고급수학 및 연습 2 중간고사

(2010년 10월 16일 오후 1:00 - 3:00)

학번:	이름:

모든 문제의 답에 풀이과정을 명시하시오. (총점 200점)

Problem 1 (20pts). Find the (minimum) distance between $\{(x,y,z) \mid x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 14\}$ and $\{(x,y,z) \mid x+y+z = 15\}$.

Problem 2 (20pts). Find the second-degree approximate polynomial of the following function $f(x,y) = \arctan(e^{x+2y} - 1)$ at the origin $(0,0)$.

Problem 3 (20pts). Let S be the boundary of the region $\{(x,y,z) \mid x^2+y^2+z \le 2\} \cap \{(x,y,z) \mid \ x^2+y^2+z^2 \le 4\}$. Find the maximum and minimum values of $f(x,y,z) = xyz^2$ on S.

Problem 4 (20pts). Let

$$B = \{X = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : |X| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}} \le 1\}.$$

Let $f : B \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous function and let f be differentiable on $B_0 = \{X \in \mathbb{R}^n : |X| < 1\}$.

(a) If $\mathrm{grad} f = 0$ on B_0, show that f is constant on B_0.

(b) Suppose $f(X) = 0$ on $\{X \in \mathbb{R}^n : |X| = 1\}$. Show that there is a point $X_0 \in B_0$ for which $\mathrm{grad} f(X_0) = 0$.

Problem 5 (20pts). Let $A = (a_{ij})$ be a nonzero symmetric 3×3 matrix, that is, $a_{ij} = a_{ji}$ for any $1 \le i,j \le 3$. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function defined by $f(X) = \frac{1}{2} XAX^t = \frac{1}{2}\sum_{i,j=1}^{3} a_{ij}x_i x_j$ for any $X = (x_1, x_2, x_3) \in \mathbb{R}^3$. Here we consider a vector X as a 1×3 matrix.

(a) What is the $\mathrm{grad} f$?

(b) Let $S^2 = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$. Show that there must be a point $X \in S^2$ and a constant $\lambda \ne 0$ such that $AX^t = \lambda X^t$.

Problem 6 (20pts). For the vector field

$$\mathbf{F}(x,y) = (xe^{x^2+y^2} + 3xy, ye^{x^2+y^2} + x^2)$$

find the values of line integral $\int_C \mathbf{F} \cdot d\mathbf{s}$, where C is the triangular curve consisting of the line segments from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$, and from $(0,1)$ to $(0,0)$.

Problem 7 (20pts). Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1-function such that $G(0,0) = (0,1)$ and $G'(0,0) = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$. For the function $F(x,y) = (x^2+y^2+x, xy+3y)$, find the Jacobian determinant of $F \circ G$ at $(0,0)$.

Problem 8 (30pts). Let m and n be positive integers. Find all pairs (m,n) for which the function

$$f(x,y) = \begin{cases} \dfrac{x^m y^n}{x^2+y^4}, & (x,y) \ne (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is continuous. Justify your answer.

Problem 9 (30pts). Let $\mathbf{F}(X) = \dfrac{X}{|X|^2}$ be a vector field defined on $\mathbb{R}^3 - \{0\}$. Answer the following questions.

(a) Find a potential function of $\mathbf{F}$ if it has a potential function.

(b) For a curve

$$Y(t) = (\cos t, \sin t, t), \quad 0 \le t \le \frac{\pi}{2},$$

evaluate the values of line integral $\int_Y \mathbf{F} \cdot d\mathbf{s}$.