Self-Organized Criticality in Sandpile Model

IDC 621 Modelling Complex Systems

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Introduction

The Sandpile model was one of the first discovered instance of a cellular automata exhibiting self-organized criticality. It was introduced by Bak, Tang and Wiesenfeld in a 1987 paper. This simple cellular automaton was shown to produce several characteristic features observed in natural complexity (power law distributions) and these features emerged in a manner that did not depend on finely tuned details of the system.

1 Sandpile Model

In this report, we will investigate the Sandpile model on a 2-D grid. Each site (x, y) on the grid has a non-negative integer valued state $z_{(x,y)}$ assigned to it which can be thought of as the height of that sand-pile. In the course of the simulation, the value of $z_{(x,y)}$ is increased by 1 at random on some point (x, y) within the grid which acts as external driving-

$$z_{(x,y)}(t+1) = z_{(x,y)}(t) + 1$$

The state $z_{(x,y)}$ can take integer values between $[0, z_c]$ where z_c is the threshold height of the pile. If some point $z_{(x,y)}$ exceeds z_c after a perturbation, then it triggers an avalanche according to the following rule-

$$\text{If } z_{(x,y)} > z_c \implies \quad \begin{cases} z_{(x,y)} \longrightarrow z_{(x,y)} - 4 \\ z_{(x,y\pm1)} \longrightarrow z_{(x,y\pm1)} + 1 \\ z_{(x\pm1,y)} \longrightarrow z_{(x\pm1,y)} + 1 \end{cases}$$

Further, we fix the following things in our study

- **Threshold Height** z_c=4
- **Boundaries** All boundaries are fixed at 0.

2 Simulations and Analysis

To start with the simulation of the model, we choose the dimensions of the grid M to generate a $M \times M$ grid and assign random integers between 0 and $(z_c - 1)$ to it. We then add a single sand particle to a random site on the grid, wait for the system to settle down after the effects of the perturbation cease, note down the relevant data, and again add another sand particle to repeat the process. We count *adding a sand particle as a single time step.*

The quantities whose data we measure on each time step are-

- **Area affected**: The number of sites affected by a single perturbation.
- **No. of Topples** The number of unstable sites generated and toppled due to the perturbation.
- **Relaxation Time**: The number of iterations before the system settles down to equilibrium.

Say we represent a general measured quantity by X. The measured quantities are expected to exhibit a power law distribution when plotted as X v/s $\mathcal{N}(X)$ where $\mathcal{N}(X)$ denotes the number of occurrences of X. Hence, we expect that if we do a log-log plot of such a power law distribution, we can obtain the power by doing a least square fit determination of the slope of log-log plot.

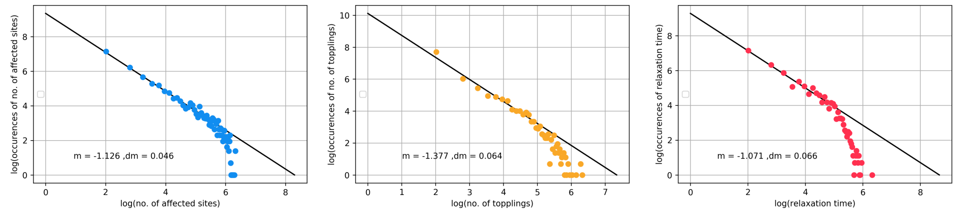
$$\mathcal{N}(X) \propto X^p \implies \log(\mathcal{N}(X)) \sim p \cdot \log(X)$$

2.1 Results

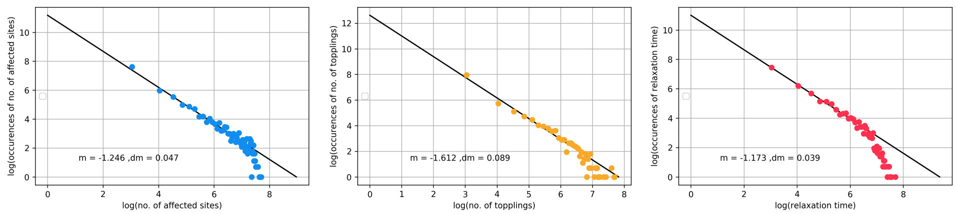
While simulating the model, we try to plot our data for various system parameters by varying the number of time steps and the size of the grid. Since we are limited by the computing power, the situations we consider in the report are as follows-

- Time steps $1 \times 10^4 \longrightarrow$ Grid Sizes- 25×25, 50×50, 75×75
- Time steps $5 \times 10^4 \longrightarrow$ Grid Sizes- 25×25, 50×50
- Time steps $1 \times 10^5 \longrightarrow$ Grid Sizes- 25×25, 50×50
- Time steps $1 \times 10^6 \longrightarrow$ Grid Sizes- 25×25

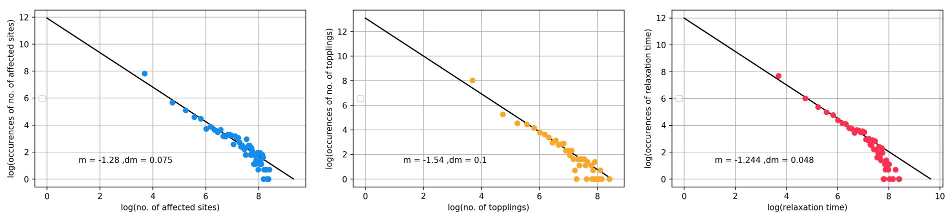
Since the raw data will have a bin size of 1 by default, we bin the data into appropriate sizes depending upon the time steps T. For T time steps, we bin the data into $\sqrt{T}$ bins.
The plots are given on the next page.



(a) $T = 10^4,\ \mathcal{M} \times \mathcal{M} = 25 \times 25$

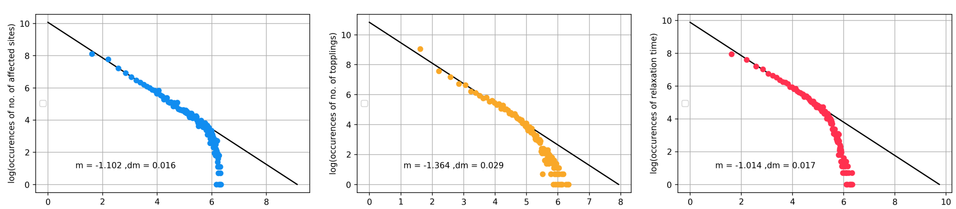


(b) $T = 10^4,\ \mathcal{M} \times \mathcal{M} = 50 \times 50$

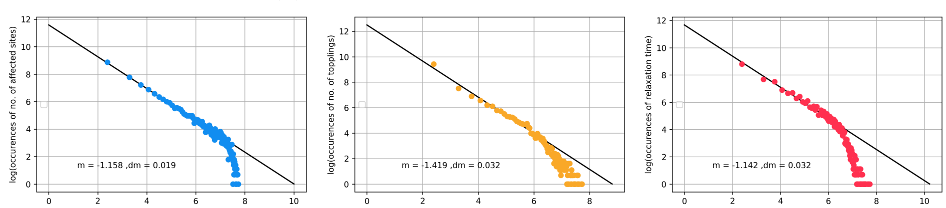


(c) $T = 10^4,\ \mathcal{M} \times \mathcal{M} = 75 \times 75$

Figure 1: <u>Time Steps $T = 10^4$</u>

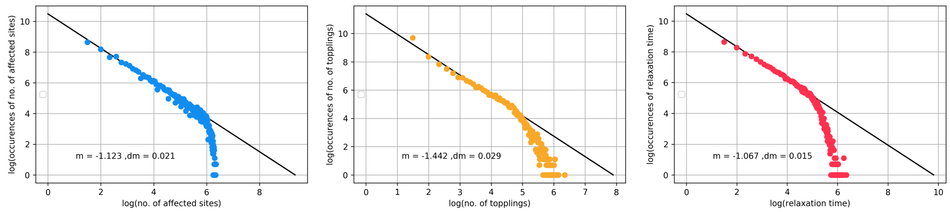


(a) $T = 5 \times 10^4,\ \mathcal{M} \times \mathcal{M} = 25 \times 25$

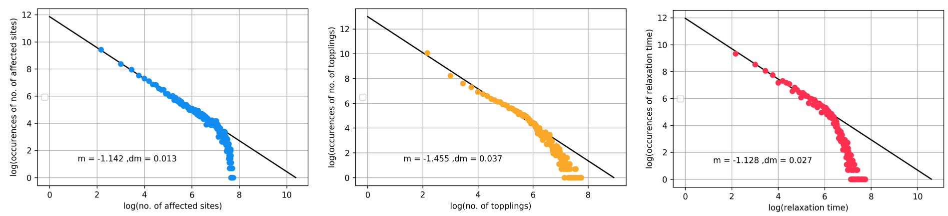


(b) $T = 5 \times 10^4,\ \mathcal{M} \times \mathcal{M} = 50 \times 50$

Figure 2: <u>Time Steps $T = 5 \times 10^4$</u>

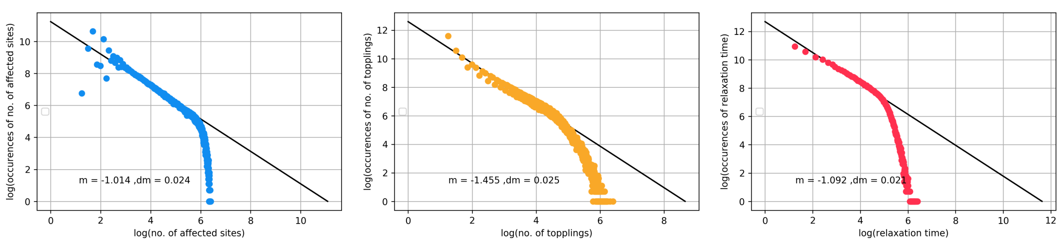


(a) $T = 10^5,\ \mathcal{M} \times \mathcal{M} = 25 \times 25$



(b) $T = 10^5,\ \mathcal{M} \times \mathcal{M} = 50 \times 50$

Figure 3: Time Steps $T = 10^5$



(a) $T = 10^6,\ \mathcal{M} \times \mathcal{M} = 25 \times 25$

Figure 4: Time Steps $T = 10^6$

2.2 Analysis

Plotting out the log-log data helps us see that there clearly exists a linear regime which shows the power law distribution. The deviation from the linear regime decreases as we increase the grid dimensions, and the number of sampling points increases as the time steps increase. The measured values of the slope of the curves in the linear regime (first 35% points) are as follows-

- **Affected Area**

-	$\mathbf{T = 10^4}$	$\mathbf{T = 5 \times 10^4}$	$\mathbf{T = 10^5}$	$\mathbf{T = 10^6}$
25 × 25	-1.126 ± 0.046	-1.102 ± 0.016	-1.123 ± 0.021	-1.014 ± 0.024
50 × 50	-1.246 ± 0.047	-1.158 ± 0.019	-1.142 ± 0.013	-
75 × 75	-1.28 ± 0.075	-	-	-

- **Number of Topplings**

-	$\mathbf{T = 10^4}$	$\mathbf{T = 5 \times 10^4}$	$\mathbf{T = 10^5}$	$\mathbf{T = 10^6}$
$\mathbf{25 \times 25}$	-1.377 ± 0.064	-1.364 ± 0.029	-1.442 ± 0.029	-1.455 ± 0.025
$\mathbf{50 \times 50}$	-1.612 ± 0.089	-1.419 ± 0.032	-1.455 ± 0.047	-
$\mathbf{75 \times 75}$	-1.54 ± 0.1	-	-	-

- **Relaxation time**

-	$\mathbf{T = 10^4}$	$\mathbf{T = 5 \times 10^4}$	$\mathbf{T = 10^5}$	$\mathbf{T = 10^6}$
$\mathbf{25 \times 25}$	-1.071 ± 0.066	-1.014 ± 0.017	-1.067 ± 0.015	-1.092 ± 0.021
$\mathbf{50 \times 50}$	-1.173 ± 0.039	-1.142 ± 0.032	-1.128 ± 0.027	-
$\mathbf{75 \times 75}$	-1.244 ± 0.048	-	-	-

The tabulated values are the exponents of the power laws that we have measured. In all the cases, the power is very close to ~ -1. The powers that we get in each table are roughly the same demonstrating that the power law distribution is independent of the parameters. However, for some reason, the power law deduced from "Number of Topplings" is tending towards ~ -1.5 as we increase the time steps as well as the grid size.

3 Conclusion

Power Law distributions were obtained for all the three measured quantities by evolving the sand pile through sand drop perturbations. The power of the exponent is deduced to be ≈ -1 for the collected data when the log-log plot is least square fitted in the linear regime. This demonstrates the self-organized criticality of the Sandpile Model, since the patterns observed are independent of the system parameters.

Code

The code for this project was written in Python 3.7.6 and can be found on the following GitHub repository: `https://github.com/kaizokugarizoro/ComplexSystems`.