# Case study
# *Union-Find*

*Please note that these slides are based in part on material originally developed by Prof. Kevin Wayne of the CS Dept at Princeton University.*

# Steps to developing a usable algorithm

- Steps to developing a usable algorithm
  - Model the problem
  - Find an algorithm to solve it
  - Fast enough? Fits in memory?
  - If not, figure out why not
  - Find a way to address the problem
  - Iterate until satisfied
- The scientific method
- Mathematical analysis

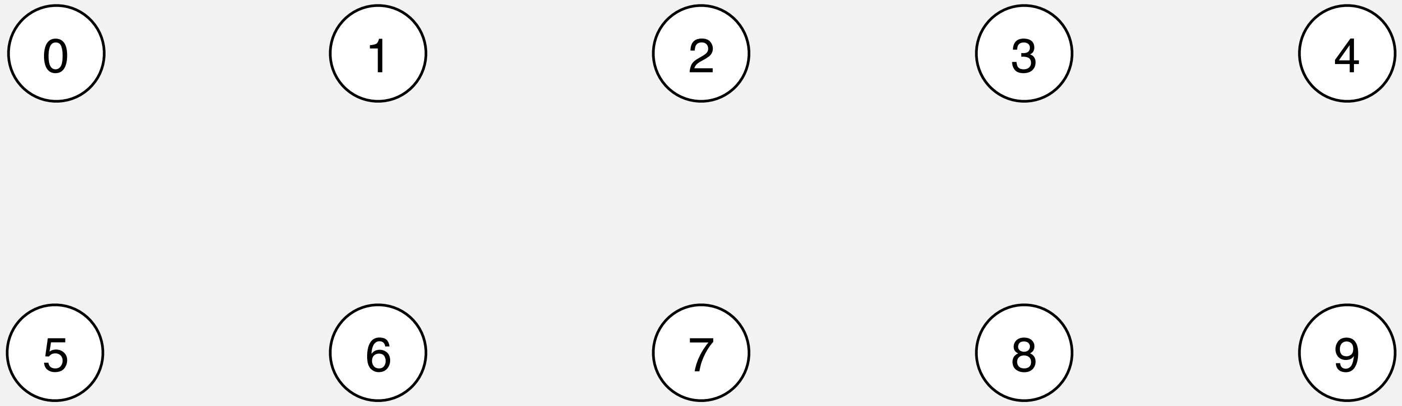
# Let’s look at an example

- Union-Find is a solution to a real problem:
    - It is a special case from graph theory (which we will look at in more detail later on)
    - Here, all we care about is whether two nodes are “connected” (directly, or indirectly).
    - Said connections have no attributes (as they probably would in a true graph).

# Dynamic connectivity problem

Given a set of N objects, we support two operations:

- Connect two objects. (mutating)
- Is there a path connecting the two objects? (non-mutating)



# Dynamic connectivity problem

Given a set of N objects, we support two operations:

- Connect two objects. (mutating)
- Is there a path connecting the two objects? (non-mutating)

*connect 4 and 3*

*connect 3 and 8*

*connect 6 and 5*

*connect 9 and 4*

*connect 2 and 1*

*are 0 and 7 connected?* ✗

*are 8 and 9 connected?* ✔

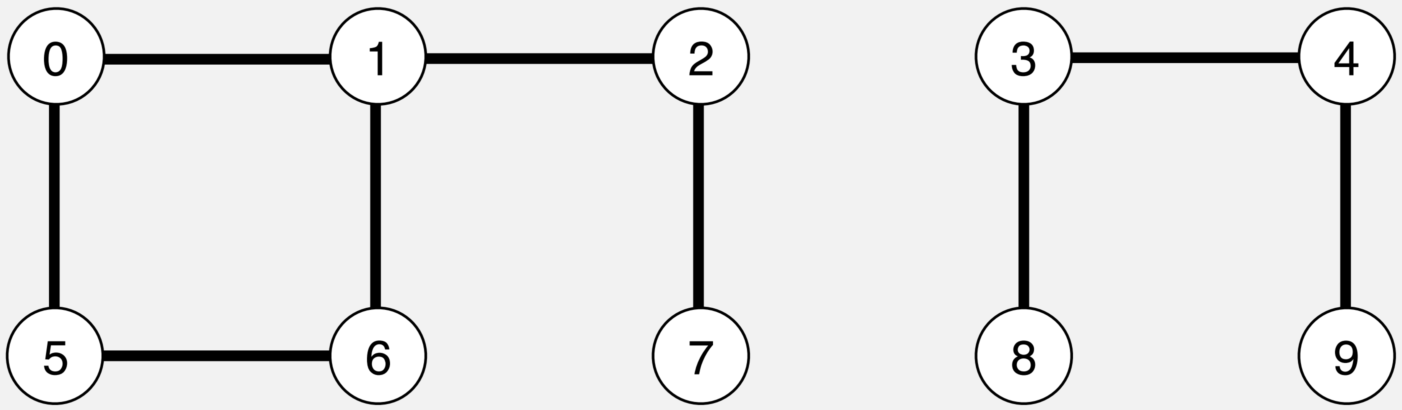
*connect 5 and 0*

*connect 7 and 2*

*connect 6 and 1*

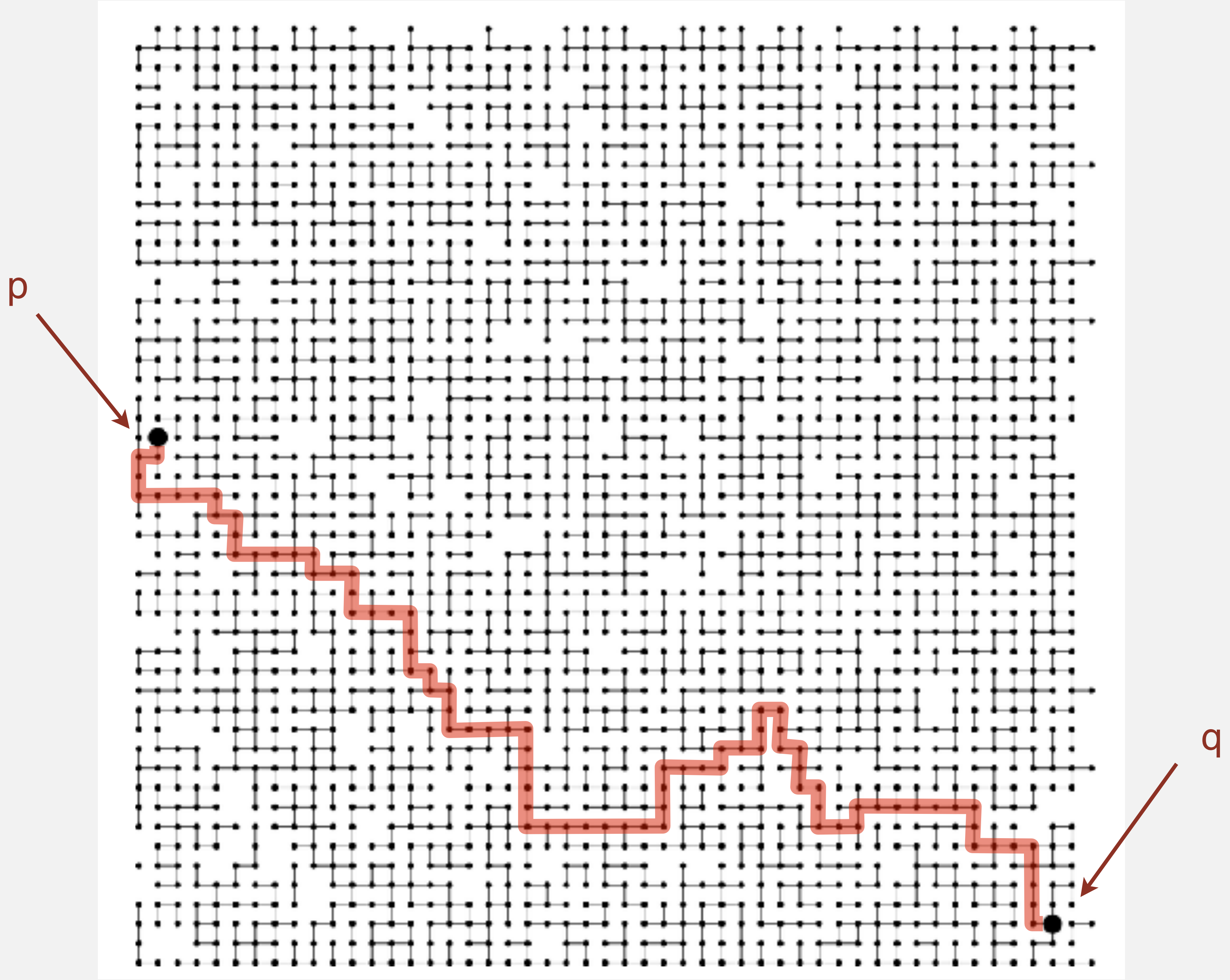
*connect 1 and 0*

*are 0 and 7 connected?* ✔



# A larger connectivity example

Q. Is there a path connecting $p$ and $q$ ?




A. Yes.

# Modeling the objects

Applications involve manipulating objects of all types.

- Pixels in a digital photo.
- Computers in a network.
- Friends in a social network.
- Transistors in a computer chip.
- Elements in a mathematical set.
- Variable names in a Fortran program.
- Metallic sites in a composite system.

When programming, convenient to name objects 0 to $N - 1$.

- Use integers as array index.
- Suppress details not relevant to union-find.

can use symbol table to translate from site names to integers: stay tuned (Chapter 3)

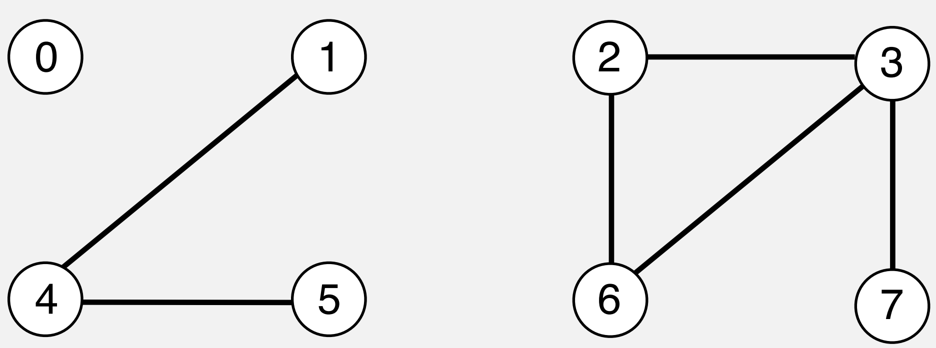
# Modeling the connections

We assume "is connected to" is an equivalence relation:

- Reflexive: $p$ is connected to $p$.
- Symmetric: if $p$ is connected to $q$, then $q$ is connected to $p$.
- Transitive: if $p$ is connected to $q$ and $q$ is connected to $r$, then $p$ is connected to $r$.

New model entity:

*Connected component.* Maximal set of objects that are mutually connected.




{ 0 } { 1 4 5 } { 2 3 6 7 }

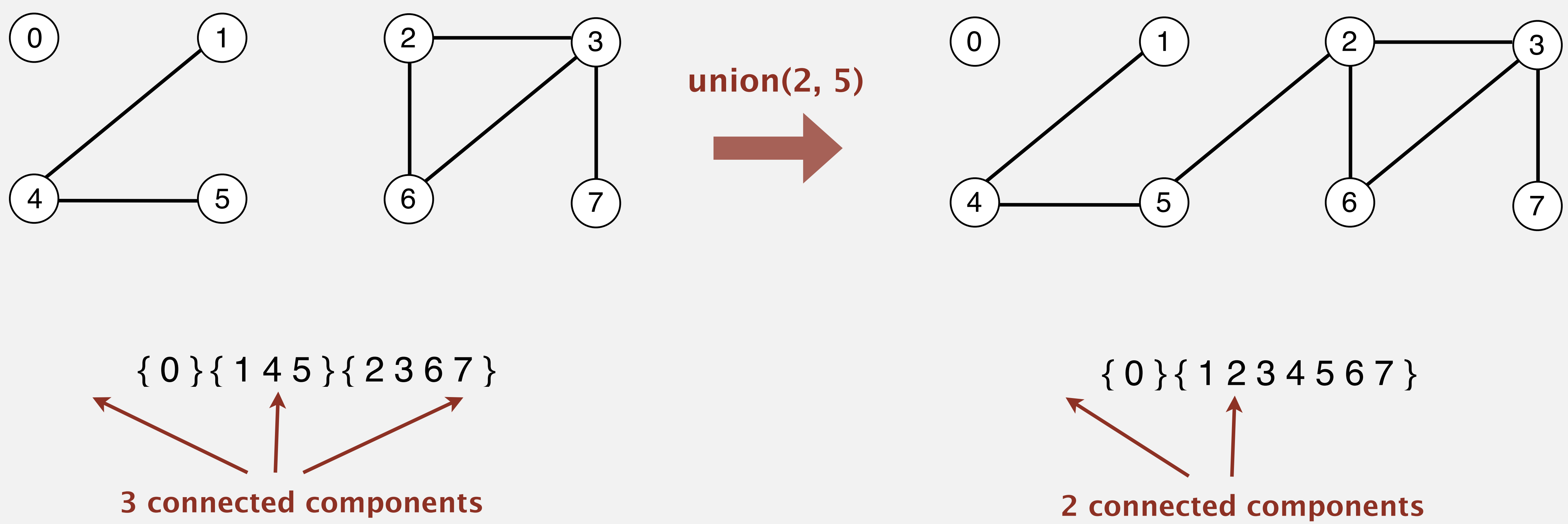
**3 connected components**

# Implementing the operations

Find. In which component is object $p$ ?

Connected. Are objects $p$ and $q$ in the same component?

Union. Replace components containing objects $p$ and $q$ with their union.



# What just happened?

- We transformed the problem that we had, i.e. to implement for a no-attribute graph of vertices and edges:

  We call this "Reduction" as you will recall

  - connect(p,q); // connect object p to object q
  - isPath(p,q); // is there a path from p to q?
- Into a slightly different problem, i.e. for a set of connected components:
  - find(p); // which component does object p belong to?
  - connected(p,q); // is p connected to q? i.e. find(p)==find(q)
  - union(p,q). // replace the components p and q with their union.

# Union-find data type (API)

**Goal.** Design efficient data structure for union-find.

- Number of objects $N$ can be huge.
- Number of operations $M$ can be huge.
- Union and find operations may be intermixed.

| public class UF | | |
|---|---|---|
| | UF(int N) | *initialize union-find data structure with $N$ singleton objects ($0$ to $N - 1$)* |
| void | union(int p, int q) | *add connection between p and q* |
| private int | find(int p) | *component identifier for p ($0$ to $N - 1$)* |
| boolean | connected(int p, int q) | *are p and q in the same component?* |

```
public boolean connected(int p, int q)
{  return find(p) == find(q);  }
```

**1-line implementation of connected()**

# Dynamic-connectivity client

- Read in number of objects $N$ from standard input.
- Repeat:
  - read in pair of integers from standard input
  - if they are not yet connected, connect them and print out pair

```
public static void main(String[] args)
{
  int N = StdIn.readInt();
  UF uf = new UF(N);
  while (!StdIn.isEmpty())
  {
    int p = StdIn.readInt();
    int q = StdIn.readInt();
    if (!uf.connected(p, q))
    {
      uf.union(p, q);
      StdOut.println(p + " " + q);
```

```
% more tinyUF.txt
10
4 3
3 8
6 5
9 4
2 1
8 9
5 0
7 2
6 1
1 0
6 7
```

already connected


http://algs4.cs.princeton.edu

# 1.5 Union-Find

- dynamic connectivity
- quick find
- quick union
- improvements
- applications

# Quick-find [eager approach]

## Data structure.

if and only if

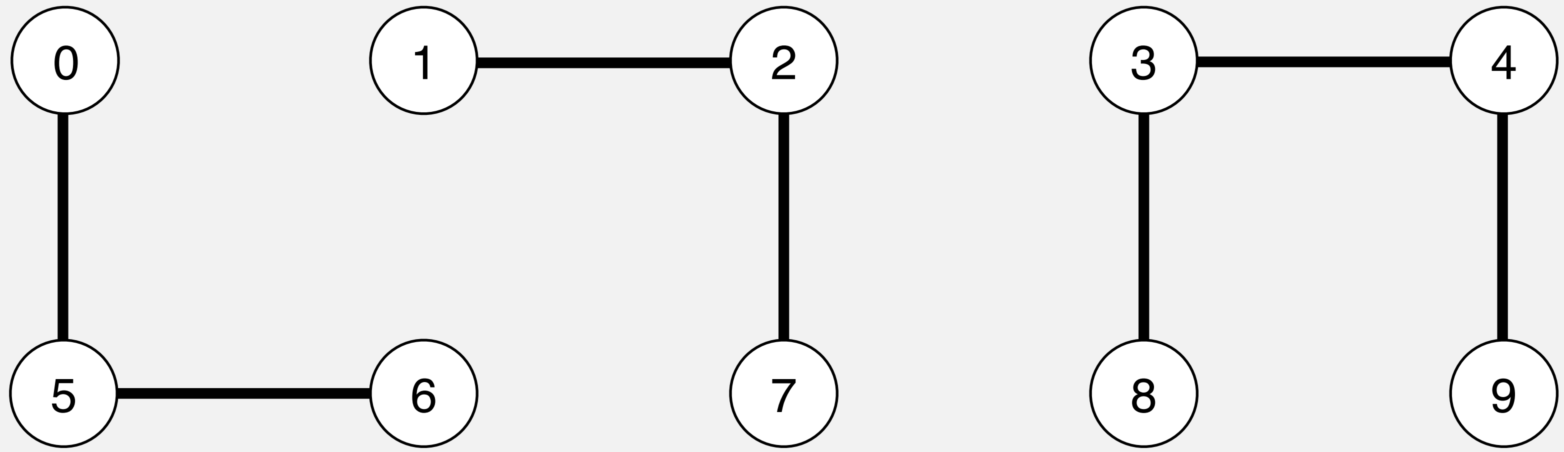
- Integer array id[] of length N.
- Interpretation: id[p] is the id of the component containing p.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| id[] | 0 | 1 | 1 | 8 | 8 | 0 | 0 | 1 | 8 | 8 |

0, 5 and 6 are connected

1, 2, and 7 are connected

3, 4, 8, and 9 are connected



# Quick-find [eager approach]

Data structure.

- Integer array id[] of length N.
- Interpretation: id[p] is the id of the component containing p.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| id[] | 0 | 1 | 1 | 8 | 8 | 0 | 0 | 1 | 8 | 8 |

Find. What is the id of p?

Connected. Do p and q have the same id?

id[6] = 0; id[1] = 1

6 and 1 are not connected

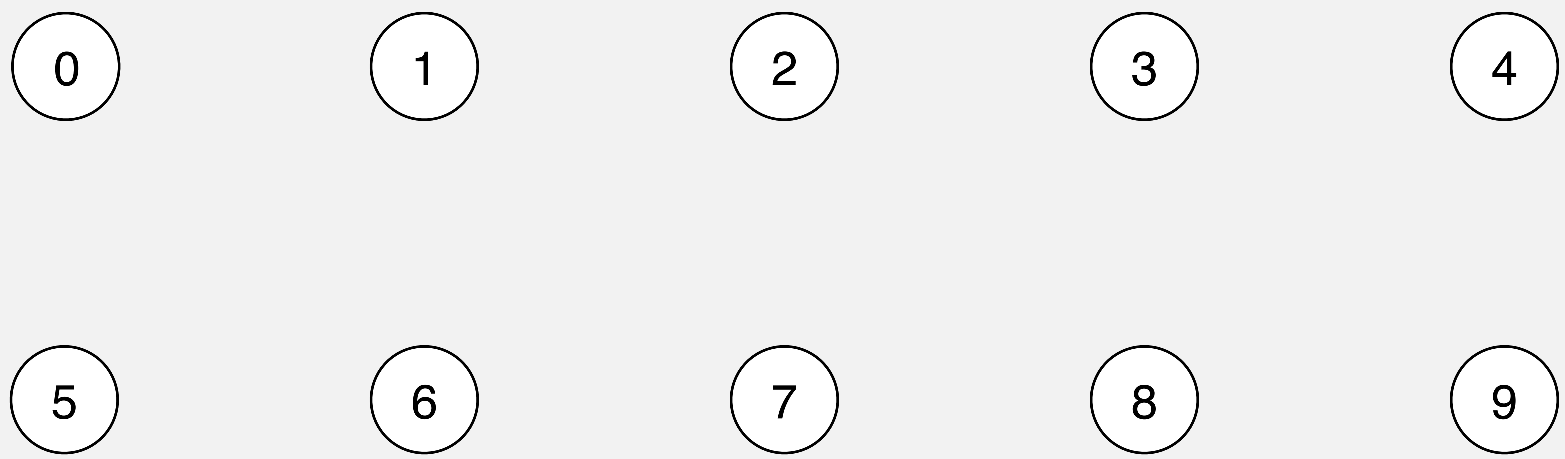
Union. To merge components containing p and q, change all entries whose id equals id[p] to id[q].

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| id[] | 1 | 1 | 1 | 8 | 8 | 1 | 1 | 1 | 8 | 8 |

**after union of 6 and 1**

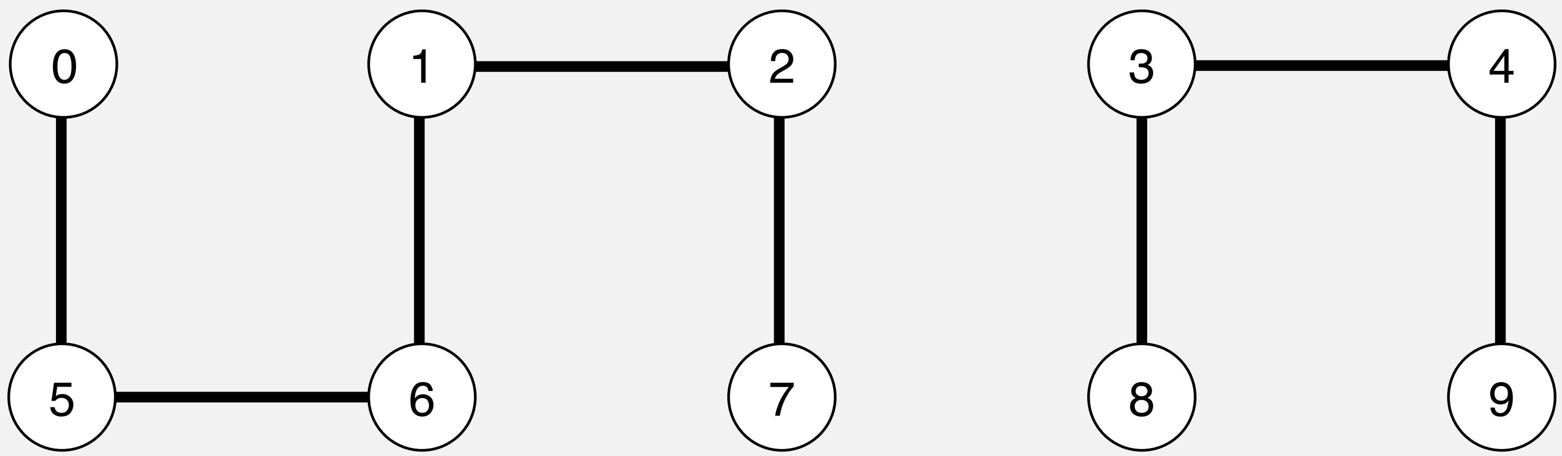
problem: many values can change

# Quick-find demo




| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| **id[]** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

# Quick-find demo




| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| **id[]** | 1 | 1 | 1 | 8 | 8 | 1 | 1 | 1 | 8 | 8 |

# Quick-find: Java implementation

```
public class QuickFindUF
{
   private int[] id;
  public QuickFindUF(int N)
   {
      id = new int[N];
      for (int i = 0; i < N; i++)
      id[i] = i;
   }
   public boolean find(int p)
   {  return id[p];  }

   public void union(int p, int q)
   {
      int pid = id[p];
      int qid = id[q];
      for (int i = 0; i < id.length; i++)
         if (id[i] == pid) id[i] = qid;
   }
}
```

set id of each object to itself

(N array accesses)

return the id of p

(1 array access)

change all entries with id[p] to id[q]

(at most 2N + 2 array accesses)

# Quick-find is too slow

Cost model. Number of array accesses (for read or write).

| algorithm | initialize | union | find | connected |
|---|---|---|---|---|
| **quick-find** | N | N | 1 | 1 |

**order of growth of number of array accesses**

quadratic!

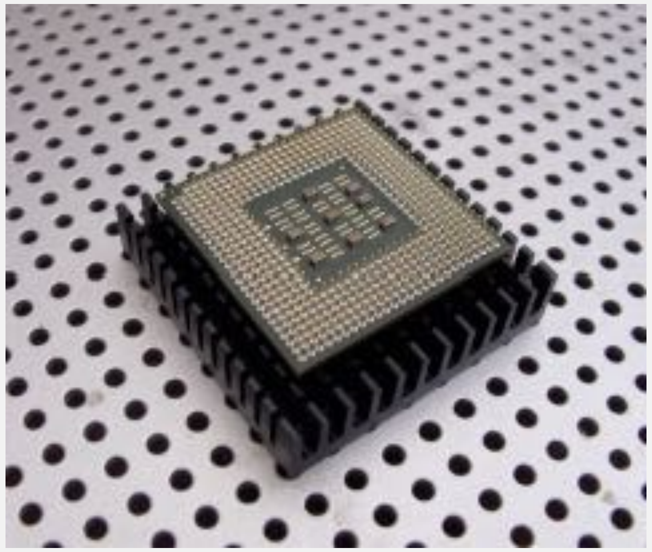
Union is too expensive. It takes $N^2$ array accesses to process a sequence of $N$ union operations on $N$ objects.

# Quadratic algorithms do not scale

Rough standard (for now).

- $10^9$ operations per second.
- $10^9$ words of main memory.
- Touch all words in approximately 1 second.

a truism (roughly) since 1950!

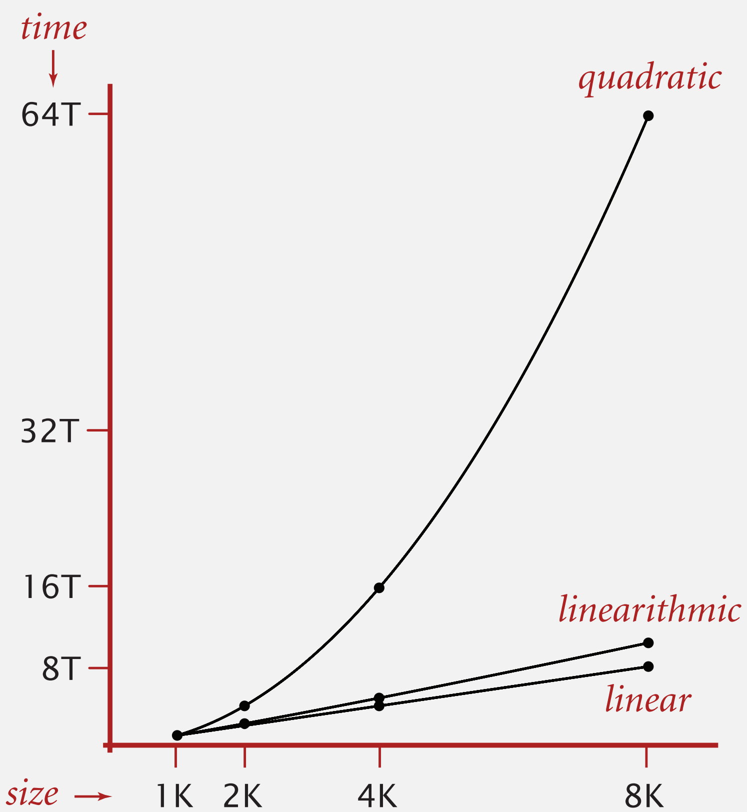


Ex. Huge problem for quick-find.

- $10^9$ union commands on $10^9$ objects.
- Quick-find takes more than $10^{18}$ operations.
- 30+ years of computer time!

Quadratic algorithms don't scale with technology.

- New computer may be 10x as fast.
- But, has 10x as much memory $\Rightarrow$ want to solve a problem that is 10x as big.
- With quadratic algorithm, takes 10x as long!




http://algs4.cs.princeton.edu

# 1.5 UNION-FIND

- *dynamic connectivity*
- *quick find*
- *quick union*
- *improvements*
- *applications*

# Quick-union [lazy approach]

Data structure.

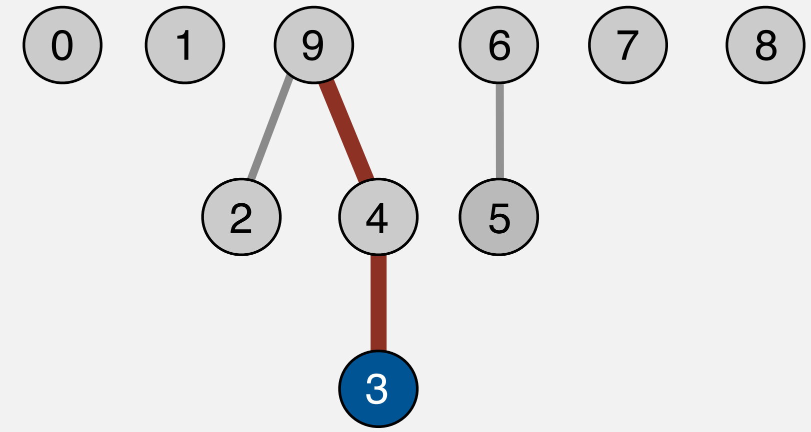
- Integer array prnt[] of length N.
- Interpretation: prnt[i] is *parent* of i.
- Root of i is prnt[prnt[prnt[...prnt[i]...]]].

*was* id[p] is the id of the component containing p

keep going until it doesn't change

(algorithm ensures no cycles)

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| **prnt[]** | 0 | 1 | 9 | 4 | 9 | 6 | 6 | 7 | 8 | 9 |




parent of 3 is 4

root of 3 is 9

# Quick-union [lazy approach]

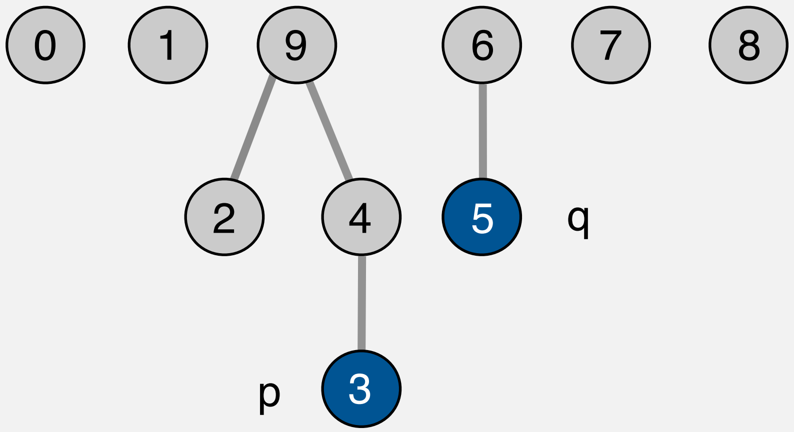
Data structure.

- Integer array id[] of length N.
- Interpretation: id[i] is parent of i.
- Root of i is id[id[id[...id[i]...]]].

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| **id[]** | 0 | 1 | 9 | 4 | 9 | 6 | 6 | 7 | 8 | 9 |

Find. What is the root of p?

Connected. Do p and q have the same root?




root of 3 is 9

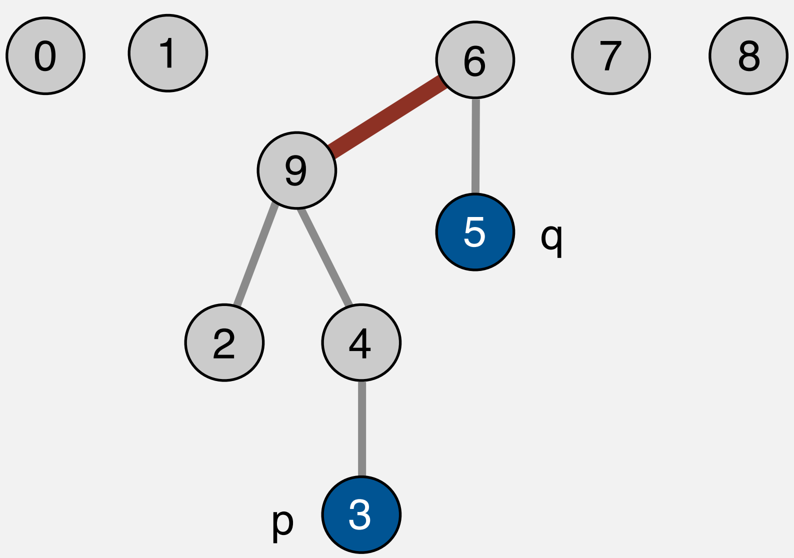
root of 5 is 6

3 and 5 are not connected

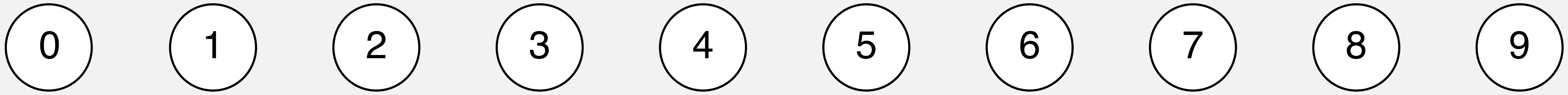
Union. To merge components containing p and q, set the id of p's root to the id of q's root.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| **id[]** | 0 | 1 | 9 | 4 | 9 | 6 | 6 | 7 | 8 | 6 |

only one value changes

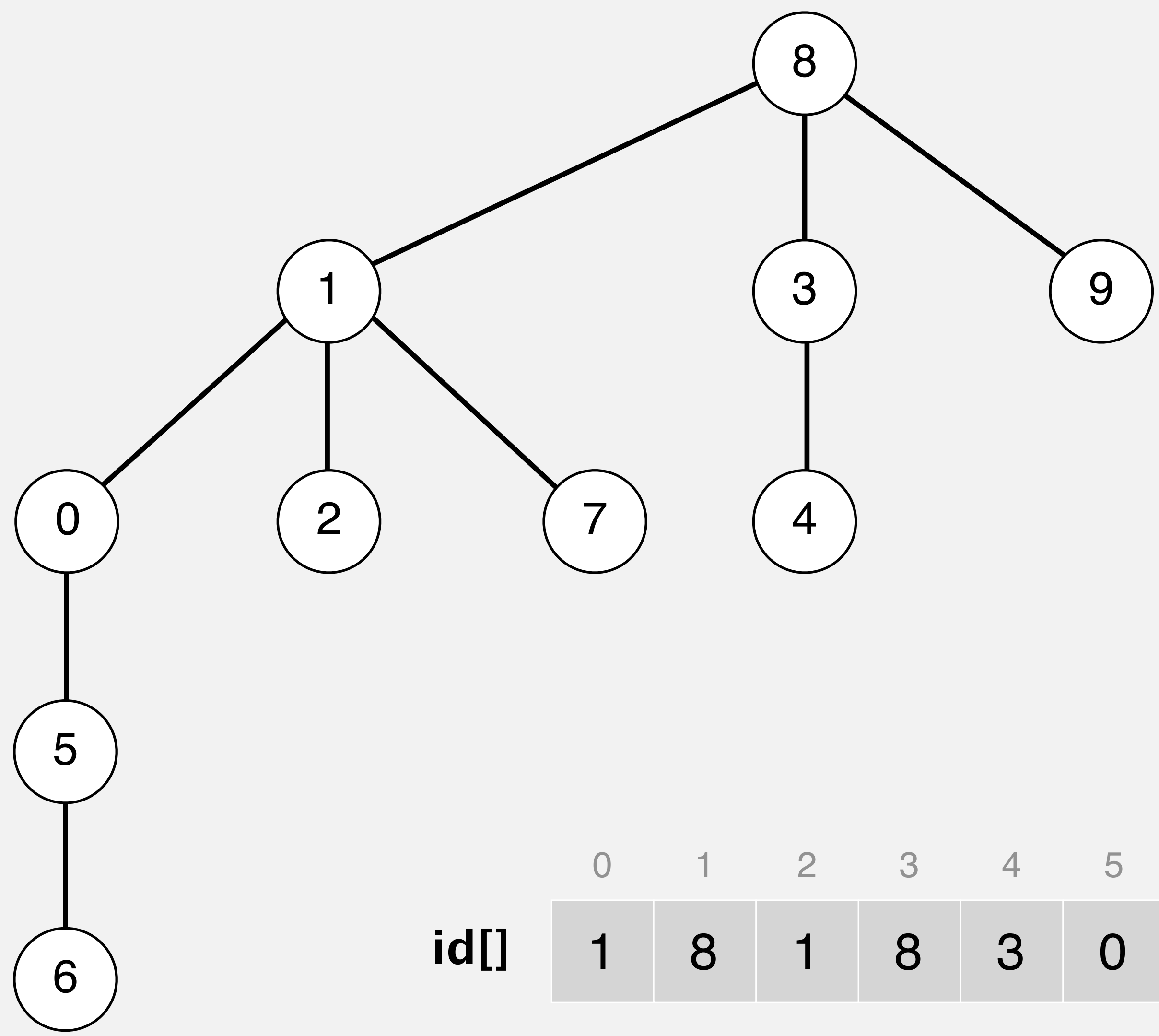


# Quick-union demo




| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| **id[]** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

# Quick-union demo




| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| **id[]** | 1 | 8 | 1 | 8 | 3 | 0 | 5 | 1 | 8 | 8 |

# Quick-union: Java implementation

```
public class QuickUnionUF {
   private int[] prnt; // array of parents
   public QuickUnionUF(int N) {
      prnt = new int[N];
      for (int i = 0; i < N; i++) prnt[i] = i;
   }
   public int find(int i) {
      while (i != prnt[i]) i = prnt[i];
      return i;
   }
   public void union(int p, int q) {
      int i = find(p);
      int j = find(q);
      prnt[i] = j;
   }
}
```

# Quick-union is also too slow

Cost model. Number of array accesses (for read or write).

| algorithm | initialize | union | find | connected |
|---|---|---|---|---|
| **quick-find** | N | N | 1 | 1 |
| **quick-union** | N | N † | N | N |

← worst case (pessimistic)

† includes cost of finding roots

Quick-find defect.

- Union too expensive ($N$ array accesses).
- Trees are flat, but too expensive to keep them flat.

Quick-union defect.

- Trees can get tall.
- Find/connected too expensive (could be $N$ array accesses).


http://algs4.cs.princeton.edu

# 1.5 Union-Find

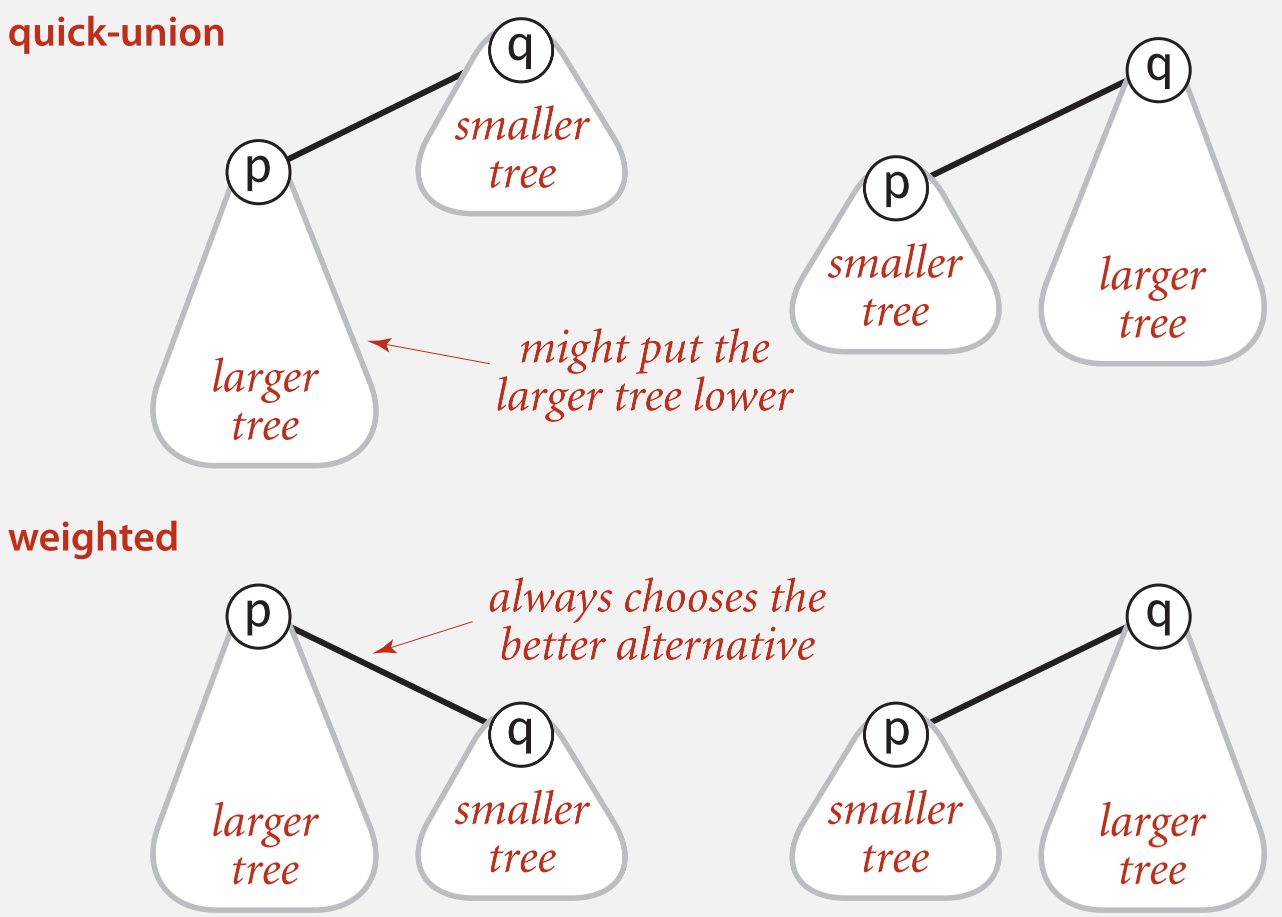
- dynamic connectivity
- quick find
- quick union
- *improvements*
- applications

# Improvement 1: weighting

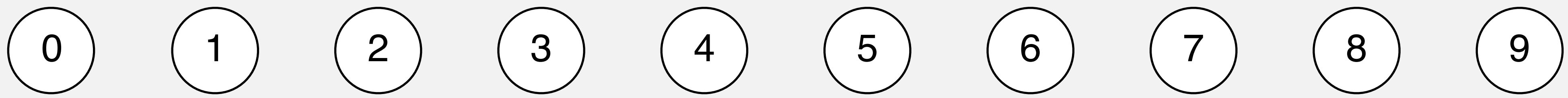
Weighted quick-union.

- Modify quick-union to avoid tall trees.
- Keep track of size of each tree (number of objects).
- Balance by linking root of smaller tree to root of larger tree.

reasonable alternatives:
union by height or "rank"

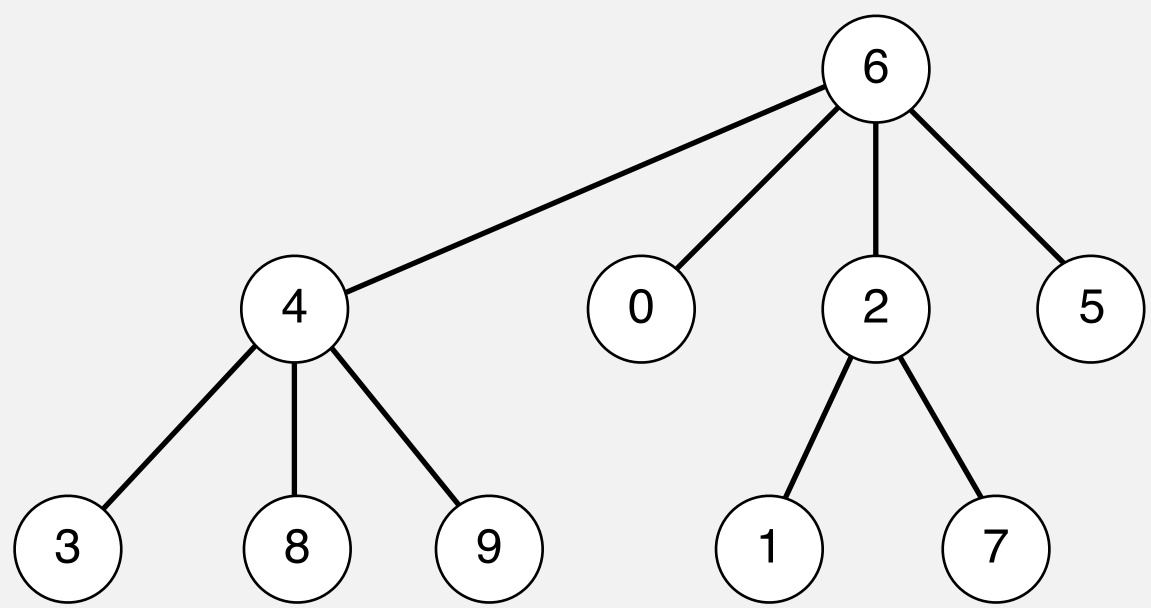


# Weighted quick-union demo




| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| **id[]** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

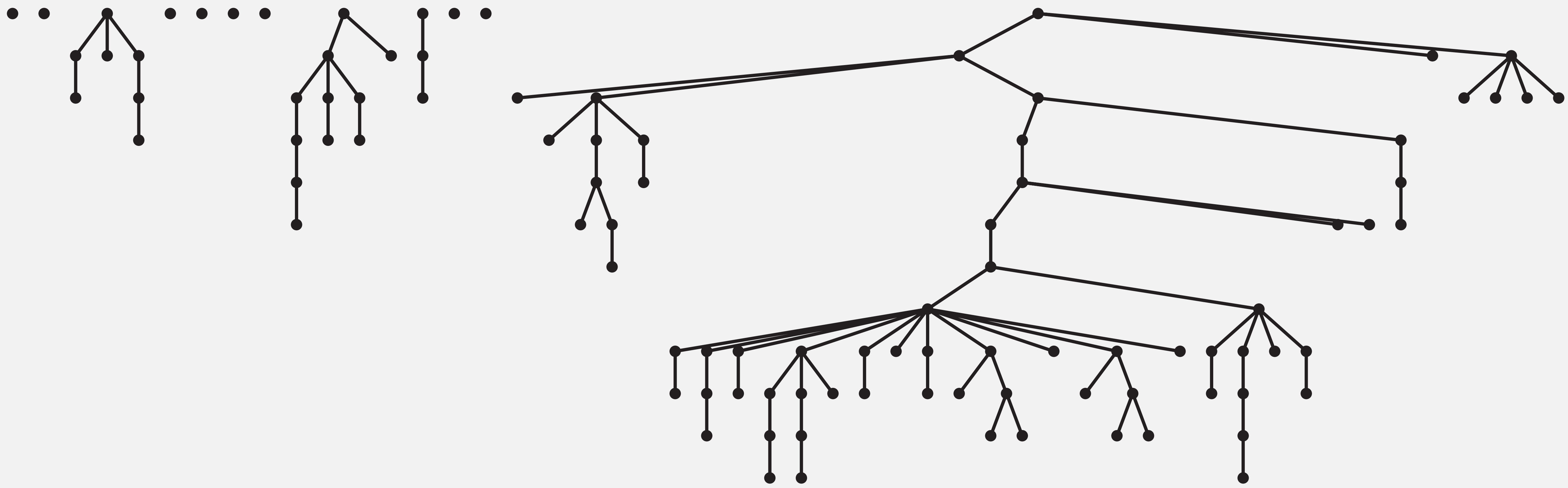
# Weighted quick-union demo




| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| **id[]** | 6 | 2 | 6 | 4 | 6 | 6 | 6 | 2 | 4 | 4 |

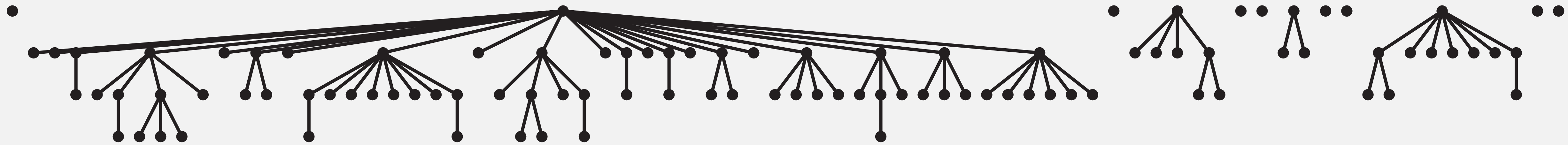
# Quick-union and weighted quick-union example

**quick-union**



*average distance to root*: 5.11

**weighted**



*average distance to root*: 1.52

**Quick-union and weighted quick-union (100 sites, 88 `union()` operations)**

# Weighted quick-union: Java implementation

Data structure. Same as quick-union, but maintain extra array sz[i] to count number of objects in the tree rooted at i.

Find/connected. Identical to quick-union.

Union. Modify quick-union to:

- Link root of smaller tree to root of larger tree.
- Update the sz[] array.

```
int i = find(p);
int j = find(q);
if (i == j) return;
if  (sz[i] < sz[j]) { id[i] = j; sz[j] += sz[i]; }
else                { id[j] = i; sz[i] += sz[j]; }
```

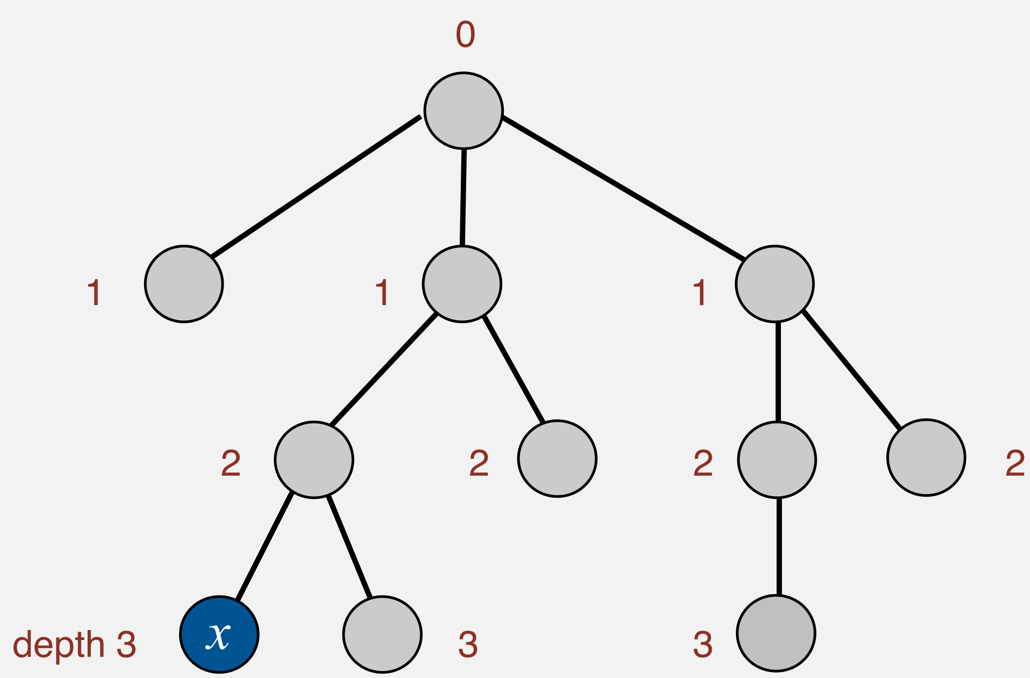
# Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of $p$.
- Union: takes constant time, given roots.

lg = base-2 logarithm

Proposition. Depth of any node $x$ is at most $\lg N$. [depth = lg N if tree is a complete binary tree; if some nodes are ternary, etc, then depth is, generally, less].




**N = 11**
**depth(x) = 3 ≤ lg N**

# Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of $p$.
- Union: takes constant time, given roots.

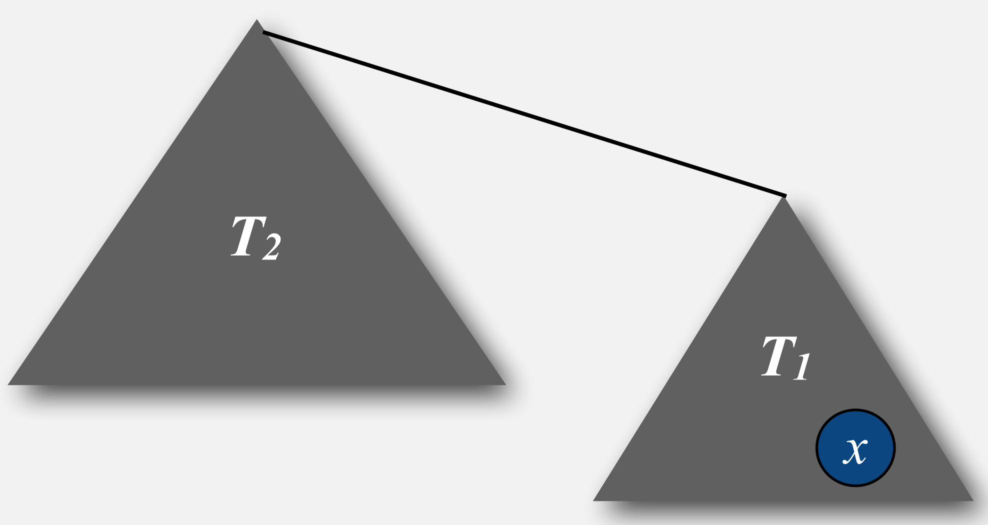
lg = base-2 logarithm

Proposition. Depth of any node $x$ is at most $\lg N$.

Pf. What causes the depth of object $x$ to increase?

Increases by $1$ when tree $T_1$ containing $x$ is merged into another tree $T_2$.

- The size of the tree containing $x$ at least doubles since $|T_2| \geq |T_1|$.
- Size of tree containing $x$ can double at most $\lg N$ times. Why?




$$\left.\begin{array}{c}1\\2\\4\\8\\16\\\vdots\\N\end{array}\right\}\ \lg N$$

# Proposition H

- Proposition: the depth $d$ of any forest built by weighted quick-union for $n$ sites is at most $lg\ n$
- Prove: for every tree of size $s$ in forest, $d <= lg\ s$
- Proof by induction:
  - Base case: when $n = 1$, $d = 0$ (d <= lg n)
  - Assume proposition is true for any tree $i$ of size $s_i$. When we combine tree $i$ of size $s_i$ with tree $j$ of size $s_j$, where $s_i <= s_j$, then
    - the *a priori* depths are: $d_i <= lg\ s_i <= lg\ s_j$ and $d_j <= lg\ s_j$
    - the *a posteriori* depths are: $d_i <= 1 + lg\ s_i$ and $d_j <= lg\ s_j$
    - but $d_i <= lg\ (s_i + s_i) <= lg\ (s_i + s_j)$
    - therefore ***all*** depths $d_k <= lg\ s_k$ where $s_k = s_i + s_j$

# Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of $p$.
- Union: takes constant time, given roots.

Proposition. Depth of any node $x$ is at most $\lg N$.

| algorithm | initialize | union | find | connected |
|---|---|---|---|---|
| **quick-find** | N | N | 1 | 1 |
| **quick-union** | N | N † | N | N |
| **weighted QU** | N | lg N † | lg N | lg N |

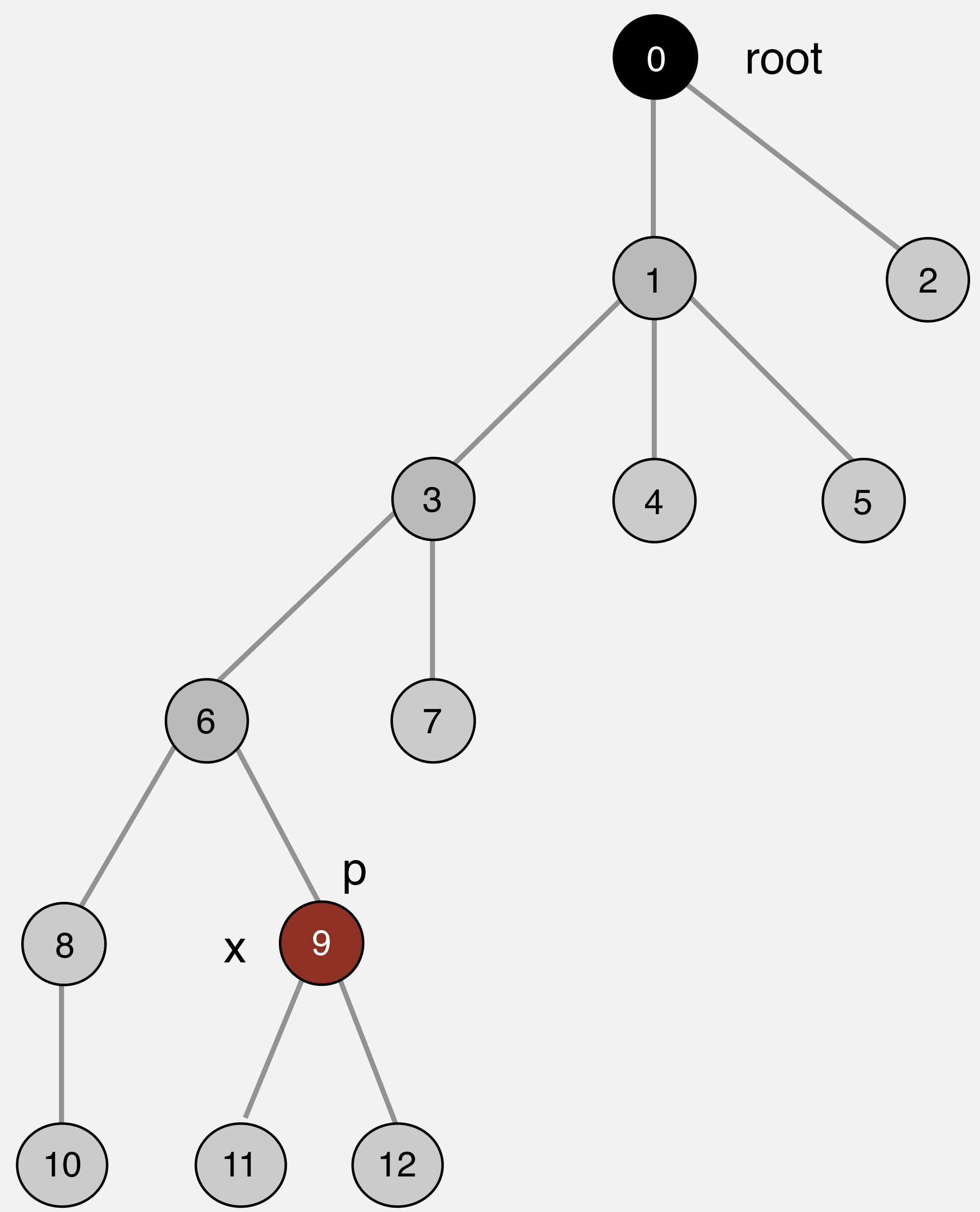
† includes cost of finding roots

Q. Stop at guaranteed acceptable performance?

A. No, easy to improve further.

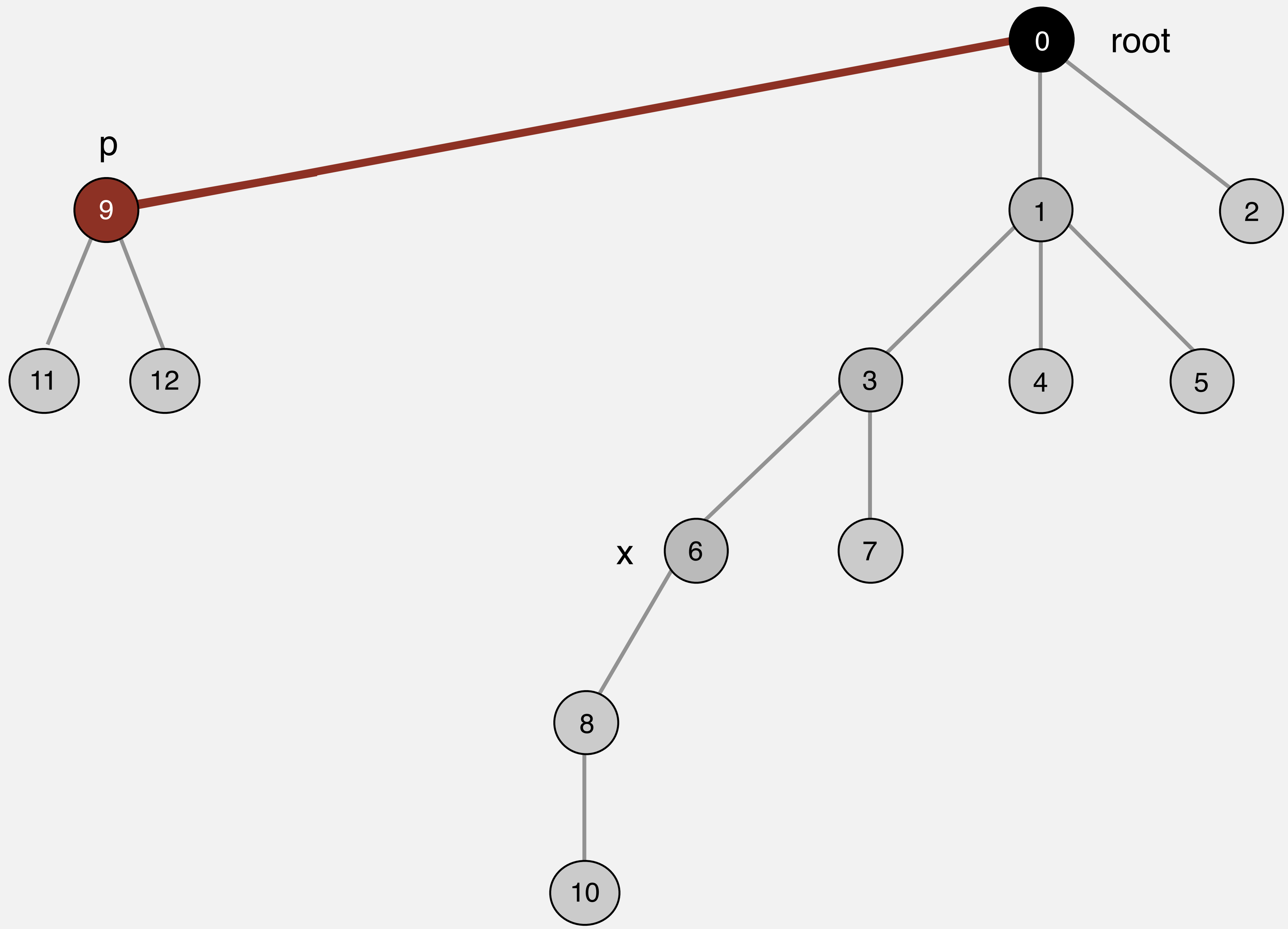
# Improvement 2: path compression

Quick union with path compression. Just after computing the root of $p$, set the id[] of each examined node to point to that root.



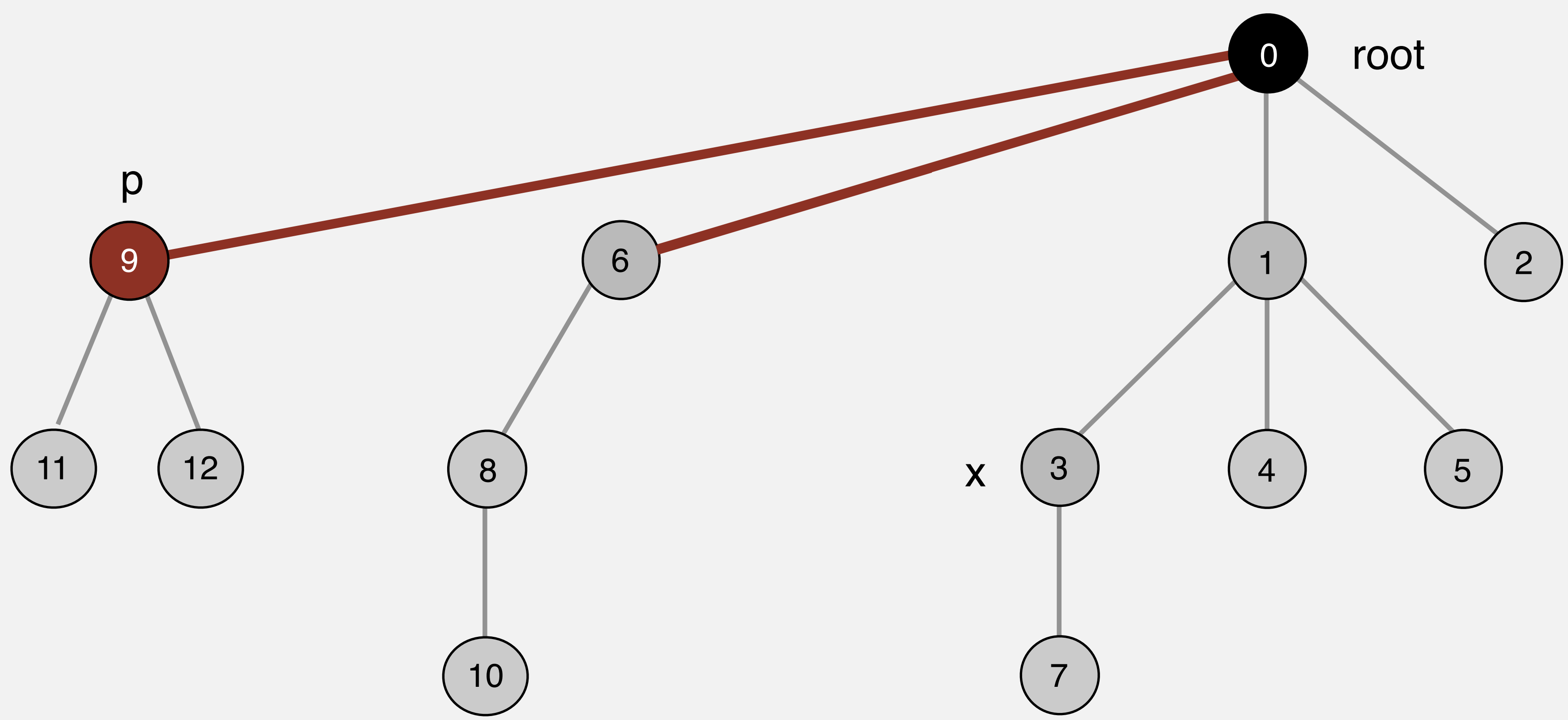
# Improvement 2: path compression

Quick union with path compression. Just after computing the root of $p$, set the id[] of each examined node to point to that root.



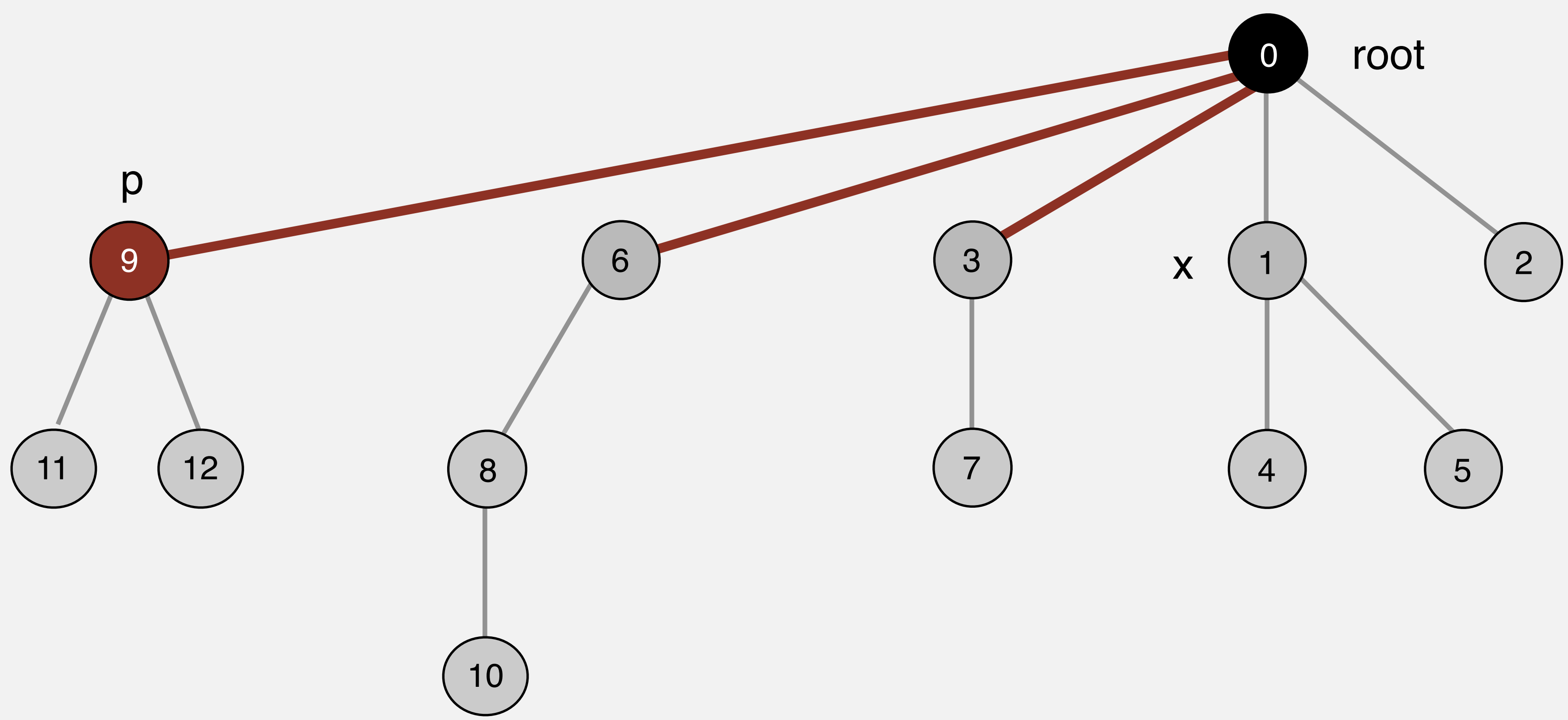
# Improvement 2: path compression

Quick union with path compression. Just after computing the root of $p$, set the id[] of each examined node to point to that root.



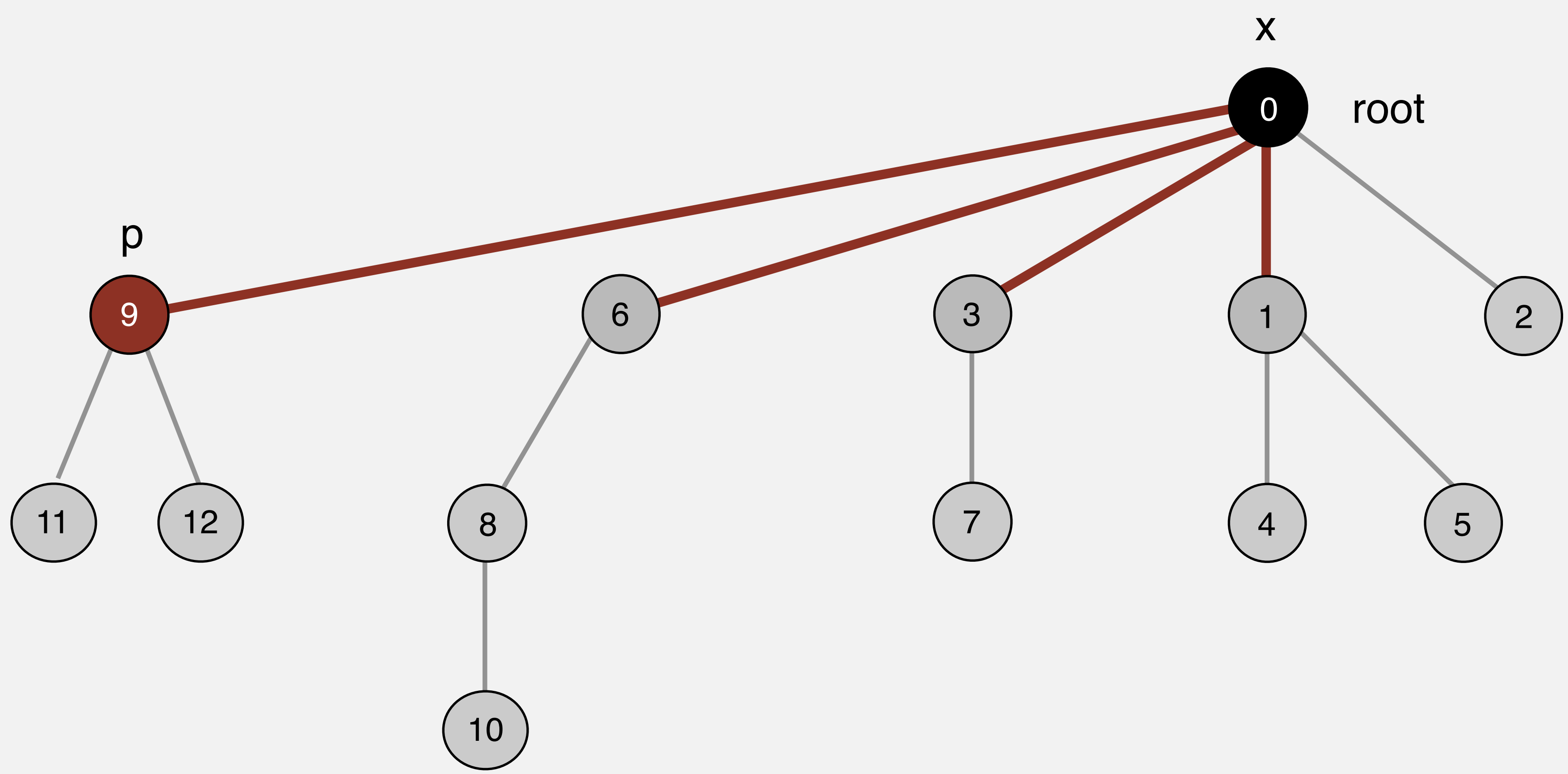
# Improvement 2: path compression

Quick union with path compression. Just after computing the root of $p$, set the id[] of each examined node to point to that root.



# Improvement 2: path compression

Quick union with path compression. Just after computing the root of $p$, set the id[] of each examined node to point to that root.




Bottom line. Now, find() has the side effect of compressing the tree.

# Path compression: Java implementation

Two-pass implementation: add second loop to find() to set the prnt[] of each examined node to the root. (Bit more complicated.)

Simpler one-pass variant (path halving): Make every other node in path point to its grandparent.

```
public int find(int i)
{
  while (i != prnt[i])
  {
    prnt[i] = prnt[prnt[i]];
    i = prnt[i];
  }
  return i;
}
```

only one extra line of code !

In practice. No reason not to! Keeps tree almost completely flat.

# Iterated log function

- lg* x is defined recursively:
- $$\lg^* n = \begin{cases} 1 & \text{if } n \leq 1 \\ 1 + \lg^*(\lg n) & \text{otherwise} \end{cases}$$

# Weighted quick-union with path compression: amortized analysis

Proposition. [Hopcroft-Ulman, Tarjan] Starting from an empty data structure, any sequence of $M$ union–find ops on $N$ objects makes ≤ $c ( N + M \lg^* N )$ array accesses.

- Analysis can be improved to $N + M \alpha(M, N)$.
- Simple algorithm with fascinating mathematics.

lg* is the iterated log function

| N | lg* N |
|---|---|
| 1 | 0 |
| 2 | 1 |
| 4 | 2 |
| 16 | 3 |
| 65536 | 4 |
| $2^{65536}$ | 5 |

**iterated lg function**

Linear-time algorithm for $M$ union-find ops on $N$ objects?

- Cost within constant factor of reading in the data.
- In theory, WQUPC is not quite linear.
- In practice, WQUPC is linear.

Amazing fact. [Fredman-Saks] No linear-time algorithm exists.

in "cell-probe" model of computation

# Summary

Key point. Weighted quick union (and/or path compression) makes it possible to solve problems that could not otherwise be addressed.

| algorithm | worst-case time |
|---|---|
| **quick-find** | M N |
| **quick-union** | M N |
| **weighted QU** | N + M log N |
| **QU + path compression** | N + M log N |
| **weighted QU + path compression** | N + M lg* N |

**order of growth for M union-find operations on a set of N objects**

Ex. [$10^9$ unions and finds with $10^9$ objects]

- WQUPC reduces time from 30 years to 6 seconds.
- Supercomputer won't help much; good algorithm enables solution.

# Subtext of today's lecture (and this course)

Steps to developing a usable algorithm.

- Model the problem.
- Find an algorithm to solve it.
- Fast enough? Fits in memory?
- If not, figure out why.
- Find a way to address the problem.
- Iterate until satisfied.

The scientific method.

Mathematical analysis.