고급수학 및 연습 2 중간고사

(2012년 10월 20일 오후 1:00-3:00)

학번:	이름:

모든 문제의 답에 풀이과정을 명시하시오. (총점 200점)

Problem 1. [25pts] Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as follows:

$$f(x,y) = \begin{cases} xy \sin \dfrac{1}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(a) (5pts) For a vector $\mathbf{v}$, find the directional derivative of f at $(0,0)$ in the direction of $\mathbf{v}$.

(b) (10pts) Determine whether f is differentiable at $(0,0)$ or not and explain your arguments.

(c) (10pts) Show that $D_1 f$ is discontinuous at $(0,0)$.

Problem 2. [20pts] Find local maximum points, local minimum points, and saddle points of the following function.

$$f(x,y) = \frac{1}{3}x^3 - x \sin y \quad (-\pi < y < \pi)$$

Problem 3. [25pts] Let a, b and c be positive real numbers.

(a) (15pts) Find the maximum value of $f(x,y,z) = x^a y^b z^c$ subject to the constraint given by $x + y + z = 1,\ x, y, z \geq 0$.

(b) (10pts) Using (a), deduce that if u, v, w are any positive real numbers, then

$$\left(\frac{u}{a}\right)^a \left(\frac{v}{b}\right)^b \left(\frac{w}{c}\right)^c \leq \left(\frac{u+v+w}{a+b+c}\right)^{a+b+c}.$$

Problem 4. [20pts] Consider the function

$$f(x,y) = \int_{\frac{\pi}{2}}^{x^2 y} \frac{\sin(xt)}{t}\, dt$$

which is defined on $\{(x,y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$.

Find the linear approximation of $f(x,y)$ at $\left(1, \dfrac{\pi}{2}\right)$.

Problem 5. [20pts] Suppose that a $\mathcal{C}^1$ function $F(x,y) = (f(x,y),\, g(x,y))$ satifies the following properties

(i) For all $(x,y) \in \mathbb{R}^2$, we have $g(x,y) = f(2x - y,\, y^2 - 4x)$.

(ii) The following table describes values of related functions at two points, $(0,0)$ and $(1,2)$.

point	f	g	$D_1 f$	$D_2 f$
$(0,0)$	2	2	5	2
$(1,2)$	1	2	4	6

Find the Jacobian matrix of F at $(1,2)$.

Problem 6. [20pts] Consider the following function $F(x,y,z) = (x^3,\, x + z^2,\, x + y^3 + z^5)$, which is written in rectangular coordinates. We transform the variables of F to spherical coordinates, and denote it as $G(\rho, \varphi, \theta)$. Find the net (infinitesimal) rate of change of volume of G at $(\rho, \varphi, \theta) = \left(1, \dfrac{\pi}{6}, \dfrac{\pi}{4}\right)$.

Problem 7. [20pts] Let $X(t)$ be a $\mathcal{C}^2$ curve on $\mathbb{R}^2$ such that $X(0) = P,\ X'(0) = \mathbf{v},\ X''(0) = \mathbf{a}$. Let $f(x,y)$ be a C^2 function with $\operatorname{grad} f(P) = \mathbf{w}$ and $f''(P)$ given by a matrix A. Calculate

$$\left.\frac{d^2}{dt^2}\right|_{t=0} f(X(t)).$$

Problem 8. [30pts] Consider the following vector fields which are defined on $\mathbb{R}^3$ except z-axis.

$$\mathbf{a}(x, y, z) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right), \quad \mathbf{F}(x, y, z) = \left(\frac{-y}{x^2+y^2} + e^x, \frac{x}{x^2+y^2}, 1\right)$$

Consider the curve $X(t) = (3\cos t,\ 2\sin t,\ t)$ $(0 \le t \le 2\pi)$ in $\mathbb{R}^3$.

(a) (10pts) Determine whether two vector fields $\mathbf{a}$, $\mathbf{F}$ are closed vector fields or not and explain your arguments.

(b) (10pts) Determine whether two vector fields $\mathbf{a}$, $\mathbf{F}$ have potential functions or not and explain your arguments.

(c) (10pts) Find the value of the line integral $\displaystyle\int_X \mathbf{F} \cdot d\mathbf{s}$.

Problem 9. [20pts] For the following vector field $\mathbf{F}$ and the parametrization X of a curve, find the value of the line integral $\int_X \mathbf{F} \cdot d\mathbf{s}$.

$$\mathbf{F}(x, y, z) = (e^x \log y,\ \frac{e^x}{y} - \cos z,\ y \sin z) \quad (y > 0)$$

$$X(t) = (\sin t \log t,\ e^t,\ \frac{t}{2})\ (\frac{\pi}{2} \le t \le \frac{3\pi}{2})$$