고급수학 및 연습 2 중간고사

(2014년 10월 18일 오후 1:00-3:00)

학번:	이름:

모든 문제의 답에 풀이과정을 명시하시오. (총점 200점)

Problem 1. [20 pts] Consider the function defined by

$$f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

(a) (5 pts) Is f continuous at $(0,0)$?

(b) (5 pts) What are $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$?

(c) (10 pts) Is f differentiable at $(0,0)$?

Problem 2. [20 pts] For the surface $z = e^x \sin y$, answer the following questions. $\left(\text{Here, } -\dfrac{\pi}{2} \leq x, y \leq \dfrac{\pi}{2}\right)$

(a) (10 pts) Find the equation of the tangent plane of the surface at the point $P\left(\log 3, \dfrac{\pi}{6}, \dfrac{3}{2}\right)$.

(b) (10 pts) Suppose the tangent plane at the point $Q(a,b,c)$ is normal to the line

$$x - 1 = y - 3 = \frac{z-5}{-\sqrt{2}}.$$

Find the point Q. Also, find the equation of the tangent plane at the point Q.

Problem 3. [20 pts] Let f, g be $\mathcal{C}^1$-functions defined on the plane. Answer the following questions.

(a) (10 pts) For $h(t) := f(tx, ty)$, Compute $h'(t)$.

(b) (10 pts) Suppose $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$ holds and put

$$\varphi(x,y) := \int_0^1 (xf(tx,ty) + yg(tx,ty))\,dt.$$

Express grad $\varphi(x,y)$ in terms of f and g.

Problem 4. [20 pts] Determine whether the function $f(x,y) = x^2 + y^2$ has a maximum or a minimum value on the constraint $x^3 + y^3 + x + y = 4$. If they exist, find them.

Problem 5. [20 pts] Answer the following questions.

(a) (10 pts) Find the third-degree Taylor polynomial of the function $f(x,y) = (\cos x)\log(1+y)$ at the origin.

(b) (10 pts) By using (a), find the third-degree approximate value of $(\cos 0.1)\log 1.1$ and show that its error is less than or equal to 4×10^{-4}.

Problem 6. [20 pts] For given $n \geq 3$, find the minimum area of an n-gon containing the unit circle.

Problem 7. [20 pts] For two $\mathcal{C}^1$-functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ and for two vectors $\mathbf{v} = (3, -2)$, $\mathbf{w} = (-2, 1)$ and for a point $P \in \mathbb{R}^2$, we have the following table.

$D_{\mathbf{v}}f(P)$	$D_{\mathbf{w}}f(P)$	$D_{\mathbf{v}}g(P)$	$D_{\mathbf{w}}g(P)$
2	0	-3	1

Find the Jacobian matrix of the function $F(x, y) = (f(x, y), g(x, y))$ at the point P.

Problem 8. [15 pts] Let S be the ellipse defined by

$$S \ : \ \frac{x^2}{9} + \frac{y^2}{4} = 1.$$

What is the distance from $P = (1, 0)$ to S ?

Problem 9. [25 pts] For the function defined on the plane

$$f(x,y) = \begin{cases} x^3 \sin\left(\dfrac{1}{x^2}\right) + y^3 \sin\left(\dfrac{1}{y^2}\right), & xy \neq 0 \\ x^3 \sin\left(\dfrac{1}{x^2}\right), & x \neq 0, y = 0 \\ y^3 \sin\left(\dfrac{1}{y^2}\right), & x = 0, y \neq 0 \\ 0, & x = y = 0 \end{cases}$$

answer the following questions and justify your answers.

(a) (8 pts) Find $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$.

(b) (8 pts) Is f differentiable at the origin ?

(c) (9 pts) Is f a $\mathcal{C}^1$-function ?

Problem 10. [20 pts] In spherical coordinates, show that

$$\nabla^2 f = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\varphi}\frac{\partial}{\partial \varphi}\left(\sin\varphi \frac{\partial f}{\partial \varphi}\right) + \frac{1}{\rho^2 \sin^2\varphi}\frac{\partial^2 f}{\partial \theta^2},$$

where $\nabla^2 := \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is a differential operator in rectangular coordinates.