1) a)

$$L = \begin{pmatrix} 0 & 0 & 0 \\ 5 & 0 & 0 \\ 4 & 2 & 0 \end{pmatrix} \quad R = \begin{pmatrix} 0 & 5 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad D^{-1} = \begin{pmatrix} \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{7} \end{pmatrix}$$

$$x^{(i+1)} = -D^{-1}\left((L+R)\,x^{(i)} - b\right)$$

$i = 0$

$$x^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad -\begin{pmatrix} \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{7} \end{pmatrix}\left(\left(\begin{pmatrix} 0 & 5 & 2 \\ 5 & 0 & 1 \\ 4 & 2 & 0 \end{pmatrix} x^0 - \begin{pmatrix} 19 \\ 5 \\ 34 \end{pmatrix}\right)\right)$$

$$= \begin{pmatrix} \frac{19}{8} \\ \frac{5}{9} \\ \frac{34}{7} \end{pmatrix}$$

$i = 1$

$$x^1 = \begin{pmatrix} \frac{19}{8} \\ \frac{5}{9} \\ \frac{34}{7} \end{pmatrix} \qquad -\begin{pmatrix} \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{7} \end{pmatrix}\left(\left(\begin{pmatrix} 0 & 5 & 2 \\ 5 & 0 & 1 \\ 4 & 2 & 0 \end{pmatrix} x^1 - \begin{pmatrix} 19 \\ 5 \\ 34 \end{pmatrix}\right)\right)$$

i	0	1	2	3	4	5	6
x^i	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 2.375 \\ 0.556 \\ 4.857 \end{pmatrix}$	$\begin{pmatrix} 0.813 \\ -1.304 \\ 3.341 \end{pmatrix}$	$\begin{pmatrix} 2.354 \\ -0.268 \\ 4.765 \end{pmatrix}$	$\begin{pmatrix} 1.351 \\ -1.282 \\ 3.588 \end{pmatrix}$	$\begin{pmatrix} 2.279 \\ -0.594 \\ 4.451 \end{pmatrix}$	$\begin{pmatrix} 1.633 \\ -1.205 \\ 3.724 \end{pmatrix}$

↳ konvergiert

b)

$$L = \begin{pmatrix} 0 & 0 & 0 \\ 5 & 0 & 0 \\ 4 & 2 & 0 \end{pmatrix} \quad R = \begin{pmatrix} 0 & 5 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad D^{-1} = \begin{pmatrix} \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{7} \end{pmatrix}$$

$$x^{(i+1)} = -D^{-1}\left((L+R)\,x^{(i)} - b\right)$$

$$x^0 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \qquad - \begin{pmatrix} \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{7} \end{pmatrix} \left(\left(\begin{pmatrix} 0 & 5 & 2 \\ 5 & 0 & 1 \\ 4 & 2 & 0 \end{pmatrix} x^i - \begin{pmatrix} 19 \\ 5 \\ 34 \end{pmatrix} \right) \right)$$

i	0	1	2	3	4
x^i	1	2.25	1.4405	2.2098	1.6882
	−1	−0.3333	−1.2024	−0.6521	−1.1585
	3	4.5714	3.6667	4.3776	3.7807

c)

$$\|x^{(3)} - \bar{x}\|_\infty \leq \frac{\|B\|_\infty}{1 - \|B\|_\infty} \cdot \|x^{(3)} - x^{(2)}\|_\infty$$

$$x^{(3)} - x^{(2)} = \begin{pmatrix} 0.7693 \\ 0.5502 \\ 0.7109 \end{pmatrix} \qquad \|x^{(3)} - x^{(2)}\|_\infty = 0.7693 = \frac{517}{672}$$

$$\frac{0.875}{1 - 0.875} \cdot \frac{517}{672} = 5.3854$$

$$x^{(3)} - \bar{x} = \begin{pmatrix} 0.2098 \\ 0.3479 \\ 0.3776 \end{pmatrix} \qquad \|x^{(3)} - \bar{x}\|_\infty = \frac{37}{98}$$

$$0.3776 \leq 5.3854$$

d)

$$\|x^{(n)} - \bar{x}\|_\infty \leq \frac{\|B\|_\infty^n}{1 - \|B\|_\infty} \cdot \|x^{(1)} - x^{(0)}\|_\infty \leq tol$$

$$B = -D^{-1}(L+R) = -\begin{pmatrix} \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{7} \end{pmatrix}\begin{pmatrix} 0 & 5 & 2 \\ 5 & 0 & 1 \\ 4 & 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\frac{5}{8} & -\frac{1}{4} \\ -\frac{5}{9} & 0 & -\frac{1}{9} \\ -\frac{4}{7} & -\frac{2}{7} & 0 \end{pmatrix}$$

$$\|B\|_\infty = 0.875$$

$$x^{(1)} - x^{(0)} = \begin{pmatrix} 1.25 \\ 0.6667 \\ 1.5714 \end{pmatrix}$$

$$\|x^{(1)} - x^{(0)}\|_\infty = 1.5714 = \frac{11}{7}$$

$$\frac{0.875^i}{1 - 0.875} \cdot \frac{11}{7} \leq 10^{-4}$$

↳ $i \geq 87.93 \rightarrow i = 88$

e)

$$x^{(3)} - x^{(2)} = \begin{pmatrix} 0.7693 \\ 0.5503 \\ 0.7109 \end{pmatrix}$$

$$\|x^{(3)} - x^{(2)}\|_\infty = 0.7693 = \frac{517}{672}$$

$$\frac{0.875^i}{1-0.875} \cdot \frac{517}{672} \leq 10^{-4}$$

$\hookrightarrow i \geq 82.88 \longrightarrow i = 83$