Chap 11. Fourier Analysis

11.1 Fourier Series

$f(x)$: periodic function $\Leftrightarrow$ $f(x+p) = f(x)$ for all x

$\nearrow$ period

$\nearrow$ the Fourier series of $f(x)$

$$\underline{f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)}: \text{periodic function}$$
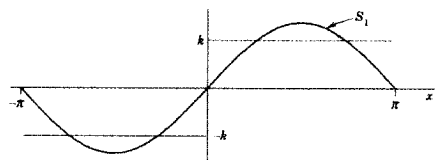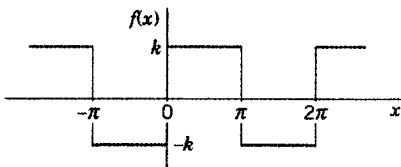
of period 2π

Euler
Formulas

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx, \quad n=1,2,\cdots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx, \quad n=1,2,\cdots$$

$\hookrightarrow$ Fourier coefficients

Ex 1: $f(x) = \begin{cases} -k & \text{if } -\pi < x < 0 \\ k & \text{if } 0 < x < \pi, \end{cases}$ and $f(x+2\pi) = f(x)$
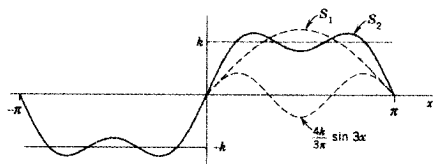
$\Rightarrow$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx = \begin{cases} \frac{4k}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$
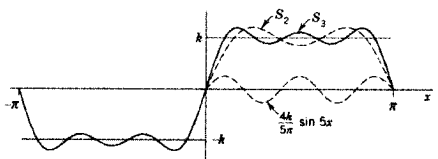

$$S_1 = \frac{4k}{\pi} \sin x$$

$$S_2 = \frac{4k}{\pi} \left(\sin x + \frac{1}{3} \sin 3x \right)$$

$\vdots$

Orthogonality

$1, \cos x, \sin x, \cos 2x, \sin 2x, \cdots$: Trigonometric system

is orthogonal

$$\langle \cos mx, \cos nx \rangle = \int_{-\pi}^{\pi} \cos mx \cdot \cos nx \, dx = 0 \quad \text{if } m \neq n$$

$$\langle \sin mx, \sin nx \rangle = \int_{-\pi}^{\pi} \sin mx \cdot \sin nx \, dx = 0 \quad \text{if } m \neq n$$

$$\langle \cos mx, \sin nx \rangle = \int_{-\pi}^{\pi} \cos mx \cdot \sin nx \, dx = 0$$

11.2 Arbitrary Period $(p = 2L)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x \right)$$

$$\begin{cases} a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx \\[2mm] a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx, \quad n=1,2,\cdots \\[2mm] b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx, \quad n=1,2,\cdots \end{cases}$$

Ex1: $f(x) = \begin{cases} 0 & \text{if } -2 < x < -1, \\ k & \text{if } -1 < x < 1, \\ 0 & \text{if } 1 < x < 2, \end{cases}$ $p = 2L = 4,$ $L = 2$

$\Rightarrow$ $\begin{cases} a_0 = \frac{1}{4} \int_{-2}^{2} f(x) \, dx = \frac{1}{4} \int_{-1}^{1} k \, dx = \frac{k}{2} \\[2mm] a_n = \frac{1}{2} \int_{-2}^{2} f(x) \cos \frac{n\pi x}{2} \, dx = \frac{1}{2} \int_{-1}^{1} k \cos \frac{n\pi x}{2} \, dx \\[2mm] \quad = \frac{2k}{n\pi} \sin \frac{n\pi}{2} = \begin{cases} 2k/n\pi & \text{if } n=1,5,9,\cdots \\ -2k/n\pi & \text{if } n=3,7,11,\cdots \end{cases} \\[2mm] b_n = 0 \end{cases}$

$$f(x) = \frac{k}{2} + \frac{2k}{\pi} \left(\cos \frac{\pi}{2}x - \frac{1}{3} \cos \frac{3\pi}{2}x + \frac{1}{5} \cos \frac{5\pi}{2}x \right.$$
$$\left. - \cdots \cdots \right)$$

Even and Odd Functions

$f(x)$: even function $\iff$ $f(-x) = f(x)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x \quad : \underline{\text{Fourier cosine series}}$$
$$\left(\text{even function of period } 2L\right)$$

$$\begin{cases} a_0 = \frac{1}{L} \int_0^L f(x)\,dx \\[2mm] a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\,dx , \quad n = 1, 2, \cdots \\[2mm] b_n = 0 \end{cases}$$

$f(x)$: odd function $\iff$ $f(-x) = -f(x)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \quad : \underline{\text{Fourier sine series}}$$
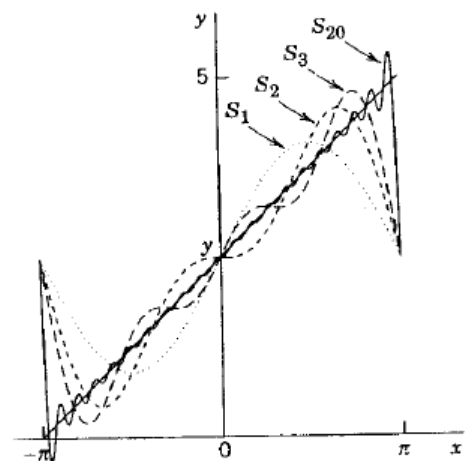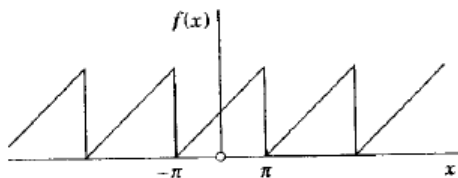$$\left(\text{odd function of period } 2L\right)$$

$$\begin{cases} a_n = 0 , \quad n = 0, 1, 2, \cdots \\[2mm] b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\,dx , \quad n = 1, 2, \cdots \end{cases}$$

Ex2: $f(x) = x + \pi$ if $-\pi < x < \pi$, $f(x + 2\pi) = f(x)$

$\Rightarrow f = f_1 + f_2$, where $\underline{f_1(x) = x}$, $f_2(x) = \pi$
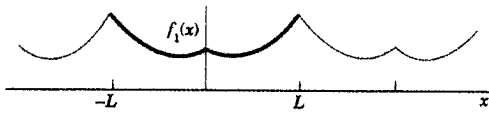$$\left(\begin{array}{l} \text{odd function} \\ \text{with } b_n = -\frac{2}{n}\cos n\pi \end{array}\right)$$

$\Rightarrow f(x) = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots \right)$
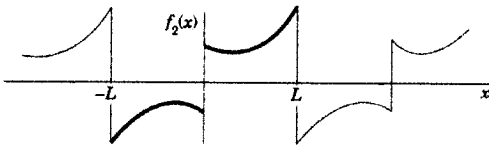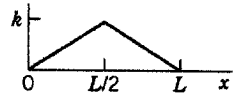
Half-Range Expansions
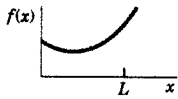
(0) The given function $f(x)$

(a) $f(x)$ continued as an **even** periodic function of period $2L$

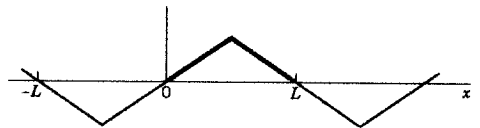(b) $f(x)$ continued as an **odd** periodic function of period $2L$

$\underline{Ex\,6:}$

$$f(x) = \begin{cases} \dfrac{2k}{L}x & \text{if} \quad 0 < x < \dfrac{L}{2} \\[3mm] \dfrac{2k}{L}(L-x) & \text{if} \quad \dfrac{L}{2} < x < L. \end{cases}$$

(a) Even extension

(b) Odd extension

$\curvearrowleft$

(Solution of Ex 6)

(a) Even periodic extension

$$a_0 = \frac{1}{L}\left[\frac{2k}{L}\int_0^{L/2} x\,dx + \frac{2k}{L}\int_{L/2}^L (L-x)\,dx\right] = \frac{k}{2}$$

$$a_n = \frac{2}{L}\left[\frac{2k}{L}\int_0^{L/2} x\cos\frac{n\pi}{L}x\,dx + \frac{2k}{L}\int_{L/2}^L (L-x)\cos\frac{n\pi}{L}x\,dx\right]$$

$$= \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right)$$

$$\Rightarrow f(x) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{L}x + \frac{1}{6^2}\cos\frac{6\pi}{L}x + \cdots\right)$$
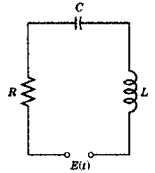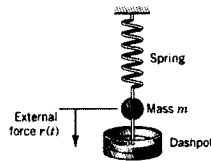
(b) Odd periodic extension

$$b_n = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\Rightarrow f(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x - \frac{1}{3^2}\sin\frac{3\pi}{L}x + \cdots\right)$$

11.3 Forced Oscillations

$$my'' + cy' + ky = r(t)$$

$$LI'' + RI' + \frac{1}{C} I = E'(t)$$

Ex1: Let $m = 1 (g)$, $c = 0.05$ (g/sec), $k = 25 (g/sec^2)$

$$y'' + 0.05 y' + 25 y = r(t)$$
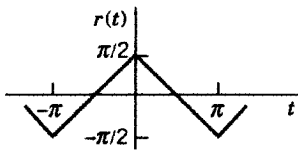
$$r(t) = \begin{cases} t + \dfrac{\pi}{2} & \text{if} \quad -\pi < t < 0. \\[2mm] -t + \dfrac{\pi}{2} & \text{if} \quad 0 < t < \pi, \end{cases} \qquad r(t + 2\pi) = r(t).$$

Representing $r(t)$ by a Fourier series:

$$r(t) = \frac{4}{\pi} \left(\cos t + \frac{1}{3^2} \cos 3t + \frac{1}{5^2} \cos 5t + \cdots \right)$$

Consider $y'' + 0.05 y' + 25 y = \dfrac{4}{n^2 \pi} \cos nt$ $(n = 1, 3, 5, \cdots)$

$$\Rightarrow \quad y_n = A_n \cos nt + B_n \sin nt,$$

$$A_n = \frac{4(25 - n^2)}{n^2 \pi D_n}, \quad B_n = \frac{0.2}{n \pi D_n},$$

where $D_n = (25 - n^2)^2 + (0.05n)^2$

$$\Rightarrow \quad y = y_1 + y_3 + y_5 + \cdots$$

11.4 Approximation by Trigonometric Polynomials
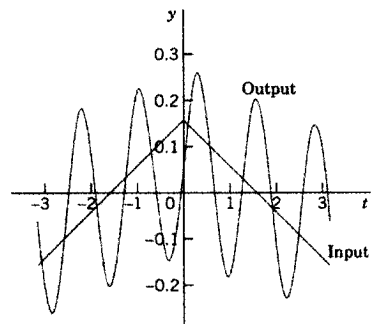
$f(x)$: a periodic function of period 2π that can be represented by a Fourier series

$$f(x) \approx a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) : \text{ Best Approximation ??}$$

$\hookrightarrow$ Fourier coefficients

Let $F(x) = A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx)$

$$\Rightarrow E = \int_{-\pi}^{\pi} (f - F)^2 dx = \int_{-\pi}^{\pi} f^2 dx - 2\int_{-\pi}^{\pi} f F dx + \int_{-\pi}^{\pi} F^2 dx$$

$$= \int_{-\pi}^{\pi} f^2 dx - 2\pi \left[2A_0 a_0 + \sum_{n=1}^{N} (A_n a_n + B_n b_n) \right]$$
$$+ \pi \left[2A_0^2 + \sum_{n=1}^{N} (A_n^2 + B_n^2) \right]$$

Take $A_n = a_n$, $B_n = b_n$,

Then $E^* = \int_{-\pi}^{\pi} f^2 dx - \pi \left[2a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2) \right]$

$$E - E^* = \pi \left\{ 2(A_0 - a_0)^2 + \sum_{n=1}^{N} \left[(A_n - a_n)^2 + (B_n - b_n)^2 \right] \right\} \geq 0$$

$\therefore E \geq E^*$

$E = E^* \iff A_0 = a_0, \ A_n = a_n, \ B_n = b_n, \ n=1,2,\cdots$

Since $E^* \geq 0$,

$$\underline{2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx} : \text{ Bessel Inequality}$$

__Ex1:__

$f(x) = x + \pi \quad (-\pi < x < \pi)$

$\Rightarrow F(x) = \pi + 2 \left(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x \right.$
$$\left. - + \cdots + \frac{(-1)^{N+1}}{N} \sin Nx \right)$$

$E^* = \int_{-\pi}^{\pi} (x+\pi)^2 dx - \pi \left(2\pi^2 + 4 \sum_{n=1}^{N} \frac{1}{n^2} \right)$

F with $N=20$

11.7 Fourier Integral

__Ex.1 (Square Wave)__

$$f_L(x) = \begin{cases} 0 & \text{if } -L < x < -1, \\ 1 & \text{if } -1 < x < 1, \\ 0 & \text{if } 1 < x < L, \end{cases} \qquad f_L(x+2L) = f_L(x)$$

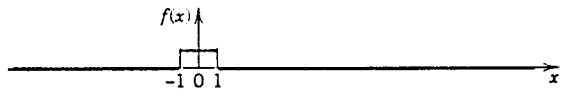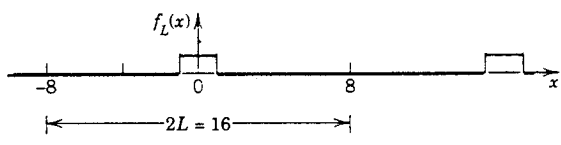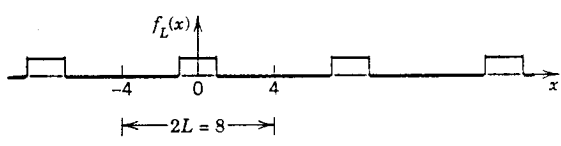$$f(x) = \lim_{L \to \infty} f_L(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$
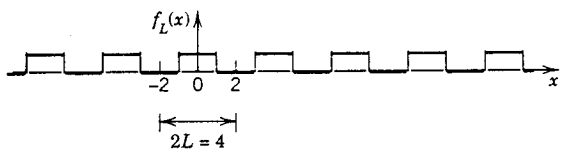
Since f_L is even, $b_n = 0$ for all n

$$a_0 = \frac{1}{2L} \int_{-1}^{1} dx = \frac{1}{L}$$

$$a_n = \frac{1}{L} \int_{-1}^{1} \cos \frac{n\pi x}{L} \, dx = \frac{2}{L} \int_{0}^{1} \cos \frac{n\pi x}{L} \, dx = \frac{2}{L} \frac{\sin(n\pi/L)}{n\pi/L}$$
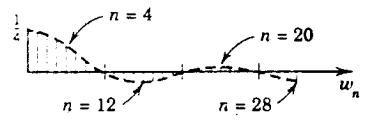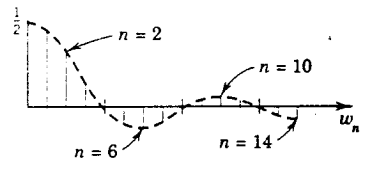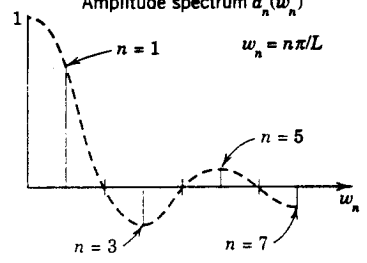
$\hookrightarrow$ amplitude spectrum of f_L

Fig. 254. Waveforms and amplitude spectra in Example 1

From Fourier Series to Fourier Integral

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n x + b_n \sin w_n x), \quad w_n = \frac{n\pi}{L}$$

$$f_L(x) = \frac{1}{2L} \int_{-L}^{L} f_L(v) dv$$

$$+ \frac{1}{L} \sum_{n=1}^{\infty} \left[\cos w_n x \int_{-L}^{L} f_L(v) \cos w_n v \, dv \right.$$

$$\left. + \sin w_n x \int_{-L}^{L} f_L(v) \sin w_n v \, dv \right]$$

Let $\Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$

$$\Rightarrow \quad \frac{1}{L} = \frac{\Delta w}{\pi}$$

$$f_L(x) = \frac{1}{2L} \int_{-L}^{L} f(v) dv$$

$$+ \frac{1}{\pi} \sum_{n=1}^{\infty} \left[(\cos w_n x) \Delta w \int_{-L}^{L} f_L(v) \cos w_n v \, dv \right.$$

$$\left. + (\sin w_n x) \Delta w \int_{-L}^{L} f_L(v) \sin w_n v \, dv \right]$$

assuming

$$f(x) = \lim_{L \to \infty} f_L(x) : \begin{array}{l} \text{absolutely} \\ \text{integrable} \end{array} \quad \left(\text{i.e., } \int_{-\infty}^{\infty} |f(x)| dx < \infty \right)$$

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \left[\cos wx \int_{-\infty}^{\infty} f(v) \cos wv \, dv \right.$$

$$\left. + \sin wx \int_{-\infty}^{\infty} f(v) \sin wv \, dv \right] dw$$

$$= \int_{0}^{\infty} \left[A(w) \cos wx + B(w) \sin wx \right] dw : \begin{array}{l} \text{Fourier} \\ \text{Integral} \end{array}$$

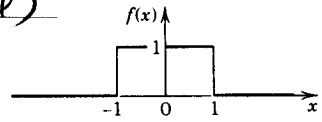where $A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv \, dv$

$$B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv \, dv$$

Applications

8x2 (Single Pulse, Sine Integral)

$$f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

$$\Rightarrow A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\, dv = \frac{2\sin w}{\pi w}$$

$$B(w) = \frac{1}{\pi}\int_{-1}^{1} 1\cdot \sin wv\, dv = 0$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_{0}^{\infty} \frac{\cos wx \, \sin w}{w}\, dw$$

$$\Rightarrow \int_{0}^{\infty} \frac{\cos wx \, \sin w}{w}\, dw = \begin{cases} \pi/2 & \text{if } 0 \le x < 1 \\ \pi/4 & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$$

$$\left[\int_{0}^{\infty} \frac{1}{2}\left(f(1-0)+f(1+0)\right) = \frac{1}{2}\right]$$

If $x=0$, $\displaystyle\int_{0}^{\infty} \frac{\sin w}{w}\, dw = \frac{\pi}{2}$.

In general,

$$\int_{0}^{a} \frac{\cos wx \, \sin w}{w}\, dw$$

$a = 8$

approximates

$$\int_{0}^{\infty} \frac{\cos wx \, \sin w}{w}\, dw$$

$a = 16$

as $a \to \infty$

$a = 32$
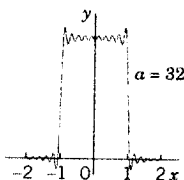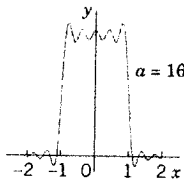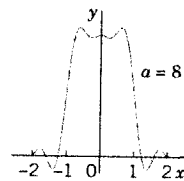
Fourier Cosine and Sine Integrals

If $f(x)$ is even, $B(w) = 0$

$$A(w) = \frac{2}{\pi} \int_0^\infty f(v) \cos wv \, dv$$

$$f(x) = \int_0^\infty A(w) \cos wx \, dw$$

If $f(x)$ is odd, $A(w) = 0$

$$B(w) = \frac{2}{\pi} \int_0^\infty f(v) \sin wv \, dv$$

$$f(x) = \int_0^\infty B(w) \sin wx \, dw$$

11.8 Fourier Cosine and Sine Transforms

Fourier Cosine Transforms

If $f(x)$ is even,

$$f(x) = \int_0^\infty A(w) \cos wx \, dw, \qquad A(w) = \frac{2}{\pi} \int_0^\infty f(v) \cos wv \, dv$$

Let $\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx$: Fourier cosine Transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(w) \cos wx \, dw$$

(Notation: $\mathcal{F}_c(f) = \hat{f}_c$)

Fourier Sine Transforms

If $f(x)$ is odd,

$$f(x) = \int_0^\infty B(w) \sin wx \, dw, \qquad B(w) = \frac{2}{\pi} \int_0^\infty f(v) \sin wv \, dv$$

Let $\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx$: Fourier Sine Transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(w) \sin wx \, dw$$

(Notation: $\mathcal{F}_s(f) = \hat{f}_s$)

Ex 1: $f(x) = \begin{cases} k & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

$\Rightarrow \hat{f_c}(w) = \sqrt{\frac{2}{\pi}} k \int_0^a \cos wx \, dx = \sqrt{\frac{2}{\pi}} k \left(\frac{\sin aw}{w} \right)$

$\hat{f_s}(w) = \sqrt{\frac{2}{\pi}} k \int_0^a \sin wx \, dx = \sqrt{\frac{2}{\pi}} k \left(\frac{1 - \cos aw}{w} \right)$

Linearity, Transforms of Derivatives

(a) $\mathcal{F}_c(af + bg) = a\mathcal{F}_c(f) + b\mathcal{F}_c(g)$

(b) $\mathcal{F}_s(af + bg) = a\mathcal{F}_s(f) + b\mathcal{F}_s(g)$

Th1: $f(x)$: conti. and absolutely integrable on the x-axis

$f'(x)$: piecewise conti. on each finite interval

$f(x) \to 0$ as $x \to \infty$

$\Rightarrow$ ⓐ $\mathcal{F}_c\{f'(x)\} = w \mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0)$,

ⓑ $\mathcal{F}_s\{f'(x)\} = -w \mathcal{F}_c\{f(x)\}$

$\Rightarrow$ ⓐ $\mathcal{F}_c\{f''(x)\} = -w^2 \mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0)$

ⓑ $\mathcal{F}_s\{f''(x)\} = -w^2 \mathcal{F}_s\{f(x)\} + \sqrt{\frac{2}{\pi}} w f(0)$

Ex 3

$f(x) = e^{-ax}$, where $a > 0$

$\Rightarrow (e^{-ax})'' = a^2 e^{-ax}$

$f''(x) = a^2 f(x)$

$\Rightarrow a^2 \mathcal{F}_c\{f(x)\} = \mathcal{F}_c\{f''(x)\} = -w^2 \mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0)$

$(a^2 + w^2) \mathcal{F}_c(f) = a \sqrt{\frac{2}{\pi}}$

$\therefore \mathcal{F}_c\{e^{-ax}\} = \sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2 + w^2} \right), \quad a > 0$

11.9. Fourier Transform

Complex Form of the Fourier Integral

$$f(x) = \int_0^\infty \left[A(\omega) \cos \omega x + B(\omega) \sin \omega x \right] d\omega$$

where
$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^\infty f(v) \cos \omega v \, dv$$

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^\infty f(v) \sin \omega v \, dv$$

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(v) \left[\cos \omega v \cos \omega x + \sin \omega v \sin \omega x \right] dv \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^\infty \left[\underbrace{\int_{-\infty}^\infty f(v) \cos(\omega x - \omega v) \, dv}_{\text{even function of } \omega} \right] d\omega$$

Note that
$$\frac{1}{2\pi} \int_{-\infty}^\infty \left[\underbrace{\int_{-\infty}^\infty f(v) \sin(\omega x - \omega v) \, dv}_{\text{odd function of } \omega} \right] d\omega = 0$$

$$\Rightarrow f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(v) \left[\cos(\omega x - \omega v) + i \sin(\omega x - \omega v) \right] dv \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(v) \cdot e^{i\omega(x-v)} \, dv \, d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(v) \cdot e^{-i\omega v} \, dv \right] e^{i\omega x} \, d\omega$$

$$\Rightarrow \text{Let}$$
$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x) \cdot e^{-i\omega x} \, dx \; : \; \text{Fourier Transform}$$

Then
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \hat{f}(\omega) \, e^{i\omega x} \, d\omega \; : \; \text{Inverse Fourier Transform}$$