고급수학 및 연습 2 중간고사

(2011년 10월 22일(토) 오후 1:00 - 3:00)

학번:	이름:

모든 문제의 답에 풀이과정을 명시하시오. (총점 200점)

Problem 1 (20pts)**.**

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

(a) (10pts) Find grad $f(0,0)$ if it exists.

(b) (10pts) Is f differentiable at the origin ?

Problem 2 (30pts)**.** Let F denote the map

$$\begin{aligned} F:\ \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (x, y+x^2) \end{aligned}$$

(a) (10pts) Compute the inverse of F.

(b) (10pts) Let $A := \{(x,y) \in \mathbb{R}^2 \mid x \in [-1,1], y \in [-1+x^2, 1+x^2]\}$. What is $F^{-1}(A)$?

(c) (10pts) Let f denote the function

$$\begin{aligned} f:\ \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x,y) &\longmapsto x^2+y^2-2x^2y+x^4. \end{aligned}$$

Find minimum and maximum of f on the set A.

Problem 3 (20pts)**.** For $t \geq 0$ compute the integral

$$\int_0^1 \frac{x^t-1}{\log x}\,dx.$$

Problem 4 (20pts)**.** Let $f(x,y,z) = x^4 + e^y + xyz^2$. Find the second-degree approximate polynomial $T_2 f\left((1,1,0),\ (x-1, y-1, z)\right)$ of f at the point $(1,1,0)$.

Problem 5 (30pts)**.** Let $f(x,y) = xy + \dfrac{1}{3}(x^3+y^3)$.

(a) (15pts) Find all critical points of f.

(b) (15pts) Determine the type (local maximum, local minimum or a saddle point) for each of the critical points in (a) .

Problem 6 (20pts)**.** If there exist, find the maximum value and the minimum value of the function $f(x,y,z) = x^2+2y^2+z^2$ restricted to the regions $xy+yz=1$.

Problem 7 (30pts)**.** Let $f : [0,1] \to \mathbb{R}$ be a C^1-function such that $f(0) = 0$, $f'(x) > 0$ for all $x \in (0,1)$ and let $g : \mathbb{R} \to \mathbb{R}$ be a positive C^1-function such that $g'(x) > 0$ for all $x \in \mathbb{R}$. Define the map

$$\begin{aligned} F : \mathbb{R} \times [0,1] &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (g(x)\cos(f(y)), g(x)\sin(f(y))). \end{aligned}$$

(a) (10pts) Compute the Jacobian matrix F'.
(b) (5pts) Show that F is locally invertible.
(c) (5pts) Is F surjective?
(d) (10pts) Give sufficient and necessary conditions for F to be an invertible map onto its image.

Problem 8 (30pts)**.** Evaluate the line integral $\int_X \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F}(x,y) = \dfrac{(-y,x)}{\sqrt{x^2+y^2}}$ and X is the following curve starting from $(1,0)$ to $(8,0)$ such that

- each path a_n is a circular arc of a circular sector of radius n with the central angle $\dfrac{\pi}{4}$,
- each b_n is a line segment of length 1.