



## 3.3 BALANCED SEARCH TREES

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- ▶ *2-3 search trees*
- ▶ *red-black BSTs*
- ▶ *B-trees*

# Improvements to BSTs?

- Like other tree/recursion situations, the problem is that not all paths are equal in length—that's to say the tree is not *balanced*;
- In the binary heap, we ensured that all levels were full except the lowest level;
- Could we possibly have all paths equal in length?
- Hint: what if some nodes can be ternary?

# Symbol table review

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implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search hit	insert	delete		
<b>sequential search (unordered list)</b>	$N$	$N$	$N$	$\frac{1}{2} N$	$N$	$\frac{1}{2} N$		<code>equals()</code>
<b>binary search (ordered array)</b>	$\lg N$	$N$	$N$	$\lg N$	$\frac{1}{2} N$	$\frac{1}{2} N$	✓	<code>compareTo()</code>
<b>BST</b>	$N$	$N$	$N$	$1.39 \lg N$	$1.39 \lg N$	$\sqrt{N}$	✓	<code>compareTo()</code>
<b>goal</b>	$\log N$	$\log N$	$\log N$	$\log N$	$\log N$	$\log N$	✓	<code>compareTo()</code>

Challenge. Guarantee performance.

This lecture. 2-3 trees, left-leaning red-black BSTs, B-trees.

# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

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## 2-3 tree

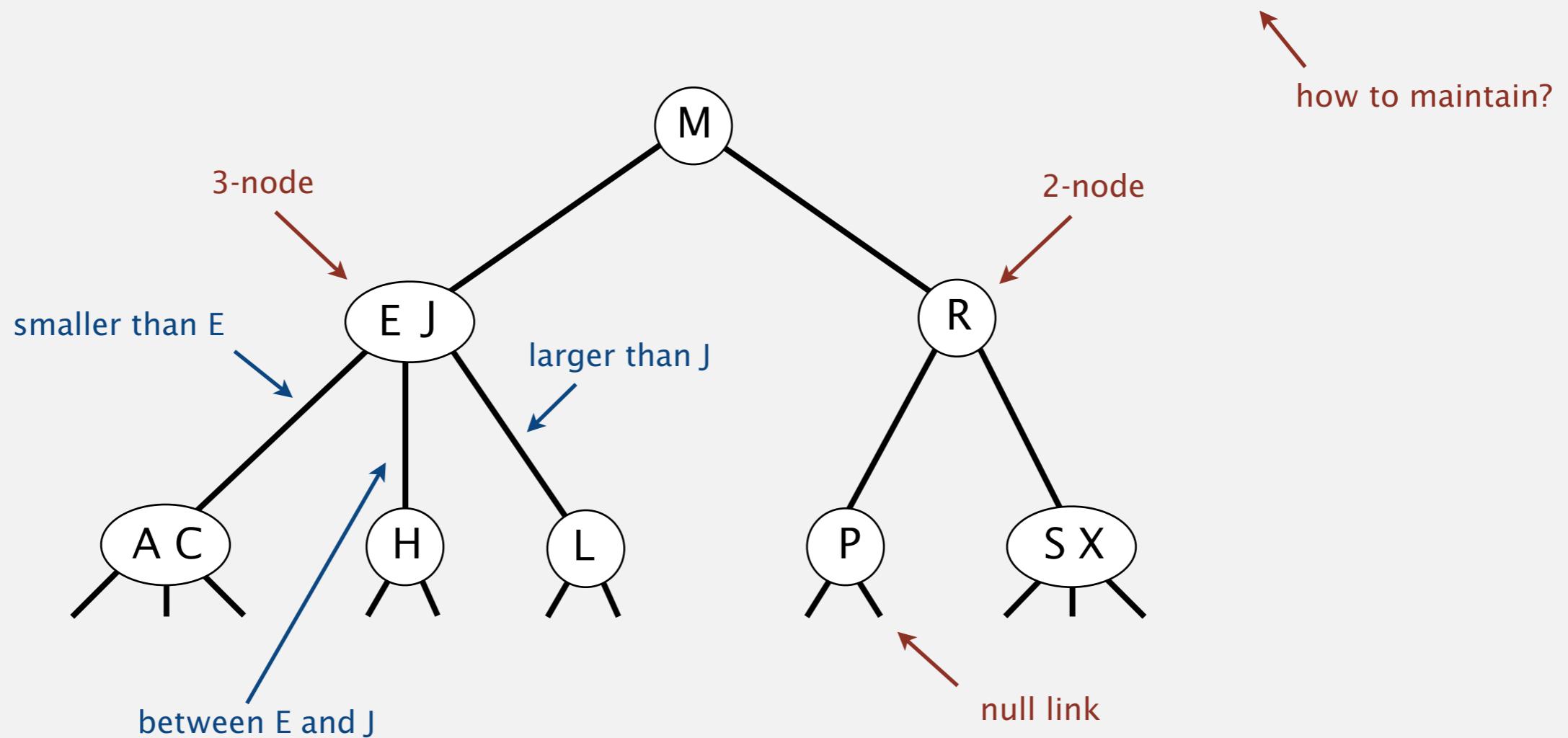
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Allow 1 or 2 keys per node.

- 2-node: one key, two children.
- 3-node: two keys, three children.

Symmetric order. Inorder traversal yields keys in ascending order.

Perfect balance. Every path from root to null link has same length.



## 2-3 tree demo

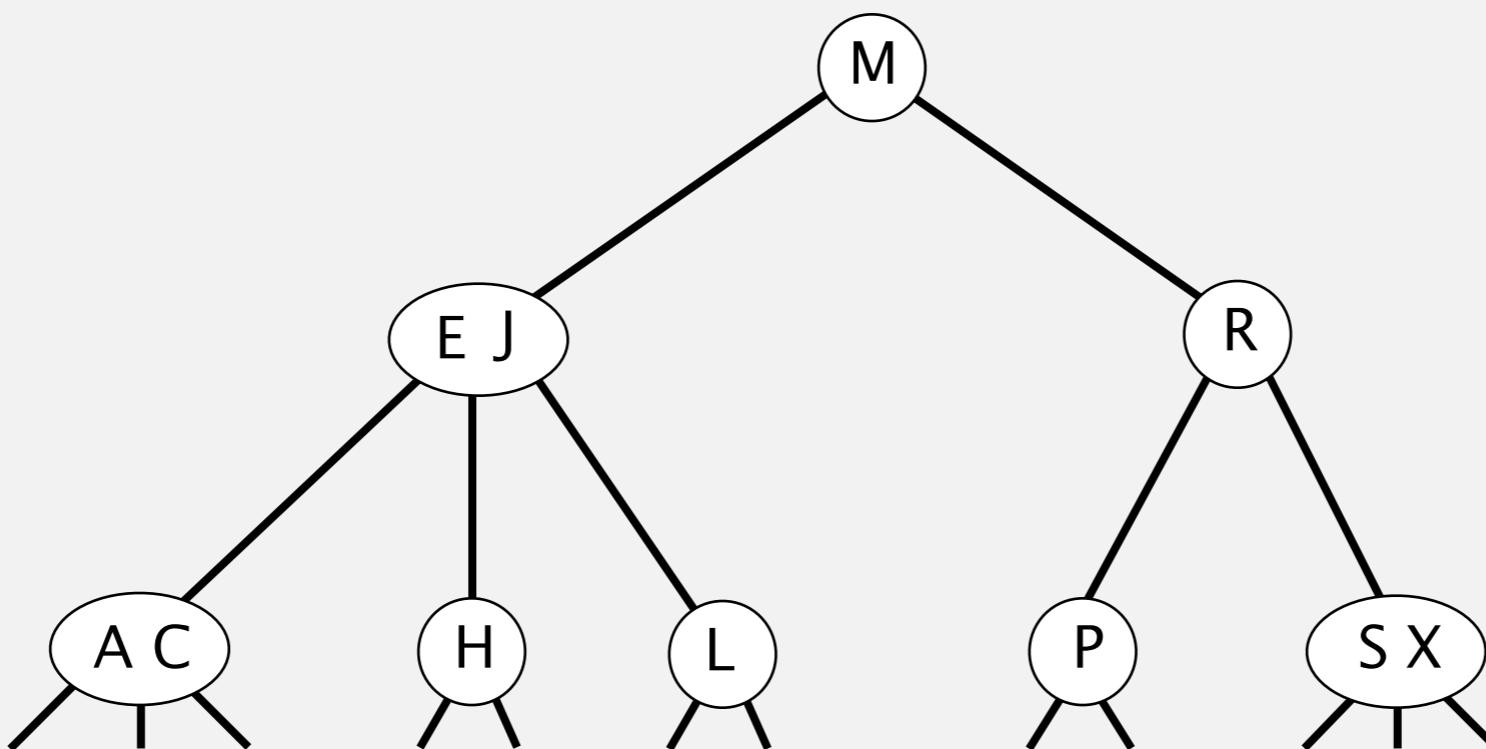
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### Search.

- Compare search key against keys in node.
- Find interval containing search key.
- Follow associated link (recursively).



search for H



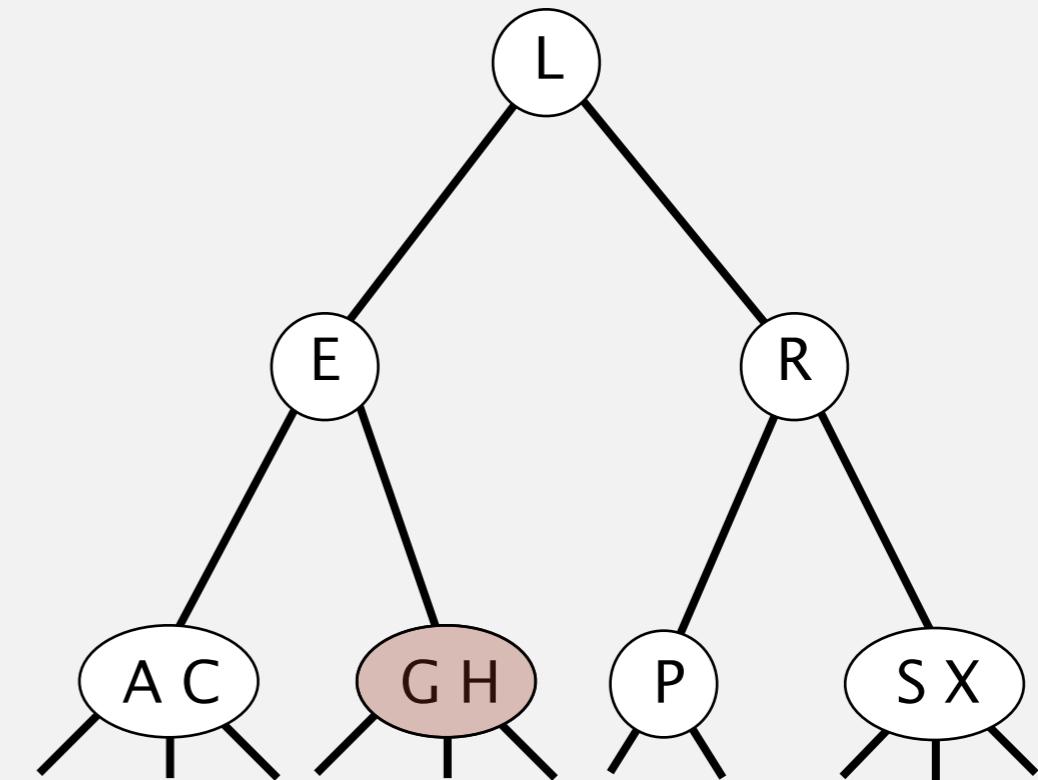
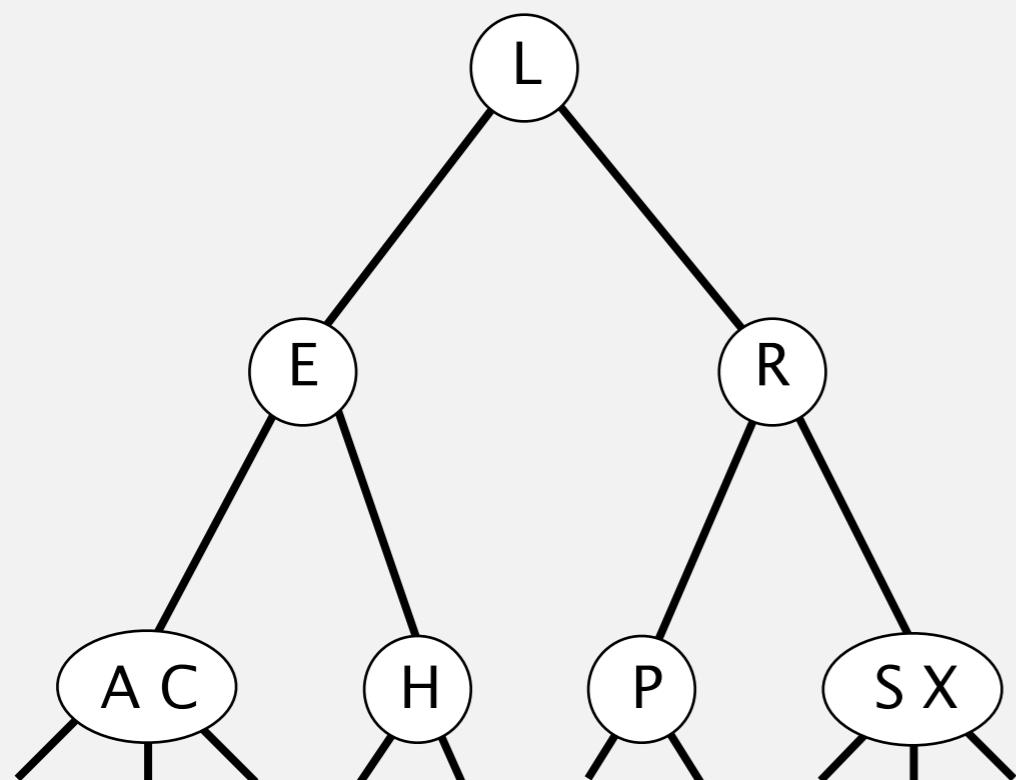
# Insertion into a 2-3 tree

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Insertion into a 2-node at bottom.

- Add new key to 2-node to create a 3-node.

**insert G**



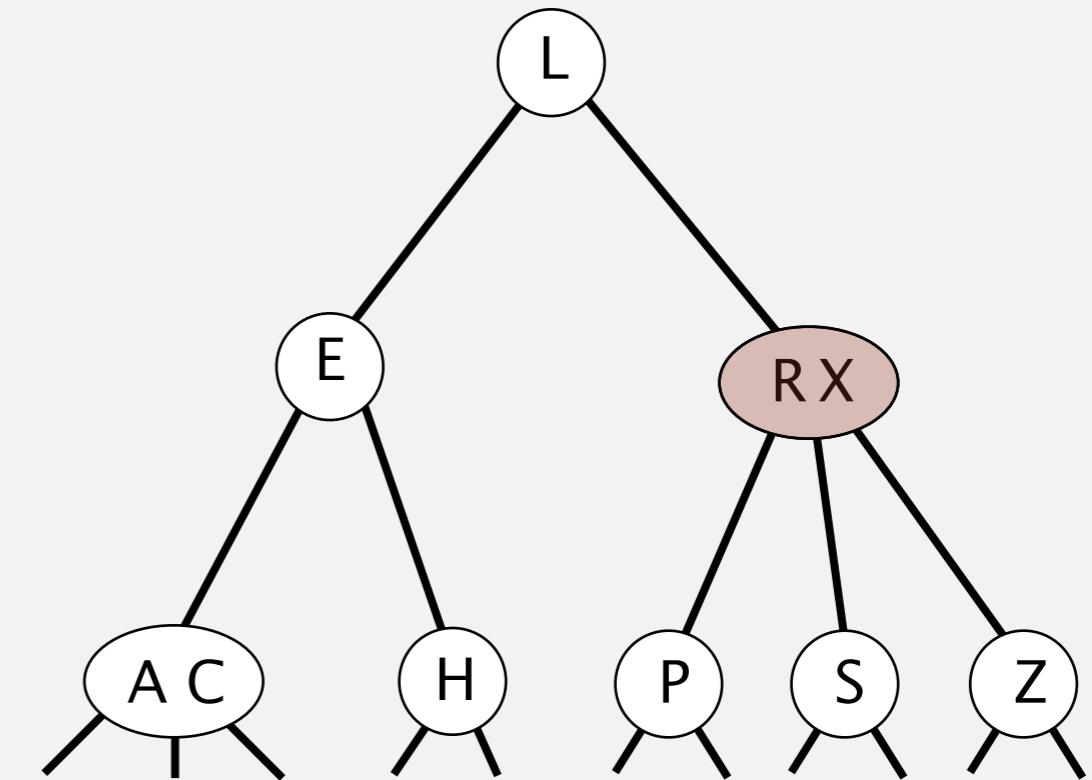
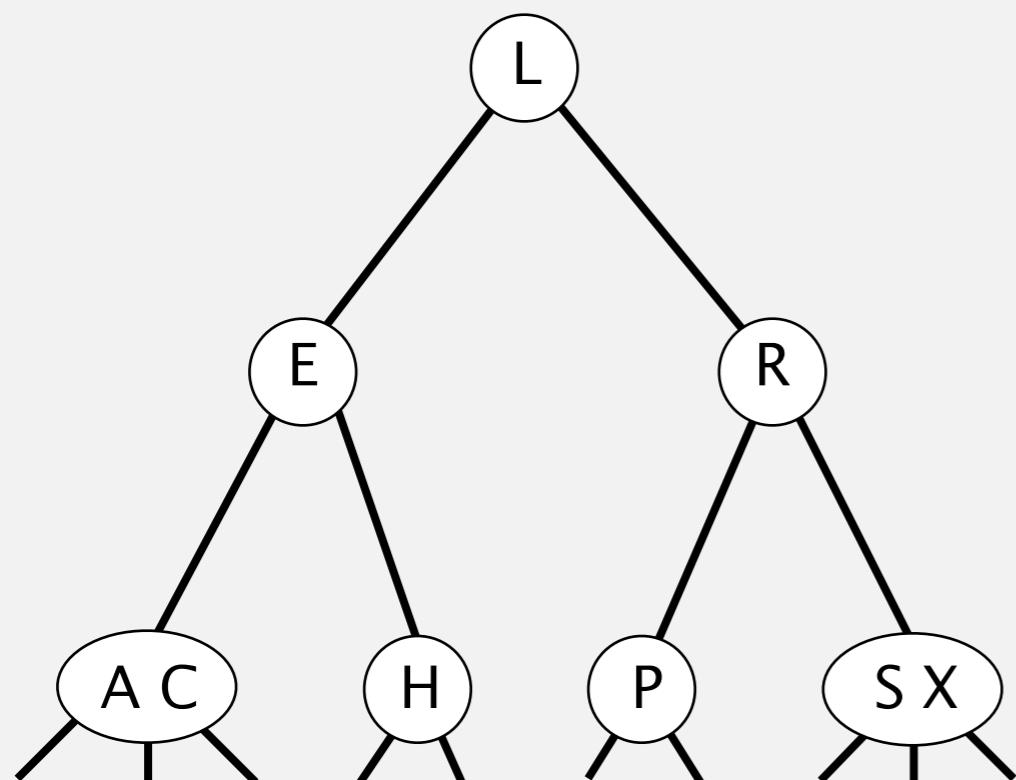
# Insertion into a 2-3 tree

---

## Insertion into a 3-node at bottom.

- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.
- Repeat up the tree, as necessary.
- If you reach the root and it's a 4-node, split it into three 2-nodes.

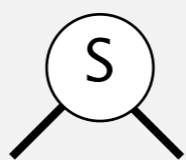
**insert Z**



## 2-3 tree construction demo

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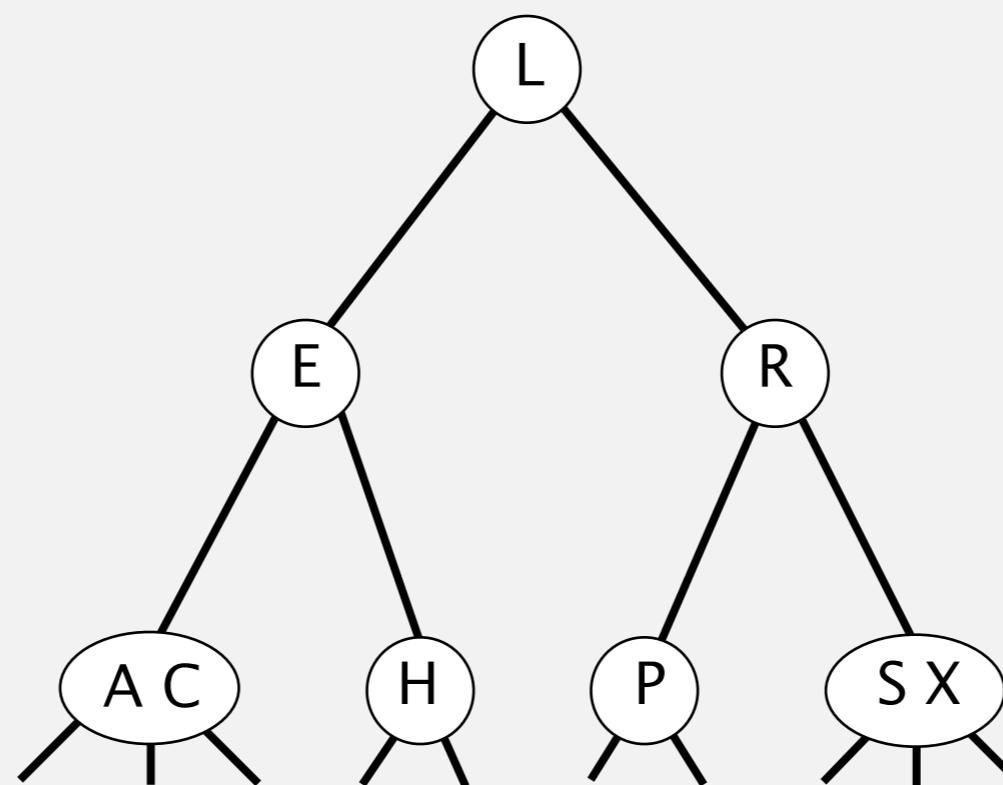
insert S



## 2-3 tree construction demo

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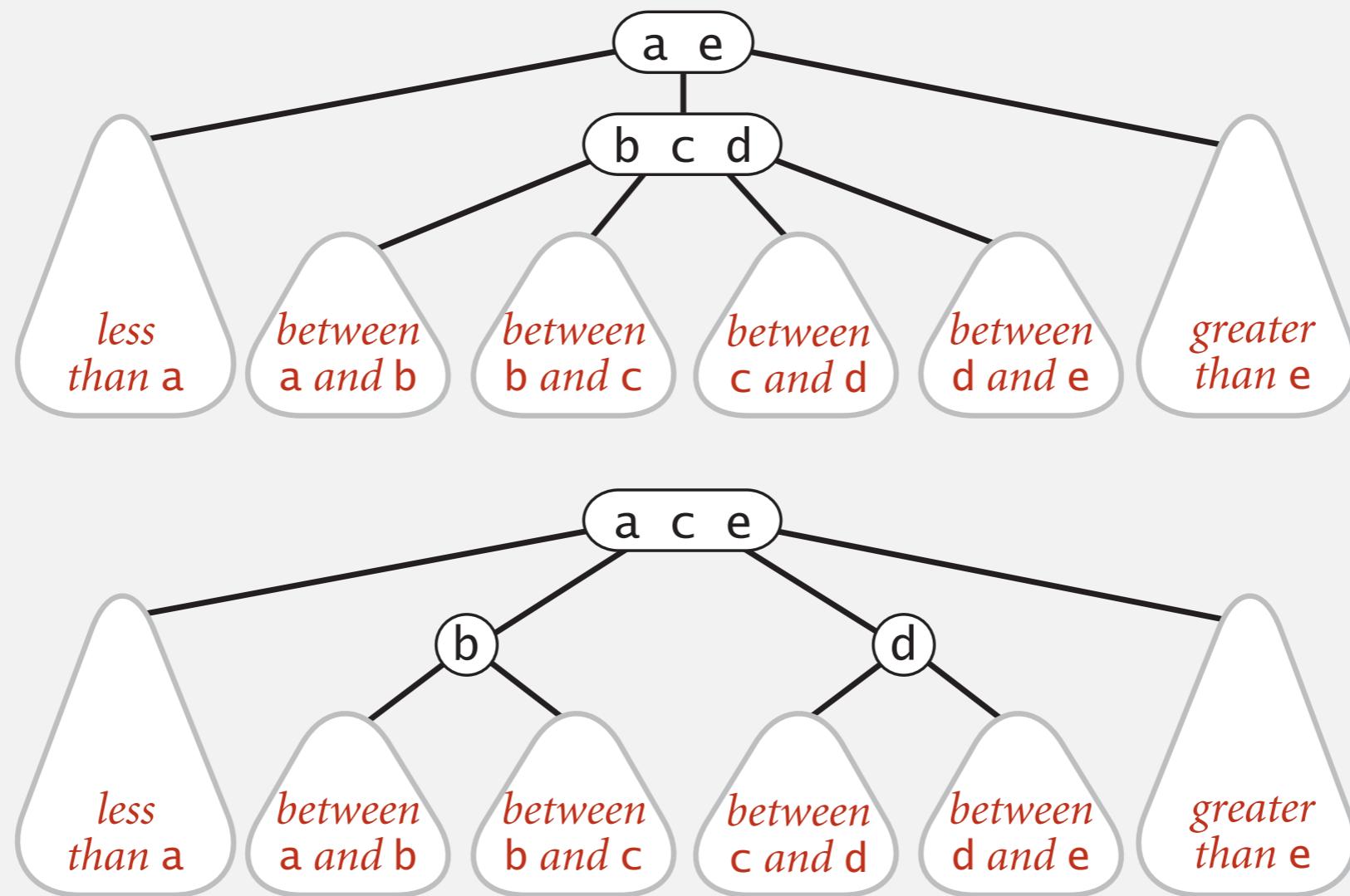
2-3 tree



# Local transformations in a 2-3 tree

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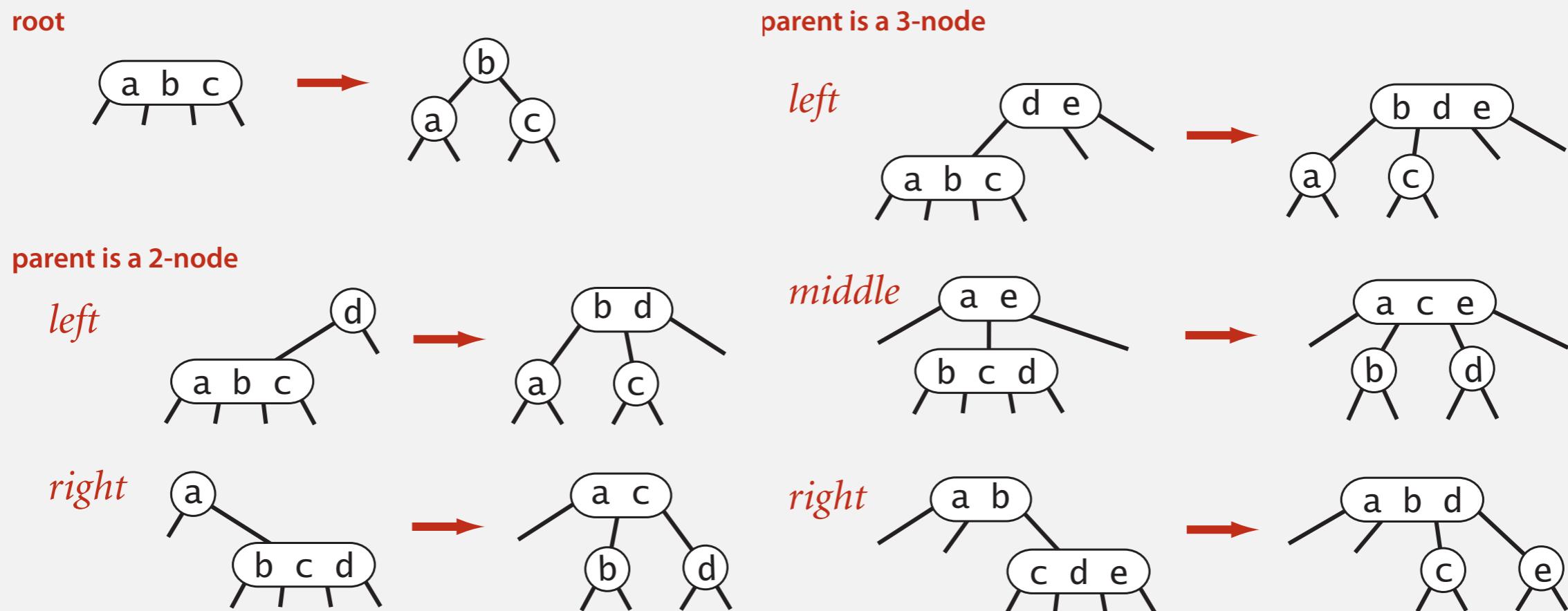
Splitting a 4-node is a **local** transformation: constant number of operations.



# Global properties in a 2-3 tree

Invariants. Maintains symmetric order and perfect balance.

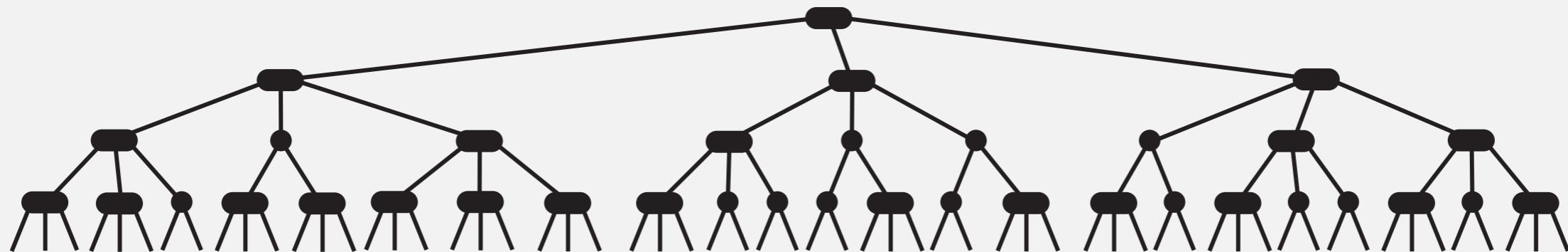
Pf. Each transformation maintains symmetric order and perfect balance.



## 2-3 tree: performance

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Perfect balance. Every path from root to null link has same length.



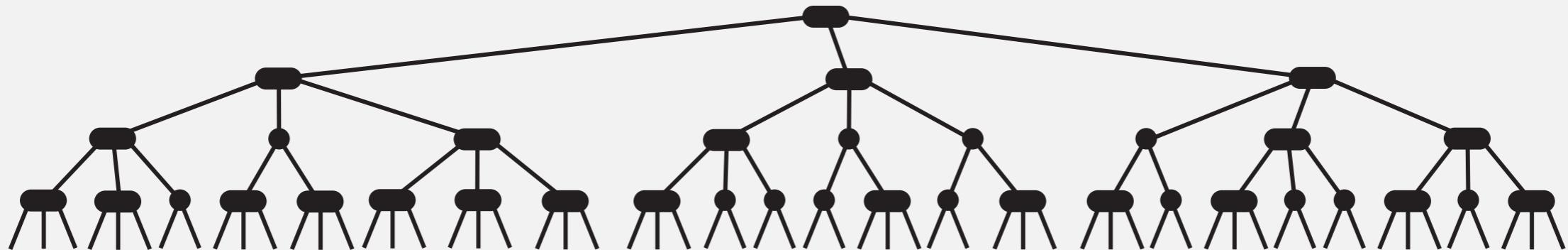
Tree height.

- Worst case:
- Best case:

## 2-3 tree: performance

---

Perfect balance. Every path from root to null link has same length.



Tree height.

- Worst case:  $\lg N$ . [all 2-nodes]
- Best case:  $\log_3 N \approx .631 \lg N$ . [all 3-nodes]
- Between 12 and 20 for a million nodes.
- Between 18 and 30 for a billion nodes.

Bottom line. Guaranteed logarithmic performance for search and

# ST implementations: summary

implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search hit	insert	delete		
<b>sequential search (unordered list)</b>	$N$	$N$	$N$	$\frac{1}{2} N$	$N$	$\frac{1}{2} N$		<code>equals()</code>
<b>binary search (ordered array)</b>	$\lg N$	$N$	$N$	$\lg N$	$\frac{1}{2} N$	$\frac{1}{2} N$	✓	<code>compareTo()</code>
<b>BST</b>	$N$	$N$	$N$	$1.39 \lg N$	$1.39 \lg N$	$\sqrt{N}$	✓	<code>compareTo()</code>
<b>2-3 tree</b>	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	✓	<code>compareTo()</code>


  
 constant  $c$  depend upon implementation

## 2-3 tree: implementation?

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Direct implementation is complicated, because:

- Maintaining multiple node types is cumbersome.
- Need multiple compares to move down tree.
- Need to move back up the tree to split 4-nodes.
- Large number of cases for splitting.

**fantasy code**

```
public void put(Key key, Value val)
{
    Node x = root;
    while (x.getTheCorrectChild(key) != null)
    {
        x = x.getTheCorrectChildKey();
        if (x.is4Node()) x.split();
    }
    if (x.is2Node()) x.make3Node(key, val);
    else if (x.is3Node()) x.make4Node(key, val);
}
```

Bottom line. Could do it, but there's a better way.

# Algorithms

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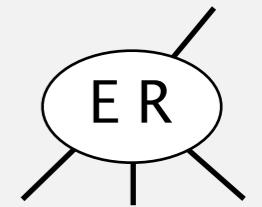
# Summary of 2-3 Trees

- A balanced 2-3 tree of depth  $h$  can store anywhere between  $2^h - 1$  and  $3^h - 1$  nodes.
- But what data structure should we use?
  - 3-tree (ternary) with some key-values unspecified?
    - Only problem is that at each node, there may be wasted space and we may have to do two key comparisons
  - Other way retaining binary search properties?

# How to implement 2-3 trees with binary trees?

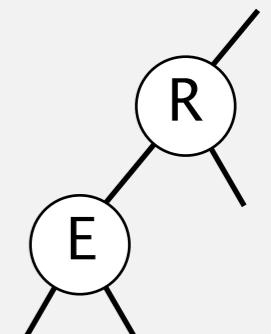
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Challenge. How to represent a 3 node?



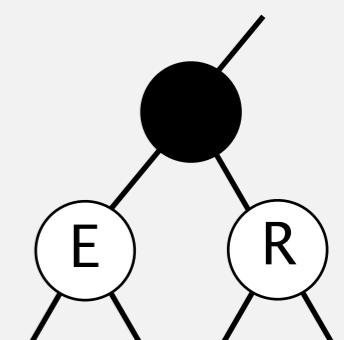
Approach 1: regular BST.

- No way to tell a 3-node from a 2-node.
- Cannot map from BST back to 2-3 tree.



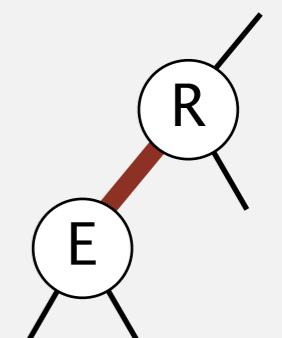
Approach 2: regular BST with "glue" nodes.

- Wastes space, wasted link.
- Code probably messy.



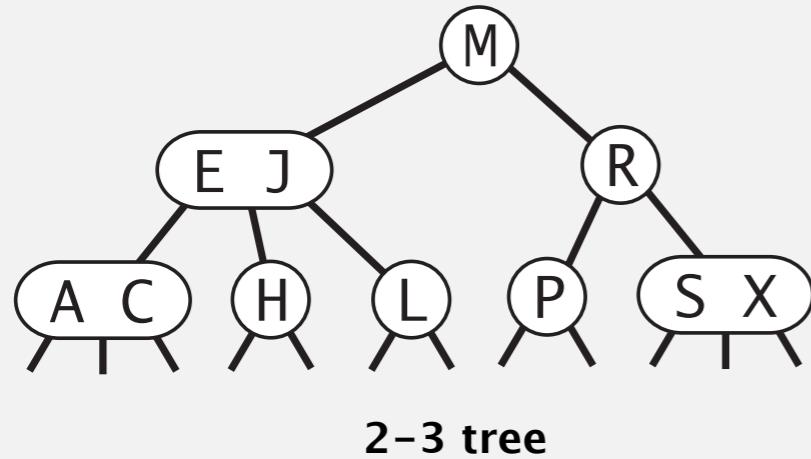
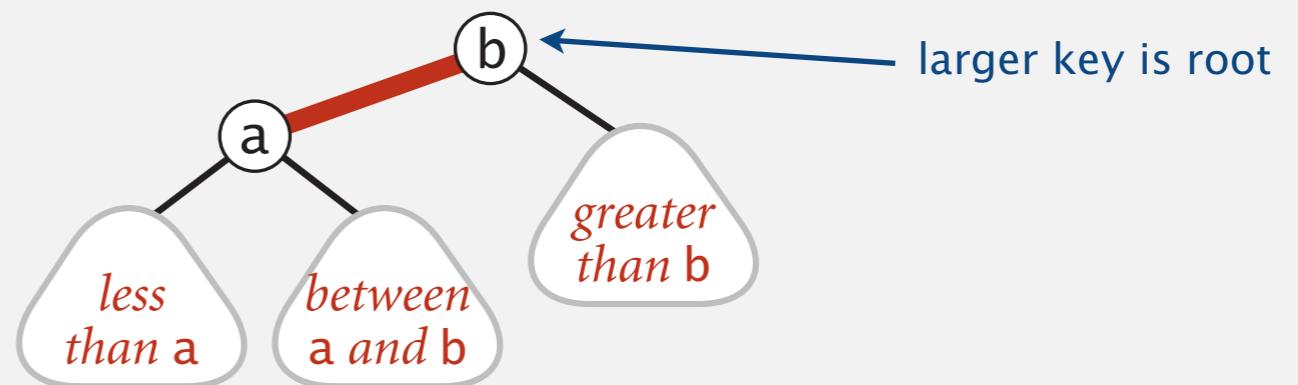
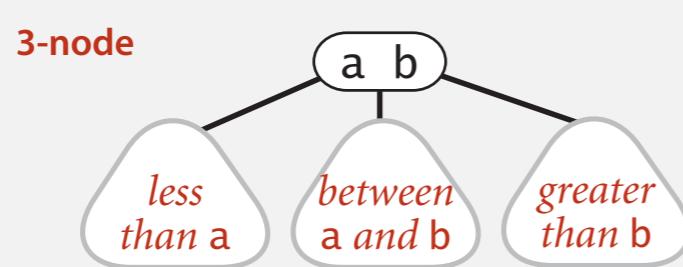
Approach 3: regular BST with red "glue" links.

- Widely used in practice.
- Arbitrary convention: red links lean left.



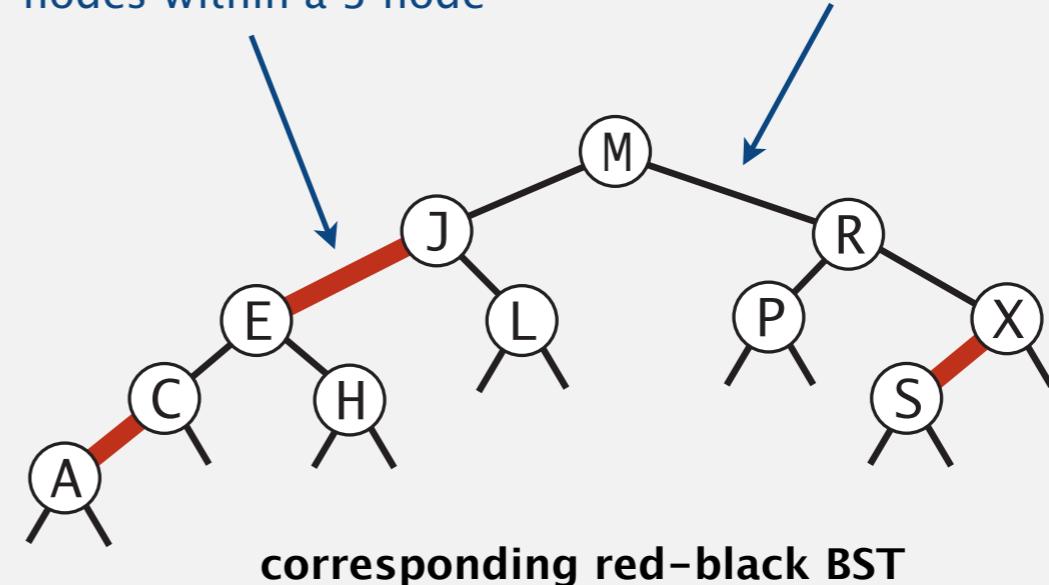
# Left-leaning red-black BSTs (Guibas-Sedgewick 1979 and Sedgewick 2007)

1. Represent 2–3 tree as a BST.
2. Use "internal" left-leaning links as "glue" for 3–nodes.



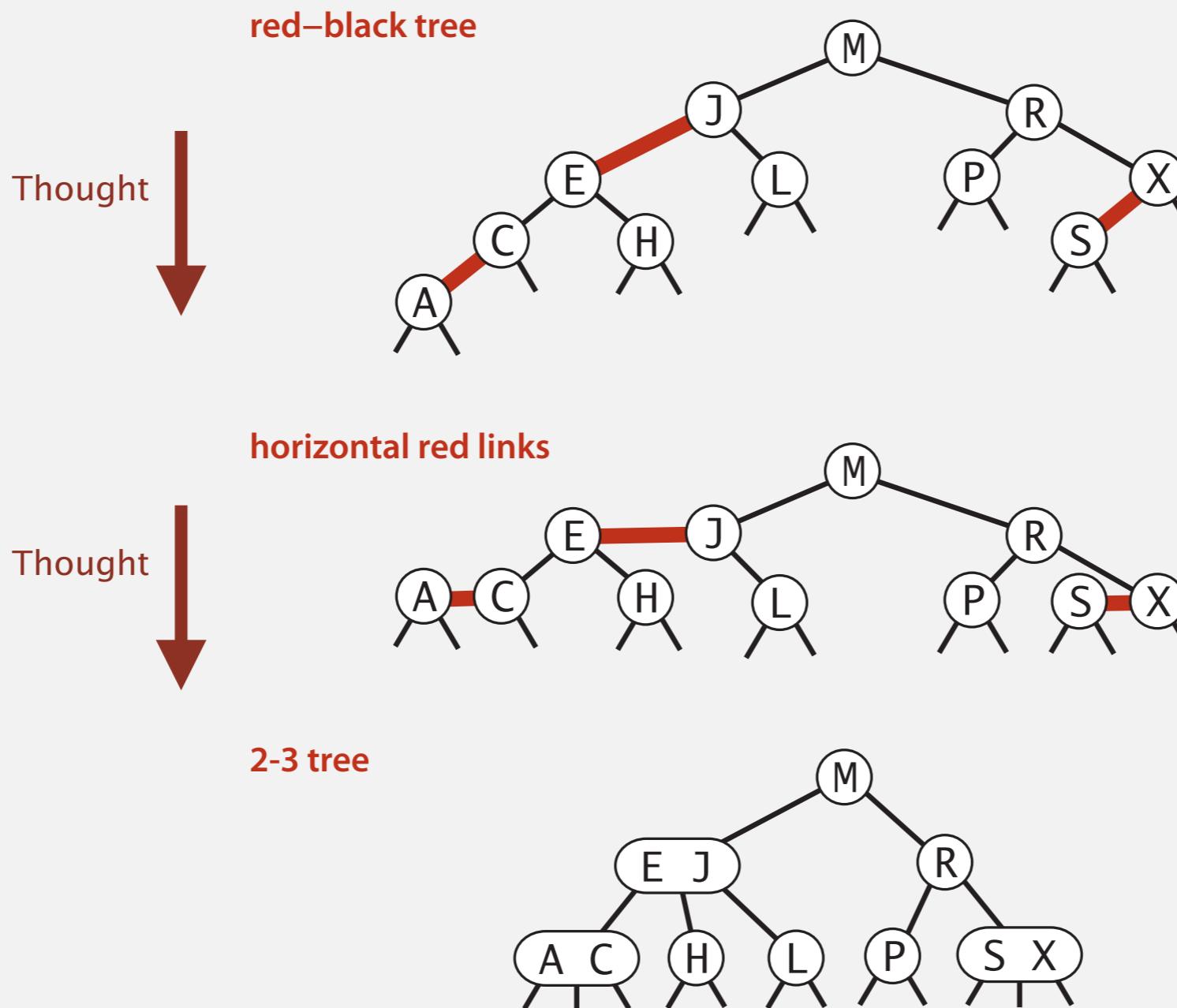
red links "glue" nodes within a 3-node

black links connect 2-nodes and 3-nodes



# Left-leaning red-black BSTs: 1-1 correspondence with 2-3 trees

Key property. 1–1 correspondence between 2–3 and LLRB.



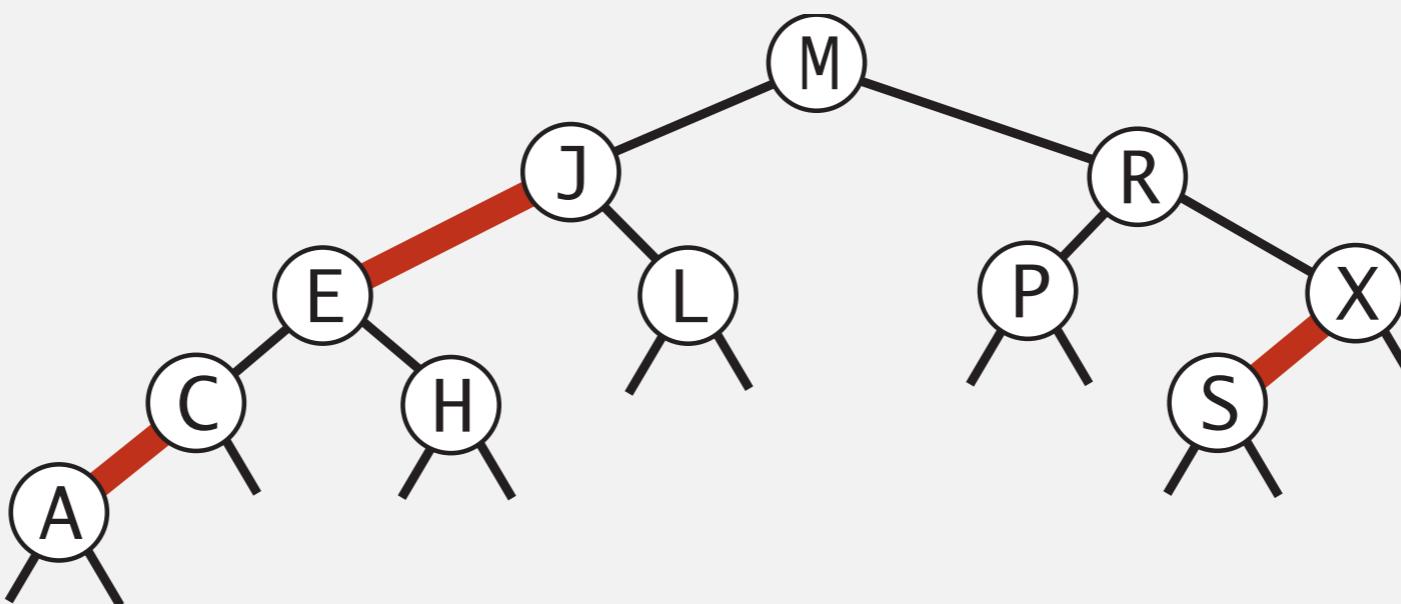
## An equivalent definition

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A BST such that (rules):

1. No node has two red links connected to it.
2. Every path from root to null link has the same number of black links.
3. Red links lean left.

"perfect black balance"

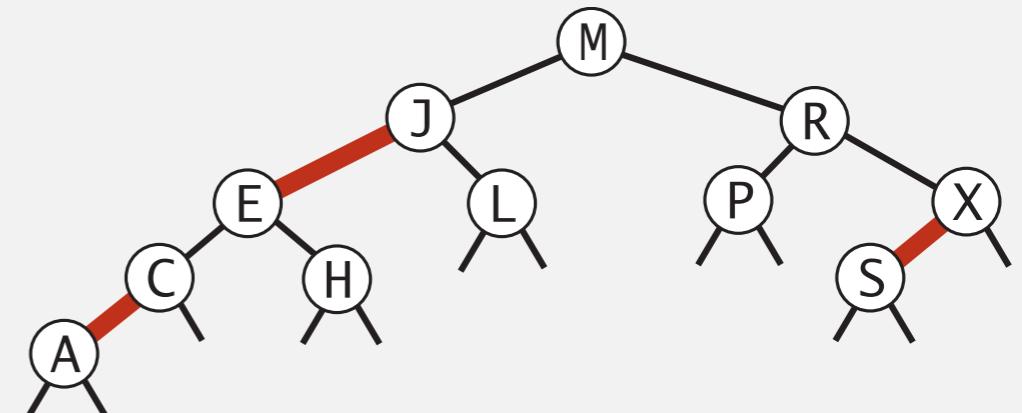


# Search implementation for red-black BSTs

**Observation.** Search is the same as for elementary BST (ignore color).

but runs faster  
because of better balance

```
public Val get(Key key)
{
    Node x = root;
    while (x != null)
    {
        int cmp = key.compareTo(x.key);
        if      (cmp < 0) x = x.left;
        else if (cmp > 0) x = x.right;
        else if (cmp == 0) return x.val;
    }
    return null;
}
```



**Remark.** Most other ops (e.g., floor, iteration, selection) are also identical.

# Red-black BST representation

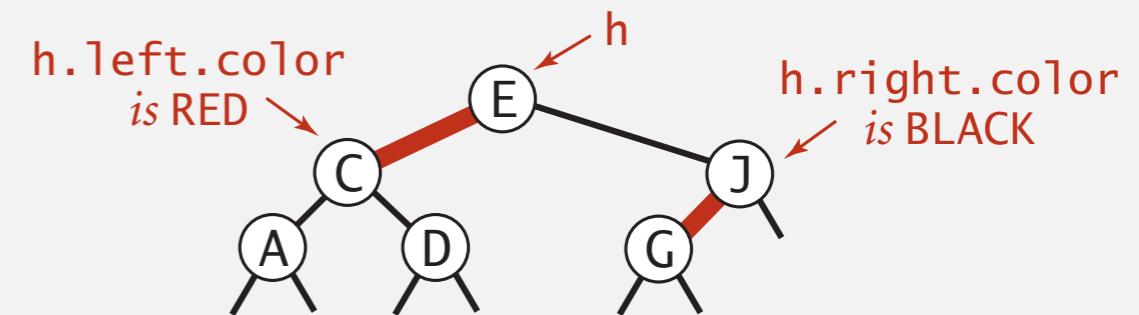
Each node is pointed to by precisely one link (from its parent)  $\Rightarrow$  can encode color of links in nodes.

```
private static final boolean RED = true;
private static final boolean BLACK = false;
```

```
private class Node
{
    Key key;
    Value val;
    Node left, right;
    boolean color; // color of parent link
}
```

```
private boolean isRed(Node x)
{
    if (x == null) return false;
    return x.color == RED;
}
```

null links are black



## Insertion in a LLRB tree: overview

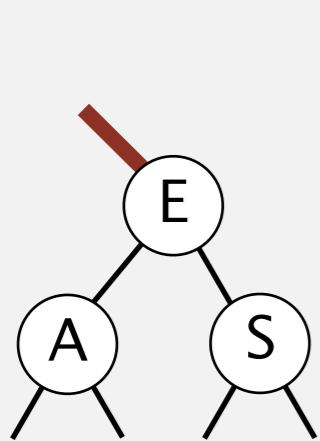
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Basic strategy. Maintain 1-1 correspondence with 2-3 trees.

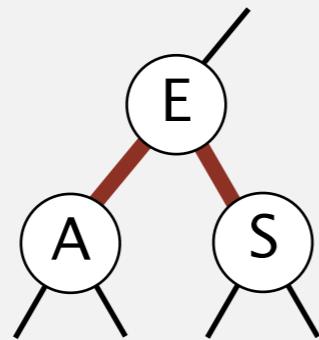
During internal operations, maintain:

- Symmetric order.
- Perfect black balance.

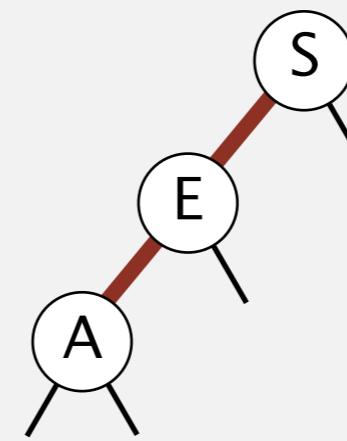
[ but not necessarily color invariants ]



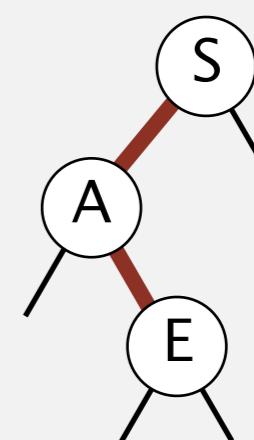
right-leaning  
red link (#3)



two red children (#1,3)  
(a temporary 4-node)



left-left red (#1)  
(a temporary 4-node)



left-right red (#1,3)  
(a temporary 4-node)

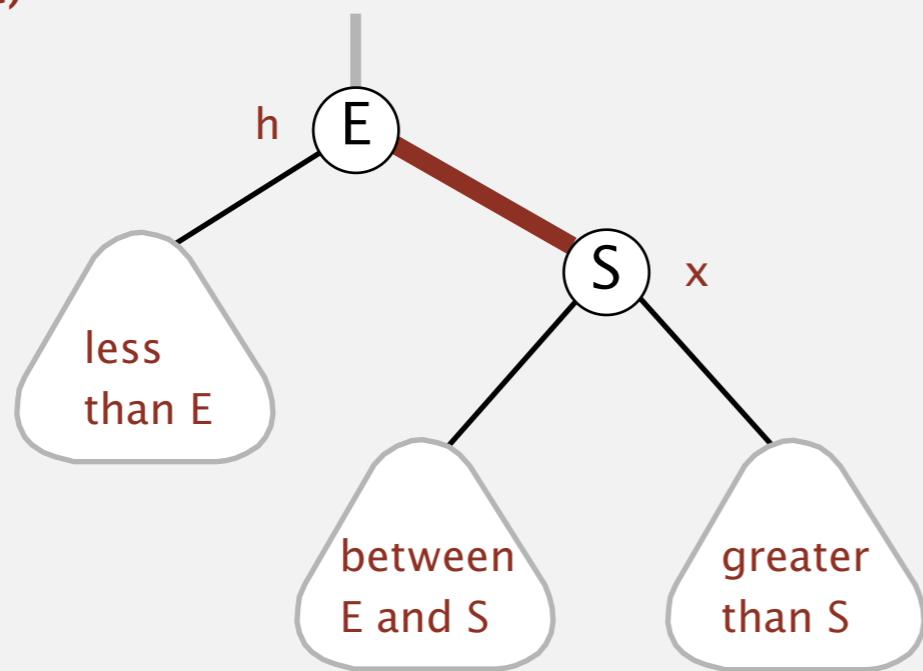
How? Apply elementary red-black BST operations: rotation and color flip.

# Elementary red-black BST operations

**Left rotation.** Orient a (temporarily) right-leaning red link to lean left—fix violation of rule #1:

rotate E left

(before)



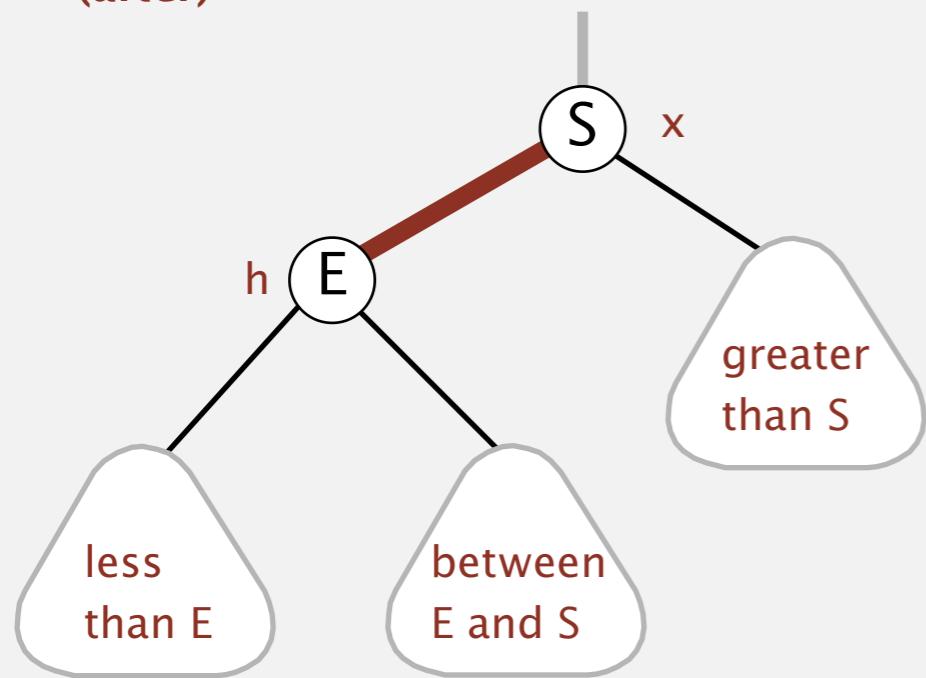
```
private Node rotateLeft(Node h)
{
    assert isRed(h.right);
    Node x = h.right;
    h.right = x.left;
    x.left = h;
    x.color = h.color;
    h.color = RED;
    return x;
}
```

# Elementary red-black BST operations

Left rotation. Orient a (temporarily) right-leaning red link to lean left.

rotate E left

(after)

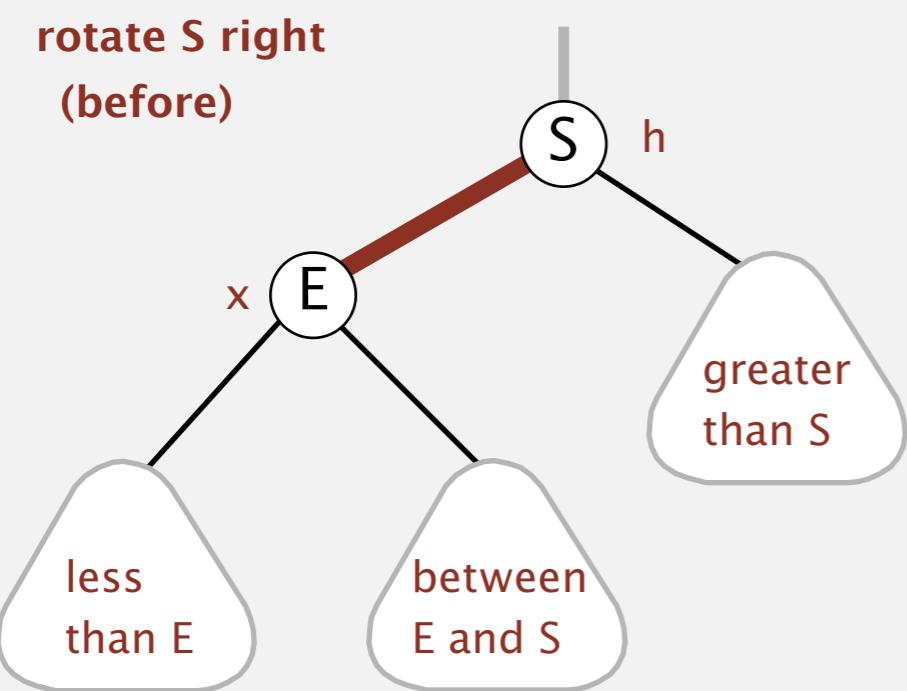


```
private Node rotateLeft(Node h)
{
    assert isRed(h.right);
    Node x = h.right;
    h.right = x.left;
    x.left = h;
    x.color = h.color;
    h.color = RED;
    return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

# Elementary red-black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right. This operation is required in certain of the insert/delete cases, particularly when we have two left-leaning (red) links connecting same element.



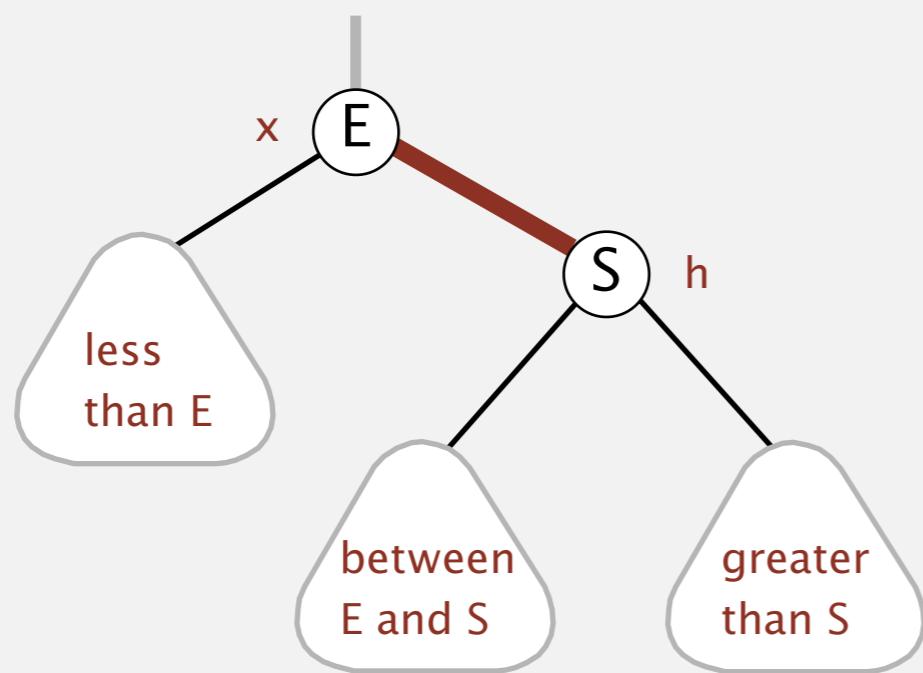
```
private Node rotateRight(Node h)
{
    assert isRed(h.left);
    Node x = h.left;
    h.left = x.right;
    x.right = h;
    x.color = h.color;
    h.color = RED;
    return x;
}
```

# Elementary red-black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.

rotate S right

(after)

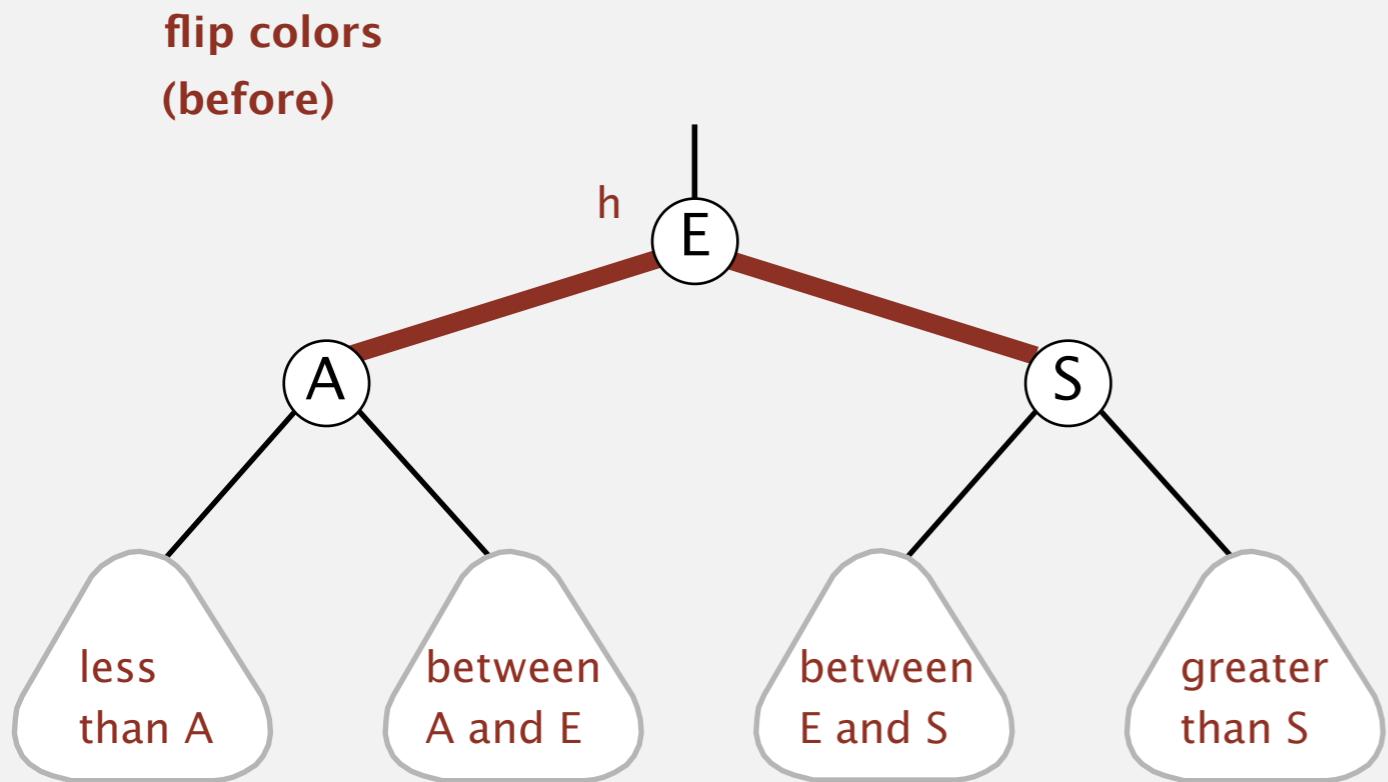


```
private Node rotateRight(Node h)
{
    assert isRed(h.left);
    Node x = h.left;
    h.left = x.right;
    x.right = h;
    x.color = h.color;
    h.color = RED;
    return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

# Elementary red-black BST operations

Color flip. Recolor to split a (temporary) 4-node (a violation of #1).

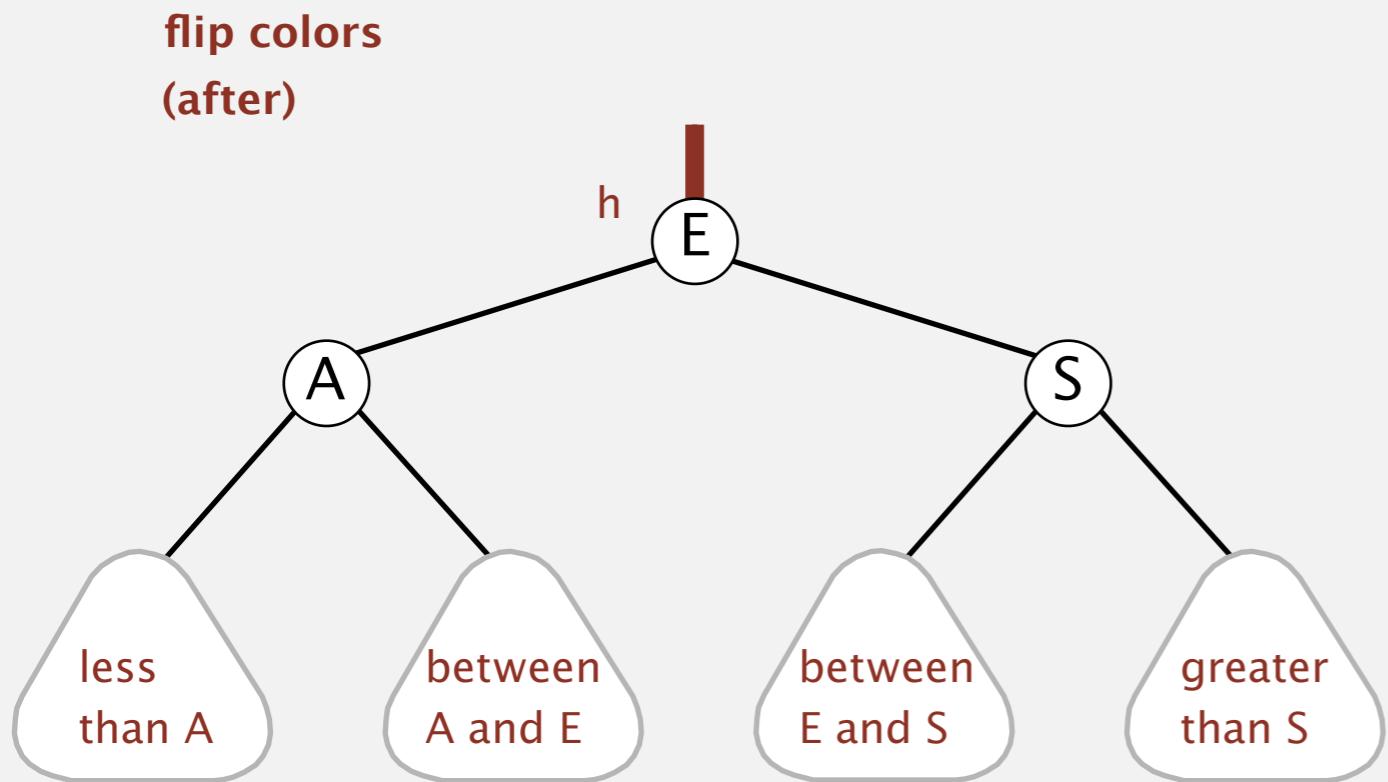


```
private void flipColors(Node h)
{
    assert !isRed(h);
    assert isRed(h.left);
    assert isRed(h.right);
    h.color = RED;
    h.left.color = BLACK;
    h.right.color = BLACK;
}
```

Invariants. Maintains symmetric order and perfect black balance.

# Elementary red-black BST operations

Color flip. Recolor to split a (temporary) 4-node.

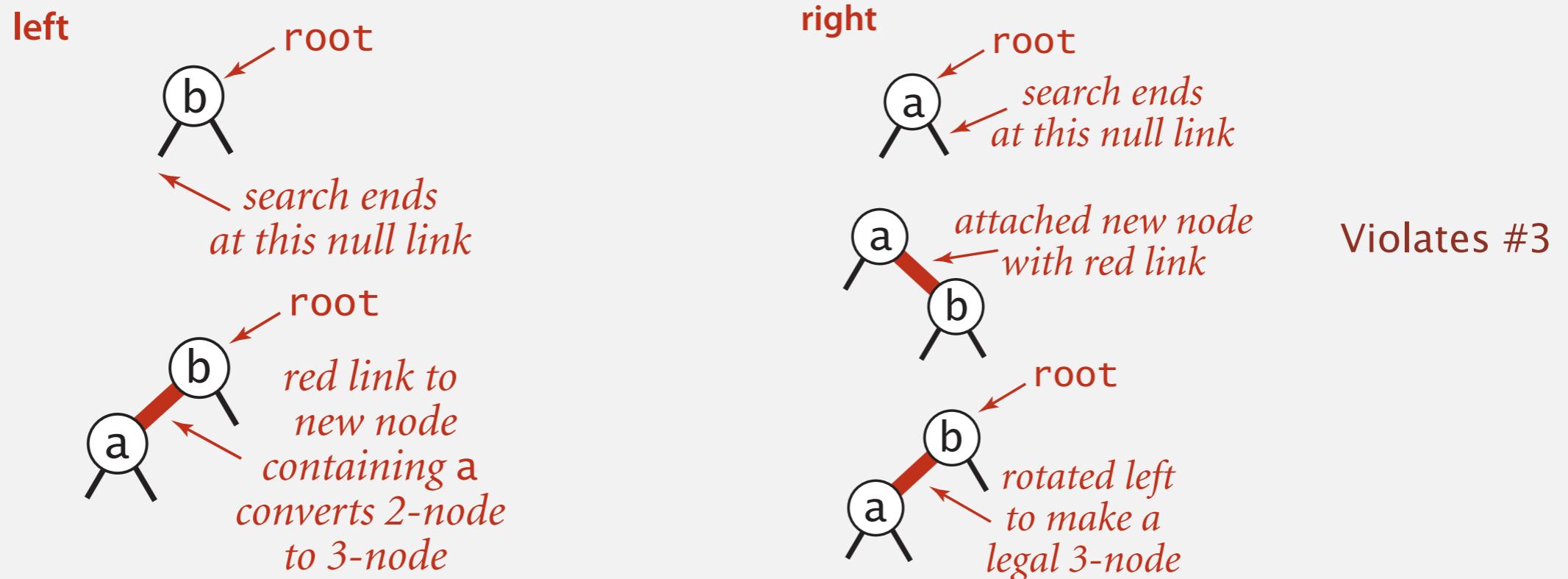


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    assert isRed(h.left);
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    h.right.color = BLACK;
}
```

Invariants. Maintains symmetric order and perfect black balance.

# Insertion in a LLRB tree

Warmup 1. Insert into a tree with exactly 1 node.



# Insertion in a LLRB tree

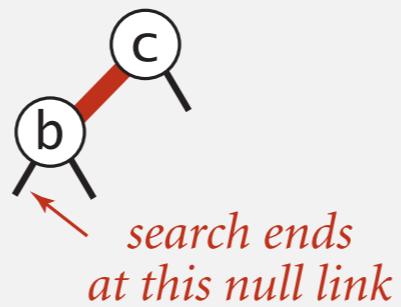
Warmup 2. Insert into a tree with exactly 2 nodes.

larger



*search ends  
at this  
null link*

smaller



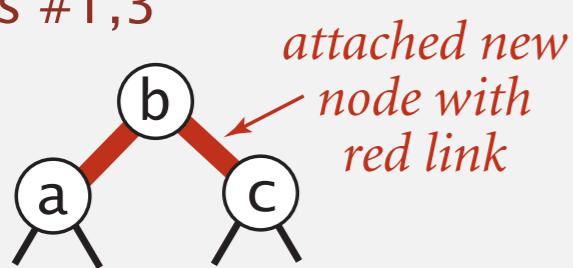
*search ends  
at this null link*

between



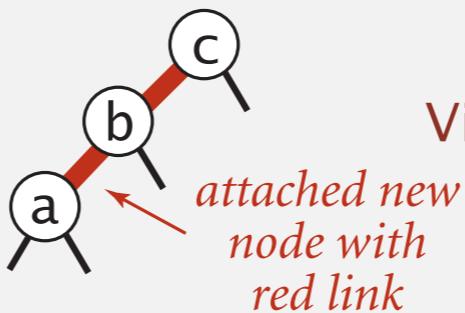
*search ends  
at this null link*

Violates #1,3



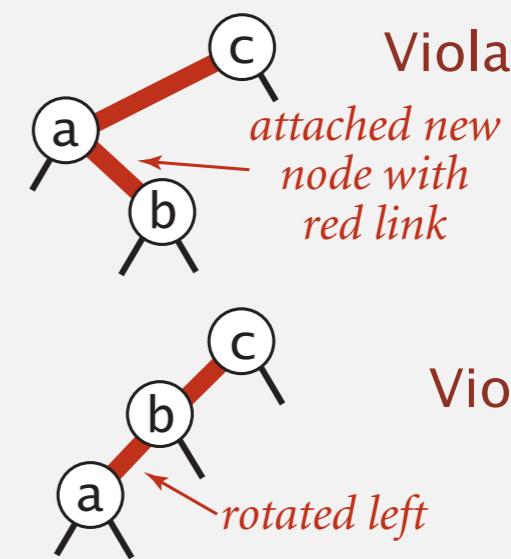
*attached new  
node with  
red link*

Violates #1



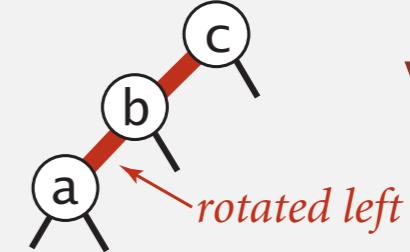
*attached new  
node with  
red link*

Violates #1,3



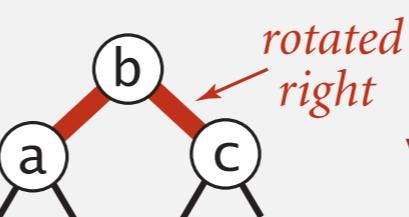
*attached new  
node with  
red link*

Violates #1



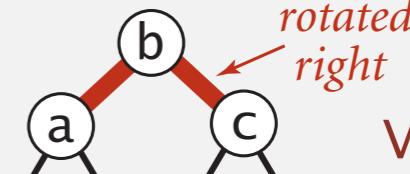
*rotated left*

Violates #1,3

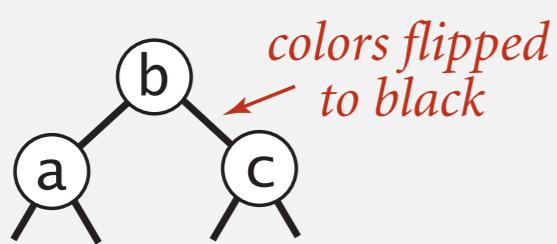


*rotated  
right*

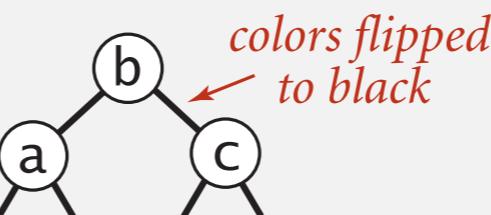
Violates #1,3



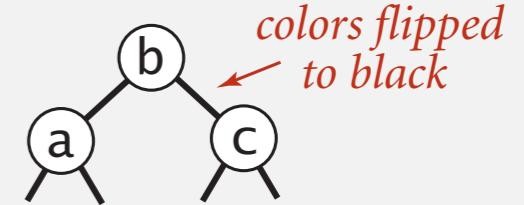
*rotated  
right*



*colors flipped  
to black*



*colors flipped  
to black*

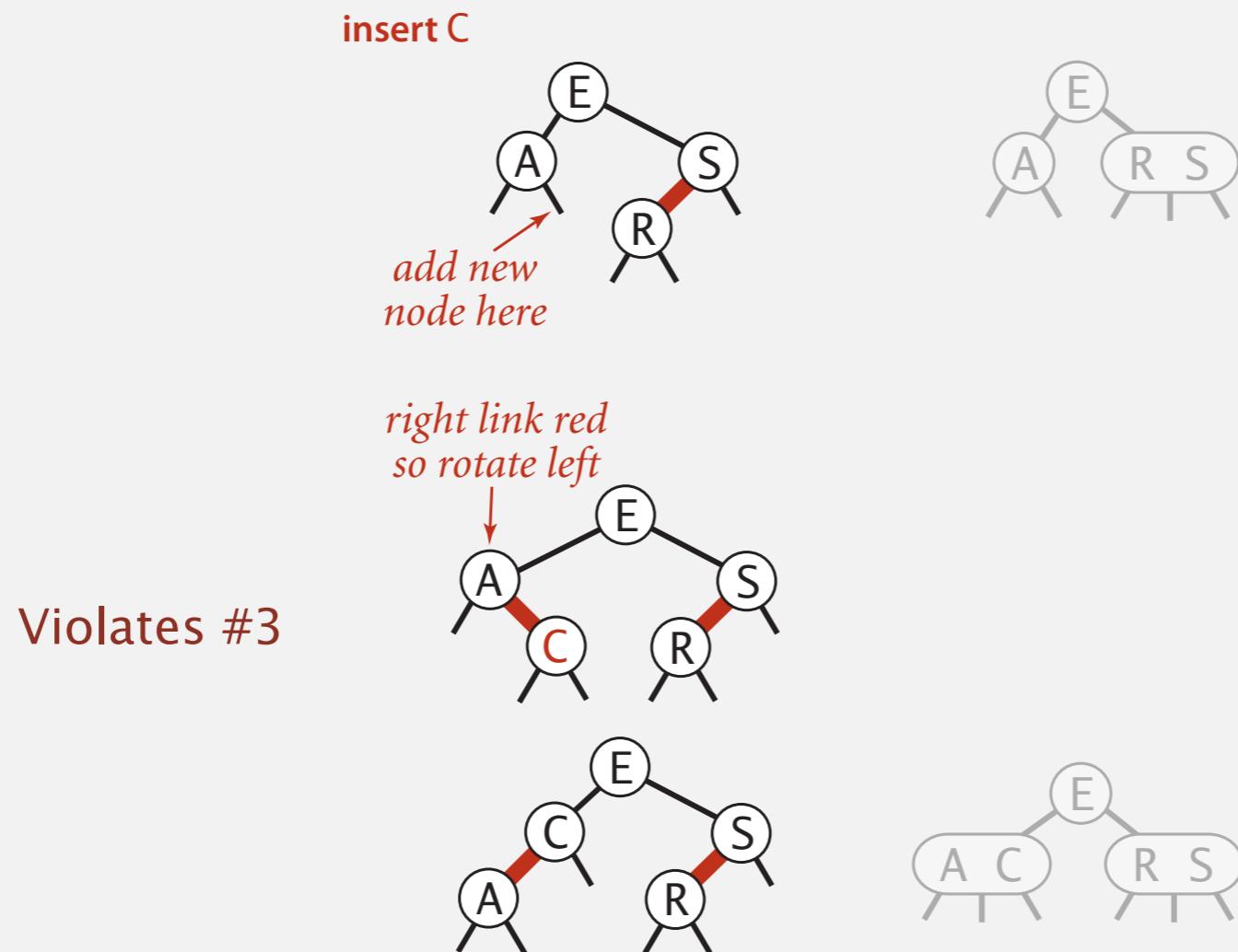


*colors flipped  
to black*

# Insertion in a LLRB tree

## Case 1. Insert into a 2-node at the bottom.

- Do standard BST insert; color new link red. ← to maintain symmetric order and perfect black balance
- If new red link is a right link, rotate left. ← to fix color invariants



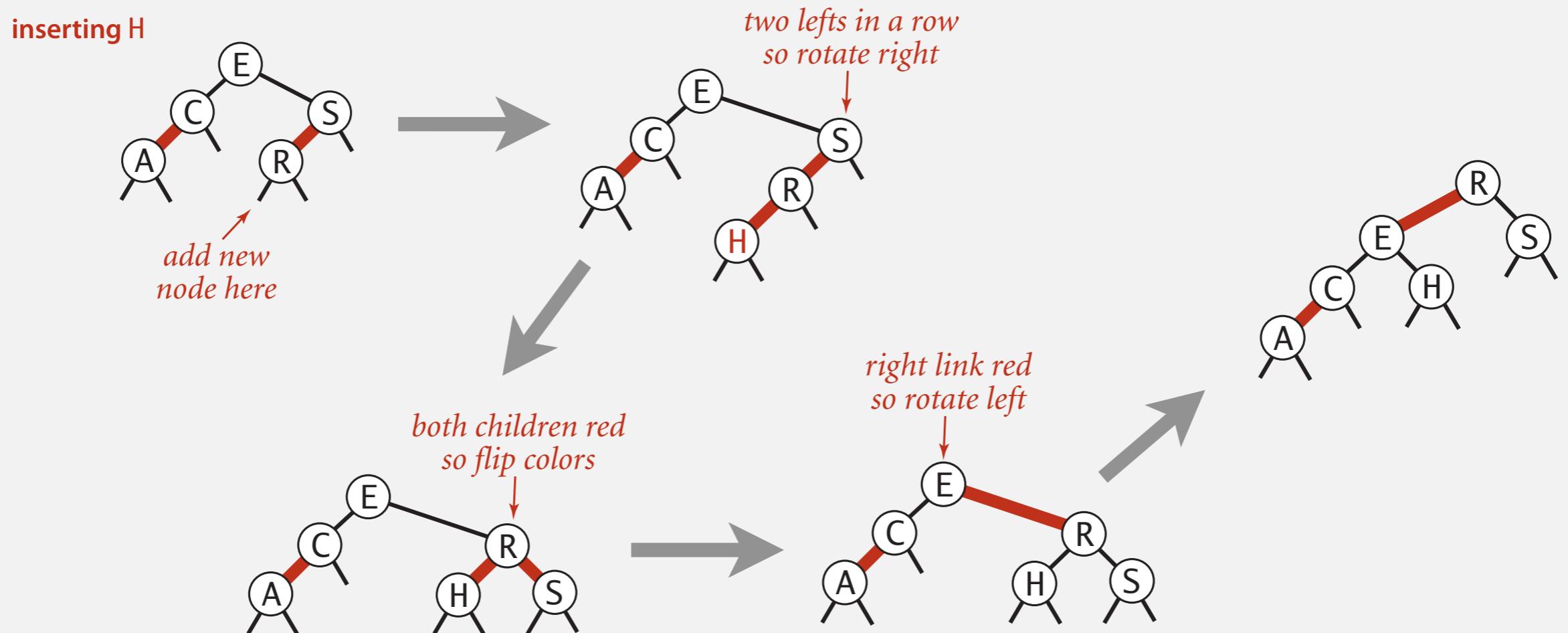
# Insertion in a LLRB tree

## Case 2. Insert into a 3-node at the bottom.

- Do standard BST insert; color new link red.
- Rotate to balance the 4-node (if needed).
- Flip colors to pass red link up one level.
- Rotate to make lean left (if needed).

to maintain symmetric order  
and perfect black balance

to fix color invariants



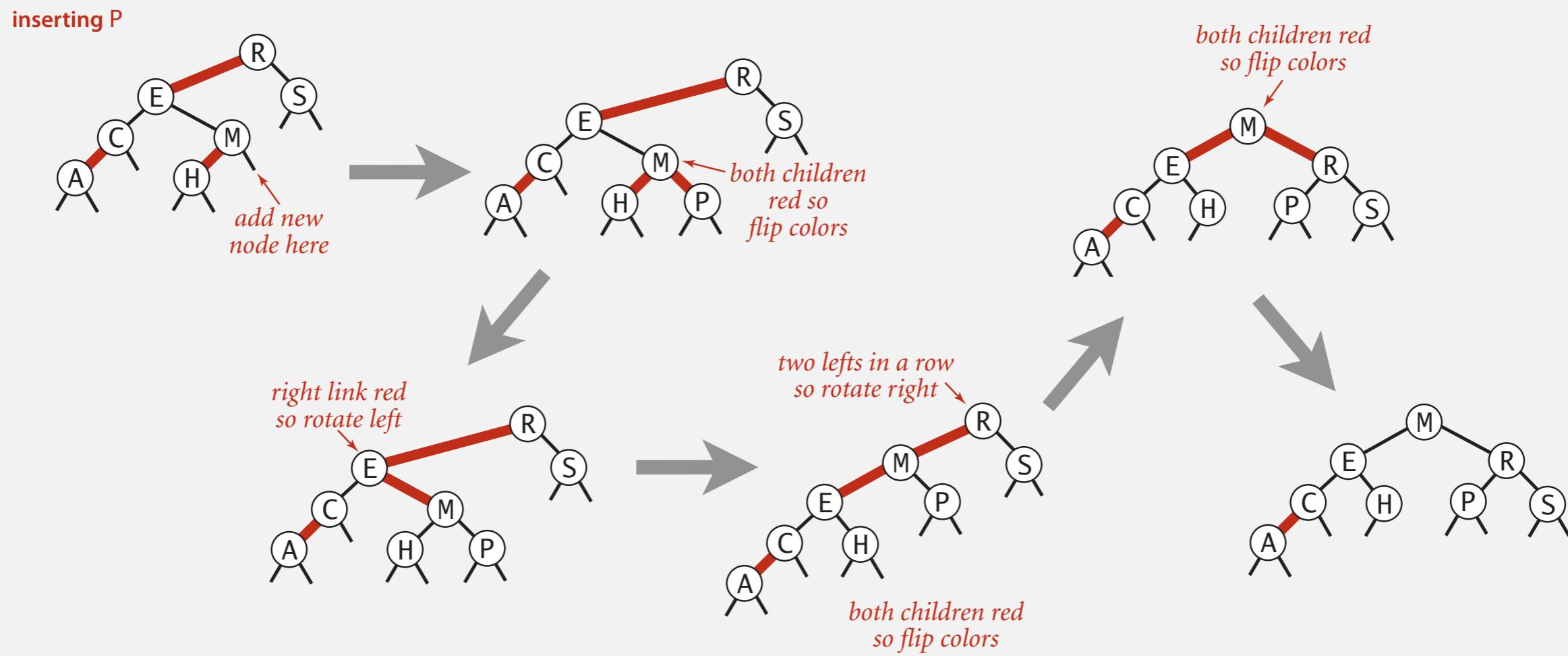
# Insertion in a LLRB tree: passing red links up the tree

## Case 2. Insert into a 3-node at the bottom.

- Do standard BST insert; color new link red.
- Rotate to balance the 4-node (if needed).
- Flip colors to pass red link up one level.
- Rotate to make lean left (if needed).
- Repeat case 1 or case 2 up the tree (if needed).

to maintain symmetric order  
and perfect black balance

to fix color invariants



# Red-black BST construction demo

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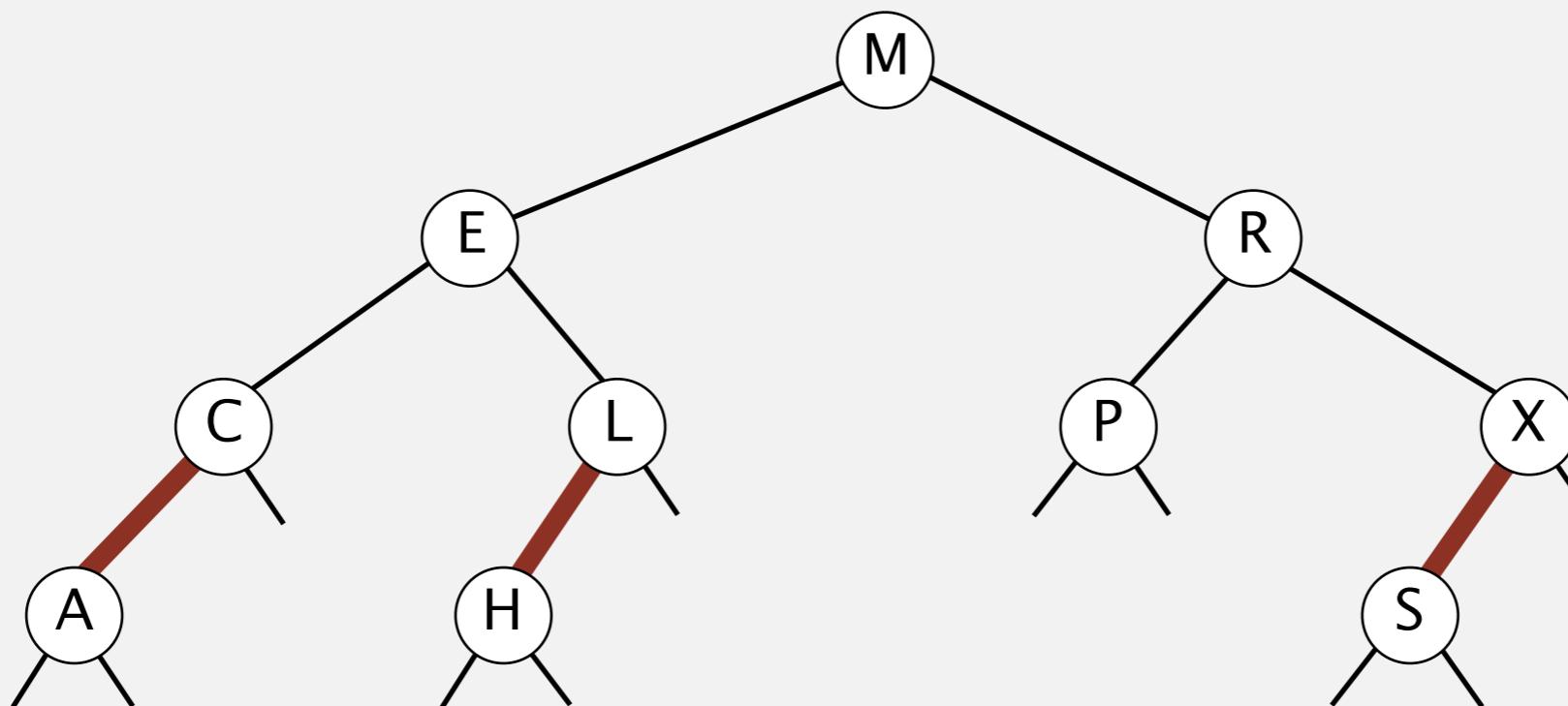
insert S



# Red-black BST construction demo

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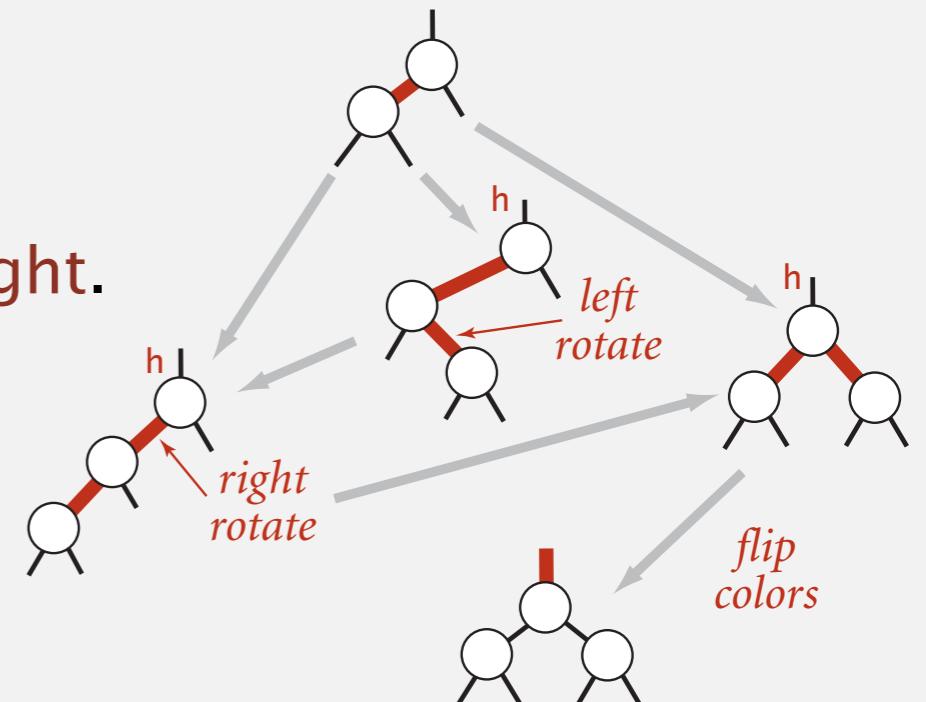
red-black BST



# Insertion in a LLRB tree: Java implementation

Same code for all cases.

- Right child red, left child black: **rotate left**.
- Left child, left-left grandchild red: **rotate right**.
- Both children red: **flip colors**.



```
private Node put(Node h, Key key, Value val)
{
    if (h == null) return new Node(key, val, RED);           ← insert at bottom
    int cmp = key.compareTo(h.key);                           (and color it red)
    if      (cmp < 0) h.left  = put(h.left,  key, val);
    else if (cmp > 0) h.right = put(h.right, key, val);
    else if (cmp == 0) h.val  = val;

    if (isRed(h.right) && !isRed(h.left))     h = rotateLeft(h);   ← lean left
    if (isRed(h.left)  && isRed(h.left.left)) h = rotateRight(h);  ← balance 4-node
    if (isRed(h.left)  && isRed(h.right))     flipColors(h);       ← split 4-node

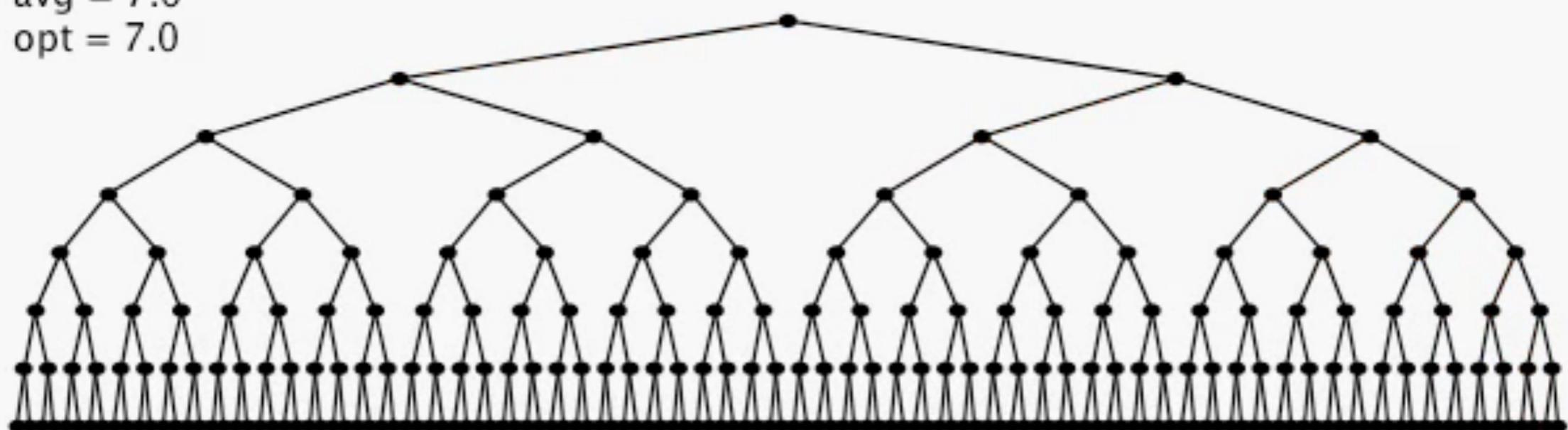
    return h;
}
```

only a few extra lines of code provides near-perfect balance

# Insertion in a LLRB tree: visualization

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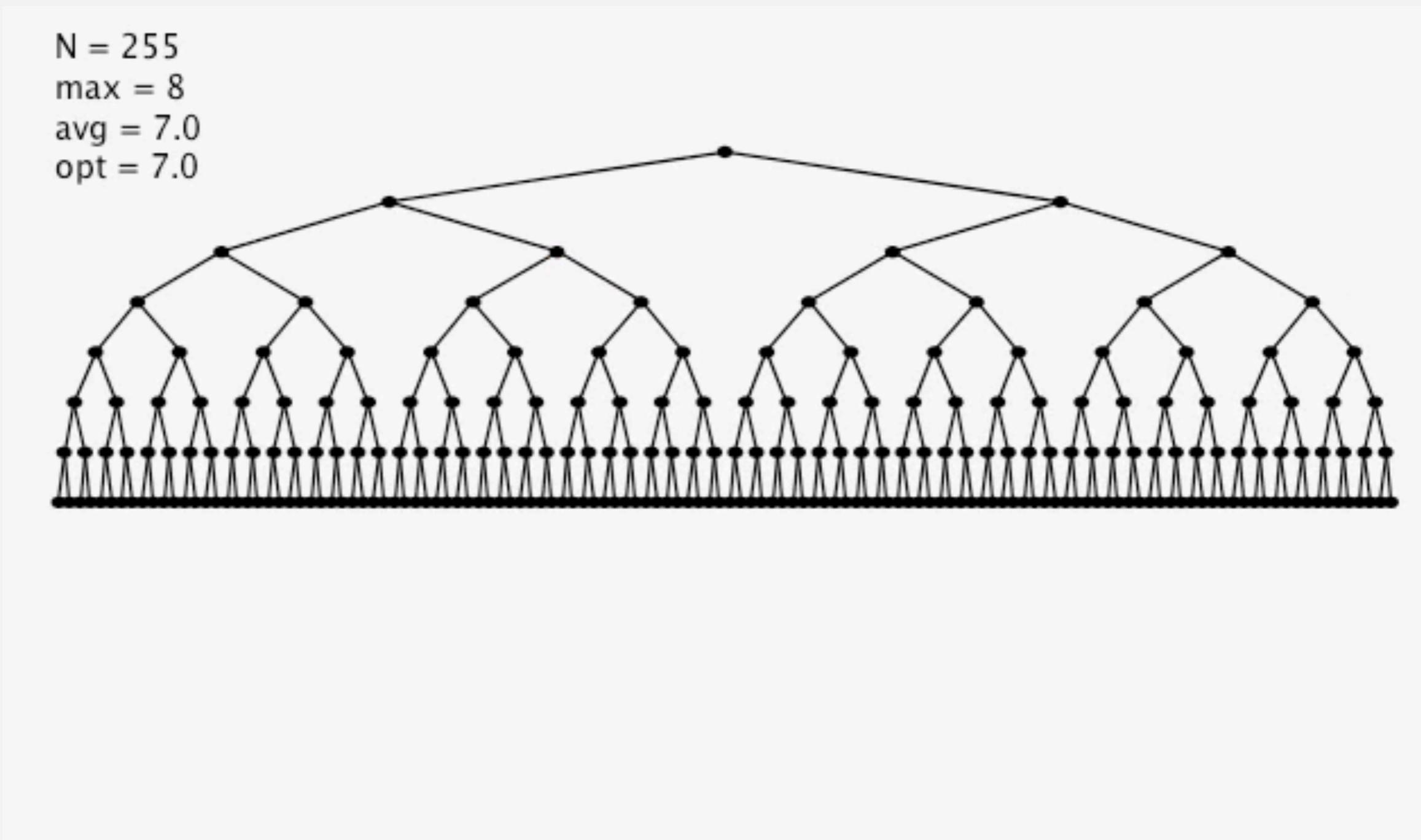
N = 255  
max = 8  
avg = 7.0  
opt = 7.0



255 insertions in ascending order

# Insertion in a LLRB tree: visualization

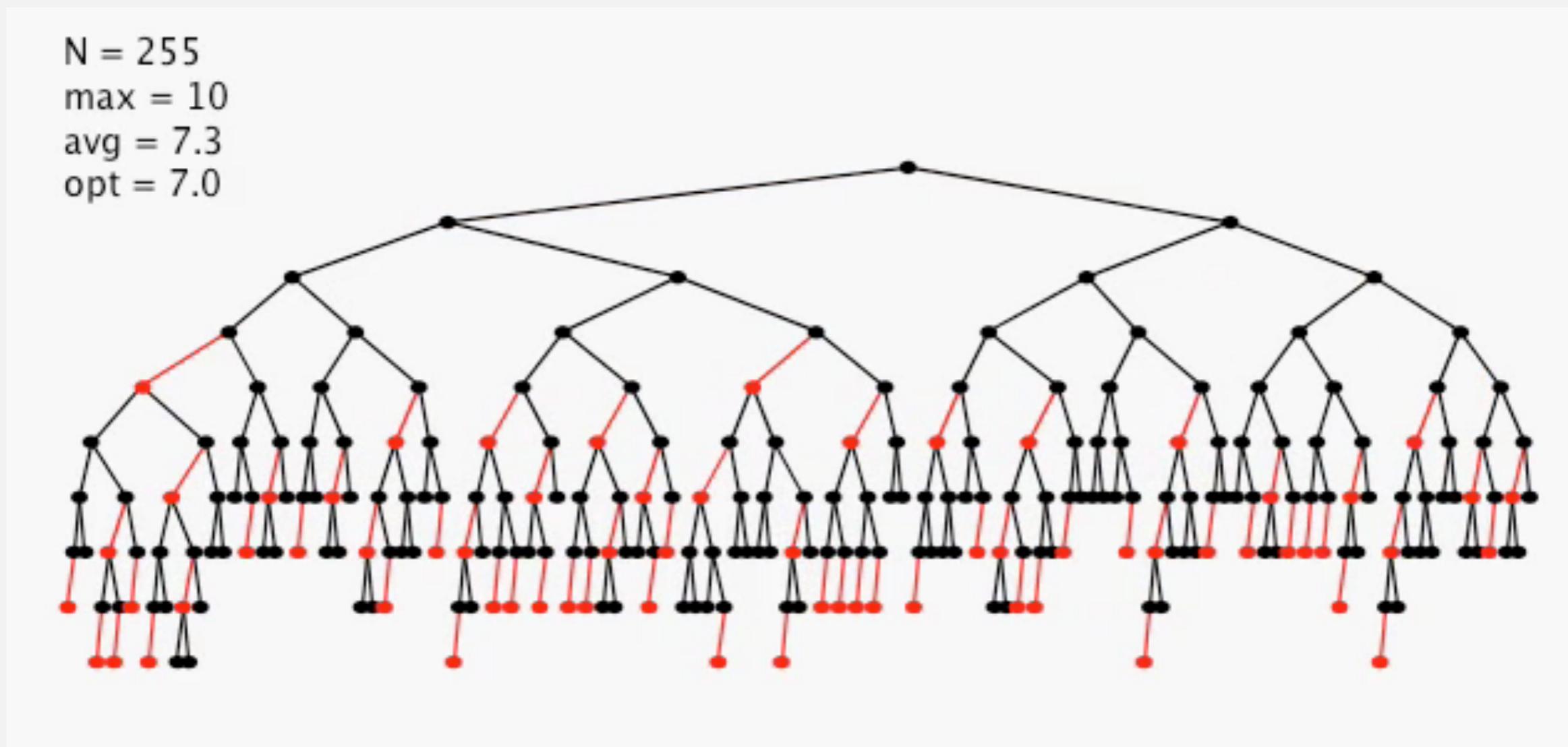
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255 insertions in descending order

# Insertion in a LLRB tree: visualization

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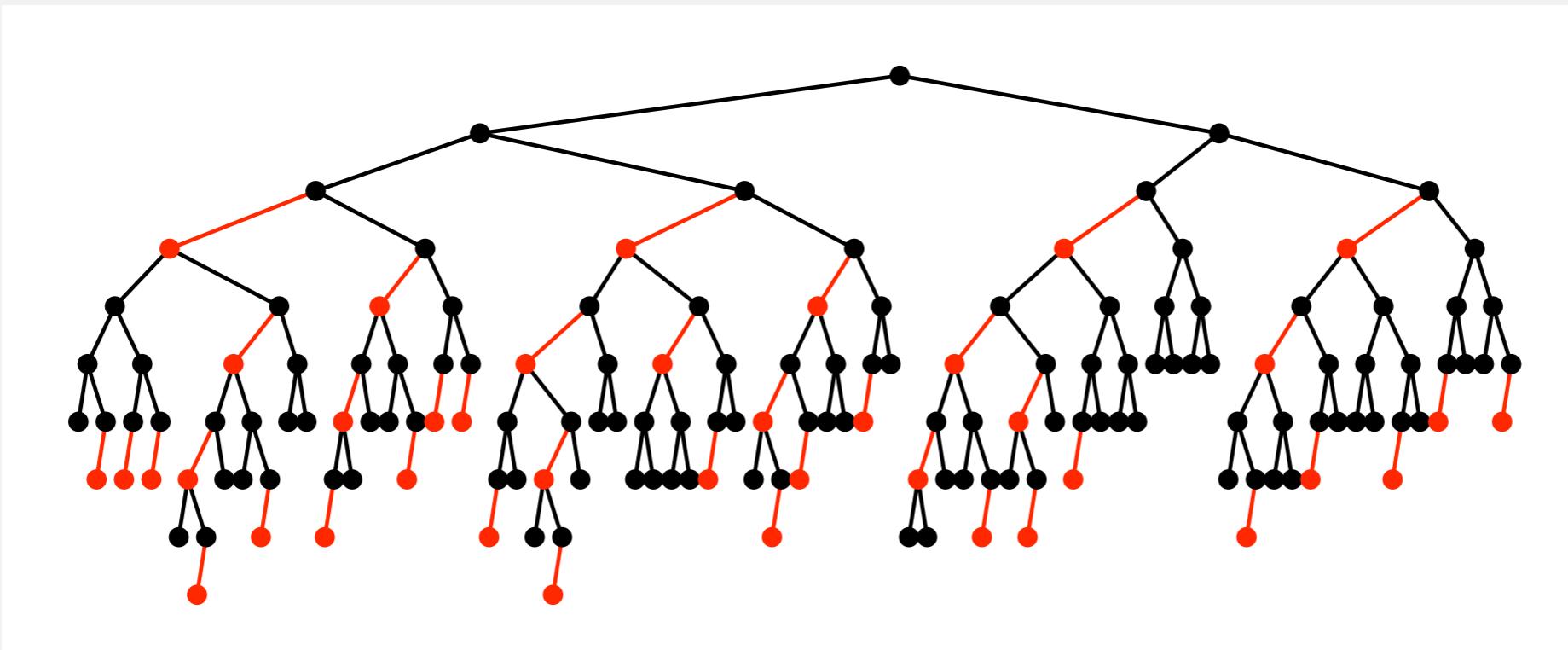
## Balance in LLRB trees

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**Proposition.** Height of tree is  $\leq 2 \lg N$  in the worst case.

Pf.

- Every path from root to null link has same number of black links.
- Never two red links in-a-row.



**Property.** Height of tree is  $\sim 1.0 \lg N$  in typical applications.

# ST implementations: summary

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implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search hit	insert	delete		
<b>sequential search (unordered list)</b>	$N$	$N$	$N$	$\frac{1}{2} N$	$N$	$\frac{1}{2} N$		<code>equals()</code>
<b>binary search (ordered array)</b>	$\lg N$	$N$	$N$	$\lg N$	$\frac{1}{2} N$	$\frac{1}{2} N$	✓	<code>compareTo()</code>
<b>BST</b>	$N$	$N$	$N$	$1.39 \lg N$	$1.39 \lg N$	$\sqrt{N}$	✓	<code>compareTo()</code>
<b>2-3 tree</b>	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	✓	<code>compareTo()</code>
<b>red-black BST</b>	$2 \lg N$	$2 \lg N$	$2 \lg N$	$1.0 \lg N^*$	$1.0 \lg N^*$	$1.0 \lg N^*$	✓	<code>compareTo()</code>

\* exact value of coefficient unknown but extremely close to 1

# War story: why red-black?

## Xerox PARC innovations. [1970s]

- Alto.
- GUI.
- Ethernet.
- Smalltalk.
- InterPress.
- Laser printing.
- Bitmapped display.
- WYSIWYG text editor.
- ...



Xerox Alto

### A DICHROMATIC FRAMEWORK FOR BALANCED TREES

Leo J. Guibas  
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Palo Alto, California, and  
Carnegie-Mellon University*

Robert Sedgewick\*  
Program in Computer Science  
*Brown University*  
Providence, R. I.

#### ABSTRACT

In this paper we present a uniform framework for the implementation and study of balanced tree algorithms. We show how to imbed in this

the way down towards a leaf. As we will see, this has a number of significant advantages over the older methods. We shall examine a number of variations on a common theme and exhibit full implementations which are notable for their brevity. One implementation is examined carefully, and some properties about its

## War story: red-black BSTs

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Telephone company contracted with database provider to build real-time database to store customer information.

### Database implementation.

- Red-black BST search and insert; Hibbard deletion.
- Exceeding height limit of 80 triggered error-recovery process.

allows for up to  $2^{40}$  keys

### Extended telephone service outage.

Hibbard deletion  
was the problem

- Main cause = height bounded exceeded!
- Telephone company sues database provider.
- Legal testimony:



*“If implemented properly, the height of a red-black BST with  $N$  keys is at most  $2 \lg N$ . ” — expert witness*



# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

## 3.3 BALANCED SEARCH TREES

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- ▶ 2-3 search trees
- ▶ red-black BSTs
- ▶ *B*-trees

# Reflections on disk-based databases

- The time to read a 1MB “block” from a hard (spindle) disk drive is dependent on:
  - Rotational latency: for a 7200 rpm drive, this is  $\sim 5$  ms
  - Seek time:  $\sim 10 - 20$  ms
  - Transfer rate:  $\sim 20$  ms
- The time to read 1MB from SDRAM memory is dependent on clock rate and memory timing:
  - $\sim 0.25$  ms
- Ratio is  $\sim 160$

# Database indexes

- Building an index with a tree:
  - following a child pointer in a tree (e.g. binary or LLRBT) will, typically, require reading a new block (compare with binary heap);
  - therefore we want to keep the number of levels to an absolute minimum;
  - therefore we want, potentially,  $m-1$  indexes ( $m$  children) at each node where  $m$  is typically much larger than 2 (binary tree) or 3 (2-3 tree)—in practice,  $m$  could be as large as, say, 1024;
  - in practice, each node (except the root) is required to have  $k >= m/2$  children;
  - external (leaf nodes) contain all of the actual keys; internal nodes have copies of keys, for guidance.
- Such a tree is called a B-tree of order  $m$ .

## File system model

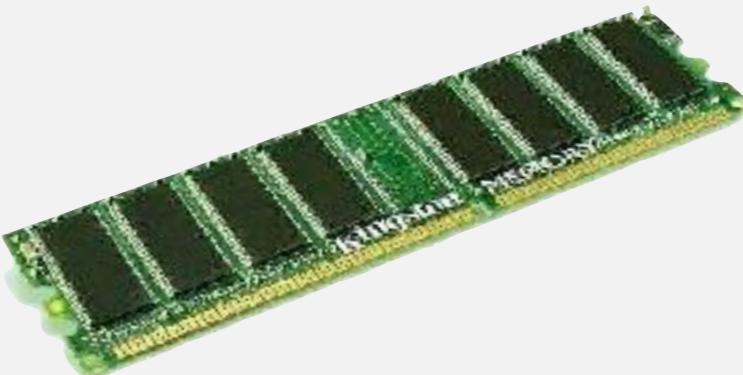
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**Page.** Contiguous block of data (e.g., a file or 4,096-byte chunk).

**Probe.** First access to a page (e.g., from disk to memory).



**slow**



**fast**

**Property.** Time required for a probe is much larger than time to access data within a page.

**Cost model.** Number of probes.

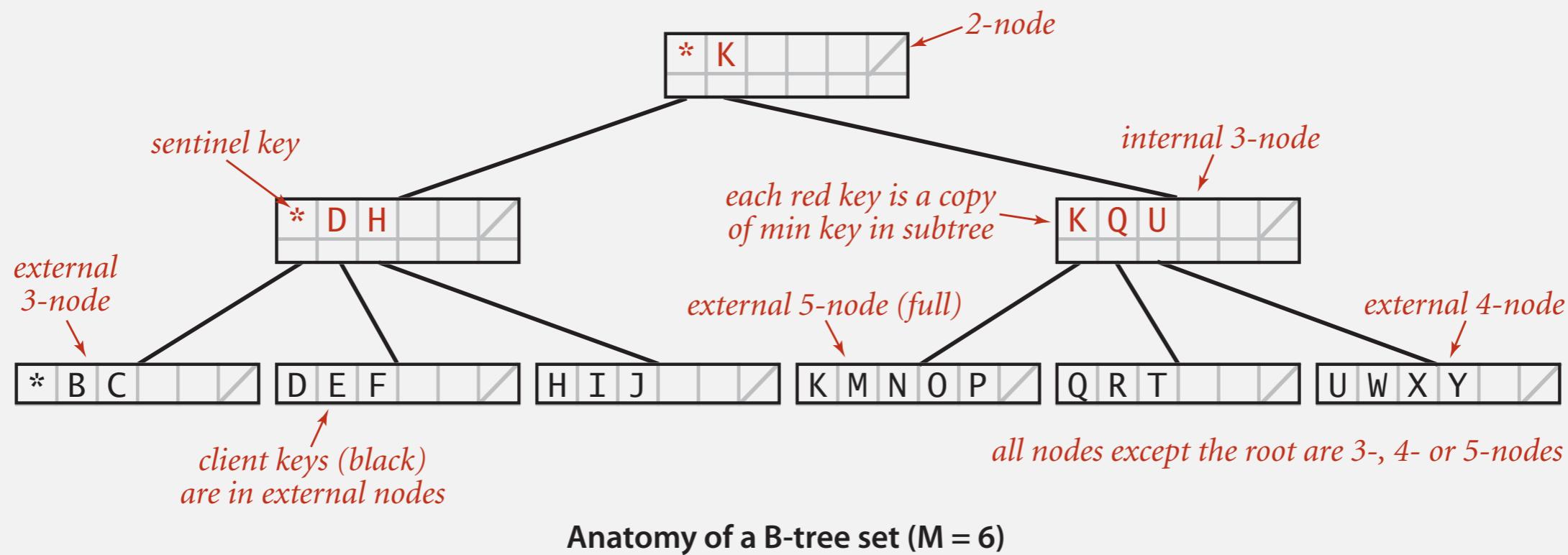
**Goal.** Access data using minimum number of probes.

# B-trees (Bayer-McCreight, 1972)

**B-tree.** Generalize 2-3 trees by allowing up to  $M - 1$  key-link pairs per node.

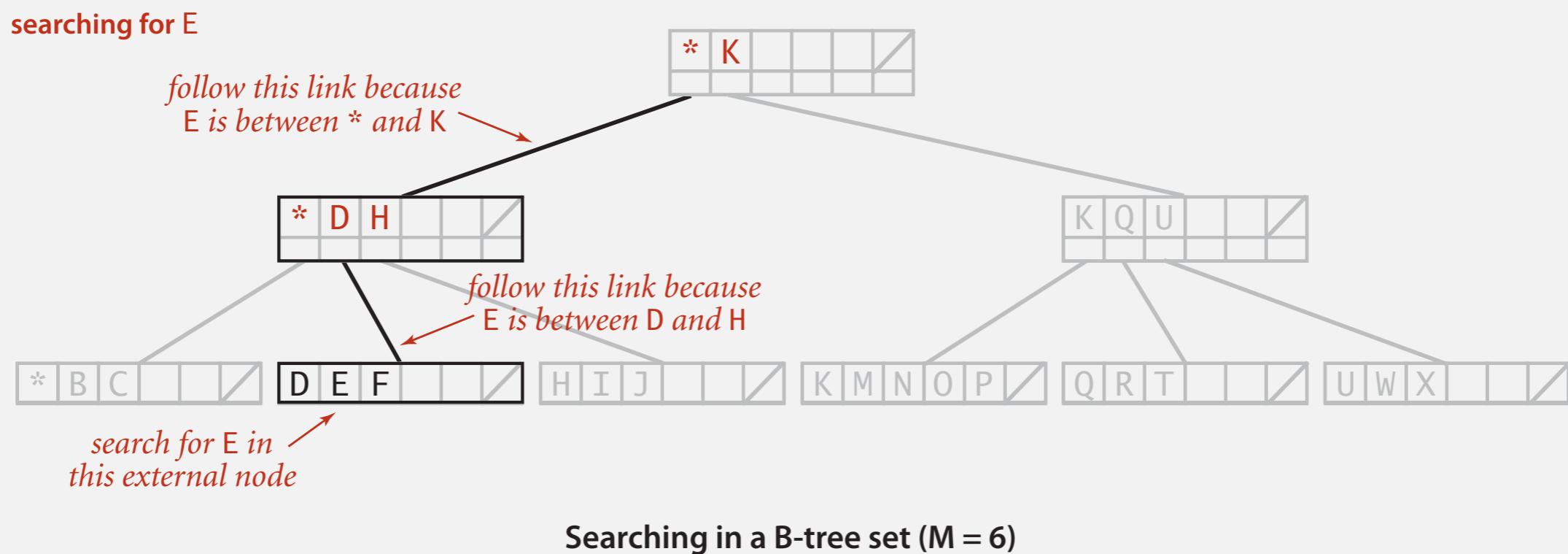
- At least 2 key-link pairs at root.
- At least  $M/2$  key-link pairs in other nodes.
- External nodes contain client keys.
- Internal nodes contain copies of keys to guide search.

choose  $M$  as large as possible so that  $M$  links fit in a page, e.g.,  $M = 1024$



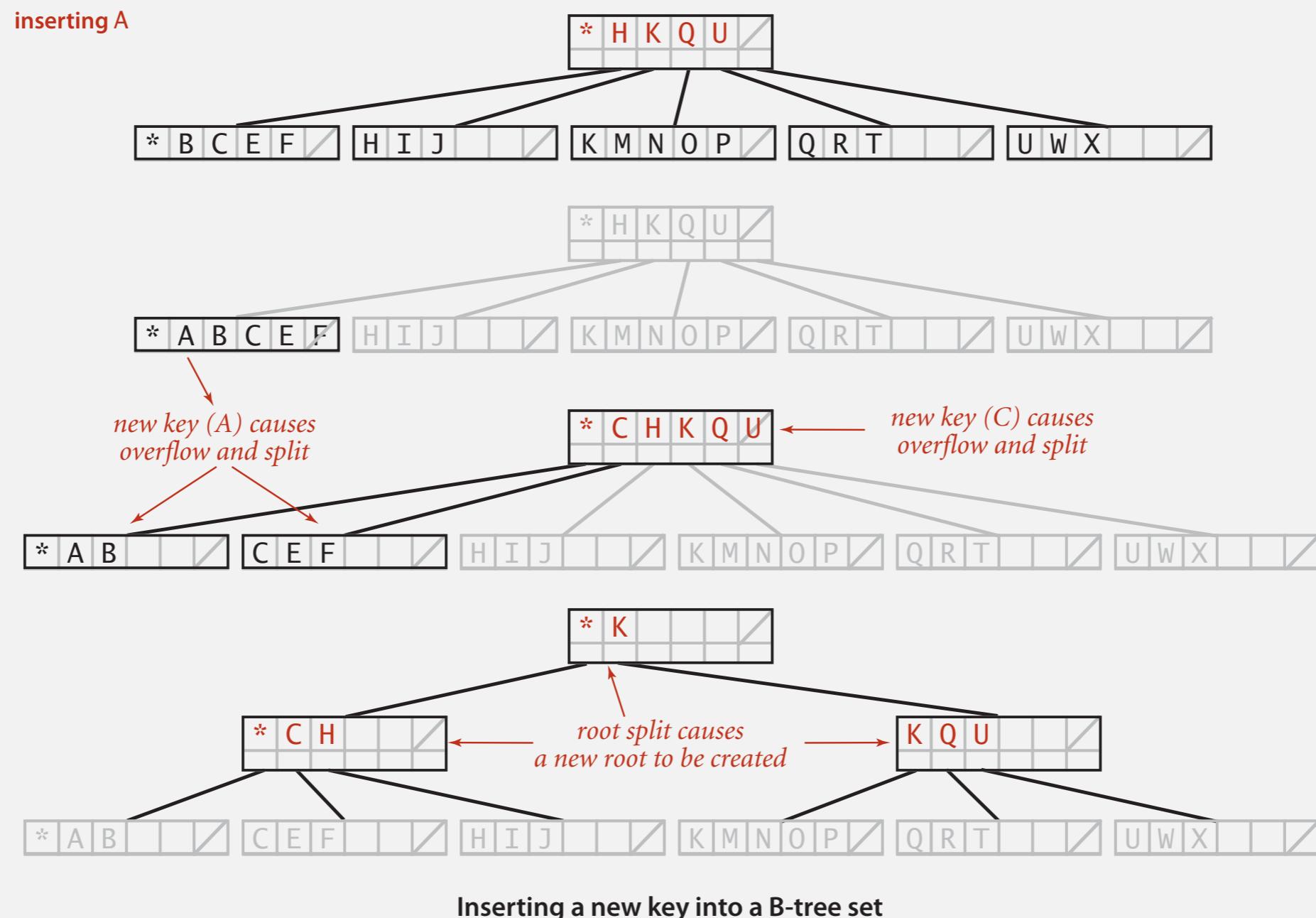
# Searching in a B-tree

- Start at root.
- Find interval for search key and take corresponding link.
- Search terminates in external node.



# Insertion in a B-tree

- Search for new key.
- Insert at bottom.
- Split nodes with  $M$  key-link pairs on the way up the tree.



## Balance in B-tree

---

**Proposition.** A search or an insertion in a B-tree of order  $M$  with  $N$  keys requires between  $\log_{M-1} N$  and  $\log_{M/2} N$  probes.

**Pf.** All internal nodes (besides root) have between  $M/2$  and  $M - 1$  links.

**In practice.** Number of probes is at most  $\leftarrow 4$ .  $M = 1024$ ;  $N = 62$  billion  
 $\log_{M/2} N \leq 4$

**Optimization.** Always keep root page in memory.

## Balanced trees in the wild

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Red-black trees are widely used as system symbol tables.

- Java: `java.util.TreeMap`, `java.util.TreeSet`.
- C++ STL: `map`, `multimap`, `multiset`.
- Linux kernel: completely fair scheduler, `linux/rbtree.h`.
- Emacs: conservative stack scanning.

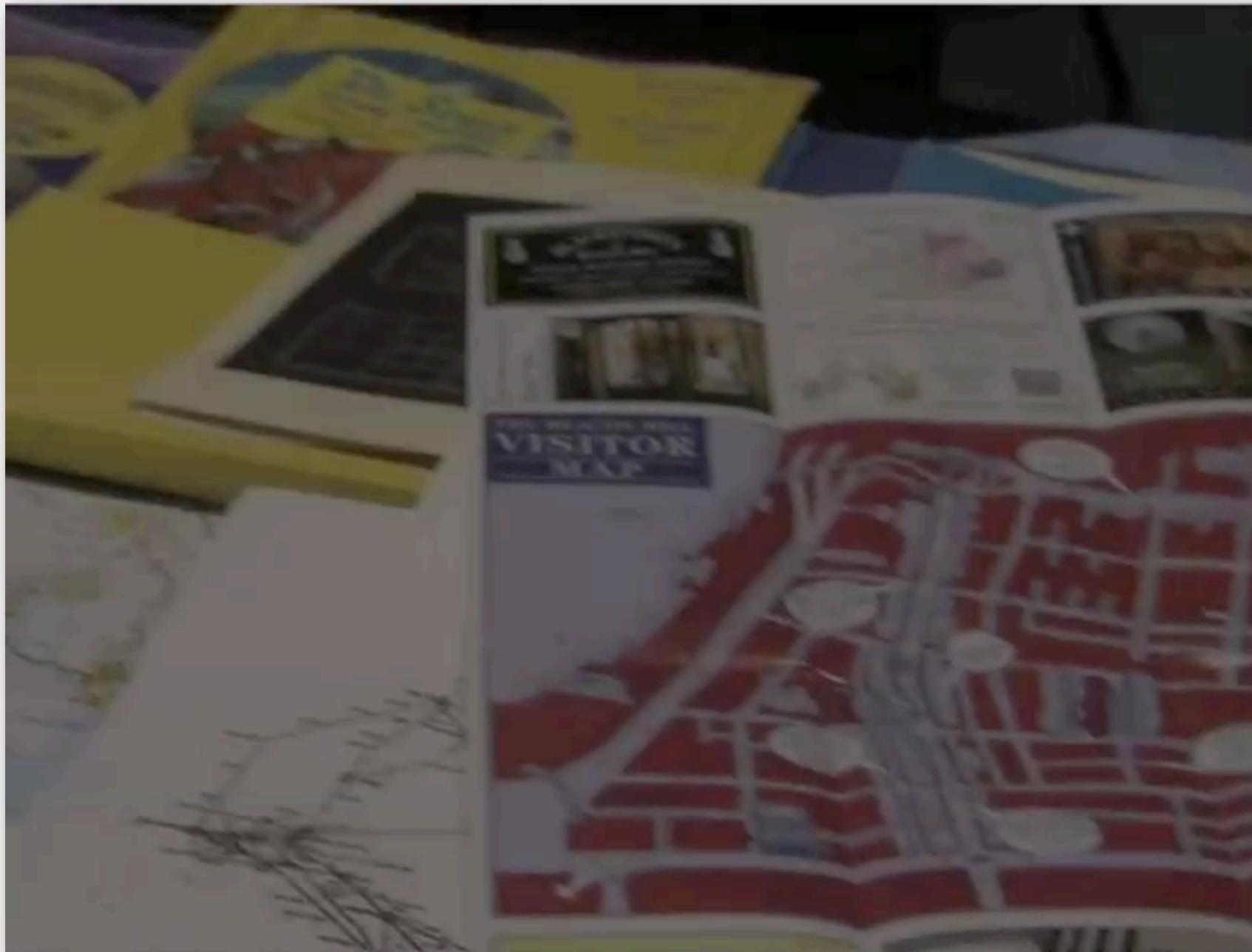
B-tree variants. B+ tree, B\*tree, B# tree, ...

B-trees (and variants) are widely used for file systems and databases.

- Windows: NTFS.
- Mac: HFS, HFS+.
- Linux: ReiserFS, XFS, Ext3FS, JFS.
- Databases: ORACLE, DB2, INGRES, SQL, PostgreSQL.

# Red-black BSTs in the wild

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*Common sense. Sixth sense.  
Together they're the  
FBI's newest team.*

# Red-black BSTs in the wild

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## ACT FOUR

FADE IN:

48 INT. FBI HQ - NIGHT

48

Antonio is at THE COMPUTER as Jess explains herself to Nicole and Pollock. The CONFERENCE TABLE is covered with OPEN REFERENCE BOOKS, TOURIST GUIDES, MAPS and REAMS OF PRINTOUTS.

JESS

It was the red door again.

POLLOCK

I thought the red door was the storage container.

JESS

But it wasn't red anymore. It was black.

ANTONIO

So red turning to black means... what?

POLLOCK

Budget deficits? Red ink, black ink?

NICOLE

Yes. I'm sure that's what it is. But maybe we should come up with a couple other options, just in case.

Antonio refers to his COMPUTER SCREEN, which is filled with mathematical equations.

ANTONIO

It could be an algorithm from a binary search tree. A red-black tree tracks every simple path from a node to a descendant leaf with the same number of black nodes.

JESS

Does that help you with girls?