'$. This implies that u is sweeping.

It remains to show (4.3.2). Denote the right hand side of (4.3.2) by S. It is easier to see that $S \geqslant e^T P(E_0^+)$. Indeed, for all $t < T$ we have $P(E_t) = e^t P(E_0^+)$ by Lemma 2.1.10(ii). On the other hand, since E_t is outward minimizing in M, for each exhaustion $\{K_i\}$ we have $P(E_t) \leqslant P(K_i)$ for all large i. Hence $S \geqslant e^t P(E_0^+)$ for all $t < T$, which implies $S \geqslant e^T P(E_0^+)$.

Now we consider other direction $S \leqslant e^T P(E_0^+)$. Note that this is trivial when u is proper or sweeping, since the sub-level sets of u serve as a candidate exhaustion. The only remaining case is when u is instantly escaping. Recall that this means $E_T(u) \Subset M$ where $T = \sup(u)$. By exponential growth, we have

$$e^T P\big(E_0^+(u)\big) = P\big(E_T(u)\big).$$

For this case we need to recall the process of constructing maximal solutions. Fix a sequence of smooth domains $E_T(u) \Subset \Omega_1 \Subset \Omega_2 \Subset \cdots \Subset M$ and $\bigcup \Omega_l = M$. By Lemma 4.1.2, we know that the solutions u_l of $\mathsf{IVP}(\Omega_l; E_0)$+$\mathsf{OBS}(\partial\Omega_l)$ converge to u in $C^0_{\mathrm{loc}}(M \backslash E_0)$. By the maximality of u_l in each Ω_l, we have $E_T(u_l) \subset E_T(u) \Subset \Omega_l$. Then by the maximum principle (Theorem 2.1.12(iii)), we actually have $E_T(u_l) = E_T(u)$ for all l. For each $\varepsilon > 0$, note the following facts:

1. $\bigcup_{l \geqslant 1} E_{T+\varepsilon}(u_l) = M$. This follows from $u_l \xrightarrow{C^0_{\mathrm{loc}}} u \leqslant T$.

2. By sub-exponential growth (Corollary 3.3.14), we have

$$P\big(E_{T+\varepsilon}(u_l)\big) \leqslant e^{\varepsilon} P\big(E_T(u_l)\big) = e^{\varepsilon} P\big(E_T(u)\big) \qquad \text{for all } l.$$

Fix $p \in E_0$. We can inductively choose a sequence $l_k \to \infty$ so that

$$E_{T+1/k}(u_{l_k}) \Supset B(p,k) \cup E_{T+1/(k-1)}(u_{l_{k-1}}).$$

Setting $F_k = E_{T+1/k}(u_{l_k})$, we find that $\{F_k\}$ is a valid candidate exhaustion that shows $S \leqslant P\big(E_T(u)\big) = e^T P\big(E_0^+(u)\big)$. $\square$

Chapter 5

IMCF and scalar curvature

This chapter is devoted to applications of the weak IMCF to scalar curvature. For a Riemannian manifold (M, g), we use R_g to denote the scalar curvature of g (and we often write R for simplicity). We will call a Riemannian manifold *PSC*, if $R > 0$ everywhere. Let us recall that all manifolds are assumed to be oriented in this thesis.

It is helpful to know the following classical theorems regarding PSC 3-manifolds. For a detailed account, we refer to the book of Dan A. Lee [69]. The following theorem is due to Schoen-Yau, and is one of the most fundamental facts about scalar curvature.

Theorem 5.0.1 ([97]). *Let $\Sigma \subset M$ be a closed two-sided stable minimal surface.*
(i) If $R \geqslant 0$, then Σ is topologically either a 2-torus or a 2-sphere.
(ii) If $R \geqslant \lambda > 0$, then Σ must be a 2-sphere, with $|\Sigma| \leqslant 8\pi\lambda^{-1}$.

To prove Theorem 5.0.1, recall that the stability condition and traced Gauss equation imply that

$$\int_\Sigma |\nabla\varphi|^2 + K_\Sigma\varphi^2 \geqslant \frac{1}{2}\lambda\int_\Sigma \varphi^2, \qquad \forall\,\varphi \in C^\infty(\Sigma).$$

Then the result follows by taking $\varphi = 1$.

The following is a topological classification of closed PSC 3-manifolds.

Theorem 5.0.2 ([69, Theorem 1.29]). *If M^3 is a closed and PSC, then we have*

$$M \cong (S^2 \times S^1)\#\cdots\#(S^2 \times S^1)\#(S^3/\Gamma_1)\#\cdots\#(S^3/\Gamma_k),$$

where each S^3/Γ_k is a spherical space form, with some finite group Γ_k acting freely on S^3.

The following theorem, known as the *macroscopic dimension theorem*, characterizes the rough geometry of closed 3-manifolds with uniformly positive scalar curvature.

Theorem 5.0.3 ([48, Corollary 10.11], see also [73]).
If M^3 is closed and satisfies $R \geqslant \lambda > 0$, then there is a graph G, and a continuous map $f : M \to G$, such that $\mathrm{diam}(f^{-1}(p)) \leqslant 12\pi\lambda^{-1/2}$ *for all $p \in G$.*

Combining Theorem 5.0.2 and 5.0.3, we have the following description of a closed 3-manifold with $R \geqslant \lambda > 0$: M looks like a web in which each edge has S^2 section. The length of each edge is unrestricted but the width of a S^2 section is at most $C\lambda^{-1/2}$. See Figure 5.1 below.

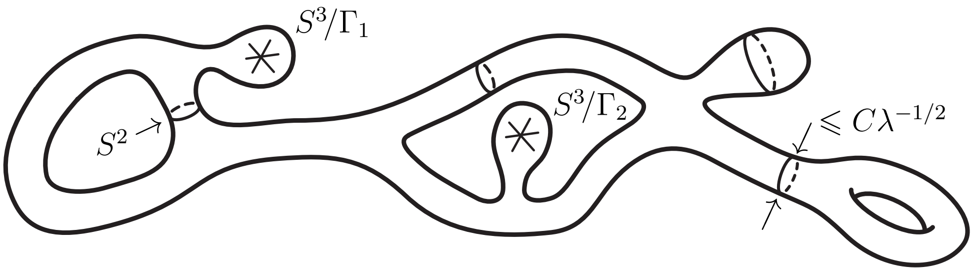

Figure 5.1: closed 3-manifolds with $R \geqslant \lambda > 0$.

5.1 More properties of the weak IMCF

This chapter concerns the weak IMCF inside 3-manifolds. By the regularity Lemma 2.1.7, each ∂E_t is a $C^{1,\alpha}$ hypersurface. Furthermore, we will not encounter delicate issues involving geometric measure theory. Hence, let us adopt the simpler notation

$$|\partial E| := P(E).$$

Recall the following property of maximal solutions. It also applies to proper solutions since proper solutions are maximal.

Lemma 5.1.1. *Let $E_0 \Subset M$ be connected, and u be the maximal solution of* $\mathsf{IVP}(M; E_0)$. *Then each E_t is connected, and $M \setminus E_t$ does not have compact connected components.*

Proof. This is a special case of Theorem 4.1.3(ii). □

This has the following topological consequence.

Corollary 5.1.2. *Suppose $E_0 \Subset M$ is connected, and u is a proper solution of* $\mathsf{IVP}(M; E_0)$. *Then for every $t > 0$, the map $H_2(\partial E_t, \mathbb{Z}) \to H_2(M \setminus E_0, \mathbb{Z})$ induced by embedding is injective.*

Proof. Denote $M_t = \overline{E_t} \setminus E_0$ and $M_t' = M \setminus E_t$. By Lemma 5.1.1 and the Lefschetz duality [50, Theorem 3.43 and Exercise 3.3.35], we have

$$H_3(M_t, \partial E_t, \mathbb{Z}) = H^0(M_t, \partial E_0, \mathbb{Z}) = 0.$$

Then, by Lemma 5.1.1 and Lefschetz duality again, we have

$$H_3(M_t', \partial E_t, \mathbb{Z}) = H_c^0(M_t', \mathbb{Z}) = 0.$$

Hence by the excision theorem [50, Theorem 2.20],

$$H_3(M \setminus E_0, \partial E_t, \mathbb{Z}) \cong H_3(M_t, \partial E_t, \mathbb{Z}) \oplus H_3(M_t', \partial E_t, \mathbb{Z}) = 0.$$

The corollary then follows from the long exact sequence of relative homology

$$0 = H_3(M \setminus E_0, \partial E_t, \mathbb{Z}) \to H_2(\partial E_t, \mathbb{Z}) \to H_2(M \setminus E_0, \mathbb{Z}) \to \cdots .$$ □

Next, we state the weak Geroch monotonicity formula:

Theorem 5.1.3.

Suppose M is a 3-manifold, $E_0 \Subset M$ is a $C^{1,1}$ initial value, and u is a proper solution of $\mathsf{IVP}(M; E_0)$. *Then each ∂E_t is a $C^{1,1}$ surface, and for all $0 \leqslant t_1 < t_2$, we have*

$$\int_{\partial E_{t_2}} H^2 \leqslant \int_{\partial E_{t_1}} H^2 + \int_{t_1}^{t_2} \Big[4\pi\chi(\partial E_s) - \int_{\partial E_s} R - \frac{1}{2}\int_{\partial E_s} H^2 \Big]\, ds. \tag{5.1.1}$$

Proof. The $C^{1,1}$ regularity is proved in Heidusch's thesis [51]. Inequality (5.1.1) is exactly [53, H^2 Growth Formula 5.7]. □

Solving the ODE inequality (5.1.1), we obtain:

Corollary 5.1.4. *Assume the same setup as in Theorem 5.1.3, and let $t > 0$. If $R \geqslant \lambda$ on $E_t \setminus E_0$, and E_0 is outward minimizing, then*

$$\int_{\partial E_t} H^2 \leqslant e^{-t/2} \int_{\partial E_0} H^2 + 4\pi e^{-t/2} \int_0^t e^{s/2}\chi(\partial E_s)\, ds - \frac{2}{3}\lambda|\partial E_0|\big(e^t - e^{-t/2}\big). \tag{5.1.2}$$

Proof. For $0 \leqslant s \leqslant t$, we denote $H(s) = \int_{\partial E_s} H^2$, $\chi(s) = \chi(\partial E_s)$, $F(s) = \int_s^t H(r)\, dr$. Since E_0 is outward minimizing, we have exponential growth $|\partial E_s| = e^s|\partial E_0|$. Applying (5.1.1) with $t_1 = s$, $t_2 = t$, and noting that $\int_{\Sigma_r} R \geqslant \lambda e^r |\partial E_0|$, for almost every s it holds

$$\begin{aligned} \frac{d}{ds}\big[e^{s/2}F(s)\big] &= e^{s/2}\Big[\frac{1}{2}F(s) - H(s)\Big] \\ &\leqslant 4\pi e^{s/2}\int_s^t \chi(r)\, dr - \lambda|\partial E_0|e^{s/2}\big(e^t - e^s\big) - e^{s/2}H(t). \end{aligned}$$

Integrating from 0 to t, this implies

$$-F(0) \leqslant 4\pi \int_0^t 2\big(e^{r/2} - 1\big)\chi(r)\, dr - \lambda|\partial E_0|\Big(\frac{4}{3}e^{3t/2} - 2e^t + \frac{2}{3}\Big) - 2\big(e^{t/2} - 1\big)H(t).$$

Combining this and another use of (5.1.1) with $t_1 = 0$, $t_2 = t$, we obtain (5.1.2). □

5.2 A topological gap theorem for the π_2-systole

Let M^3 be closed, connected, and oriented. If M has nonvanishing second homotopy group, then we define the π_2-*systole* of M by

$$\operatorname{sys}\pi_2(M, g) = \inf\Big\{|S^2|_{f^*g} : f : S^2 \to M \text{ is an immersion with } [f] \neq 0 \in \pi_2(M)\Big\},$$

where $\pi_2(M)$ denotes the set of free homotopy classes of maps $S^2 \to M$. By a theorem of Meeks and Yau [85], the π_2-systole is always achieved by a smooth minimizer (which is either an embedded sphere or a two-fold cover of an embedded $\mathbb{RP}^2$).

In [18], Bray, Brendle and Neves considered π_2-systolic inequalities in the context of uniformly positive scalar curvature. For manifolds M with nonvanishing second homotopy group, one has the sharp inequality

$$\operatorname{sys}\pi_2(M, g) \cdot \min_M R_g \leqslant 8\pi, \tag{5.2.1}$$

where R_g denotes the scalar curvature of g. For the rigidity case, it is proved in [18] that

$$\operatorname{sys}\pi_2 \cdot \min(R) = 8\pi \quad \Leftrightarrow \quad M \text{ is isometrically covered by a round } S^2 \times S^1. \tag{5.2.2}$$

Here, a round $S^2 \times S^1$ means the product of an S^1 with an S^2 with constant curvature. Inequality (5.2.1) follows by applying Theorem 5.0.1 to the area minimizer.

Analogues of (5.2.1) were obtained in some other contexts. In Bray-Brendle-Eichmair-Neves [17], the "$\mathbb{RP}^2$-systole" of a closed 3-manifold is considered, which is defined by $\mathcal{A} := \inf\{\text{area of embedded } \mathbb{RP}^2\}$. There it is proved that $\mathcal{A} \cdot \min(R) \leqslant 12\pi$, with the model case being a round $\mathbb{RP}^3$. The non-compact version of (5.2.1) was proved by Zhu [116] using Gromov's μ-bubble technique [47]. For the case of negative scalar curvature bounds, Nunes [94] and Lowe-Neves [75] proved sharp area lower bounds for high-genus minimizing surfaces. The rigidity cases there are characterized respectively by product metrics $\Sigma \times \mathbb{R}$ (where Σ has genus at least 2) and hyperbolic metrics.

The main theorem of this section is the following:

Theorem 5.2.1 (= Theorem F)**.** *Suppose M is a closed 3-manifold such that $\pi_2(M) \neq 0$ and M is not covered by $S^2 \times S^1$. Then for any metric g on M we have*

$$\operatorname{sys}\pi_2(M,g) \cdot \min_M R_g \leqslant 24\pi \cdot \frac{2-\sqrt{2}}{4-\sqrt{2}} \qquad (\approx 5.44\pi). \tag{5.2.3}$$

Thus, (5.2.1) is improved when we topologically exclude its rigidity case (5.2.2). In particular, the comparison between (5.2.1) and (5.2.3) shows that a universal gap is present in the π_2 –systolic inequality. To supply a further geometric understanding, we state the following local version:

Theorem 5.2.2. *Let M be diffeomorphic to the 3-sphere with k balls removed ($k \geqslant 3$). Suppose g is a smooth metric on M such that*

(i) all the components of ∂M are stable minimal surfaces;

(ii) in the interior of M, there is no embedded stable minimal surface representing a nontrivial element in the integral homology $H_2(M,\mathbb{Z})$.

Let A_0 be the minimum of the area of all connected components of ∂M. Then

$$A_0 \cdot \min_M(R_g) \leqslant 24\pi \cdot \frac{2-\sqrt{2}}{4-\sqrt{2}}. \tag{5.2.4}$$

Theorem 5.2.2 suggests that Theorem 5.2.1 is localized at the "non-cylindrical locus" of the manifold. Let M have positive scalar curvature and satisfy the hypotheses of Theorem 5.2.1. For simplicity, let us assume $M = (S^2 \times S^1)\#(S^2 \times S^1)$. By cutting along a maximal disjoint collection of stable minimal surfaces, we decompose M into building pieces (this is possible for generic metrics [111]; see similar arguments in [73, 103]). Each building piece is diffeomorphic to a sphere with disks removed, and its boundary consists of stable minimal spheres. See Figure 5.2 for an example of decomposition, in comparison with a decomposition of $S^2 \times S^1$. The topology of M implies the following: there must exist a

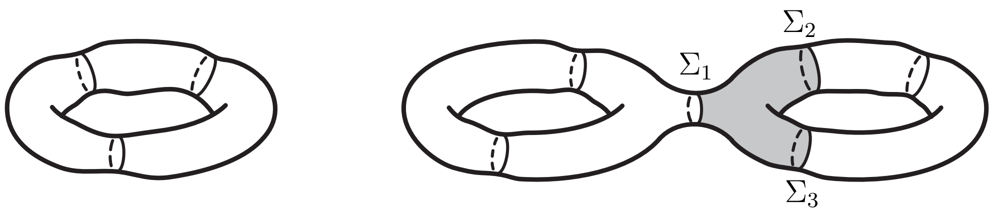

Figure 5.2: Decompositions of $S^2 \times S^1$ and $(S^2 \times S^1)\#(S^2 \times S^1)$.

non-cylindrical piece, namely, a piece with three or more boundary components (such as the shaded piece in Figure 5.2). Theorem 5.2.2 then yields $\min\{|\Sigma_1|, |\Sigma_2|, |\Sigma_3|\} \leqslant 5.44\pi$

for such a piece. Furthermore, we can arrange the decomposition such that all the Σ_i are nontrivial in $\pi_2(M)$. Thus this shows a way to recover the global inequality (5.2.3) from the local one (5.2.4). See Subsection 5.2.3 for the full argument. We remind that the formal proof of the main theorems goes in the opposite direction: we first prove Theorem 5.2.1, then obtain Theorem 5.2.2 via a doubling argument.

This result is also partially motivated by the stability problem for scalar curvature. Concerning the inequality (5.2.1), the stability problem asks the following: if we relax the rigidity condition in (5.2.2) to almost rigidity, i.e. if we impose $\operatorname{sys}\pi_2 \cdot \min(R) \geqslant 8\pi - \varepsilon$, whether the manifold M remains close to a round cylinder (or its quotient) in some sense. In this respect, Theorem 5.2.1 confirms that M must be topologically the same as rigidity, thus establishing a topological stability. On the other hand, the metric stability of (5.2.1) remains an open question. The weak control of scalar curvature on the metric allows various pathological phenomena. We refer the reader to the survey of Sormani [103], and references therein, for an overview of this topic.

Let us briefly sketch the proof of Theorem 5.2.1. Let $\Sigma \subset M$ an area minimizer in $\pi_2(M)$. Consider the universal cover of M (denoted by $\widetilde{M}$), and a lift of Σ onto $\widetilde{M}$ (denoted by $\widetilde{\Sigma}$). Then we solve a weak IMCF with initial value $\widetilde{\Sigma}$. Figure 5.3 below shows the behavior of the flow in the universal cover of $(S^2\times S^1)\#(S^2\times S^1)$, which resembles the shape of a binary tree. So the IMCF starts with $\widetilde{\Sigma}$ which sits at the bottom of the figure, and the flow moves upwards. The jumpings are marked with shadows in the figure. At

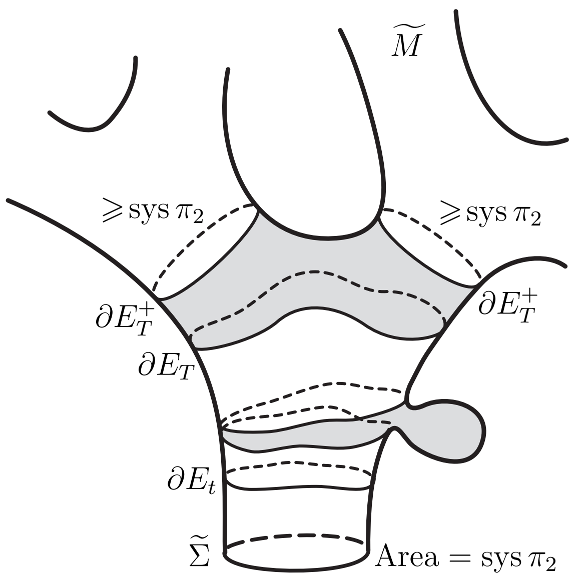

Figure 5.3: The weak IMCF in $\widetilde{M}$ starting with $\widetilde{\Sigma}$.

a certain time T, the surface splits into two components which will individually continue evolving in their own branches. There are several key points to be mentioned.

1. The existence of proper solution is a consequence of Theorem A and the exponential growth of $\widetilde{M}$. Meanwhile, one may observe that there does not exist any proper IMCF on the universal cover of $S^2\times S^1$. This is where the topological condition gets involved.

2. The splitting time T satisfies $T \geqslant \log 2$. Roughly speaking, this is because the IMCF starts from a surface with area $\operatorname{sys}\pi_2(M)$, and when it splits, the area is at least $2\operatorname{sys}\pi_2(M)$ (each spherical component is nontrivial in π_2).

Having these two observations, the main theorem follows by using the Geroch monotonicity formula for $t \in [0, \log 2]$.

The remaining part of this section is organized as follows. In Subsection 5.2.1 we prove Theorem 5.2.1, and in Subsection 5.2.2 we prove Theorem 5.2.2. Finally, in Subsection 5.2.3 we make precise the decomposition mentioned above.

5.2.1 Proof of Theorem 5.2.1

Since Theorem 5.2.1 is vacuous when $R \leqslant 0$ somewhere, we may assume $\min R = \lambda > 0$. Denote $A_0 = \operatorname{sys}\pi_2(M)$. Taking a double cover if necessary, we assume that M has no $\mathbb{RP}^3$ factors. Then by a theorem of Meeks and Yau [85], there exists an embedded sphere $\Sigma \subset M$ with $|\Sigma| = A_0$. Let us fix this choice of Σ.

By Theorem 5.0.2, M can topologically be written as

$$M = (S^2 \times S^1)\#\cdots\#(S^2 \times S^1)\#(S^3/\Gamma_1)\#\cdots\#(S^3/\Gamma_k). \tag{5.2.5}$$

Based on this classification, there are three possible cases of the growth of $\pi_1(M)$:

(1) M is a spherical space form, for which $\pi_1(M)$ is finite.
(2) M is either $S^2 \times S^1$ or $\mathbb{RP}^3\#\mathbb{RP}^3$, for which $\pi_1(M)$ is virtually cyclic.
(3) All the remaining cases, where $\pi_1(M)$ has exponential growth.

Therefore, the topological condition in Theorem 5.2.1 implies case (3).

Let $\widetilde{M}$ be the universal cover of M. By a result of Coulhon and Saloff-Coste [35], the growth of M implies that $\widetilde{M}$ supports a Euclidean isoperimetric inequality

$$|\partial E| \geqslant c|E|^{2/3}, \qquad \forall E \Subset \widetilde{M}.$$

Thus by Theorem A, proper IMCF exists for all initial data $E_0 \Subset \widetilde{M}$.

Before running the weak IMCF, we make the following setups. Let $\widetilde{\Sigma}$ be an isometric lift of Σ onto $\widetilde{M}$. Note that $\widetilde{\Sigma}$ must be separating. Otherwise, there is a loop in $\widetilde{M}$ that intersects $\widetilde{\Sigma}$ once, violating $\pi_1(\widetilde{M}) = 0$. Denote the two connected components of $\widetilde{M} \setminus \widetilde{\Sigma}$ by $\widetilde{M}_1$ and $\widetilde{M}_2$.

Claim 1. Both $\widetilde{M}_1$, $\widetilde{M}_2$ are non-compact.

Proof. It follows from van Kampen's theorem that $\pi_1(\widetilde{M}_1) = 0$. If $\widetilde{M}_1$ is compact, then by the relative Poincare duality and the long exact sequence of relative cohomology, we have $H_2(\widetilde{M}_1, \mathbb{Z}) = H^1(\widetilde{M}_1, \partial\widetilde{M}_1, \mathbb{Z}) = H^1(\widetilde{M}_1, \mathbb{Z}) = 0$. Hence $\pi_2(\widetilde{M}_1) = 0$ by Hurewicz's theorem. This contradicts with the fact that $[\widetilde{\Sigma}] \neq 0 \in \pi_2(\widetilde{M})$. For the same reason, $\widetilde{M}_2$ is noncompact. □

Let $E_0' \subset \widetilde{M}_1$ be a small (one-sided) collar neighborhood of $\widetilde{\Sigma}$ in $\widetilde{M}_1$, and u' be the proper solution of $\mathsf{IVP}(\widetilde{M}; E_0')$. As the part of the solution in $\widetilde{M}_1$ is not utilized and causes inconvenience, we excise it from the manifold in the following way. Consider a new Riemannian manifold $N = \widetilde{M}_2 \cup_{\widetilde{\Sigma}} D$ (D denotes a 3-disk), where we arbitrarily extend the metric on $\widetilde{M}_2$ into D. See Figure 5.4 for a depiction. Let $u \in \mathrm{Lip}_{\mathrm{loc}}(N)$ be such that $u|_{\widetilde{M}_2} = u'$ and $u|_D < 0$. By checking Definition 2.1.8(1), one verifies that u is a proper solution of $\mathsf{IVP}(N; E_0)$ with $E_0 := D$. We shall be working with the solution u.

Claim 2. E_0 is outward minimizing in N.

Proof. By Theorem 2.5.3, E_0' admits a precompact minimizing hull F_0' in $\widetilde{M}$. Then by Theorem A.4.5, E_0' is a least area solution outside E_0 in $\widetilde{M}$ (see Definition A.4.3). Directly

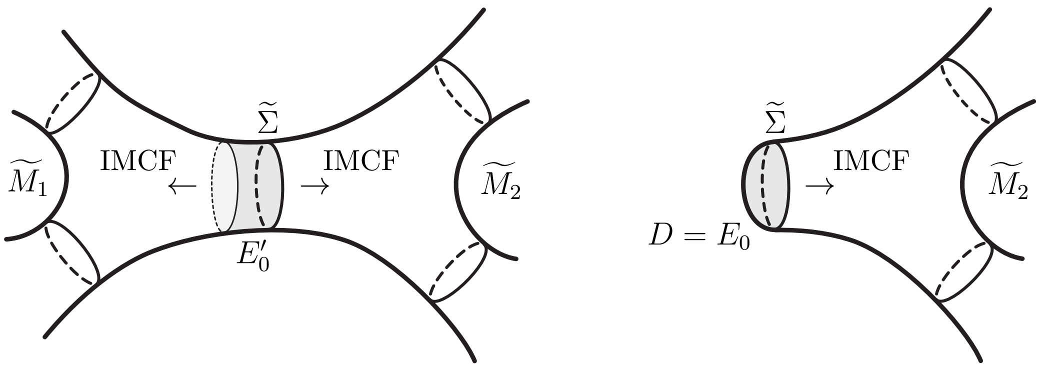

Figure 5.4: Before and after the excision.

from this definition, it follows that $F_0 := (F_0' \cap \widetilde{M}_2) \cup E_0$ solves the least area outside E_0 in N. The claim follows if we can show that $|\partial F_0| \geqslant |\partial E_0|$.

By the strong maximum principle, F_0 either coincides with E_0 or has a stable minimal boundary in $N \setminus \overline{E_0} \cong \widetilde{M}_2$. The former case immediately implies the claim. Suppose that the latter holds. Then we have $[\partial F_0] = [\widetilde{\Sigma}]$ viewed as elements in $H_2(\widetilde{M}, \mathbb{Z})$, which is nonzero by the Hurewicz theorem. Since $\widetilde{M}$ has positive scalar curvature, all connected components of ∂F_0 are spherical. Hence at least one component of ∂F_0 is nonzero in $\pi_2(\widetilde{M})$. Finally, we obtain $|\partial F_0| \geqslant \operatorname{sys} \pi_2(\widetilde{M}) = \operatorname{sys} \pi_2(M) = A_0 = |\partial E_0|$. This shows that E_0 is outward minimizing. □

Claim 2 and exponential growth then imply that $|\partial E_t| = e^t A_0$ for all $t > 0$. Define

$$T = \inf\big\{t > 0 : \partial E_t \text{ has at least two spherical connected components}\big\}.$$

Claim 3. Each spherical component of ∂E_t ($t > 0$) has area at least A_0.

Proof. Let Σ' be a spherical component of ∂E_t, which we may also view as a surface in $\widetilde{M}_2$ or in $\widetilde{M}$. By Corollary 5.1.2, $[\Sigma']$ is nonzero in $H_2(N \setminus E_0, \mathbb{Z}) \cong H_2(\widetilde{M}_2, \mathbb{Z})$. By the fact that $\widetilde{M}_1$ is noncompact (Claim 1) and the long exact sequence of relative homology

$$0 = H_3(\widetilde{M}, \widetilde{M}_2, \mathbb{Z}) \to H_2(\widetilde{M}_2, \mathbb{Z}) \to H_2(\widetilde{M}, \mathbb{Z}),$$

$[\Sigma']$ is nonzero in $H_2(\widetilde{M}, \mathbb{Z})$. Since Σ' is spherical, it is a nonzero element in $\pi_2(\widetilde{M})$. Hence $|\Sigma'| \geqslant \operatorname{sys} \pi_2(\widetilde{M}) = \operatorname{sys} \pi_2(M) = A_0$. □

By the definition of T, there is a sequence $t_i \to 0$ such that ∂E_{T+t_i} has more than one spherical component. Therefore $e^{T+t_i} A_0 = |\partial E_{T+t_i}| \geqslant 2A_0$. Letting $i \to \infty$ it follows that $T \geqslant \log 2$. Therefore, $\chi(\partial E_t) \leqslant 2$ for all $0 \leqslant t < \log 2$. Finally, we utilize the monotonicity formula (5.1.2) to obtain

$$0 \leqslant \int_{\partial E_t} H^2 \leqslant 16\pi\big(1 - e^{-t/2}\big) - \frac{2}{3}\lambda A_0\big(e^t - e^{-t/2}\big), \quad \forall\, t \leqslant \log 2.$$

Taking $t = \log 2$ we have

$$0 \leqslant 16\pi\big(1 - \frac{1}{\sqrt{2}}\big) - \frac{2}{3}\lambda A_0\big(2 - \frac{1}{\sqrt{2}}\big),$$

which implies (5.2.3). This completes the proof of Theorem 5.2.1.

5.2.2 Proof of Theorem 5.2.2

We deduce Theorem 5.2.2 from Theorem 5.2.1, using a doubling argument. We can assume $\min_M R = \lambda > 0$. Let Σ_i $(1 \leqslant i \leqslant k)$ be all the connected components of ∂M. Denote $h_i = g|_{\Sigma_i}$ the restricted metrics on Σ_i, and choose $\varphi_i > 0$ as the first eigenfunctions of the stability operator for Σ_i. Therefore φ_i satisfy

$$\Delta_{h_i}\varphi_i \leqslant \Big(K_{h_i} - \frac{1}{2}\lambda\Big)\varphi_i,$$

where K denotes the Gauss curvature. For T a sufficiently large constant to be chosen, let P_i be diffeomorphic to the cylinders $\Sigma_i \times [-T, T]$ and equipped with the warped product metrics $g_i = h_i + \varphi_i^2 dt^2$ $(-T \leqslant t \leqslant T)$. The scalar curvature of P_i is given by

$$R_{g_i} = 2\Big(K_{h_i} - \frac{\Delta_{h_i}\varphi_i}{\varphi_i}\Big) \geqslant \lambda,$$

see [69, Proposition 1.13]. Let $M^{\pm}$ be identical copies of M, whose metrics are still denoted by g. Set

$$N = M^- \cup_{\partial M^-} \Big(\bigsqcup_{i=1}^{k} P_i\Big) \cup_{\partial M^+} M^+,$$

equipped with the Lipschitz metric g_N that agrees with g on $M^{\pm}$ and agrees with g_i on P_i. Topologically, N is a connected sum of $k-1$ copies of $S^2 \times S^1$.

Given any $0 < \varepsilon < 1/100$, we claim the following:

Claim 1. There is a sufficiently large T for which the following holds. Suppose g_N' is any smooth metric on N, such that $g_N' = g_i$ in each P_i, and $||g_N' - g_N||_{C^0(g_N)} \leqslant \varepsilon^3$. Then $\operatorname{sys}\pi_2(N, g_N') \geqslant (1-\varepsilon)A_0$.

Claim 2. There exists a smooth metric g_N' that satisfies the hypotheses of Claim 1, and moreover has $R_{g_N'} \geqslant \lambda - \varepsilon$.

The smoothness of g_N' in the claims makes sense as a set of coordinate charts across $\partial M^{\pm}$ will be specified in the proof of Claim 2. With the two claims, it follows from Theorem 5.2.1 that $(\lambda - \varepsilon)\cdot(1-\varepsilon)A_0 \leqslant c$, which by $\varepsilon \to 0$ proves Theorem 5.2.2.

Proof of Claim 1.

We start with proving two technical facts.

Claim 3. There exists a constant c_0 independent of T such that: if $t \in (-T+2, T-2)$, $1 \leqslant i \leqslant k$, and $S \subset N$ is a closed g_N'-minimal surface that intersects with $\Sigma_i \times (t-1, t+1) \subset P_i$, then $\big|S \cap \big(\Sigma_i \times (t-2, t+2)\big)\big|_{g_i} \geqslant c_0$.

Proof. Let $r_0 > 0$ be such that $B_{g_i}(x, r_0) \subset \Sigma_i \times (t-1.5, t+1.5)$ for all $x \in \Sigma_i \times (t-1, t+1)$. By the classical monotonicity formula [85, Lemma 1], there exists a constant $c > 0$ such that $|S \cap B(x,r)|_{g_i} \geqslant cr^n$ for all $x \in S \cap \big(\Sigma_i \times (t-1, t+1)\big)$ and $r \in [0, r_0]$. The constants c, r_0 depend only on the geometry of $\Sigma_i \times (t-2, t+2)$, thus not on T by the translation symmetry. Now $c_0 = c_1 r_0^n$ has the desired property.

Claim 4. Set $T = 16A_0/c_0$. Then an g_N'-area minimizer in $\pi_2(N)$ either coincides with $\Sigma_i \times \{t\} \subset P_i$ for some i, or does not intersect any P_i.

Proof. Let S be such a minimizer. Suppose S intersects with the interior of some P_i. By the strong maximum principle, S either intersects with $\Sigma_i \times \{t\}$ for every $-T < t < T$, or coincides with $\Sigma_i \times \{t\}$ for some t. If the former case happens, then applying Claim 1 to each $\Sigma_i \times [4j, 4j+4]$, $|j| \leqslant \lfloor T/4 \rfloor$, we obtain

$$|S|_{g'_N} \geqslant |S \cap \big(\Sigma_i \times (-T, T)\big)|_{g_i} \geqslant c_0 \cdot 2\lfloor \frac{T}{4} \rfloor > 2A_0.$$

This contradicts the minimality of S, since one of the Σ_i has area A_0.

Back to the proof of claim 1, we choose T as in Claim 4. Let S be a g'_N-area minimizer in $\pi_2(N)$, which is a smoothly embedded sphere. Thus Claim 4 applies to S. If S is a horizontal slice in some P_i, then $|S|_{g'_N} = |\Sigma_i|_{h_i} \geqslant A_0$, implying Claim 1. Now we may assume that S does not intersect all the P_i.

By the connectedness of S, we can assume without loss of generality that $S \subset M^-$. Since M^- is topologically a sphere with disks removed, and S does not bound a 3-ball in M^-, it follows that S represents a nonzero element in $H_2(M^-, \mathbb{Z})$. Now let S' be a g-area minimizer in the homology class of S in $H_2(M^-, \mathbb{Z})$. We note that S' must be a union of components of ∂M^-. Indeed, any interior component of S' is a smooth minimal surface, hence must be homologically trivial by the theorem's assumption. Thus we can decrease the area by removing this component, contradicting the minimizing property of S'. We then have $|S'|_g \geqslant A_0$. Combining what we obtained above, we have

$$A_0 \leqslant |S'|_g \leqslant |S|_g = |S|_{g_N} \leqslant (1+\varepsilon^2)|S|_{g'_N} = (1+\varepsilon^2)\operatorname{sys}\pi_2(N, g'_N),$$

which proves the claim. In the third inequality, we used the assumption $||g'_N - g_N||_{C^0(g_N)} \leqslant \varepsilon^3$. $\square$

Proof of Claim 2.

The proof involves smoothing g_N near $\partial M^\pm$ while preserving scalar curvature lower bounds. By symmetry, it suffices to perform the smoothing near ∂M^-. We invoke the following gluing theorem in Brendle-Marques-Neves [23].

Theorem 5.2.3 ([23, Theorem 5]). *Let M be a compact manifold with boundary ∂M, and $g, \widetilde{g}$ be two smooth metrics such that $g - \widetilde{g} = 0$ at each point on ∂M. Moreover, assume that $H_g - H_{g'} > 0$ at each point on ∂M. Then given any number $\varepsilon > 0$ and any neighborhood U of ∂M, there is a smooth metric $\hat{g}$ on M with the following property:*

(1) the scalar curvature of $\hat{g}$ satisfies $R_{\hat{g}}(x) \geqslant \min\{R_g(x), R_{\widetilde{g}}(x)\} - \varepsilon$ for every $x \in M$.

(2) $\hat{g}$ agrees with g outside U.

(3) $\hat{g}$ agrees with $\widetilde{g}$ in some neighborhood of ∂M.

(4) $||\hat{g} - g||_{C^0(g)} \leqslant \varepsilon^3$.

Item (4) is not included in the original statement but follows directly from the proof. To achieve this, we choose the tensor T to vanish outside a sufficiently small neighborhood of ∂M in [23, p.189], then choose the parameter λ to be sufficiently large in [23, p.190].

To apply the theorem, we need to perturb g so that ∂M^- is strictly mean convex. Let δ be a small constant to be chosen. Choose any function $u \in C^\infty(\overline{M})$ such that $||u||_{C^2(g)} \leqslant \delta$, $u|_{\partial M} = 0$ and $\frac{\partial u}{\partial \nu} > 0$ on ∂M. Set $g' = e^{2u}g$. The last condition of u ensures that ∂M^- is mean convex with respect to g'. We have $||g' - g||_{C^0(g)} \leqslant \varepsilon^3$ and $R_{g'} \geqslant \lambda - \varepsilon/2$ when δ is sufficiently small.

Next, we need to construct the metric $\widetilde{g}$ which extends g_i smoothly into the interior of M^-. We slightly enlarge P_i to the cylinders $Q_i = \Sigma_i \times (-T-\delta, T]$, on which the metrics are still of warped product form $g_i = h_i + \varphi_i^2 dt^2$. Let $\Phi_i : \Sigma_i \times (-T-\delta, -T] \to M^-$ be a regular smooth embedding, such that Φ_i maps $\Sigma_i \times \{-T\}$ identically to $\Sigma_i \subset \partial M^-$, and

$\Phi_i^* g' = g_i$ on $\Sigma_i \times \{-T\}$. Such a map can be constructed using normal exponential maps. Let $V_i \subset M^-$ be the image of Φ_i. Thus, the metrics $\widetilde{g}_i = (\Phi_i^{-1})^* g_i$ are defined in V_i and coincides with g' on $\Sigma_i \subset \partial M^-$. Moreover, Σ_i is totally geodesic with respect to $\widetilde{g}_i$, hence $H_{g'} - H_{\widetilde{g}_i} > 0$ on Σ_i. Let $\widetilde{g}$ be an arbitrary metric on M^- that is equal to $\widetilde{g}_i$ in a smaller neighborhood $U_i \subset V_i$ of Σ_i. Note that $R_{\widetilde{g}} = R_{\widetilde{g}_i} = (\Phi_i^{-1})^* R_{g_i} \geqslant \lambda$ in U_i. Now we apply Theorem 5.2.3 in the neighborhood $\bigcup_i U_i \supset \partial M^-$ and with g' in place of g. We obtain a new metric $\hat{g}$ on M^-, such that $||\hat{g} - g'||_{C^0(g')} \leqslant \varepsilon^3$ and $R_{\hat{g}} \geqslant \lambda - \varepsilon$.

Finally, set g'_N to be equal to $\hat{g}$ on M^-, and equal to g_i on P_i. Thus g'_N satisfies the requirements of Claim 2 (on the side of M^-) once we specify a smooth structure across ∂M^- for which g'_N is smooth. We express the space $X = M^- \cup_{\partial M^-} (\bigcup P_i)$ as

$$X = \Big(\mathrm{int}(M^-) \sqcup \big(\bigcup Q_i\big)\Big) \Big/ \sim,$$

where $\sim$ is the equivalence relation $x \sim \Phi_i(x)$, $\forall x \in \Sigma_i \times (-T - \delta, -T) \subset Q_i$. Since the relation $\sim$ is given by diffeomorphism, a smooth structure on X is naturally induced. It follows from Theorem 5.2.3 (3) that g'_N is smooth under this smooth structure. This completes the smoothing near ∂M^-, and the smoothing near ∂M^+ follows by symmetry. This proves the claim. $\square$

5.2.3 A decomposition argument

We previously deduced Theorem 5.2.2 from 5.2.1. Now we make the reverse implication, by using the decomposition argument mentioned at the beginning of this section.

Assume that Theorem 5.2.2 holds. Let M satisfy the assumptions of Theorem 5.2.1. Following the idea in the introduction, we shall show that (5.2.3) holds for any metric g on M. We may assume without loss of generality that g has positive scalar curvature; thus the topological classification (5.2.5) is available. The argument is divided into several steps. We use $\#^k(S^2 \times S^1)$ to denote the connected sum of k copies of $S^2 \times S^1$.

Step 1: we show that M is finitely covered by $\#^k(S^2 \times S^1)$ for some k. Suppose first that M does not contain $S^2 \times S^1$ prime factors, thus $M \cong (S^3/\Gamma_1)\# \cdots \#(S^3/\Gamma_m)$ for some finite groups Γ_i. By Kurosh's subgroup theorem [80, Theorem VII.5.1], the kernel of the quotient map $\pi_1(M) \mapsto \Gamma_1 \times \cdots \times \Gamma_m$ is a free group with finite index. Hence M is finitely covered by a closed 3-manifold $\overline{M}$ with free fundamental group. By the classification (5.2.5) again, we see that $\overline{M} \cong \#^k(S^2 \times S^1)$ for some k. For the general case, we may write $M = \#^k(S^2 \times S^1)\#N$, where N is a connected sum of spherical space forms. From the special case, N is covered by $\overline{N} = \#^m(S^2 \times S^1)$ for some m. This implies that M is convered by $\#^{k\deg(\overline{N}\to N)}(S^2 \times S^1)\#\overline{N}$.

Step 2: we decompose M into building pieces. Since the π_2 −systole remains unchanged when passing to covering spaces, we may assume $M = \#^k(S^2 \times S^1)$. By slightly perturbing the metric g, we assume that it is bumpy [111]. Let $\{\Sigma_1, \cdots, \Sigma_m\}$ be a maximal pairwise disjoint collection of stable minimal surfaces in M, such that each Σ_i is nontrivial in $\pi_2(M)$. Such collection exists and is finite by the bumpiness hypothesis. We denote by $\{U_1, \cdots, U_n\}$ the set of connected components of $M \setminus \bigcup_{i\leqslant m} \Sigma_i$. The boundary of each U_i consists of stable minimal spheres. Let V_i be the closed manifold obtained by filling the boundaries of U_i with 3-balls. Notice that M can be recovered from $\bigsqcup_{i\leqslant n} V_i$ by performing 0-surgeries. Here, a 0-surgery means removing two disks and gluing the common sphere boundaries. If the two disks are in different (resp. the same) connected components of the manifold, then the 0-surgery is equivalent to a connected sum of the two components (resp. a connected sum of that component with $S^2\times S^1$). Therefore, from the 0-surgery we obtain

$$M\cong\#^{m+1-n}(S^2\times S^1)\#V_1\#\cdots\#V_n.$$

By the uniqueness of prime decomposition, each U_i is diffeomorphic to either a 3-sphere with punctures or a $\#^s(S^2\times S^1)$ with punctures (for some $s\leqslant k$).

We argue that the interior of each U_i does not contain stable minimal surfaces that are nontrivial in $H_2(U_i,\mathbb{Z})$. Suppose otherwise that there is a counterexample $\Sigma'\subset U_1$. In particular, Σ' does not intersect any Σ_i. By the maximality of $\{\Sigma_i\}$, Σ' must be trivial in $\pi_2(M)$. Then Σ' must bound a simply-connected region (otherwise, any lift of Σ' to the universal cover $\widetilde{M}$ will be nontrivial in $H_2(\widetilde{M},\mathbb{Z})$, contradicting the homotopy triviality of Σ'). By the uniqueness of prime decomposition, Σ' bounds a 3-disk $D\subset M$. By the defining property of Σ', the disk D must contain one of the surfaces Σ_i. However, this implies that Σ_i bounds a 3-disk, contradicting its homotopy triviality. This proves our claim.

Note that, our claim implies that each U_i must be a punctured S^3. Otherwise, one finds an area minimizer in $H_2(U_i,\partial U_i,\mathbb{Z})$ to obtain a contradiction.

Step 3: there exists a piece U_1 with at least three boundary components. Otherwise, all the U_i are diffeomorphic to $S^2\times[0,1]$, which implies $M\cong S^2\times S^1$ and contradicts the assumption of Theorem 5.2.1. Suppose $\Sigma_1,\cdots,\Sigma_l$ ($l\geqslant 3$) are the boundaries of U_1. By the previous step, Theorem 5.2.2 is applicable to U_1 and yields $\min_{i\leqslant l}|\Sigma_i|_g\cdot\min_{U_1}R_g\leqslant c_0$, where c_0 is the constant on the right hand side of (5.2.3). In particular, we have

$$\operatorname{sys}\pi_2(M,g)\cdot\min_M R_g\leqslant\min_{i\leqslant l}|\Sigma_i|_g\cdot\min_{U_1}R_g\leqslant c_0.$$

This shows Theorem 5.2.1.

5.3 PSC 3-manifolds with bounded curvature

This section is based on joint work with O. Chodosh and Y. Lai [30].

Theorem 5.3.1 (= Theorem G, = [30, Theorem 1.1])**.**
Let (M^3,g) be complete, connected, contractible, satisfy $R\geqslant 0$, and has bounded geometry:

$$|\operatorname{Rm}|\leqslant\Lambda^2,\qquad \operatorname{inj}\geqslant\Lambda^{-1}. \tag{5.3.1}$$

Then M is diffeomorphic to $\mathbb{R}^3$.

This result fits into the general theme of classifying the topology of PSC 3-manifolds. As in Theorem 5.0.2, the compact case is well understood. What remains widely open is the noncompact case. In this direction, the following results are relevant:

- Schoen-Yau [98, Theorem 4]: if $\pi_1(M)$ contains the fundamental group of a closed surface of genus $\geqslant 1$, then M does not admit complete PSC metrics.
- J. Wang [109]: the Whitehead manifold does not admit complete PSC metrics. In general, if M is contractible and is a nested union of solid tori, and M admits complete PSC metrics, then $M\cong\mathbb{R}^3$ [30, Lemma 2.4].

The proof of these results involve the minimal surface technique which dates back to Schoen-Yau [97]. Another notable work is:

- J. Wang [108]: if M^3 admits a complete metric with $R \geqslant 1$, then M is diffeomorphic to a (possibly infinite) connected sum of spherical space forms and $S^2 \times S^1$. In particular, if M is further contractible, then $M \cong \mathbb{R}^3$. This result was earlier obtained by Bessiéres–Besson–Maillot [16] assuming an additional bounded geometry condition (the tool in [16] is Ricci flow).

Despite these results, it remain open whether $\mathbb{R}^3$ is the only contractible 3-manifold admitting complete PSC metrics (see [115, Question 27]). Our Theorem 5.3.1 resolves this question assuming (5.3.1).

It turns out that the key to proving Theorem 5.3.1 is to find exhaustions of M by spheres or tori. Indeed, we shall prove the following key intermediate result:

Theorem 5.3.2. *Let M^3 be complete, connected, contractible, and satisfies $R > 0$ and (5.3.1). Then there is an exhaustion $\Omega_1 \Subset \Omega_2 \Subset \cdots \Subset M$, where either each $\partial\Omega_i$ is a 2-sphere, or each $\partial\Omega_i$ is a 2-torus.*

The deduction of Theorem 5.3.1 from Theorem 5.3.2 is not related to IMCF. Thus, we direct the reader to [30, Lemma 2.3, 2.4] for its proof. The rest of this section is devoted to the proof of Theorem 5.3.2.

The following topological lemma will be useful in the proof.

Lemma 5.3.3. *Suppose M is contractible and with dimension $\geqslant 2$. Then:*
(i) M has only one end.
(ii) If $E \Subset M$ is a connected C^1 domain such that $M \setminus E$ does not have compact connected components, then ∂E is connected.

Proof. (i) If M has more than one end, then there is a (possibly non-connected) closed hypersurface $\Sigma \subset M$ so that $M \setminus \Sigma$ has two connected components which are both noncompact. This implies $[\Sigma] \neq 0 \in H_{n-1}(M;\mathbb{Z})$, contradiction.

(ii) Since M is one-ended, $M \setminus E$ must be connected. If ∂E is disconnected, then there is a component of ∂E that has nonzero algebraic intersection with a closed loop γ. But this contradicts the contractibility of M. □

Now we start proving Theorem 5.3.2. In the work of Wang [108] mentioned above, the main tool is the μ-bubble introduced by Gromov [47]. Under the assumption $R \geqslant 1$ in [108], μ-bubbles can be used to effectively find exhaustions with spherical boundaries. However, μ-bubbles are not applicable to the case of Theorem 5.3.2 since the scalar curvature is not uniformly positive. Here, the exhaustion in Theorem 5.3.2 is found by running an innermost IMCF u from a small geodesic ball.

Proof of Theorem 5.3.2.

Without loss of generality, we may further assume that

$$\mathrm{inj} \geqslant \Lambda_0^{-1}, \qquad |\nabla^k \operatorname{Rm}| \leqslant \Lambda_k, \qquad \forall\, k \geqslant 0, \tag{5.3.2}$$

for a sequence of constants $\{\Lambda_k\}$. This follows from Shi's classical Ricci flow result [99, Theorem 1.2], and the preservation of PSC under complete Ricci flow with bounded curvature [33, Section 12.5].

So let us assume that M is complete, connected, contractible, satisfying $R > 0$ and (5.3.2). Fix $p \in M$. Let $E_0 = B(p, r_0)$ and u be the innermost solution of $\mathsf{IVP}(M; E_0)$. By Lemma 5.3.3(i) above, M is one-ended. Then by Theorem 4.3.2, we may choose r_0

sufficiently small so that u is either proper, sweeping, or instantly escaping (see Definition 4.3.1). We prove the main theorem in three cases.

Case 1: Suppose u is proper. Since ∂E_0 is connected and M is contractible, by Lemma 5.1.1 and 5.3.3(ii) we see that all the ∂E_t ($t > 0$) are connected. By the monotonicity formula (5.1.1), we have

$$\int_0^t 4\pi\chi(\partial E_s)\,ds \geqslant \int_{\partial E_t} H^2 - \int_{\partial E_0} H^2 \geqslant -\int_{\partial E_0} H^2\,d\mu_0, \quad \forall\, t > 0.$$

Thus there exists a sequence $t_i \to \infty$ such that each ∂E_{t_i} is either an S^2 or T^2, for all i. Since $u \in \mathrm{Lip}_{\mathrm{loc}}(M)$, the sets E_{t_i} must form an exhaustion of M. The result follows by taking a subsequence.

Case 2: Suppose u is sweeping. Then recall from Definition 4.3.1 that there exists $T \in (0,\infty)$ such that $E_t \Subset M$ for all $t < T$, and $E_T = \bigcup_{t\in[0,T)} E_t = M$. By Lemma 5.1.1 and 5.3.3(ii), each ∂E_t is connected. Then by Lemma 4.3.3(ii) and 4.3.4(ii), for all $t \in [0,T)$ we have that ∂E_t is uniformly $C^{1,\alpha}$-bounded and has uniformly bounded diameters. Thus, we can select a sequence $t_i \nearrow T$ such that

$$\lim_{i\to\infty} d\big(\partial E_{t_{i-1}}, \partial E_{t_i}\big) = \infty. \tag{5.3.3}$$

Next, recall that each E_{t_i} is outward area-minimizing. Hence for each i and set F with $E_{t_{i-1}} \subset F \Subset M$, we have

$$|\partial F| \geqslant \big|\partial E_{t_{i-1}}(u)\big| = e^{t_{i-1}-t_i}\big|\partial E_{t_i}(u)\big|. \tag{5.3.4}$$

Now fix basepoints $p_i \in \partial E_{t_i}$, so $p_i \to \infty$ as $i \to \infty$. By (5.3.2), we may pass to a subsequence and assume that

- (M, g, p_i) converge smoothly to a manifold $(M_\infty, g_\infty, p_\infty)$ with $R \geqslant 0$,
- E_{t_i} converge in L^1_{loc} to a set $E_\infty \subset M_\infty$,
- ∂E_{t_i} converge in the $C^{1,\beta}$ ($\beta < \alpha$) sense to a connected closed surface $\Sigma_\infty \subset M_\infty$.

So $\Sigma_\infty = \partial E_\infty$. Combining (5.3.4) and (5.3.3) and the set-replacing argument, it follows that E_∞ is perimeter-minimizing. Thus Σ_∞ must be topologically S^2 or T^2. Hence there is a subsequence of $\{\partial E_{t_i}\}$ which either consists of S^2 or consists of T^2.

Case 3: Suppose u is escaping. Let T be the escape time of u as in Definition 4.3.1. We wish to argue as in Case 2, but we encounter the trouble that there is no level set outside E_T. To resolve this issue, we need to slightly perturb the metric and create some level sets outside E_T. We have the following statement:

Lemma 5.3.4. *Let M^3 be complete, connected, contractible, and satisfy (5.3.2). Suppose $E_0 \Subset M$, and u is an instantly escaping maximal solution of* $\mathsf{IVP}(M; E_0)$. *Let $T \in (0,\infty)$ be the escape time of u. Then there exists a constant $C > 0$ such that: for any domain K with $E_T(u) \Subset K \Subset M$ and all $\delta > 0$, there exists a set E such that*

(i) $K \Subset E \Subset M$, and ∂E is a connected $C^{1,\alpha}$ surface, with $\mathrm{diam}(\partial E) \leqslant C$, and the $C^{1,\alpha}$ norm of ∂E is controlled by C;

(ii) For any F with $E_T(u) \Subset F \Subset M$, we have $|\partial E|_g \leqslant (1+\delta)|\partial F|_g$.

We postpone the proof to the next subsection, and here let us assume it. Taking a sequence $\delta_i \to 0$ and an exhaustion $\{K_i\}$ of M, the lemma provides a sequence of sets $\{F_i\}$ from the lemma. Taking a further subsequence, we find that $\{F_i\}$ form an exhaustion, and has uniformly $C^{1,\alpha}$ boundaries with uniformly bounded diameters. Thus we are able to perform the same limiting argument as in Case 2. Condition (ii) in Lemma 5.3.4 and the set-replacing argument implies that the limit set E_∞ is perimeter-minimizing. The theorem follows by the same reason as in Case 2. $\square$

5.3.1 Modifying instantly escaping IMCF

The aim of this subsection is to prove Lemma 5.3.4. Consider an instantly escaping maximal solution u of $\mathsf{IVP}(M; E_0)$, for some $E_0 \Subset M$. Thus there exists $T \in (0, \infty)$ such that

$$E_T(u) \Subset M, \qquad u \equiv T \text{ in } M \setminus E_T(u).$$

To prove Lemma 5.3.4, we perturb the metric far outside K to slightly enlarge it at infinity. In view of Theorem 4.3.2, the escape time of an innermost IMCF is related to the circumference at infinity. Thus our perturbation will delay the escape time, and as a result, new sub-level sets will appear in the edited region. These new sub-level sets turn out to be the set E in Lemma 5.3.4. See Figure 5.5 for a description of this procedure.

Figure 5.5: The perturbation argument.

The following lemma is the technical description of the above procedure. Since there are multiple Riemannian metrics and maximal solutions appearing in this subsection, we will keep the full notation $E_t(u)$ and $|\partial E|_g$ for clarity.

Lemma 5.3.5. *Suppose (M, g) is one-ended and satisfies (5.3.2), and u is an instantly escaping maximal solution of $\mathsf{IVP}(M, g; E_0)$. Let $T \in (0, \infty)$ be the escape time of u.*

Then for any $K \Subset M$ and $\varepsilon \in (0, 1)$, there is a smooth metric $\widetilde{g}$ with $\|\widetilde{g} - g\|_{C^{10}(g)} \leqslant \varepsilon$ and with the following property: if $\widetilde{u}$ is the maximal solution of $\mathsf{IVP}(M, \widetilde{g}; E_0)$, then:

(i) $E_T(\widetilde{u}) = E_T(u)$,

(ii) there exists $t \in (T, T + \varepsilon)$ such that $E_t(\widetilde{u}) \Subset M$ and $\partial E_t(\widetilde{u}) \cap (M \setminus K) \neq \emptyset$.

Proof of Lemma 5.3.4 assuming Lemma 5.3.5.

Note that M is one-ended. Thus for any $K' \Subset M$ and $\varepsilon < 10^{-n}$, we may apply Lemma 5.3.5 to obtain a new metric $\widetilde{g}$. Let $\widetilde{u}$ be the maximal solution of $\mathsf{IVP}(M, \widetilde{g}; E_0)$.

Combining Lemma 5.1.1 and 5.3.3(ii), $\partial E_t(\widetilde{u})$ is connected. Combined with Lemma 4.3.4(ii) and $\|\widetilde{g} - g\|_{C^{10}} \leqslant \varepsilon$, we have

$$\operatorname{diam}_g\big(\partial E_t(\widetilde{u})\big) \leqslant C\big(T, \Lambda, |\partial E_0|_g, \operatorname{diam}_g(\partial E_0)\big).$$

In particular, it is independent of K'. Choosing K' large enough, it follows that we can achieve $E_t(\widetilde{u}) \Supset K$ (where K is the domain given in Lemma 5.3.4). Moreover, by Lemma 4.3.3(ii) and $\|\widetilde{g} - g\|_{C^{10}} \leqslant \varepsilon$, $\partial E_t(\widetilde{u})$ has controlled $C^{1,\alpha}$ norm with respect to $\widetilde{g}$, hence with respect to g as well. This proves Lemma 5.3.4(i) by setting $E = E_t(\widetilde{u})$.

Since $E_T(\widetilde{u})$ is outward minimizing in $(M, \widetilde{g})$, for any $F \Supset E_T(\widetilde{u})$ we have

$$(1 + \varepsilon)^{n-1}|\partial F|_g \geq |\partial F|_{\widetilde{g}} \geqslant |\partial E_T(\widetilde{u})|_{\widetilde{g}} = e^{-(t-T)}|\partial E|_{\widetilde{g}} \geqslant e^{-\varepsilon}|\partial E|_g.$$

By taking ε sufficiently small, this implies item (ii) since $E_T(\widetilde{u}) = E_T(u)$. $\square$

Proof of Lemma 5.3.5.

Let K, ε be as stated in the lemma. Since the statement is stronger when K is larger, we have the flexibility of arbitrarily enlarging K. Recall by Theorem 4.3.2 that the escape time T satisfies

$$|\partial E_T(u)|_g = \inf\Big\{ \liminf_{i\to\infty} |\partial F_i|_g : \ \{F_i\} \text{ is a } C^1 \text{ exhaustion of } M\Big\}.$$

By choosing an almost optimal exhaustion, and letting K be a sufficiently large element in it, we may assume that K satisfies

$$K \Supset E_T(u), \qquad |\partial K|_g < (1+\varepsilon)|\partial E_T(u)|_g. \tag{5.3.5}$$

Fix a point $p \in E_0$. Assume $K \Subset B_g(p, R_1)$ for some $R_1 > 0$. We use $C > 0$ to denote a generic constant that only depends on finitely many Λ_k (where Λ_k is as in (5.3.2)).

Claim 1. There exists $R_2 > R_1$ and a smooth Riemannian metric $\widetilde{g}$, such that
(i) $\widetilde{g} \geqslant g$ and $\|\widetilde{g} - g\|_{C^{10}(g)} \leqslant C\varepsilon$;
(ii) $\widetilde{g} = g$ in $B_g(p, 2R_1)$, and $\widetilde{g} = (1+\varepsilon)g$ in $M \setminus B_g(p, R_2/2)$.

We postpone its proof to the end. In the following, we fix the choice of $R_2 > R_1$ and $\widetilde{g}$ as in Claim 1. Then we see that $\widetilde{g}$ satisfies

$$\widetilde{\mathrm{inj}} \geqslant (\Lambda')^{-1}, \qquad |\widetilde{\nabla}^k \widetilde{\mathrm{Rm}}| \leqslant (\Lambda')^{k+2} \tag{5.3.6}$$

for all $k \leqslant 8$ for some other constant Λ'. Thus Lemma 4.3.3 and 4.3.4 hold in $(M, \widetilde{g})$ with a weaker constant. It is helpful to recall the following chain of inclusions:

$$E_T(u) \Subset K \Subset B_g(p, R_1) \Subset \big\{\widetilde{g} = g\big\} \Subset \big\{\widetilde{g} \neq (1+\varepsilon)g\big\} \Subset B_g(p, R_2).$$

Claim 2. u is a solution of $\mathsf{IVP}(M, \widetilde{g}; E_0)$.

Proof. It suffices to show that u solves $\mathsf{IMCF}(M\setminus\overline{E_0}, \widetilde{g})$. Suppose $t \in \mathbb{R}$, F is a competitor set with $F\Delta E_t \Subset K' \Subset M \setminus \overline{E_0}$. Note that $\widetilde{g} \geqslant g$, and $\partial E_t(u) \subset \{\widetilde{g} = g\}$, and $\int_A |\nabla_{\widetilde{g}} u|\, dV_{\widetilde{g}} = \int_A |\nabla_g u|\, dV_g$ for all set A (since $u \equiv T$ wherever $\widetilde{g} \neq g$). As a result,

$$\begin{aligned} J^{K'}_{u,\widetilde{g}}(F) &= P(F; K')_{\widetilde{g}} - \int_{F\cap K'} |\nabla_{\widetilde{g}} u|\, dV_{\widetilde{g}} \geqslant P(F; K')_g - \int_{F\cap K'} |\nabla_g u|\, dV_g \\ &= J^{K'}_{u,g}(F) \geqslant J^{K'}_{u,g}(E_t) = J^{K'}_{u,\widetilde{g}}(E_t), \end{aligned}$$

proving the claim. □

Let $\widetilde{u}$ be the maximal solution of $\mathsf{IVP}(M, \widetilde{g}; E_0)$. Hence $\widetilde{u} \geqslant u$. This implies $E_T(\widetilde{u}) \subset E_T(u) \Subset M$, thus by the maximum principle (Theorem 2.1.12(iii)), $E_T(\widetilde{u}) = E_T(u)$.

Claim 3. For all $t < T + (n-1)\log(1+\varepsilon)$, we have $E_t(\widetilde{u}) \Subset M$.

Proof. Fix such a t. We find a smooth precompact exhaustion $\{\Omega_l\}$ with

$$B_g(p, R_2) \Subset \Omega_1 \Subset \Omega_2 \Subset \cdots \Subset M, \qquad \bigcup \Omega_l = M.$$

Apply Lemma 4.1.2 to this exhaustion: we obtain a sequence of functions $\{u_l\}$ where each u_l solves $\mathsf{IVP}(\Omega_l, \widetilde{g}; E_0)$+$\mathsf{OBS}(\partial\Omega_l)$. Furthermore, $\{\widetilde{u}_l\}$ is a descending sequence, and

$\widetilde{u}_l \to \widetilde{u}$ in $C^0_{\mathrm{loc}}(M \setminus E_0)$. In particular, note that $E_T(\widetilde{u}_l) \subset E_T(\widetilde{u}) \Subset \Omega_l$. By the interior maximum principle, we have $E_T(\widetilde{u}_l) = E_T(\widetilde{u}) = E_T(u)$ for each l.

By Lemma 4.3.4(i) and (5.3.6) and the one-endedness of M, to prove the claim it suffices to prove that $E_t(\widetilde{u}) \neq M$. Suppose this is not the case. As $E_t(\widetilde{u}) = \bigcup_{l\geqslant 1} E_t(\widetilde{u}_l)$, we would have $E_t(\widetilde{u}_l) \Supset B_g(p, R_2)$ for some sufficiently large l. This implies

$$\partial E_t(\widetilde{u}_l) \subset M \setminus B_g(p, R_2) \subset \big\{\widetilde{g} = (1+\varepsilon)g\big\}.$$

So by our choice of t and the sub-exponential growth of area (Corollary 3.3.14), we have

$$|\partial E_t(\widetilde{u}_l)|_g = \frac{|\partial E_t(\widetilde{u}_l)|_{\widetilde{g}}}{(1+\varepsilon)^{n-1}} \leqslant e^{t-T}\frac{|\partial E_T(\widetilde{u}_l)|_{\widetilde{g}}}{(1+\varepsilon)^{n-1}} = e^{t-T}\frac{|\partial E_T(u)|_g}{(1+\varepsilon)^{n-1}} < |\partial E_T(u)|_g.$$

So we find a surface outside $E_T(u)$ that has a strictly smaller g-perimeter, but this contradicts the outward minimizing of $E_T(u)$. □

Claim 4. If $T + \log(1+\varepsilon) < t < T + (n-1)\log(1+\varepsilon)$, then $\overline{E_t(\widetilde{u})} \cap (M \setminus K) \neq \emptyset$.

Proof. Suppose the claim is false at such a time t. So $E_t(\widetilde{u}) \subset K \Subset M$. By the exponential growth of area (see Remark 2.1.4(v); we remind that obstacles are not involved in this claim) and noting that $\widetilde{g} = g$ in K, we have

$$|\partial E_t(\widetilde{u})|_{\widetilde{g}} = e^{t-T}|\partial E_T(\widetilde{u})|_{\widetilde{g}} > (1+\varepsilon)|\partial E_T(u)|_g.$$

Then recalling (5.3.5) and $\widetilde{g} = g$ on K we have

$$|\partial K|_{\widetilde{g}} = |\partial K|_g < (1+\varepsilon)|\partial E_T(u)|_g \leqslant |\partial E_t(\widetilde{u})|_{\widetilde{g}},$$

which contradicts the outward minimization of $\partial E_t(\widetilde{u})$. □

Combining Claim 3, 4, the lemma is proved. □

Proof of Claim 1.

Fix the bump function $\eta \in C^\infty(\mathbb{R}_+)$ defined by

$$\eta(\rho) = \begin{cases} \exp\big(1/(4\Lambda^2\rho^2 - 1)\big), & \rho < 1/2\Lambda, \\ 0, & \rho > 1/2\Lambda. \end{cases}$$

Then consider

$$\widetilde{d}(x) = \int_M \eta\big(d(x,y)\big) d(y,p)\, dy.$$

By our higher bounded geometry condition (5.3.2), we have the uniform bound

$$\big|\nabla_x^k d(x,y)^2\big| \leqslant C, \qquad \forall\, k \leqslant 20,\ \forall\, x, y \text{ with } d(x,y) \leqslant 1/2\Lambda,$$

where the constant C depends on finitely many Λ_k and is independent of x, y. It follows that the C^{20} norm of $\widetilde{d}$ is uniformly bounded on M. Furthermore, we have the pointwise estimates

$$\widetilde{d}(x) \geqslant \big[d(x,p) - 1/2\Lambda\big] \int_M \eta\big(d(x,y)\big)\, dy \geqslant C^{-1}\big[d(x,p) - 1/2\Lambda\big], \tag{5.3.7}$$

and

$$\widetilde{d}(x) \leqslant \big[d(x,p) + 1/2\Lambda\big] \int_M \eta\big(d(x,y)\big)\, dy \leqslant C\big[d(x,p) + 1/2\Lambda\big], \quad \forall\, x \in M. \tag{5.3.8}$$

Next, fix a cutoff function $\sigma \in C^\infty(\mathbb{R})$ with

$$0 \leqslant \sigma \leqslant 1, \quad \sigma|_{(-\infty,0]} \equiv 0, \quad \sigma|_{[1,\infty)} \equiv 1.$$

For sufficiently large ρ, consider the metric

$$\widetilde{g} = \big(1 + \varepsilon\sigma(\widetilde{d} - \rho)\big)g.$$

Thus $\|\widetilde{g} - g\|_{C^{20}(g)} \leqslant C\varepsilon$ where C is independent of ε, ρ. By (5.3.8), for some $\rho \gg 1$ we have $\widetilde{g} = g$ in $B_g(p, 2R_1)$. Fix this choice of ρ. Then by (5.3.7), we have $\{\widetilde{g} \neq (1+\varepsilon)g\} \Subset M$, and item (ii) follows by choosing large enough R_2. □

Appendix A

Sets with locally finite perimeter

This appendix contains some background materials in geometric measure theory. In particular, we review the notion of sets with locally finite perimeter, and its classical properties. Some useful auxiliary results are also stated in this appendix. The main reference is Maggi's textbook [77].

We will fix a background manifold M without boundary, and an underlying smooth Riemannian metric g on M.

A.1 Basic notions

Let E be a measurable set in M. We define its *measure-theoretic interior* to be

$$E^{(1)} := \Big\{x \in M : \lim_{r\to 0} \frac{|E \cap B(x,r)|}{\omega_n r^n} = 1\Big\}, \tag{A.1.1}$$

and its *measure-theoretic exterior* to be

$$E^{(0)} := \Big\{x \in M : \lim_{r\to 0} \frac{|E \cap B(x,r)|}{\omega_n r^n} = 0\Big\}, \tag{A.1.2}$$

By Lebesgue's differentiation theorem, we have $\big|E\Delta E^{(1)}\big| = 0$ and $\big|(M \setminus E)\Delta E^{(0)}\big| = 0$.

For a domain $\Omega \subset M$, we define the *perimeter* of E inside Ω to be

$$\begin{aligned} P(E;\Omega) := \sup\Big\{ \int_E \operatorname{div} X : X \text{ is a } C^1 \text{ vector field with} \\ \operatorname{spt} X \Subset \Omega,\ |X| \leqslant 1\Big\}. \end{aligned} \tag{A.1.3}$$

When $\Omega = M$, we denote the *total curvature* of E by

$$P(E) := P(E;M)$$

We call E a *set with locally finite perimeter* in a domain Ω, if $P(E;K) < \infty$ for all $K \Subset \Omega$. Also, we call E a *set with finite perimeter* in Ω, if $P(E;\Omega) < \infty$.

We say that a sequence of sets E_i converges to a set E in $L^1_{\mathrm{loc}}(\Omega)$, if $\chi_{E_i} \to \chi_E$ in $L^1_{\mathrm{loc}}(\Omega)$. A fundamental property of the perimeter is its lower semi-continuity:

Theorem A.1.1 ([77, Proposition 12.15])**.** *Suppose E_i have locally finite perimeter in Ω, and $E_i \to E$ in $L^1_{\mathrm{loc}}(\Omega)$. Then for all sub-domains $\Omega' \subset \Omega$ we have*

$$P(E;\Omega') \leqslant \liminf_{i\to\infty} P(E_i;\Omega'). \tag{A.1.4}$$

If in addition $\sup_i P(E_i;K) < \infty$ for all $K \Subset \Omega$, then E has locally finite perimeter in Ω.

The following compactness theorem is also useful:

Theorem A.1.2 ([77, Corollary 12.27]). *Suppose E_i have locally finite perimeter in Ω, such that*

$$\sup_i P(E_i; K) < \infty \qquad \forall\, K \Subset \Omega.$$

Then up to a subsequence, we have $E_i \to E$ in $L^1_{\mathrm{loc}}(\Omega)$ for some set E with locally finite perimeter in Ω.

The following result shows that every set with finite perimeter is smoothable.

Theorem A.1.3 ([77, Theorem 13.8]). *Suppose $E \Subset \Omega$ has finite perimeter. Then there is a sequence of sets $E_i \Subset \Omega$ with smooth boundaries, such that*
(i) $E_i \Subset K\ \forall\, i$, for a common domain $K \Subset \Omega$,
(ii) $E_i \to E$ in L^1, and $\partial E_i \to \partial E$ in Hausdorff topology,
(iii) $|\mu_{E_i}| \rightharpoonup |\mu_E|$. In particular, $P(E_i) \to P(E)$.

Suppose E has locally finite perimeter in Ω. By the Riesz representation theorem, there is a vector valued Radon measure in Ω, denoted by μ_E, such that

$$\int_E \operatorname{div} X = \int X \cdot d\mu_E \tag{A.1.5}$$

for all C^1 vector field X with $\operatorname{spt} X \Subset \Omega$. The measure μ_E is called the *Gauss-Green measure* of E. By the polar decomposition of vector-valued measures, we can write

$$\mu_E = \nu_E \cdot |\mu_E|,$$

where $|\mu_E|$ is a (scalar-valued) Radon measure, and ν_E is a $|\mu_E|$-measurable vector field with $|\mu_E|$-a.e. unit length. We call $|\mu_E|$ the *perimeter measure* of E, and ν_E the *(measure-theoretic) outer unit normal* of E. By the general decomposition theorem [77, Theorem 4.7], we have

$$P(E; K) = |\mu_E|(K), \qquad \forall\, \text{domain } K \Subset \Omega.$$

This justifies the name of $|\mu_E|$ as "perimeter measure".

Let μ_i, μ be scalar or vector valued measures in Ω. We say that $\mu_i \rightharpoonup \mu$ weakly, if

$$\int \phi\, d\mu_i \to \int \varphi\, d\mu$$

for all C^1 function or vector field ϕ with $\operatorname{spt} \phi \Subset \Omega$.

By Riesz representation, the convergence of sets $E_i \xrightarrow{L^1_{\mathrm{loc}}} E$ implies the weak convergence of the Gauss-Green measures $\mu_{E_i} \rightharpoonup \mu_E$. We warn the reader that the convergence $|\mu_{E_i}| \not\rightharpoonup |\mu_E|$ is in general not true. The following result is often helpful:

Lemma A.1.4 ([77, Proposition 4.26]).
Suppose μ_i, μ are (scalar valued) measures in Ω, with $\mu_i \rightharpoonup \mu$. Then we have:
(i) $\mu(U) \leqslant \liminf_{i\to\infty} \mu_i(U)$ for all open sets U,
(ii) $\mu(S) \geqslant \limsup_{i\to\infty} \mu_i(S)$ for all compact sets S,
(iii) $\mu(A) = \lim_{i\to\infty} \mu_i(A)$ for all precompact sets A with $\mu(\partial A) = 0$.

For a set E with locally finite perimteter, we define the *reduced boundary* of E to be

$$\partial^* E = \Big\{x \in \operatorname{spt}\mu_E : \lim_{r\to 0}\frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r))} \text{ exists and belongs to } S^{n-1}\Big\}. \tag{A.1.6}$$

A classical result of De Giorgi states that:

Theorem A.1.5 ([77, Theorem 15.9])**.** *Suppose E has locally finite perimeter in Ω. Then*

$$|\mu_E| = \mathcal{H}^{n-1}\llcorner\partial^* E \qquad \textit{as measures in } \Omega, \tag{A.1.7}$$

and

$$\nu_E(x) = \lim_{r\to 0}\frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r))} \qquad \forall\, x\in\partial^* E. \tag{A.1.8}$$

It is important to note that: if two sets E, F differ by a set with zero measure, then we have $\mu_E = \mu_F$ and $\partial^* E = \partial^* F$. This is readily checked from the definition. In fact, all the objects defined above are unaffected by modifications with zero measure.

Next, we review several decomposition formulas for the perimeter. In what follows, E, F are sets with locally finite perimeter in Ω, and $K \Subset \Omega$ is a domain.

First, we note the following *cup-cap inequality* [77, Lemma 12.22]:

$$P(E\cap F;K) + P(E\cup F;K) \leqslant P(E;K) + P(F;K). \tag{A.1.9}$$

Next, we note the following set operation formulas [77, Theorem 16.3]. Here, we recall (A.1.1) (A.1.2) for the notions $E^{(1)}$, $E^{(0)}$, and the last terms $\{\nu_E = \pm\nu_F\}$ are shorthands for the sets $\big\{x\in\partial^* E\cap\partial^* F : \nu_E(x) = \pm\nu_F(x)\big\}$.

$$\begin{aligned} P(E\cap F;K) = \mathcal{H}^{n-1}\big(\partial^* E\cap F^{(1)}\cap K\big) + \mathcal{H}^{n-1}\big(\partial^* F\cap E^{(1)}\cap K\big)& \\ + \mathcal{H}^{n-1}\big(\{\nu_E = \nu_F\}\cap K\big)&, \end{aligned} \tag{A.1.10}$$

$$\begin{aligned} P(E\cup F;K) = \mathcal{H}^{n-1}\big(\partial^* E\cap F^{(0)}\cap K\big) + \mathcal{H}^{n-1}\big(\partial^* F\cap E^{(0)}\cap K\big)& \\ + \mathcal{H}^{n-1}\big(\{\nu_E = \nu_F\}\cap K\big)&, \end{aligned} \tag{A.1.11}$$

$$\begin{aligned} P(E\setminus F;K) = \mathcal{H}^{n-1}\big(\partial^* E\cap F^{(0)}\cap K\big) + \mathcal{H}^{n-1}\big(\partial^* F\cap E^{(1)}\cap K\big)& \\ + \mathcal{H}^{n-1}\big(\{\nu_E = -\nu_F\}\cap K\big)&, \end{aligned} \tag{A.1.12}$$

When two sets E, F interacts, we can decompose the perimeter of E according to how it overlaps with F [77, Theorem 16.2]:

$$\begin{aligned} P(E;K) \geqslant \mathcal{H}^{n-1}\big(\partial^* E\cap F^{(1)}\cap K\big) + \mathcal{H}^{n-1}\big(\partial^* E\cap F^{(0)}\cap K\big)& \\ + \mathcal{H}^{n-1}\big(\{\nu_E = \nu_F\}\cap K\big) + \mathcal{H}^{n-1}\big(\{\nu_E = -\nu_F\}\cap K\big)&. \end{aligned} \tag{A.1.13}$$

A direct consequence of these formulas is the following:

$$P(E;K) - P(E\cap F;K) \geqslant \mathcal{H}^{n-1}\big(\partial^* E\cap F^{(0)}\cap K\big) - \mathcal{H}^{n-1}\big(\partial^* F\cap E^{(1)}\cap K\big). \tag{A.1.14}$$

$$P(E;K) - P(E\setminus F;K) \geqslant \mathcal{H}^{n-1}\big(\partial^* E\cap F^{(1)}\cap K\big) - \mathcal{H}^{n-1}\big(\partial^* F\cap E^{(1)}\cap K\big). \tag{A.1.15}$$

When $\mathcal{H}^{n-1}(\partial^* E\cap\partial^* F) = 0$, the set $\{\nu_E = \pm\nu_F\}$ makes no contribution. This makes the above equations much simpler.

The topological and reduced boundary are in general different. Consider the set $E = \{y < 0,\, x \neq 0\} \subset \mathbb{R}^2$. Then $\partial E = \{y = 0\} \cup \{x = 0, y < 0\}$ while $\partial^* E = \operatorname{spt}|\mu_E| = \{y = 0\}$. This general subtlety can be effectively removed by considering the measure-theoretic interior of a set (see also [77, Proposition 12.19]):

Lemma A.1.6. *If E has locally finite perimeter in Ω, then*

$$\partial E^{(1)} \cap \Omega = \operatorname{spt}|\mu_E| \cap \Omega = \overline{\partial^* E} \cap \Omega.$$

In particular, if E has locally finite perimeter in M, then $\partial E^{(1)} = \operatorname{spt}|\mu_E| = \overline{\partial^* E}$.

Proof. Replacing M by Ω, we may assume $\Omega = M$. The fact $\operatorname{spt}|\mu_E| = \overline{\partial^* E}$ follows from De Giorgi's formula. If $x \in M \setminus \partial E^{(1)}$, then E occupies either full measure or zero measure in some neighborhood of x. Thus we have $\operatorname{spt}|\mu_E| \subset \partial E^{(1)}$. It remains to show that $\partial E^{(1)} \subset \operatorname{spt}|\mu_E|$. Suppose $x \in \partial E^{(1)}$. Then for every $r > 0$, the ball $B(x, r)$ contains points in both $E^{(1)}$ and $\Omega \setminus E^{(1)}$, hence

$$|E \cap B(x, r)| > 0, \qquad |B(x, r) \setminus E| > 0.$$

When r is sufficiently small (depending on the local geometry near x), we have the Neumann isoperimetric inequality

$$P\big(E; B(x, r)\big) \geqslant c(n) \min\Big\{|E \cap B(x, r)|, |B(x, r) \setminus E|\Big\} > 0.$$

This implies $x \in \operatorname{spt}|\mu_E|$. □

A.2 Almost perimeter minimizers

Let E be a set with locally finite perimeter in a domain Ω. We say that E is almost perimeter minimizing in Ω, if for any $K \Subset \Omega$, there is a constant $\Lambda_E(K) > 0$ such that

$$P(E; K) \leqslant P(F; K) + \Lambda_E(K) \cdot |E \Delta F| \tag{A.2.1}$$

for all competitors F with $E \Delta F \Subset K$.

This is close to the notion of "(Λ, r_0)-perimeter minimizer" introduced in [77, Chapter 21]. In particular, if E is almost perimeter minimizing as defined here, with $E = E^{(1)}$, then for all $K \Subset \Omega$, E is $(\Lambda_E(K), r_0)$-perimeter minimizing in K, for all $r_0 > 0$. Thus, the results in [77] are applicable to almost perimeter-minimizing sets. (We remind that in [77, p.278], the author assumed that $\operatorname{spt}|\mu_E| = \partial E$. In view of Lemma A.1.6 above, our setup is consistent with [77] if we assume $E = E^{(1)}$ in the theorems below.)

We have the following well-known regularity theorem:

Theorem A.2.1 (regularity, [77, Theorem 26.5, 28.1])**.**
If E is an almost perimeter-minimizer in Ω, with $E = E^{(1)}$, then $\partial^* E$ is a $C^{1,\alpha}$ ($\alpha < 1/2$) hypersurface, and $\partial E \setminus \partial^* E$ has Hausdorff dimension at most $n - 8$.

The following convergence result and density bound follows by combining (the Riemannian analogue of) Theorem 21.11, 21.14 and 26.6 in [77]. See also [104, 105].

Theorem A.2.2 (convergence)**.** *Let (M, g) be a Riemannian manifold, and $\Omega \subset M$ is a domain, and g be a smooth metric in Ω. Assume the following data:*

(1) Ω_i is a sequence of domains that converge to Ω locally,
(2) g_i are smooth metrics that converge to g locally smoothly,
(3) E_i are almost perimeter minimizers in (Ω_i, g_i),
(4) the constants $\Lambda_{E_i}(K)$ in (A.2.1) are uniformly bounded for all large i, for all $K \Subset \Omega$,
(5) E has locally finite perimeter in Ω, and $E_i \to E$ in $L^1_{\mathrm{loc}}(\Omega)$,
(6) $x_i \in \operatorname{spt}|\mu_{E_i}|$, with $x_i \to x \in \Omega$.

Then we have:

(i) E is almost perimeter minimizing in Ω, with $\Lambda_E(K) \leqslant \liminf_{i\to\infty} \Lambda_{E_i}(K)$ for all $K \Subset \Omega$. Furthermore, $|\mu_{E_i}| \rightharpoonup |\mu_E|$ weakly as measures in Ω,
(ii) $x \in \operatorname{spt}|\mu_E|$ (in particular, $\operatorname{spt}|\mu_{E_i}| \to \operatorname{spt}|\mu_E|$ in local Hausdorff sense in Ω),
(iii) there is a constant $c = c(n) < 1$ so that

$$\limsup_{r\to 0} \frac{|E \cap B(x,r)|}{\omega_n r^n} \leqslant c.$$

In particular, $x \notin E^{(1)}$.
(iv) If $x \in \partial^* E$, then $x_i \in \partial^* E_i$ for sufficiently large i, and we have $\nu_{E_i}(x_i) \to \nu_E(x)$.

When $n \leqslant 7$, the regularity statement can be made uniform. More precisely, it depends only on the constants $\Lambda_E(K)$ and the underlying metric. In what follows, we fix a domain Ω (thus all the norms depend implicitly on Ω).

Let g be a smooth Riemannian metric in Ω, and $x \in \Omega$. Suppose $k \geqslant 0$. The C^k norm of g near x, denoted by $\|g\|_{C^k(x)}$, is defined to be the smallest constant L so that:
(1) $B_g(x, 3L^{-1}) \Subset \Omega$, and the injectivity radius at x is at least $2L^{-1}$,
(2) in any geodesic normal coordinate in $B_g(x, L^{-1})$, the metric tensor g_{ij} satisfies

$$\|g_{ij} - \delta_{ij}\|_{C^0} + \sum_{|\gamma|=1}^{k} L^{-|\gamma|} \|\partial^\gamma g_{ij}\|_{C^0} \leqslant 10^{-n}, \qquad \forall\, 1 \leqslant i, j \leqslant n. \tag{A.2.2}$$

This C^k norm is scale invariant: one may check that $\|\lambda^2 g\|_{C^k(x)} = \lambda^{-1}\|g\|_{C^k(x)}$ for all $\lambda > 0$. Denoting by g_{euc} the Euclidean metric in $\mathbb{R}^n$, we have $\|g_{\mathrm{euc}}\|_{C^k(0)} = 0$ for all k.

The C^k norm of a metric can be bounded in terms of the injectivity radius, the Riemannian tensor and its derivatives. See [38]: if $k \geqslant 0$ and g satisfies

$$\mathrm{inj}(x) \geqslant \Lambda^{-1} \qquad \text{and} \qquad |\nabla^l \operatorname{Rm}| \leqslant \Lambda^{l+2} \text{ in } B(x, \Lambda^{-1}),\ \forall\, l \leqslant k, \tag{A.2.3}$$

then $\|g\|_{C^k(x)} \leqslant C(n,k)\Lambda$. A particular case is bounded geometry, namely, the condition (A.2.3) with $k = 0$. In this case, we obtain $\|g\|_{C^0(x)} \leqslant C(n)\Lambda$.

Let E be a subset of M, and $x \in \partial E$. Suppose $k \geqslant 1$, $\alpha \in (0,1)$. The $C^{k,\alpha}$ norm of E at x is defined to be the smallest constant L so that:
(1) $B_g(x, 3L^{-1}) \Subset \Omega$, and the injectivity radius at x is at least $2L^{-1}$,
(2) in some geodesic normal coordinate $(x_1, \cdots, x_n)$ centered at x, and in the region $\big\{|x_n| < L^{-1}, x_1^2 + \cdots + x_{n-1}^2 < L^{-2}\big\}$, the set E is represented as the sub-graph of a C^1 function f such that

$$\|f\|_{C^0} + \sum_{|\gamma|=1}^{k} L^{-\gamma} \|\partial^\gamma f\|_{C^0} + L^{-k-\alpha} \sum_{|\gamma|=k} \|\partial^\gamma f\|_{C^{0,\alpha}} \leqslant 10^{-n} L^{-1}. \tag{A.2.4}$$

Since this norm also depends on the underlying metric g, we denote it by $\|E\|_{C^{k,\alpha}(g,x)}$. Note its scaling: $\|E\|_{C^{k,\alpha}(\lambda^2 g,x)} = \lambda^{-1}\|E\|_{C^{k,\alpha}(g,x)}$.

Theorem A.2.3. *Suppose $n \leqslant 7$, $\alpha \in (0, 1/2)$, and E is an almost perimeter minimizer in Ω, with $E = E^{(1)}$. Suppose $x \in \partial E$. Then for any $R \leqslant d(x, \partial\Omega)/2$, we have*

$$\|E\|_{C^{1,\alpha}(g,x)} \leqslant R^{-1} C\Big(n, \alpha, R\,\|g\|_{C^4(x)}, R\,\Lambda_E\big(B(x,R)\big)\Big),$$

where $\Lambda_E(\cdot)$ is the constant appearing in (A.2.1).

Proof. Since the statement is scale-invariant, we can assume without loss of generality that $R = 1$. By Theorem A.2.1, ∂E is a $C^{1,\alpha}$ surface near x, so $\|E\|_{C^{1,\alpha}(g,x)} < \infty$. Suppose that the theorem does not hold. Then there is a sequence of domains (Ω_i, g_i), almost perimeter-minimizing sets E_i in Ω_i, points $x_i \in \partial E_i \cap \Omega_i$, such that:

$$d_i(x_i, \partial\Omega_i) > 2, \qquad \|g_i\|_{C^4(x_i)} \leqslant L, \qquad \Lambda_{E_i}\big(B_i(x_i, 1)\big) \leqslant L,$$

but

$$Q_i := \|E_i\|_{C^{1,\alpha}(g_i,x_i)} \to \infty.$$

Consider the rescaled metrics $\widetilde{g}_i = Q_i g_i$. We have $\widetilde{d}_i(x_i, \partial\Omega_i) \to \infty$ and

$$\|\widetilde{g}_i\|_{C^4(x)} \leqslant LQ_i^{-1/2}, \qquad \Lambda_{E_i}\big(\widetilde{B}_i(x_i, Q_i^{1/2})\big) \leqslant LQ_i^{-1/2}, \qquad \|\partial E_i\|_{C^{1,\alpha}(\widetilde{g}_i,x_i)} = Q_i^{1/2}.$$

Let us argue that for any constant $\mu > 0$, the small excess condition $P\big(E_i; \widetilde{B}_i(x_i, 1)\big) \leqslant (1+\mu)\omega_{n-1}$ holds for all sufficiently large i. Then the classical ε-regularity theorem [104, 3] (see also [77, Theorem 26.3]) would imply that $\|E_i\|_{C^{1,\alpha}(\widetilde{g}_i,x_i)} \leqslant C$ for all large i, leading to a contradiction.

Now suppose that our claim is false. Thus we may pick a subsequence so that $P\big(E_i; \widetilde{B}_i(x_i, 1)\big) > (1 + \mu)\omega_{n-1}$ for all i. Since the C^4 norm of $\widetilde{g}_i$ converges to zero, we may choose a large geodesic normal coordinate chart around each x_i and take a limit

$$\widetilde{g}_i \xrightarrow{C^3_{\mathrm{loc}}} g_{\mathrm{euc}}, \quad E_i \xrightarrow{L^1_{\mathrm{loc}}} E \quad \text{in} \quad \mathbb{R}^n.$$

Theorem A.2.2 then implies that E is perimeter minimizing in $\mathbb{R}^n$, with $0 \in \operatorname{spt}|\mu_E|$. By the Bernstein theorem [77, Theorem 28.17], E must be a half-space. Finally, by the convergence of perimeter measure and the fact that $\mathcal{H}^{n-1}\big(E \cap \partial B^{\mathbb{R}^n}(0,1)\big) = 0$, we obtain

$$\omega_{n-1} = P\big(E; B^{\mathbb{R}^n}(0,1)\big) = \lim_{i\to\infty} P\big(E_i; \widetilde{B}_i(x_i, 1)\big) \geqslant (1 + \mu)\omega_{n-1},$$

contradiction. □

The following result, which holds in all dimensions, provides effective density estimate for almost perimeter minimizers. Recall from above that a uniform upper bound on $\|g\|_{C^0(x)}$ can be obtained if the manifold has bounded geometry.

Theorem A.2.4. *Suppose E is an almost perimeter minimizer in Ω with $E = E^{(1)}$, and $x \in \partial E$. For any radius $R < d(x, \partial\Omega)/2$, there is a constant*

$$c = c\Big(n, R\,\|g\|_{C^0(x)}, R\,\Lambda_E\big(B(x,R)\big)\Big) > 0,$$

such that that

$$P\big(E; B(x, r)\big) \geqslant c(n) r^{n-1} \qquad \forall\, r \in (0, cR].$$

Proof. By scale invariance, we may assume $R = 1$. Denote $\Lambda_1 = \|g\|_{C^0(x)}$, $\Lambda_2 = \Lambda_E\big(B(x,1)\big)$. Set $c = \min\{1/4\Lambda_1, 1/4\Lambda_2\}$. By the definition of C^0 norm, we have

$$\big(1-10^{-n}\big)g_{\mathrm{euc}} \leqslant g \leqslant \big(1+10^{-n}\big)g_{\mathrm{euc}}$$

in any geodesic normal coordinates in $B(x,2c)$. This implies the isoperimetric inequality

$$P(E) \geqslant \frac{2}{n\omega_n^{1/n}}|E|^{\frac{n}{n-1}}, \qquad \forall\, E \Subset B(x,c).$$

One may then argue verbatim as in [77, Theorem 21.11] to obtain the desired density bound (where we note that $E = E^1$ implies $x \in \operatorname{spt}|\mu_E|$). □

In the main text, we frequently employed the fundamental *set-replacing argument*. It first appeared in Fact 1.2.10 with technical simplifications. Here, we take the chance to present the full argument. We will omit the details elsewhere when this technique is used, since the arguments are largely the same with only minor modifications.

Recall that for $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ and $K \Subset \Omega$, we have defined the energy

$$J_u^K(E) = P(E;K) - \int_{E\cap K} |\nabla u|.$$

Lemma A.2.5 (set-replacing argument)**.** *If E_i are local minimizers of J_u in Ω, and $E_i \to E$ in $L^1_{\mathrm{loc}}(\Omega)$, then E is a local minimizer of J_u in Ω.*

Proof. We closely follow [77, Theorem 21.14]. Let F be a competitor, so that $E\Delta F \Subset \Omega$. We may assume that E, E_i, F are all equal to their measure-theoretic interiors.

We claim that there is a choice of K such that $E\Delta F \Subset K \Subset \Omega$, and

$$\mathcal{H}^{n-1}\big(\partial K \cap \partial^* E\big) = \mathcal{H}^{n-1}\big(\partial K \cap \partial^* F\big) = \mathcal{H}^{n-1}\big(\partial K \cap \partial^* E_i\big) = 0 \qquad \forall\, i, \tag{A.2.5}$$

and

$$\liminf_{i\to\infty} \mathcal{H}^{n-1}\big(\partial K \cap (E\Delta E_i)\big) = 0. \tag{A.2.6}$$

To achieve this, we first choose any smooth K_0 with $E\Delta F \Subset K_0 \Subset \Omega$. For sufficiently small ε, denote $N_\varepsilon = \big\{0 < d(\cdot, K_0) < \varepsilon\big\}$ the open collar neighborhood of K_0. For $0 < t < \varepsilon$, we denote $K_t = \big\{d(\cdot, K_0) = t\big\}$.

It is clear that (A.2.5) holds with $K = K_t$ for all but countably many t. Next, combining Fatou's lemma and the coarea formula, we have

$$\begin{aligned}\int_0^\varepsilon \Big(\liminf_{i\to\infty} \mathcal{H}^{n-1}\big(\partial K_t \cap (E\Delta E_i)\big)\Big)\,dt &\leqslant \liminf_{i\to\infty} \int_0^\varepsilon \mathcal{H}^{n-1}\big(\partial K_t \cap (E\Delta E_i)\big)\,dt \\ &= \liminf_{i\to\infty} \big|N_\varepsilon \cap (E\Delta E_i)\big| = 0.\end{aligned}$$

Hence (A.2.6) holds with $K = K_t$ for almost every t. This proves our claim.

Now, choose any K' with $K \Subset K' \Subset \Omega$. We compare each E_i with $F_i = (F\cap K) \cup (E_i \setminus K)$. By [77, Theorem 16.16], we have

$$P(F_i;K') = P(F;K) + P(E_i;K'\setminus \overline{K}) + \mathcal{H}^{n-1}\big(\partial K \cap (E\Delta E_i)\big).$$

Here we have used (A.2.5). Using (A.2.5) again, we have

$$P(E_i;K') = P(E_i;K) + P(E_i;K'\setminus\overline{K}).$$

Therefore, the comparison $J_u^{K'}(E_i) \leqslant J_u^{K'}(F_i)$ implies

$$P(E_i;K) - \int_{E_i\cap K} |\nabla u| \leqslant P(F;K) - \int_{F\cap K} |\nabla u| + \mathcal{H}^{n-1}\big(\partial K \cap (E\Delta E_i)\big). \tag{A.2.7}$$

Taking the limit as $i \to \infty$, using (A.2.6) and the lower semi-continuity of perimeter, we obtain

$$P(E;K) - \int_{E_i\cap K} |\nabla u| \leqslant P(F;K) - \int_{F\cap K} |\nabla u|.$$

This proves the lemma. □

A.3 The isoperimetric profile

Recall from Section 2.5 that the isoperimetric profile of a manifold is defined as

$$\mathrm{ip}(v) := \inf\Big\{P(E) : E \Subset M, |E| = v\Big\}.$$

We define the strong isoperimetric profile

$$\mathrm{sip}(v) = \inf\Big\{P(E) : E \Subset M, |E| \geqslant v\Big\},$$

for all $0 < v < |M|$. Its formal inverse, denoted by sip^{-1} for convenience, is defined as

$$\mathrm{sip}^{-1}(a) = \sup\Big\{|E| : E \Subset M, P(E) \leqslant a\Big\}.$$

Clearly $\mathrm{sip}(v) = \inf_{v'\geqslant v} \mathrm{ip}(v')$. The strong isoperimetric profile has less pathologic behavior to be concerned. In particular, $\mathrm{sip}(v)$ is always continuous when M is connected, while the continuity of $\mathrm{ip}(v)$ is a delicate problem. See [41, 92] and more recently [7, Corollary 4.16] which address the latter problem. On the other hand, the strong isoperimetric profile is a non-trivial quantity only for manifolds that are non-degenerate at infinity. For example, a manifold with finite volume has $\mathrm{sip}(v) \equiv 0$. It is easy to examine that both $\mathrm{sip}(v)$ and $\mathrm{sip}^{-1}(a)$ are non-decreasing, and sip^{-1} is defined with finite value on the interval $\big(0, \lim_{v\to\infty} \mathrm{sip}(v)\big) = \big(0, \liminf_{v\to\infty} \mathrm{ip}(v)\big)$.

Lemma A.3.1. *Suppose M is complete, connected, and has infinite volume. Then* $\mathrm{sip}(v)$ *is continuous on* $(0,\infty)$.

Proof. Since $\mathrm{sip}(v)$ is non-decreasing, its right continuity at a value v is equivalent to

$$\inf_{|E|\geqslant v} P(E) \geqslant \inf_{|E|>v} P(E). \tag{A.3.1}$$

Given any precompact set E with $|E| \geqslant v$ and $P(E) \leqslant \mathrm{sip}(v)+\varepsilon$, we choose any geodesic ball B of perimeter at most ε, such that $|B \setminus E| > 0$. Then note that $|E \cup B| > v$ and $P(E\cup B) \leqslant \mathrm{sip}(v) + 2\varepsilon$. Letting $\varepsilon \to 0$ this proves (A.3.1). It remains to prove left continuity at any fixed $v > 0$. Denote

$$a := \lim_{v'\to v^-} \mathrm{sip}(v') = \sup_{v'<v} \mathrm{sip}(v').$$

Fix a basepoint $x_0 \in M$. Let R be sufficiently large such that $|B(x_0,R)| > 2\cdot 10^n v$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that: for all $x \in B(x_0,R)$, the unique geodesic ball centered at x and with volume δ has perimeter $< \varepsilon$. Denote this geodesic ball by $B(x, r_x)$. By further decreasing δ, we may assume that $|B(x, 5r_x)| \leqslant 10^n |B(x, r_x)|$ for all $x \in B(x_0, R)$. By the definition of $\mathrm{sip}(v)$, for any $v - \delta/2 < v' < v$ there exists a precompact set E with $|E| \geqslant v'$ and $P(E) \leqslant \mathrm{sip}(v') + \varepsilon \leqslant a + \varepsilon$. If $|E| \geqslant v$, then this already implies $\mathrm{sip}(v) \leqslant a + \varepsilon$. Now assume $|E| \leqslant v$. We claim that there is a point $x \in B(x_0, R)$ such that $|E \cap B(x, r_x)| \leqslant \frac{1}{2}|B(x, r_x)|$. Otherwise, we would have $|E \cap B(x, r_x)| > \frac{1}{2}|B(x, r_x)|$ for all $x \in B(x_0, R)$. By Vitali's covering lemma, there is a disjoint countable collection of balls $B(x_i, r_{x_i})$ such that $B(x_i, 5r_{x_i})$ covers $B(x_0, R)$. Thus

$$|E| \geqslant \sum |E \cap B(x_i, r_{x_i})| \geqslant \frac{1}{2} \cdot 10^{-n} \sum |B(x_i, 5r_{x_i})| \geqslant \frac{1}{2} \cdot 10^{-n} |B(x_0, R)| > v,$$

which is a contradiction. After finding the ball $B(x, r_x)$, consider the new set $E' = E \cup B(x, r_x)$. We have $|E'| \geqslant |E| + \frac{1}{2}|B(x, r_x)| > (v - \delta/2) + \delta/2 \geqslant v$ and $P(E') \leqslant P(E) + P(B(x, r_x)) \leqslant a + 2\varepsilon$, hence $\mathrm{sip}(v) \leqslant a + 2\varepsilon$. Thus either $|E| \geqslant v$ or $|E| \leqslant v$ we have obtained $\mathrm{sip}(v) \leqslant a + 2\varepsilon$. Taking $\varepsilon \to 0$ this implies $\mathrm{sip}(v) \leqslant a$, which proves the left continuity. □

Lemma A.3.2. *Suppose M is complete, connected, and has infinite volume. Then either* $\mathrm{sip} \equiv 0$ *or* $\mathrm{sip}(v) > 0$ *for all* $v > 0$.

Proof. Suppose $\mathrm{sip}(v)$ is not identically zero but is zero somewhere. By monotonicity and continuity, there exists a value $v_0 > 0$ such that $\mathrm{sip}(v_0) = 0$ but $\mathrm{sip}(v) > 0$ for all $v > v_0$. By definition, there is a sequence of bounded sets E_k such that $|E_k| \geqslant v_0$ and $P(E_k) \leqslant 1/k$. By the maximality of v_0, we can assume $|E_k| \leqslant \frac{3}{2} v_0$ for all k. For a fixed $k \in \mathbb{N}$, E_k is contained in some bounded connected smooth domain Ω. Enlarging Ω if necessary, we may assume that $|\Omega| \geqslant 3v_0$. Note that this selection of Ω requires the connectedness and infinite volume of M. By the relative isoperimetric inequality, for all $l > k$ we have $|E_l \cap \Omega| \leqslant C(\Omega) P(E_l, \Omega)^{n/(n-1)} \leqslant C(\Omega) l^{-n/(n-1)}$, therefore $|E_l \cap \Omega| \leqslant v_0/2$ for sufficiently large l. Now the set $E_k \cup E_l$ has volume $\geqslant 3v_0/2$ and perimeter $\leqslant 2/k$, hence $\mathrm{sip}(3v_0/2) \leqslant 2/k$. Letting $k \to \infty$ this yields $\mathrm{sip}(3v_0/2) = 0$, which contradicts the maximality of v_0. □

Corollary A.3.3. *Suppose M is complete, connected, and has infinite volume. If it holds* $\liminf_{v \to \infty} \mathrm{ip}(v) > 0$, *then* $\mathrm{ip}(v) > 0$ *for all* $v > 0$.

Proof. The condition implies that $\mathrm{sip}(v) > 0$ for some $v > 0$. Then Lemma A.3.2 implies that $\mathrm{sip}(v) > 0$, hence $\mathrm{ip}(v) > 0$, for all $v > 0$. □

A.4 Strictly outward minimizing hull

In this section, all the equality and inclusion of sets are up to an error with zero measure. The role of this section is to prove Lemma 1.2.12. Recall that a set $E \subset \Omega$ is (strictly) locally outward minimizing, if for all F with $E \subset F \subset \Omega$ (resp. for all such F with $|F \setminus E| > 0$), and domain K satisfying $F \setminus E \Subset K \Subset \Omega$, it holds $P(E; K) \leqslant P(F; K)$ (resp. $P(E; K) < P(F; K)$). Also, recall the definition of minimizing hulls:

Definition A.4.1. Given a set $Q \subset \Omega$ with locally finite perimeter, denote

$$\mathcal{F}(Q) = \Big\{ F : Q \subset F \subset \Omega \text{ and } F \text{ is strictly outward minimizing} \Big\}.$$

A set $E \in \mathcal{F}(Q)$ is called the minimizing hull of Q in Ω, if $E \subset E'$ for all $E' \in \mathcal{F}(Q)$.

Here we remark that: the set $\mathcal{F}(Q)$ is nonempty, since Ω is an element. The following lemma shows the uniqueness of minimizing hull.

Lemma A.4.2. *If E_1, E_2 are both strictly outward minimizing, then so is $E_1 \cap E_2$. Therefore, the minimizing hull of a set Q is unique up to measure zero, if exists.*

Proof. Suppose $F \supset E_1 \cap E_2$ with $F \setminus (E_1 \cap E_2) \Subset K \Subset \Omega$. We may compare

$$P(E_1; K) \leqslant P(F \cup E_1; K) \qquad \Rightarrow \qquad P(F \cap E_1; K) \leqslant P(F; K). \tag{A.4.1}$$

Then we may compare

$$P(E_2; K) \leqslant P\big((F \cap E_1) \cup E_2; K\big) \quad \Rightarrow \quad P(F \cap E_1 \cap E_2; K) \leqslant P(F \cap E_1; K). \tag{A.4.2}$$

Combining (A.4.1) (A.4.2) we obtain $P(E_1 \cap E_2; K) \leqslant P(F; K)$. If $|F \setminus (E_1 \cap E_2)| > 0$, then either $|F \setminus E_1| > 0$ or $\big|(F \cap E_1) \setminus E_2\big| > 0$. Hence one of the inequalities in (A.4.1) (A.4.2) must be strict. This shows the strict minimization of $E_1 \cap E_2$. □

If the minimizing hull E happens to be precompact, then it obviously minimizes the volume among all elements in $\mathcal{F}(Q)$. Note that Lemma A.4.2 implies the converse: if $E \Subset \Omega$ and E is the least volume element in $\mathcal{F}(Q)$, then E is the minimizing hull of Q.

Now we restrict ourselves the setup of Lemma 1.2.12 and consider case where $E \Subset \Omega$. The following definition follows [42, Definition 2.6]:

Definition A.4.3 (least area problem and maximal volume solution)**.**
Given a set $Q \Subset \Omega$ with finite perimeter. We say that a set E with $Q \subset E \Subset \Omega$ is a least area solution outside Q in Ω, if

$$P(E) = \inf\big\{P(F) : Q \subset F \Subset \Omega\big\}.$$

We say that a set $E \Subset \Omega$ is a maximal volume least area solution (outside Q in Ω), if

$$|E| = \sup\Big\{|F| : F \text{ is a least area solution outside } Q \text{ in } \Omega\Big\}.$$

The following facts are straightforward to verify, by repeatedly using (A.1.9).

Lemma A.4.4. *If E_1, E_2 are both least area solutions outside Q, so is $E_1 \cup E_2$. Therefore, the maximal volume least area solution is unique up to measure zero, if it exists. The maximal volume solution is always strictly outward minimizing. If E is the maximal volume solution, and E' is another least area solution, then $E' \subset E$.*

The following is our main result of this section, strengthening [42, Theorem 2.16]:

Lemma A.4.5 (= Lemma 1.2.12)**.**
Given $Q \Subset \Omega$ with finite perimeter. The following statements are equivalent:
(i) There exists a maximal volume least area solution E_1 with $E_1 \Subset \Omega$.
(ii) There exists a minimizing hull E_2 of Q in Ω, with $E_2 \Subset \Omega$.
Moreover, we have $E_1 = E_2$ up to measure zero, if either of them exists.

Proof. (i) $\Rightarrow$ (ii). Assume E_1 exists as in (i). By Lemma A.4.4, E_1 is strictly outward minimizing. Suppose $F \supset Q$ is another strictly outward minimizing set. Note that $P(F \cap E_1) \geqslant P(E_1)$ by the minimization of E_1, which implies $P(F \cup E_1; K) \leqslant P(F; K)$ for all K with $E_1 \Subset K \Subset \Omega$. This implies $E_1 \subset F$ up to measure zero, so (ii) follows.

(ii) $\Rightarrow$ (i). Suppose that E_2 exists as in (ii). We first show that E_2 is a least area solution outside Q in Ω. Suppose this is not true, so that there is another F with $Q \subset F \Subset \Omega$ and $P(F) < P(E_2)$. Since $P(F \cup E_2) \geqslant P(E_2)$ by the minimization of E_2, we have $P(F \cap E_2) < P(F)$. Consider the following least area problem with obstacles

$$A = \inf \Big\{ P(G) : Q \subset G \subset E_2 \Big\},$$

and the volume maximizing problem among least area sets

$$V = \sup \Big\{ |G| : Q \subset G \subset E_2, P(G) = A \Big\}.$$

By the classical compactness theorem, there exists a maximal volume solution G_0 with $|G_0| = V$, $P(G_0) = A$ and $Q \subset G_0 \subset E_2$. We claim that G_0 is strictly outward minimizing in Ω. Suppose H is such that $G_0 \subset H \Subset \Omega$. By the minimizing property of E_2, we have

$$P(H \cup E_2) \geqslant P(E_2) \;\; \Rightarrow \;\; P(H \cap E_2) \leqslant P(H). \tag{A.4.3}$$

By the minimizing property of G_0, we have

$$P(G_0) \leqslant P(H \cap E_2). \tag{A.4.4}$$

These imply that $P(G_0) \leqslant P(H)$. Moreover, if equality holds then (A.4.3) (A.4.4) also attains equality. By the strict minimizing property of G_0 (in E_2) and E_2 (in Ω), it holds $H = G_0$ up to measure zero. This proves the strict outward minimizing of G_0. We have now found a strictly outward minimizing set G_0 inside E_2, with $P(G_0) \leqslant P(F \cap E_2) \leqslant P(F) < P(E_2)$. Hence $|E_2 \setminus G_0| > 0$, contradicting the volume minimization of E_2.

We have proved that E_2 is a least area solution. Suppose there is another solution F with $Q \subset F \Subset \Omega$ and $P(F) = P(E_2)$. Since $P(F \cap E_2) \geqslant P(F)$, we conclude $P(F \cup E_2) \leqslant P(E_2)$. This implies $F \subset E_2$ by the minimizing property of E_2, hence E_2 is the maximal volume solution. The proof of the theorem is complete. $\square$

A.5 Locally Lipschitz domains

Fix a Riemannian manifold (M, g). We say that a domain Ω is *locally Lipschitz*, if for each $x \in \partial\Omega$ there is a cylindrical geodesic coordinate chart centered at x, in which Ω is the sub-graph of a Lipschitz function. By Rademacher's theorem, the outer unit normal of Ω (denoted by ν_Ω) exists $\mathcal{H}^{n-1}$-almost everywhere on $\partial\Omega$. When a locally Lipschitz domain is precompact, we will simply call it *Lipschitz*.

We prove the following auxiliary lemmas about locally Lipschitz domains.

Lemma A.5.1 (truncation)**.**

Suppose Ω is locally Lipschitz. Then for all $K \Subset M$ there exists a Lipschitz domain $\Omega' \subset \Omega$ with $\Omega' \Subset M$, such that $\Omega \cap K = \Omega' \cap K$.

Lemma A.5.2 (inner approximation).

Suppose Ω is locally Lipschitz. Then there is a sequence of Lipschitz domains $\Omega_1 \Subset \Omega_2 \Subset \cdots \Subset \Omega$, with $\bigcup \Omega_i = \Omega$ and $|\mu_{\Omega_i}| \rightharpoonup |\mu_\Omega|$ weakly as measures.

In addition, for any set E with locally finite perimeter, we can choose Ω_i in a way such that $\mathcal{H}^{n-1}\big(\partial^* E \cap \partial^* \Omega_i\big) = 0$ for all i.

Lemma A.5.3 (existence of collar neighborhoods).

Suppose $\Omega \Subset M$ is Lipschitz. Then there is a neighborhood $N \subset \overline{\Omega}$ of $\partial\Omega$ and a Lipschitz map $\Phi : N \to \partial\Omega$ with $\Phi|_{\partial\Omega} = \mathrm{id}$.

We first introduce some preliminary setups. A smooth vector field X is said to be (inward) transverse to Ω, if $-\langle \nu_\Omega, X\rangle$ is locally uniformly positive on $\partial\Omega$. By patching the local vector fields $-\partial/\partial x_n$ via a partition of unity, it is not hard to show that transverse vector fields always exist.

We say that a transverse vector field X is complete, if the family of diffeomorphisms generated by X (denoted by $\{\Phi_t^X\}$) exists for $0 \leqslant t \leqslant 1$. From any transverse vector field X we can always produce a complete one. For example, we can choose a complete metric g' (which may be different from the original one) and replace X by $X/\big(1 + |X|_{g'}^2\big)^{1/2}$.

Given a complete transverse vector field X. The following properties of Φ_t^X are not hard to verify. First, for all $t > 0$ the domain $\Phi_t^X(\Omega)$ is locally Lipschitz, and we have $\Phi_t^X(\Omega) \subset \Omega$ and $\partial\Phi_t^X(\Omega) \cap \partial\Phi_s^X(\Omega) = \emptyset$ for all $s \neq t$. Next, since Φ_t^X locally C^1 converges to the identity as $t \to 0$, we have

$$|\mu_{\Phi_t^X(\Omega)}| \rightharpoonup |\mu_\Omega|.$$

Proof of Lemma A.5.1.

Find domains $K \Subset K_1 \Subset K_2 \Subset K_3 \Subset K_4 \Subset K_5 \Subset M$. We first push Ω slightly inward. Choose a cutoff function η with $0 \leqslant \eta \leqslant 1$, $\operatorname{spt}\eta \subset M \setminus K_1$, and $\eta|_{M\setminus K_2} > 0$. Consider the family of diffeomorphisms $\{\Phi_t^{\eta X}\}$ generated by ηX, which exists for all $t \leqslant 1$. Choose $\Omega'' = \Phi_t^{\eta X}(\Omega) \subset \Omega$. For sufficiently small t, we have $\Omega'' \cap K = \Omega \cap K$. Also, it is clear that $\partial\Omega \cap \partial\Omega' \subset K_2$.

For the next step, let us perturb Ω'' in $K_5 \setminus \overline{K_2}$, so that it has smooth boundary in $K_4 \setminus \overline{K_3}$. Denote $U = \partial\Omega'' \cap (K_5 \setminus \overline{K_2})$ and $N = \bigcup_{-\varepsilon<t<\varepsilon} \Phi_t^X(U)$, for some small enough ε. Note that we have topologically $N \cong U \times (-\varepsilon, \varepsilon)$, where the vertical direction is generated by X. Let $\pi : N \to U$ be the natural projection, and $t : N \to (-\varepsilon, \varepsilon)$ be the vertical coordinate function. Denote the set $V = \pi\big(N \cap (K_4 \setminus \overline{K_3})\big)$. Note that for sufficiently small ε, we have $V \Subset U$ and $N \Subset \Omega$.

Note that N is diffeomorphic to an abstract line bundle, hence has a smooth section. This means that there is a smooth open submanifold $\Sigma \subset N$, such that $\pi|_\Sigma : \Sigma \to U$ is bijective and locally bi-Lipschitz. Denote $f = t \circ (\pi|_\Sigma)^{-1} : U \to (-\varepsilon, \varepsilon)$. Let $\varphi : U \to [0, 1]$ be a Lipschitz function such that $\operatorname{spt}\varphi \Subset U$ and $\varphi \equiv 1$ on V. The function φ is used to merge Σ with the original boundary $\partial\Omega''$.

Having these setups, consider the domain

$$\Omega''' = (\Omega \setminus N) \cup \big\{x \in N : t(x) > \varphi(\pi(x)) \cdot f(\pi(x))\big\}.$$

By our way of construction, it follows that $\Omega''' \subset \Omega$, and Ω''' is locally Lipschitz, and $\partial\Omega''' \cap (K_4 \setminus \overline{K_3}) = \Sigma \cap (K_4 \setminus \overline{K_3})$ is a smooth hypersurface.

Having found Ω''', it is easy to find a smooth domain L such that $K_3 \Subset L \Subset K_4$, and ∂L intersects $\partial\Omega'''$ transversely. It follows that $\Omega''' \cap L$ is the desired bounded Lipschitz domain. $\square$

Proof of Lemma A.5.2.

By Lemma A.5.1, there is a sequence of bounded Lipschitz domains $\overline{\Omega}_1 \subset \overline{\Omega}_2 \subset \cdots \subset \Omega$, such that for each $K \Subset M$ we have $\overline{\Omega}_i \cap K = \Omega \cap K$ for all large i. Let X_i be an transverse vector field of $\overline{\Omega}_i$. Recall that such X_i are constructed by patching the local vector fields $\partial/\partial z$. Hence, we can constructed X_i in the way such that: for each $K \Subset M$, it holds $X_i|_K = X_{i+1}|_K = X_{i+2}|_K = \cdots$ for all large i. Then for some sequence $t_1 > t_2 > \cdots \to 0$, the choice $\Omega_i = \Phi^{X_i}_{t_i}(\overline{\Omega}_i)$ satisfies the requirement of the lemma.

The fact that we can achieve $\mathcal{H}^{n-1}(\partial^* E \cap \partial^* \Omega_i) = 0$ for a given E follows from the general fact that for a measurable set X and a family of mutually disjoint sets $\{Y_t\}$, we have $\mu(X \cap Y_t) = 0$ for all but countably many t. Thus in the above construction, we may slightly decrease each t_i so that $\mathcal{H}^{n-1}(\partial^* E \cap \partial^* \Omega_i) = 0$. ☐

Proof of Lemma A.5.3.

Let X be a complete inward transversal vector field of Ω, and $\{\Phi^X_t\}_{0 \leqslant t \leqslant 1}$ be the family of diffeomorphisms generated by X. Then take

$$N = \bigsqcup_{0 \leqslant t \leqslant 1/2} \Phi^X_t(\partial\Omega)$$

and define Φ so that it maps $\Phi^X_t(y)$ to y, for all $y \in \partial\Omega$, $t \in [0, 1/2]$. It is easily verified that N and Φ defined in this way satisfy the desired conditions. ☐

A.6 Miscellaneous useful statements

This section contains several technical lemmas that are useful in the main text.

Lemma A.6.1 (inner approximation of sets)**.**

Fix a background metric on M. Suppose $\Omega_i, \Omega \subset M$ are sets with locally finite perimeter, such that $\chi_{\Omega_i} \to \chi_\Omega$ in $L^1_{\mathrm{loc}}(M)$ and $|\mu_{\Omega_i}| \rightharpoonup |\mu_\Omega|$ weakly as measures in M. Then for any $K \Subset M$ with $\mathcal{H}^{n-1}(\partial^* \Omega \cap \partial K) = 0$ and any set $A \subset \Omega$ with locally finite perimeter, we have

$$P(A; K) = \lim_{i \to \infty} P\big(A \cap \Omega_i; K\big).$$

Proof. By the lower semi-continuity of perimeter,

$$P(A; K) \leqslant \liminf_{i \to \infty} P\big(A \cap \Omega_i; K\big). \tag{A.6.1}$$

Moreover, by Lemma A.1.4(i)(iii) we have

$$P(\Omega; K) \leqslant \liminf_{i \to \infty} P\big(A \cup \Omega_i; K\big), \qquad P(\Omega; K) = \lim_{i \to \infty} P(\Omega_i; K). \tag{A.6.2}$$

To show the reverse direction of (A.6.1), we note that

$$P(A; K) + P\big(\Omega_i; K\big) \geqslant P\big(A \cap \Omega_i; K\big) + P\big(A \cup \Omega_i; K\big).$$

Taking $i \to \infty$ and using (A.6.2), we obtain

$$P(A; K) + P\big(\Omega; K\big) \geqslant \limsup_{i \to \infty} P\big(A \cap \Omega_i; K\big) + P(\Omega; K),$$

which implies the desired result. ☐

Remark A.6.2 (rectifiability of geodesic spheres)**.** Let M be a complete manifold. Consider the family of geodesic balls $B_r = B(x_0, r)$. By the main theorem of [58], for almost every $r \in \mathbb{R}$ the geodesic sphere ∂B_r is a smooth hypersurface except at a singular set of zero $(n-1)$-Hausdorff measure. Hence for these r, we have

$$B_r = (B_r)^{(1)}, \qquad M \setminus B_r = (M \setminus B_r)^{(1)}$$

up to zero $(n-1)$-Hausdorff measure.

Lemma A.6.3 (criterion for containing a cone)**.** *Let $\Omega \subset \mathbb{R}^n$ be a convex domain. Fix $\theta \in (0, \pi/2)$. Suppose $E \subset \mathbb{R}^n$ has locally finite perimeter. Moreover, assume that*

$$\operatorname*{ess\,inf}_{\partial^* E \cap \Omega} \langle \nu_E, e_n \rangle \geqslant \cos\theta,$$

If E has nonzero lower density at a point $x = (x', x_n) \in \Omega$, then

$$E^{(1)} \supset \Omega \cap \Big\{ (y', y_n) \in \mathbb{R}^n : y_n < x_n - |y' - x'| \tan\theta \Big\}. \tag{A.6.3}$$

Here, we denote $e_n = (0, \cdots, 0, 1) \in \mathbb{R}^n$, and define the lower density of a set E at a point x as

$$\underline{\Theta}(E; x) = \liminf_{r \to 0} \frac{|E \cap B(x, r)|}{\omega_n r^n}.$$

Proof of Lemma A.6.3.

Let ρ_ε $(\varepsilon > 0)$ be a family of standard mollifiers, and set $f_\varepsilon = \chi_E * \rho_\varepsilon$. Let $\Omega' \Subset \Omega$ be another convex domain. For any smooth vector field X with $\operatorname{supp}(X) \subset \Omega'$ and any $\varepsilon < d(\partial\Omega', \partial\Omega)$, we compute

$$-\int_{\mathbb{R}^n} \nabla f_\varepsilon \cdot X = \int_{\mathbb{R}^n} f_\varepsilon \operatorname{div} X = \int_E \operatorname{div}(\rho_\varepsilon * X) = \int_{\partial^* E} (\rho_\varepsilon * X) \cdot \nu_E \, d\mathcal{H}^{n-1}.$$

Let w be any constant vector field with $\langle w, e_n \rangle \geqslant \sin\theta$ (thus it holds $\langle \nu_E, w \rangle \geqslant 0$ a.e.). Setting $X = \varphi w$ for arbitrary $\varphi \in C_0^\infty(\Omega')$, $\varphi \geqslant 0$, we obtain

$$-\int_{\mathbb{R}^n} \varphi \frac{\partial f_\varepsilon}{\partial w} = \int_{\partial^* E} (\rho_\varepsilon * \varphi) \langle w, \nu_E \rangle \geqslant 0.$$

This implies $\frac{\partial f_\varepsilon}{\partial w} \leqslant 0$ in Ω'. Let $\Theta > 0$ be the lower density of E at the given point x. For all sufficiently small ε we have $f_\varepsilon(x) \geqslant \frac{1}{2}\Theta$, and it follows by the convexity of Ω' that $f_\varepsilon \geqslant \frac{1}{2}\Theta$ in the region

$$Q := \Omega' \cap \Big\{ (y', y_n) : y_n < x_n - |y' - x'| \tan\theta \Big\}.$$

Taking $\varepsilon \to 0$, we conclude that E has lower density at least $\frac{1}{2}\Theta$ almost everywhere in Q. Therefore E occupies full measure in Q. The result follows by taking $\Omega' \to \Omega$. □

Lemma A.6.4 (Bernstein theorem in a half space)**.**

Let $E \subset \{x_n < 0\}$ be nonempty and locally perimeter-minimizing in $\mathbb{R}^n$. Then E must be a half space, i.e. there exists $c > 0$ such that $E = \big\{(x', x_n) : x_n < -c\big\}$.

Proof. Choose $x \in \partial^* E$. As a consequence of the classical monotonicity formula, there exists a sequence $\lambda_i \to \infty$ such that the limit $E_\infty = \lim_{i \to \infty} \lambda_i (E - x)$ exists and is a perimeter-minimizing cone. We refer to [77, Theorem 28.17] for the precise argument. Since $E_\infty \subset \{x_n \leqslant 0\}$ and $0 \in \operatorname{spt}\big(|D\chi_{E_\infty}|\big)$, it follows by a strong maximum principle [110, Theorem 4] that $\partial E_\infty = \{x_n = 0\}$. By the monotonicity formula again, ∂E itself is a cone centered at x. Thus ∂E must be a flat hyperplane. □

Appendix B

Notations

The following is a list of frequent notations. When the symbols in this list are used, they always have the meaning indicated here unless specified by the context.

- M: a smooth, connected, oriented manifold. A background Riemannian metric g always comes equipped on M, which we often make implicit.
- Ω: a domain (i.e. open set) in M, which we always assume to be connected.
- K: a domain that is usually precompact.
- E, F: a set which is usually of locally finite perimeter.
- Σ: a two-sided embedded hypersurface.
- A, H, ν: the second fundamental form, mean curvature and unit normal of a (two-sided) hypersurface.
- $E_t; E_t^+$: we set $E_t(u) = \{u < t\}$, $E_t^+(u) = \{u \leqslant t\}$. When u is defined in a domain Ω, these sets are viewed as subsets of Ω.
- u^∂: used in Chapter 3 to denote the boundary trace of BV functions.

Our sign conventions are as follows:

- When a hypersurface is the boundary of a domain, its unit normal points outward.
- For a hypersurface, we have $A(X,Y) = \langle \nabla_X \nu, Y\rangle$ and $H = \mathrm{tr}_\Sigma A$. Hence, the unit sphere $\big\{|x| = 1\big\} \subset \mathbb{R}^n$ has mean curvature $n-1$.
- The Riemannian curvature is defined as $R(X,Y)Z = (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Z$.
- The Laplacian is defined as $\Delta = \sum_i \nabla_{e_i}\nabla_{e_i}$, for any orthonormal frame $\{e_i\}$.

We include the definition of several function spaces:

- $u \in \mathrm{Lip}(\Omega)$, if there is a constant Λ so that $|u(x) - u(y)| \leqslant \Lambda d(x,y)$, $\forall\, x, y \in \Omega$.
- $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, if $u \in \mathrm{Lip}(K)$ for all $K \Subset \Omega$;
- $u \in \mathrm{Lip}_{\mathrm{loc}}(\overline{\Omega})$, if $u \in \mathrm{Lip}(K \cap \Omega)$ for all $K \Subset M$. Namely, the regularity holds up to $\partial\Omega$ but remains local in M.
- $u \in \mathrm{BV}_{\mathrm{loc}}(\overline{\Omega})$, if $u \in \mathrm{BV}(K \cap \Omega)$ for all $K \Subset M$. (Note: the space $\mathrm{BV}(\Omega)$ is still in the traditional sense: $u \in \mathrm{BV}(\Omega)$ means $u \in L^1(\Omega)$ and $||Du||(\Omega) < \infty$.)
- $u \in \mathrm{Lip}_0(\Omega)$, if $u \in \mathrm{Lip}(\Omega)$ and $\mathrm{spt}(u) \Subset \Omega$.

The terminologies in the following list always have their special meanings:

- $\sigma(x; \Omega, g)$, $\sigma(x; \Omega)$: the regular radius, see Definition 2.4.2.
- $\mathsf{IMCF}(\Omega)$, $\mathsf{IMCF}(\Omega, g)$: see Definition 2.1.2.
- $\mathsf{IVP}(\Omega; E_0)$, $\mathsf{IVP}(\Omega, g; E_0)$: see Definition 2.1.8.

- $\mathsf{IMCF}(\Omega)$+$\mathsf{OBS}(\partial\Omega)$, $\mathsf{IMCF}(\Omega, g)$+$\mathsf{OBS}(\partial\Omega)$: see Definition 3.3.2.
- $\mathsf{IVP}(\Omega; E_0)$+$\mathsf{OBS}(\partial\Omega)$, $\mathsf{IVP}(\Omega, g; E_0)$+$\mathsf{OBS}(\partial\Omega)$: see Definition 3.3.6.
- The set-replacing argument: this refers to the standard technique appearing in the proof of Lemma A.2.5.

We recall the following standard notations:

- $A \Subset B$: A is precompact in B, meaning that the closure of A is compact in B. Reminder: $A \Subset B$ does not mean that A is compact, since A need not be closed.
- $B(x, r)$: the open ball of radius r centered at x.
- $|\operatorname{Rm}|$ and inj: the norm of Riemannian tensor and the global injectivity radius. These are used in contexts related to bounded geometry.
- χ_E: the indicator function of E.
- $|E|$: the volume of E. In some contexts, we also use $|\Sigma|$ to denote the area of Σ.
- $\mathcal{H}^k(A)$: the k-dimensional Hausdorff measure of A.
- $\mu_i \rightharpoonup \mu$: the weak convergence of measures, in duality with the space of continuous functions with compact support.
- $\mathbb{S}^n$: the round n-sphere.
- ω_n: the volume of $\mathbb{S}^n$.

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