고급수학 및 연습 1 기말고사

(2013년 6월 8일 오후 1:00-3:00)

학번:	이름:

모든 문제의 답에 풀이과정을 명시하시오. (총점 200점)

Problem 1 (25 pts)**.** Let $P_l(\mathbf{x}), P_m(\mathbf{x})$ be the orthogonal projections of the point $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ onto the lines $l : x = \dfrac{y}{2} = \dfrac{z}{3}$ and $m : x = \dfrac{y}{2} = \dfrac{z-4}{3}$, respectively.

(a) (10pts) Find the 3×3 matrix A satisfying $P_l(\mathbf{x}) = A\mathbf{x}$.

(b) (15pts) Show that $P_m(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for some constant vector $\mathbf{b}$ and find $\mathbf{b}$.

Problem 2 (20pts)**.** Let A be an $n \times n$ matrix. Suppose that $|A\mathbf{x}| = 2013|\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^n$.

(a) (10pts) Show that $A\mathbf{x} \cdot A\mathbf{y} = 2013^2\ \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

(b) (10pts) Find the value of $|\det A|$. (Hint : Use $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^t\mathbf{y}$.)

Problem 3 (20pts)**.** Let $\mathcal{M}$ be the set of all 2×2 matrices with real entries. When we identify a matrix

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}$$

with a vector $(a, b, c, d) \in \mathbb{R}^4$, show that the map

$$T : \mathcal{M} \to \mathcal{M}, \quad T(X) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} X$$

is linear and find the matrix corresponding to T.

Problem 4 (20pts)**.** Find the constant t satisfying the following equation.

$$\det \begin{pmatrix} 3a_1 + 5b_1 & 3a_2 + 5b_2 & 3a_3 + 5b_3 \\ 4b_1 + 5c_1 & 4b_2 + 5c_2 & 4b_3 + 5c_3 \\ 8c_1 + 5a_1 & 8c_2 + 5a_2 & 8c_3 + 5a_3 \end{pmatrix} = t\ \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

Problem 5 (20pts)**.** For points $A = (1, 1, 3),\ B = (2, 3, 2),\ C = (0, 2, 5) \in \mathbb{R}^3$, consider two line segments $L_1 = \overline{AB}$ and $L_2 = \overline{AC}$.

(a) (10pts) Find the equation of the plane containing L_1 and L_2.

(b) (10pts) Find the area of the parallelogram obtained by orthogonal projecting the parallelogram with two sides L_1, L_2 onto the plane $3x - 5y + z = 1$.

Problem 6 (20pts)**.** For a curve $X(t) = (t, t^2, t^3, t^4),\ (t > 0)$ on $\mathbb{R}^4$, let X_1, X_2, X_3, X_4 be mutually distinct points on this curve. Show that vectors $\overrightarrow{OX_i}\ (i = 1, 2, 3, 4)$ are linearly independent. $(O = (0, 0, 0, 0))$

Problem 7 (30pts)**.** For a curve on $\mathbb{R}^2$ given by $l : r = 1 + \cos\theta,\ (0 \le \theta \le 2\pi)$ in polar coordinates, consider the curve l' obtained by restricting l to $0 \le \theta < \pi$.

(a) (10pts) Parametrize the curve l' by arc length.

(b) (10pts) Find the point on the curve l' which divides l' into the two connected parts with the same length.

(c) (10pts) Find the center of the curve l.

Problem 8 (20pts)**.** Let a curve be given by $r = f(\theta)$ in polar coordinates. Show that the curvature of this curve is

$$\kappa(\theta) = \frac{|2(r')^2 - rr'' + r^2|}{\{(r')^2 + r^2\}^{3/2}}.$$

Problem 9 (25pts)**.** In a graph of the function $y = e^x$, find the point whose radius of curvature is minimal and the center of the osculating circle at that point.