Proof by Induction

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Induction

- Often in mathematics or computer science, we need to prove a relationship involving some number n:
  - $\boldsymbol{S}_n = f(n)$ where:
    - $\boldsymbol{S}_n$ represents the left-hand-side of the relationship and where
    - *f(n)*, some polynomial function of *n*, represents the right-hand side.
- Induction allows us to prove our relationship using two steps:
  - Base case
  - Inductive step

Base case

- If, for example, we want to prove $\boldsymbol{S}_n = f(n)$ for all positive n, then we should choose for our base case $n=0$ or perhaps $n=1$.
- We simply verify that the relationship holds in the base case.

Inductive step

- If, for example, we want to prove $\boldsymbol{S}_n = f(n)$ for all positive n, then we assume that $\boldsymbol{S}_n = f(n)$ and then, using that as a relationship as if it were fact, we show that $\boldsymbol{S}_{n+1} = f(n+1)$.
- Our ability to prove the inductive step will depend on our knowledge of the behavior of $\boldsymbol{S}_n$.

The proof

- If we confirm the base case (*n=0*) and we confirm the inductive step such that if the relationship is true for *n*, it is true for *n+1*, then we can combine these “facts” and assert that the relationship is *true for all positive integers*.

A simple example

- We will prove a formula for the sum of all integers 1 through *n*.
- Relationship to prove: $\boldsymbol{S}_n = n\,(n+1)\,/\,2$
- Base case (n=1): 1 = 1(2)/2 (confirmed)
- Inductive step: $\boldsymbol{S}_{n+1} - \boldsymbol{S}_n = n+1$ (by definition of $\boldsymbol{S}_n$)
    - Using the given relationship, we have:
    - $\boldsymbol{S}_{n+1} - \boldsymbol{S}_n = (n+1)(n+2)/2 - n(n+1)/2 = (n+1)/2 * (n+2-n)$
    - $\boldsymbol{S}_{n+1} - \boldsymbol{S}_n = n+1$
    - *QED*, 证毕, इति सिद्धम