Form of continuous-time, DC-motor model

We will do some simple modeling of a DC-motor in class. For now, you can take the equations as given below. The variables are

- applied voltage, V, volts
- current in windings, I, amperes
- angular velocity of motor, ω, rads/sec
- Winding resistance, R, ohms
- External applied torque, T_e, N-m
- Viscous bearing friction, α
- Load moment of inertia, J, kg-m^2
- motor back-EMF constant, K_v, Volt-Secs
- Motor torque constant, K_a, N-m/ampere

The governing equations are

$$V(t) - I(t)R - K_v\omega(t) = 0$$

$$J\dot{\omega}(t) = K_aI(t) + T_e(t) - \alpha\omega(t)$$

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Eliminate I from model

Solving for I gives

$$I(t) = \frac{1}{R}\left(V(t) - K_v\omega(t)\right)$$

Substituting this into the governing equation for $\dot{\omega}$ gives

$$J\dot{\omega}(t) = \frac{K_a}{R}\left(V(t) - K_v\omega(t)\right) - \alpha\omega(t)$$

giving

$$\dot{\omega}(t) = \frac{\alpha R - K_aK_v}{JR}\omega(t) + \frac{K_a}{JR}V(t)$$

This is a first-order system, with a specific time-constant (tau), and steady-state gain (γ), so rewritten as

$$\dot{\omega}(t) = \frac{1}{\tau}\left(-\omega(t) + \gamma V(t)\right)$$

We will determine these two parameters, approximately, from several simple experiments. Unfortunately, since there are 5 physical parameters, our limited experiments will only allow us to infer the time-constant and steady-state gain, and not the individual physical parameters themselves.

Experiment description

We can easily apply a constant input $V(t) = \bar{V}$, and zero external torque, $T_e(t) = 0$. With the built-in angle encoder, we measure the resultant angle $\theta(t)$. The crude angle encoder that has a quantization level of 1 degree. The measurement is sampled (at 50 measurements/second).

Converting experimental output

The experimental setup easily allows you to apply a contant voltage to the system with initial condition $\omega(0) = 0$ and $\theta(0) = 0$. The only output that is available is the angular displacement, sampled at 50 samples/second, and quantized at 1-degree. We need to convert this θ measurement to ω, through some form of numerical differentiation.

Instead of radians for the angular displacement, we will use degrees, as that is the units of the encoder. We will work on identifying τ and γ in these units (note that τ is unaffected by a unit-change in ω or θ, but the steady=state gain is affected by unit choices there, and in V as well).

Let's pretend like the system has $\tau = 0.1$ and $\gamma = 14$, so the response to $V(t) = \bar{V}$ is $\omega(t) = 14\bar{V}(1 - e^{-10t})$. Integrating this, and getting the initial conditions right means we will observe

$$\theta(t) = 14\bar{V}\left(t - \tfrac{1}{10}\left(1 - e^{-10t}\right)\right)$$

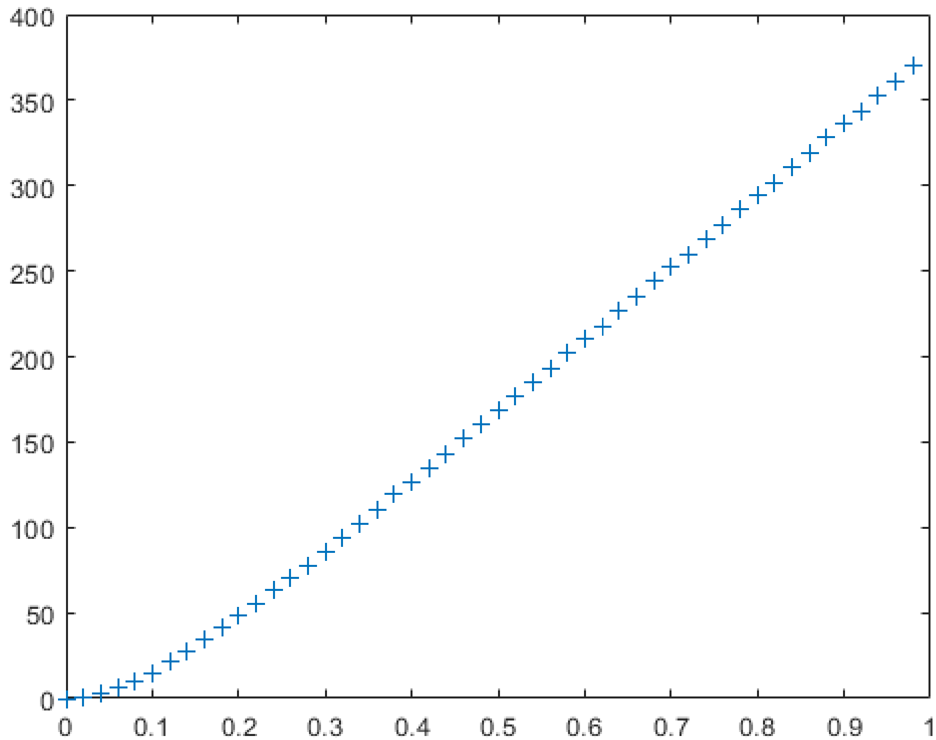
But the measurement is sampled, at time-interval $T = 0.02$ so the value at the k'th measurement, $k = 0, 1, \ldots$ is

$$\theta^k := \theta(t = kT)$$

Furthermore the measurement is quantized, with quantization level 1, so

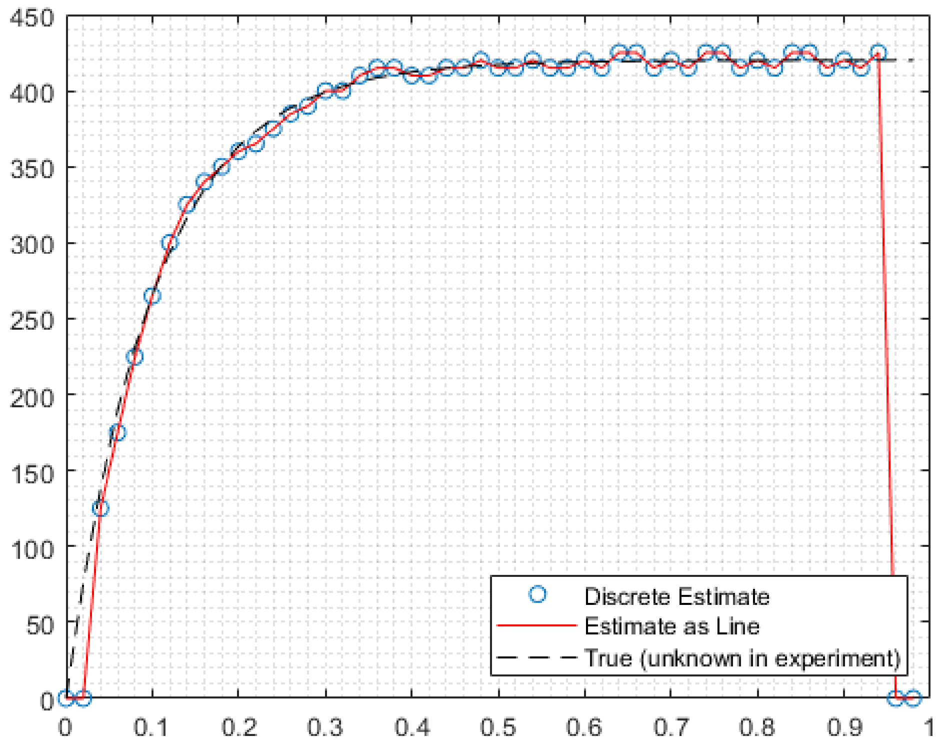
$$\theta^k_{\text{meas}} := \text{round}(\theta^k)$$

```
barV = 30;
DeltaT = 0.02;
nSamples = 50;
OmegaTrue = @(t) 14*barV*(1-exp(-10*t));
ThetaTrue = @(t) 14*barV*(t-(1/10).*(1-exp(-10*t)));
tMeas = (0:nSamples-1)*DeltaT;
ThetaSampled = ThetaTrue(tMeas);
ThetaSampledQuantized = round(ThetaSampled);
plot(tMeas,ThetaSampledQuantized,'+')
```



Estimating the derivative using least-squares polynomial fits

```
wEst = zeros(1,numel(tMeas));
% We will use 5 points, centered at |tMeas(i)| to fit a quadratic function
% through the data, using POLYFIT.  The means we use |tMeas(i-2:i+2)|, so
% our estimate needs to start at tMeas(3) and end at tMeas(end-2).  The
% "estimates" at |tMeas(1)|, |tMeas(2)|, |tMeas(end-1)|, |tMeas(end)| are
% simply zero, since we don't have all of the necessary points, and will be
% ignored (even though they are plotted).
for i=(1+2):(numel(tMeas)-2)
   % Take 5 points, centered at tMeas(i), fit with quadratic
   P = polyfit(tMeas(i-2:i+2),ThetaSampledQuantized(i-2:i+2),2);
   % Take derivative of quadratic
   W = polyder(P);
   % Evaluate derivative at tMeas(i), to get estimate of angular velocity
   wEst(i) = polyval(W,tMeas(i));
end
plot(tMeas,wEst,'o',tMeas,wEst,'r',tMeas,OmegaTrue(tMeas),'k--');
legend('Discrete Estimate','Estimate as Line','True (unknown in experiment)',...
   'location','southeast')
grid minor
```

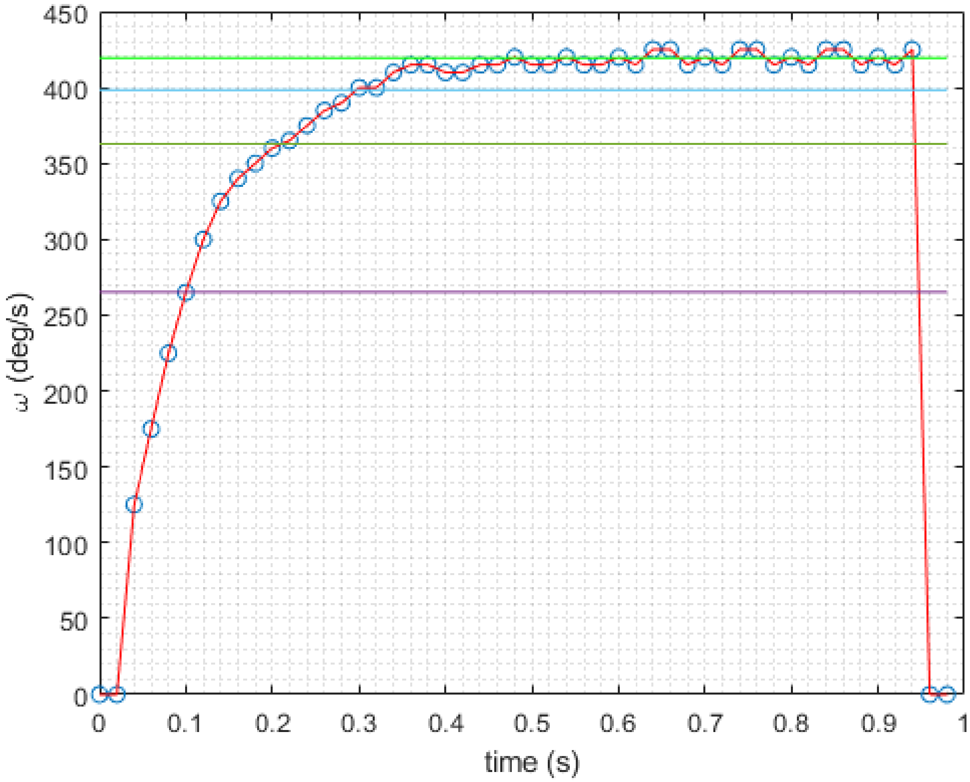


Estimate the steady-state value

Take the mean of the last 9 estimates, and call that the steady-state for ω. Divide that by `barV` to get an estimate of the steady-state gain. Plot the estimated response, and the 3 values associated with one, two and 3 time constants (ie., e^{-1}, e^{-2}, e^{-3} times the final value, so that you can estimate the time-constant by examining the graph.

```
steadyStateValue = mean(wEst(end-10:end-2));
steadyStateGain = steadyStateValue/barV;
disp(['Estimate of Steady-State Gain = ' num2str(steadyStateGain)]);
plot(tMeas,wEst,'o',tMeas,wEst,'r',...
   tMeas,repmat(steadyStateValue,[1 numel(tMeas)]),'g',...
   tMeas,repmat((1-exp(-1))*steadyStateValue,[1 numel(tMeas)]),...
   tMeas,repmat((1-exp(-2))*steadyStateValue,[1 numel(tMeas)]),...
   tMeas,repmat((1-exp(-3))*steadyStateValue,[1 numel(tMeas)]))
hold off
grid minor
ylabel('\omega (deg/s)');
xlabel('time (s)');
```

```
Estimate of Steady-State Gain = 13.9815
```



Task #1

Apply these ideas to the p-coded file `SyntheticMotor.p`. The syntax is `[ThetaSampledQuantized,tVec] = syntheticMotor(barV)`. The outout is the sampled (at 0.02 seconds), quantized (at 1-degree) measurement for 50 samples, starting at t=0. Both arguments are returned as row-vectors. Apply the ideas we used above, using 6 values for `barV`, namely 5, 10, 20, 40, 60, 80. **At which input-levels does the quantization cause the most difficulty?** In each case, estimate the steady-state gain, and the time-constant. Collect all of your estimates into a single estimate for the time-constant, and a single estimate for the steady-state gain, along with an assessment of the uncertainty you are observing.

Turn in (as a group) your answer to the question above, your 6 plots (ω vs t), and table of time constants, τ, and steady state gain constants, γ.

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