Free Fall with Fixed or Varying Drag

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INTRODUCTION

Felix Baumgartner set the world record for falling from a great height in October 14th, 2012. He jumped from a helium balloon at a height of 39045m, fell for 4 minutes and 19 seconds and reached a maximum speed of 373 m/s. This exercise solves the equations of motion for a free-falling object by exploring the tools to solve ODEs. In this problem, Euler's method is used to solve the differential equations describing a one-dimensional free fall under gravity where the object experiences a free fall under constant air density. Then, a modified Euler method is used, and the accuracy of both methods will be tested against an analytical predicted solution. Finally, parameters are varied to predict whether Baumgartner breaks the sound barrier or not.

The problem describes a free falling's body displacement and velocity from the starting height. The velocity is determined by resolving forces; therefore, Newton's second law gives:

$$m\frac{dv}{dt} = W + F_{drag} = -mg - kv^2$$

[Eq.1]

Assuming a constant drag force, k can be given by the equation:

$$k = \frac{C_d \rho_0 A}{2}$$

[Eq.2]

where Cd is the drag co-efficient of the projectile, ρ0 is the air density, and A is the cross-sectional area of the projectile. If equation 1 is rearranged with respect to dv/dt, the velocity is determined to then find the displacement of the object.

$$\frac{dy}{dt} = v$$

[Eq.3]

THEORY

PART A: ANALYTICAL PREDICTION

For the first part of the exercise, the analytical predictions for height y and vertical speed v_y for a free-falling object under constant gravity and constant drag factor, k, are plotted as a function of time to then test how well Euler's method works.

$$v = -\sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{kg}{m}}t\right)$$

[Eq.4]

$$y = y_0 - \frac{m}{2k}log_e[cosh^2\left(\sqrt{\frac{kg}{m}}t\right)]$$

[Eq.5]

These equations can be derived from equations 1 and 2. The initial height y_0 will be 1km.

PART B: EULER'S METHOD

Euler's method is used to solve ODEs and is like the numerical method to differentiation and integration. The differential is replaced by a ratio of a finite difference [1]. Therefore, in the limit of small differences the differential can be evaluated:

$$\frac{dy}{dx} \rightarrow \frac{\Delta y}{\Delta x}$$

[Eq.6]

By defining the difference $\Delta y = y_{n+1} - y_n$, using this method gives the following equations using varying t. For velocity:

$$v_{n+1} = v_n - \Delta t(g + \frac{k}{m}v_n^2)$$

[Eq.7]

For height:

$$y_{n+1} = y_n + \Delta t v_n$$

[Eq.8]

The time at certain points is then obtained by:

$$t_{n+1} = t_n + \Delta t$$

[Eq.9]

The parameters used are v_i = 0m/s and a y_i = 1000m. The time increment taken is Δt = 1s and the mass of the body is approximately m = 100kg, as he weighs 73kg and the suit was 45kg.. The parameters used to calculate the drag factor are C_0 = 1, ρ_0 = 1200g/m³ and A = 0.5m³.

The second part of this section uses a modified Euler's method which consists in connecting two points and using the gradient of their midpoint instead of the gradient at each point [1]. This reduces the overshoot effect and gives a more accurate approximation of a curve as it takes derivatives at more points. The parameters used are the same.

PART C: MODIFIED EULER'S METHOD WITH A VARYING AIR DENSITY

Using the modified Euler's method used at the end of part B, this problem can be analysed in a more realistic way. This is done by varying the air density, because in real life air density is lower the higher the altitude. Baumgartner jumped from an altitude of yi = 39045m, a very high altitude, where the air density is very low, and so the drag coefficient k is replaced with a function k(y). To do this the scale height for the atmosphere h is used. The rest of the parameters stay the same, but the air density is changed to make it as a function of height,

$$\rho(y) = \rho_0 e^{\frac{-y}{h}}$$

[Eq.10]

where h is approximately $7.64km^2$.

PART D: FURTHER INVESTIGATION

This part of the problem investigates the effect of varying the jump parameters (initial velocity, initial height, drag coefficient, cross sectional area and mass). The drag coefficient has a range of 1 to 1.3, the cross-sectional area has a range of $0.4m^3$ to $0.8m^3$ for the average skydiver, and the initial height used is Baumgartner's. The smaller the ratio C_dA/m, the higher the peak in the velocity graph, so the maximum velocity is higher.

RESULTS

PART A: ANALYTICAL PREDICTION

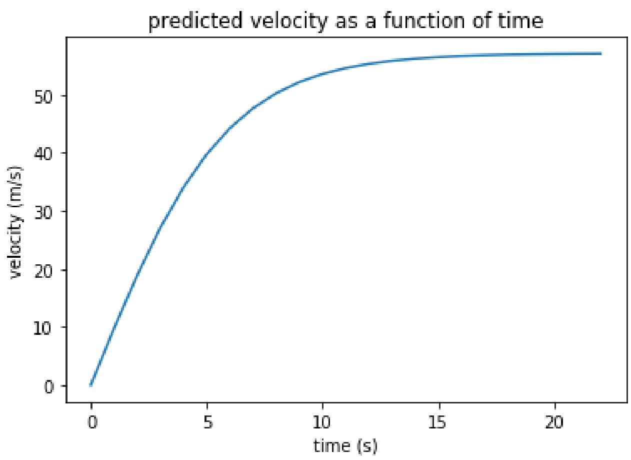

Figure 1: Predicted velocity of the body against time.

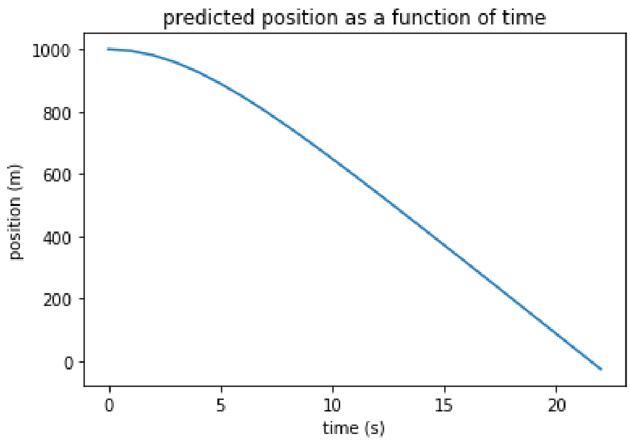

Figure 2: Predicted position of the body against time.

PART B: EULER'S METHOD

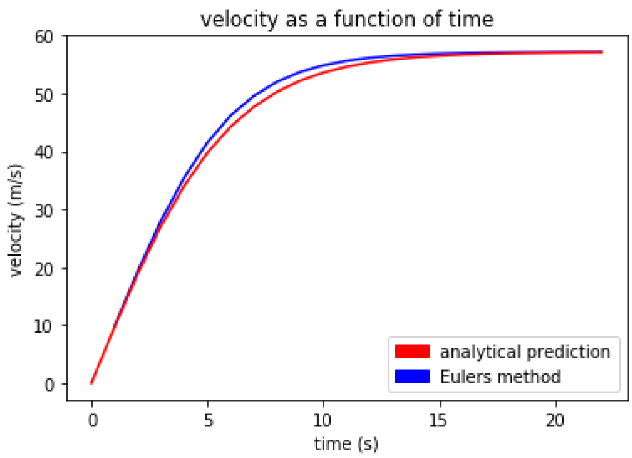

Figure 3: Velocity of the body against time. Analytical prediction plotted in red and Euler's method plotted in blue.

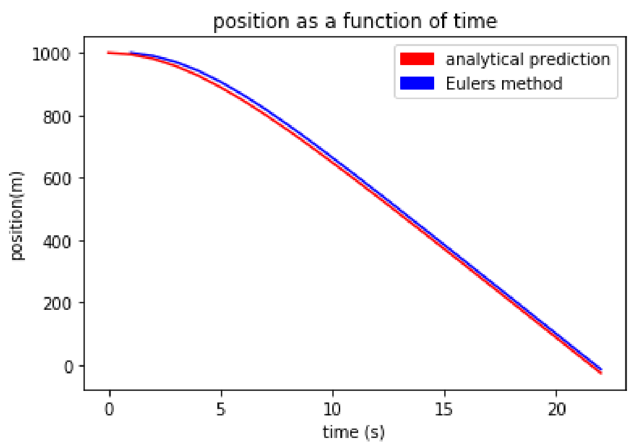

Figure 4: Position of the body against time. Analytical prediction plotted in red and Euler's method plotted in blue.

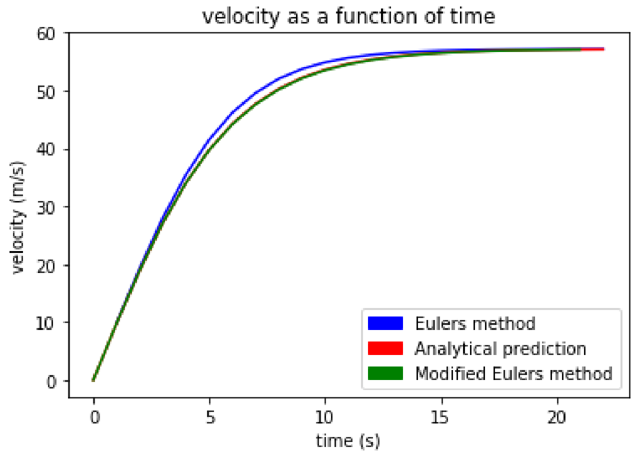

Figure 5: Velocity of the body against time. Same as figure 3 with modified Euler's method in green.

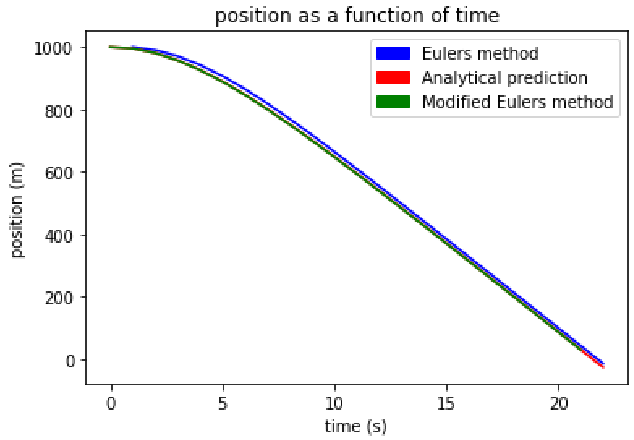

Figure 6: Position of the body against time. Same as figure 4 with modified Euler's method in green.

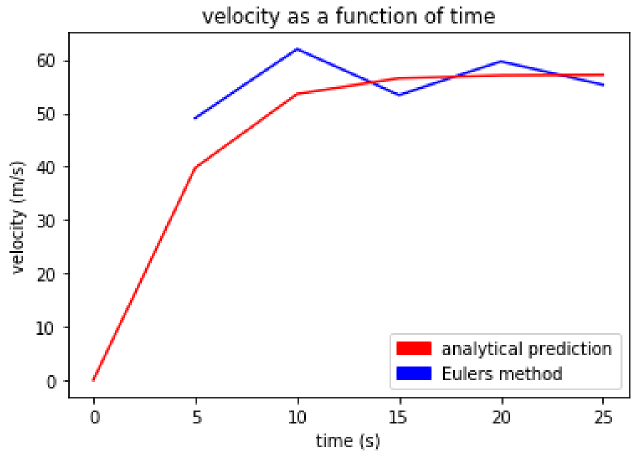

Figure 7: Velocity of the body against time. Analytical prediction plotted in red and Euler's method plotted in blue. Same as figure 3 with a step size of $\Delta t = 5s$.

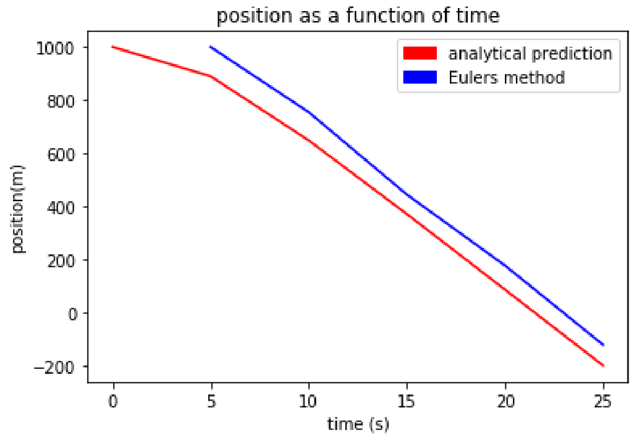

Figure 8: Position of the body against time. Analytical prediction plotted in red and Euler's method plotted in blue. Same as figure 4 with a step size of $\Delta t = 5s$.

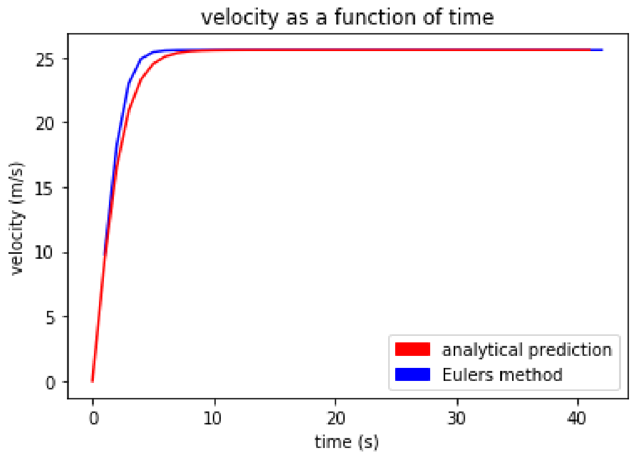

Figure 9: Velocity of the body against time. Analytical prediction plotted in red and Euler's method plotted in blue. Same as figure 3 with a bigger k/m ratio.

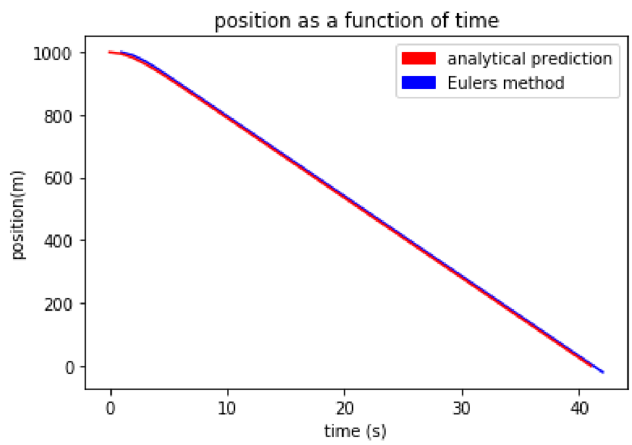

Figure 10: Position of the body against time. Analytical prediction plotted in red and Euler's method plotted in blue. Same as figure 4 with a bigger k/m ratio.

PART C: MODIFIED EULER'S METHOD WITH A VARYING AIR DENSITY

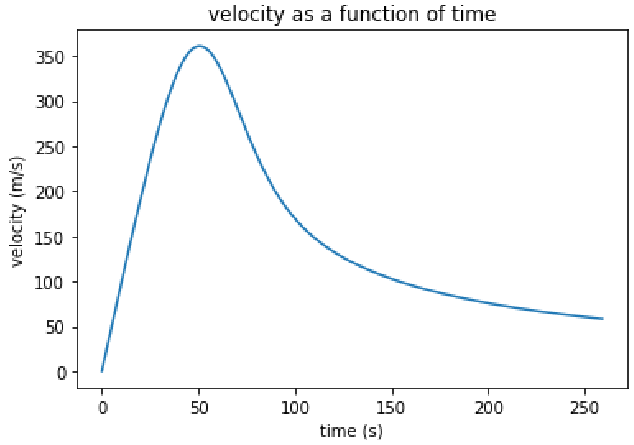

Figure 11: Velocity of the body plotted against time for a varying air density using modified Euler's method.

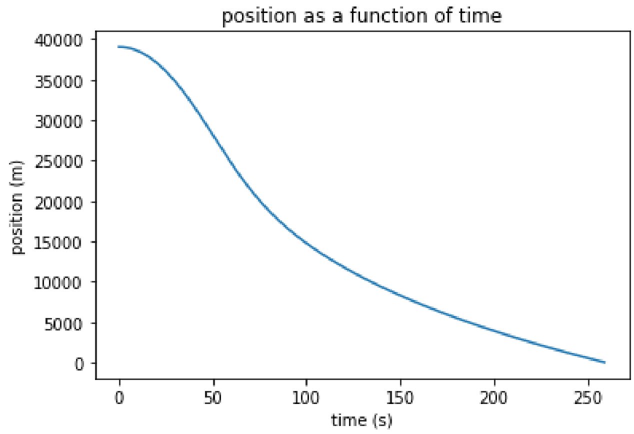

Figure 12: Position of the body plotted against time for a varying air density using modified Euler's method.

PART D: FURTHER INVESTIGATION

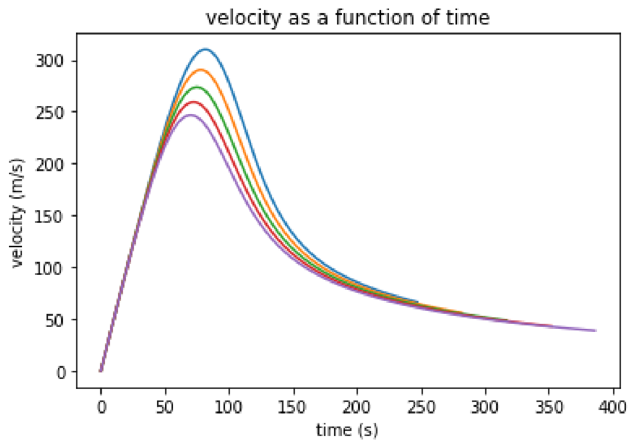

Figure 13: Velocity of the body plotted against time for a varying air density.

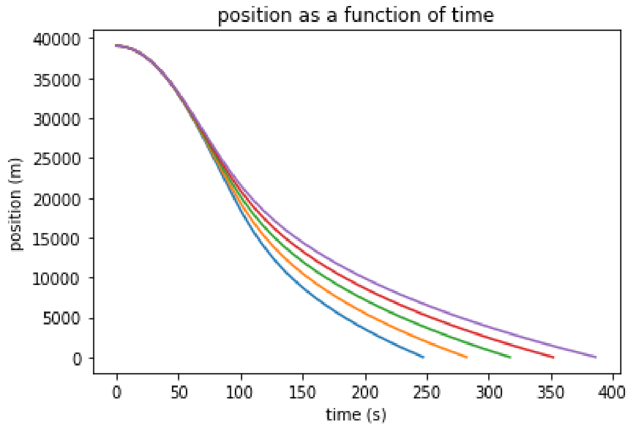

Figure 14: Position of the body plotted against time for a varying air density.

DISCUSSION

In figure 1, the velocity increases with time and gradually increases less as the body's acceleration approaches 0. It then reaches a constant terminal velocity. Figure 2 steadily decreases with time until it reaches 0, then it hits the ground. The time step used is $\Delta t = 1s$ and throughout the whole problem.

Euler's method matches nicely with the analytical prediction, as observed in figures 3 and 4. Euler's method overshoots the prediction, which is reduced by decreasing the time step.

The modified Euler's method reduces the overshoot in figures 1 and 3, as observed in figure 5. The overshoot is reduced, and the prediction has more undershooting, but is only noticeable with a bigger step size. It gives a better approximation to Euler's method because the undershoot is less than the overshoot.

In figures 9 and 10 the factor k/m is increased, which reduces overshoot. In figures 7 and 8, the step size is increased to 5s. As observed, the graphs don't describe the problem properly.

REFERENCES

[1] Dr. Simon Hanna. Level 5 Computational Physics. Lectures 3 and 4, 2019/2020.