고급수학 및 연습 2 중간고사

(2013년 10월 19일 오후 1:00-3:00)

학번:	이름:

모든 문제의 답에 풀이과정을 명시하시오. (총점 200점)

Problem 1. [30 pts] For $a > 0$, consider the one parameter family of parabolas $y = ax^2 - 1$ in the plane $\mathbb{R}^2$.

(a) (10 pts) Compute the distance from the parabola to the origin as a function of the parameter a. (Denote this function by f.)

(b) (5 pts) Draw the graph of f.

(c) (5 pts) Is f continuous?

(d) (10 pts) Is f differentiable?

Problem 2. [15 pts] Let $f(a, b)$ be the length of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Find the directional derivative $D_{\mathbf{v}} f(2, 2)$ for $\mathbf{v} = (1, 1)$.

Problem 3. [25 pts] Answer the following question.

(a) (10 pts) Find the third-degree Taylor polynomial of the function $f(x, y) = \cos x \sin y$ at the origin $(0, 0)$.

(b) (15 pts) Using (a), find the approximate value of $\cos 0.02 \sin 0.01$ within an error of less than 4×10^{-8}.

Problem 4. [20 pts] Let $\mathcal{F} = \{f \in C^2[0, 1] \mid f(0) = 0,\ f(1) = 2\}$ and consider the functional $A\colon \mathcal{F} \to \mathbb{R}$ given by

$$A(f) = \int_0^1 \sqrt{1 + f'(x)^2}\, dx.$$

(a) (15 pts) Using **Calculus of Variations**, find the minimum point of the functional A.

(b) (5 pts) Interpret your result geometrically.

Problem 5. [30 pts] Answer the following problems.

(a) (10 pts) Show that for any real number $t \neq 0$, there exists a unique real number λ such that

$$(1 - \lambda^2)^2 = t\lambda, \qquad -1 < \lambda < 1.$$

(b) (10 pts) Denote the above number by $\lambda(t)$. Find $\lambda\left(-\dfrac{9}{8}\right)$ and its derivative $\lambda'\left(-\dfrac{9}{8}\right)$.

(c) (10 pts) Find the point on the surface

$$S:\ x^2 + y^2 + 16 = z^2, \quad z > 0$$

which has the shortest distance from the point $P = \left(\dfrac{3}{2}, \dfrac{3}{2}, \dfrac{3}{\sqrt{2}}\right)$.

Problem 6. [15 pts] Find local maximum points, local minimum points, and saddle points of the following function.

$$f(x, y) = y \sin x + xy^2 - y^2$$

Problem 7. [20 pts] Answer the following problems.

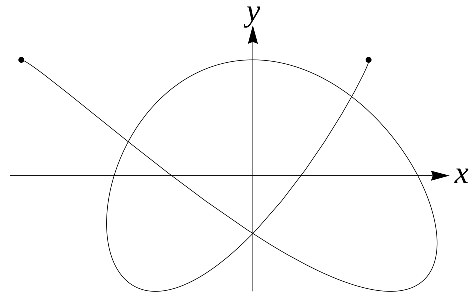
(a) (10 pts) Find the Jacobian determinant of the function $G(r, \theta) = r^2(\cos 2\theta, \sin 2\theta)$.

(b) (10 pts) Show that G has a local inverse F at $(1, 0)$ and find the Jacobian determinant of F at $G(1, 0)$.

Problem 8. [30 pts]

(a) (10 pts) Show that there is a unique C^1 function $f : (0, \infty) \to \mathbb{R}$ such that $f(1) = 1$ and $\mathbf{F}(x, y) = f(r)\,(-y, x)$ is a closed vector field on $\mathbb{R}^2 - \{(0,0)\}$, where $r = \sqrt{x^2 + y^2}$.

(b) (10 pts) For the above vector field $\mathbf{F}$, find the integral $\int_X \mathbf{F} \cdot d\mathbf{s}$, where $X(t) = (2^t \cos 3\pi t, \cos 4\pi t)$ for $0 \le t \le 1$. (The graph of the curve X is shown below)



(c) (5 pts) Does $\mathbf{F}$ have a potential function on $\mathbb{R}^2 - \{(0,0)\}$? If it has, find all potential functions.

(d) (5 pts) Does $\mathbf{F}$ have a potential function on $\{(x, y) \mid x > 0\}$? If it has, find all potential functions.

Problem 9. [15 pts] Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a linear function such that $D_i f(\mathbf{e}_i) = \dfrac{(-1)^i}{i}$ for $i = 1, \ldots, n$, where $\{\mathbf{e}_1, \ldots \mathbf{e}_n\}$ is the standard basis for $\mathbb{R}^n$. For a curve $X_n(t) = (t, t^2, \ldots, t^n)$, $0 \le t \le 1$, find the value of $\lim_{n \to \infty} \int_{X_n} df$.