Engineering Mathematics I
(Comp 400.001)

Midterm Exam, October 26, 2011

< Solution >

Problem	Score
1	
2	
3	
4	
5	
Total	

Name: ______________________

ID No: ______________________

Dept: ______________________

E-mail: ______________________

1. (25 points) A body of mass m is projected vertically upward with an initial velocity v_0 in a medium offering a resistance $k|v|$, where k is a constant. Assume that the gravitational force is constant.

   (a) (10 points) Find the velocity $v(t)$ of the body at any time.

   (b) (7 points) Find the time t_m at which the velocity vanishes: $v(t_m) = 0$.

   (c) (8 points) Find the maximum height h_m attained by the body.

(a) $m \cdot v'(t) = -kv - mg$ (+5), $\frac{dv}{dt} = -\frac{k}{m}\left(v + \frac{mg}{k}\right)$

$\ln\left|v + \frac{mg}{k}\right| = -\frac{k}{m}t + c^*$ (+2)

$v(t) + \frac{mg}{k} = c \cdot e^{-\frac{k}{m}t}, \quad (c = e^{c^*})$

$\therefore c = v_0 + \frac{mg}{k}$ (+2)

$v(t) = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-\frac{k}{m}t}$ (+1)

(b)

$0 + \frac{mg}{k} = \left(v_0 + \frac{mg}{k}\right)e^{-\frac{k}{m}\cdot t_m}$ (+4)

$\ln\left(\frac{mg}{kv_0 + mg}\right) = -\frac{k}{m} \cdot t_m$ (+2)

$\therefore t_m = \frac{m}{k}\ln\left(1 + \frac{kv_0}{mg}\right)$ (+1)

(c) $h_m = \int_0^{t_m} v(t)\,dt$ (+4)

$= -\frac{mg}{k} \cdot \frac{m}{k}\ln\left(1 + \frac{kv_0}{mg}\right) + \int_0^{t_m}\left(v_0 + \frac{mg}{k}\right)e^{-\frac{k}{m}t}\,dt$ (+2)

$= -\frac{m^2 g}{k^2}\ln\left(1 + \frac{kv_0}{mg}\right) + \frac{mv_0}{k}$ (+2)

2. (15 points) Show that the solution of the initial value problem:

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0, \quad y(0) = y_0, \ y'(0) = y_0',$$

can be expressed as the sum

$$y(x) = u(x) + v(x),$$

where

(a) $u(x)$ satisfies the initial conditions: $u(0) = y_0, u'(0) = 0$,

(b) $v(x)$ satisfies the initial conditions: $v(0) = 0, v'(0) = y_0'$, and

(c) both $u(x)$ and $v(x)$ satisfy the same differential equation as $y(x)$.

Let $y(x) = u(x) + v(x)$, +3

$\Rightarrow \quad y(0) = u(0) + v(0) = y_0 + 0 = y_0$ +3

$y'(0) = u'(0) + v'(0) = 0 + y_0' = y_0'$ +3

$$\begin{aligned} & y''(x) + p(x)y'(x) + q(x)y(x) \\ &= u''(x) + v''(x) + p(x)\left(u'(x) + v'(x)\right) + q(x)\left(u(x) + v(x)\right) \\ &= u''(x) + p(x)u'(x) + q(x)u(x) + v''(x) + p(x)v'(x) + q(x)v(x) \\ &= 0 + 0 = 0. \end{aligned}$$

+3

+3

3. (20 points) Solve the following initial value problem

$$xy'' + y' - \frac{y}{x} = \ln x, \quad y(1) = \frac{1}{8}, \; y'(1) = \frac{1}{8}.$$

$$x^2y'' + xy' - y = x\ln x \qquad (+3)$$

Let $y = x^m$, then $m(m-1) + m - 1 = 0$

$$\therefore m = \pm 1$$

$$y_1 = x, \quad y_2 = \frac{1}{x}. \qquad (+2)$$

$$y_h = c_1 x + c_2 \cdot \frac{1}{x}. \qquad (+2)$$

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix} = -\frac{1}{x} - \frac{1}{x} = -\frac{2}{x}. \qquad (+2)$$

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = \frac{\ln x}{x} = r(x). \qquad (+3)$$

$$y_p = -y_1 \int \frac{y_2 \cdot r}{W}\,dx + y_2 \int \frac{y_1 \cdot r}{W}\,dx$$

$$= \frac{1}{4}x(\ln x)^2 - \frac{1}{4}x\ln x + \frac{1}{8}x \qquad (+3)$$

$$y = c_1 x + c_2 \cdot \frac{1}{x} + \frac{1}{4}x(\ln x)^2 - \frac{1}{4}x\ln x + \frac{1}{8}x \qquad (+1)$$

$$y(1) = c_1 + c_2 + \frac{1}{8} = \frac{1}{8} \Rightarrow c_1 + c_2 = 0$$

$$y'(1) = c_1 - c_2 - \frac{1}{8} = \frac{1}{8} \Rightarrow c_1 - c_2 = \frac{2}{8} \qquad (+3)$$

$$\therefore y = \frac{1}{4}x - \frac{1}{8} \cdot \frac{1}{x} + \frac{1}{4}x(\ln x)^2 - \frac{1}{4}x\ln x$$

(+1)

4. (20 points) Solve the following system of ODEs:

$$y_1' = 5y_1 - y_2 + e^{2t}$$
$$y_2' = 3y_1 + y_2 + 3e^{2t}$$

$$\mathbf{y}' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2t} \quad +2$$

$$\det(A - \lambda I) = (\lambda - 5)(\lambda - 1) + 3 = \lambda^2 - 6\lambda + 8 = 0 \quad +2$$

$$\lambda_1 = 2, \ \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} ; \quad \lambda_2 = 4, \ \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad +2$$

$$\mathbf{y}^{(h)} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} \quad +2$$

$$\mathbf{y}^{(p)} = \mathbf{u} t e^{2t} \quad +3$$

$$\mathbf{y}^{(p)\prime} = \mathbf{u} e^{2t} + 2\mathbf{u} t e^{2t}$$
$$= A\mathbf{u} t e^{2t} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2t} \quad +3$$

$$A\mathbf{u} = 2\mathbf{u} \Rightarrow \mathbf{u} = \begin{bmatrix} a \\ 3a \end{bmatrix}, \ a \neq 0$$

$$\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad +4$$

$$\therefore \mathbf{y}^{(p)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} t e^{2t}$$

$$\therefore \mathbf{y} = \mathbf{y}^{(h)} + \mathbf{y}^{(p)} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} t e^{2t}$$

+2

4. (20 points) Solve the following system of ODEs:

$$y_1' = 5y_1 - y_2 + e^{2t}$$
$$y_2' = 3y_1 + y_2 + 3e^{2t}$$

$$y' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} y + \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2t}$$ +2

+2

$$\det(A - \lambda I) = (\lambda-5)(\lambda-1)+3 = \lambda^2 - 6\lambda + 8 = 0$$

$$\lambda_1 = 2,\ x^{(1)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix};\ \lambda_2 = 4,\ x^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ +2

$$y^{(1)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2t},\quad y^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$$ +2

+2

$$Y = \begin{bmatrix} e^{2t} & e^{4t} \\ 3e^{2t} & e^{4t} \end{bmatrix},\quad Y^{-1} = \frac{1}{-2e^{6t}} \begin{bmatrix} e^{4t} & -e^{4t} \\ -3e^{2t} & e^{2t} \end{bmatrix}$$

+2

$$u' = Y^{-1} g = \frac{1}{-2e^{6t}} \begin{bmatrix} e^{4t} & -e^{4t} \\ -3e^{2t} & e^{2t} \end{bmatrix} \begin{bmatrix} e^{2t} \\ 3e^{2t} \end{bmatrix}$$

+2

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$u(t) = \int_0^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} d\tilde{t} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$ +2

+2

$$Y u(t) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} t e^{2t}$$: particular solution

$$\therefore\ \underline{y = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} t e^{2t}}$$

+2

5. (20 points) The following Table compares the results of applying the Euler, improved Euler, and Runge-Kutta methods to the following initial value problem with $h = 0.2$:

$$y' = 4 - x + 2y, \quad y(0) = 1.$$

Fill in the three blanks in (A), (B), and (C), and show your work for partial credit.

x_i	Euler	Improved Euler	Runge-Kutta	Exact
0.0	1.0000	1.0000	1.0000	1.0000
0.2	2.2000	(B)	(C)	2.4525
0.4	(A)	4.4736	4.5695	4.5702
0.6	6.0960	7.4649	7.6786	7.6803
0.8	9.2144	11.8411	12.2675	12.2708

(A) $k_1 = hf(x_1, y_1) = 0.2[4 - 0.2 + 2 \times 2.2000]$
$= 1.6400$ +2

$\therefore y_2 = y_1 + k_1 = 3.8400$ +2

+2

(B) $k_1 = hf(x_0, y_0) = 0.2[4 - 0.0 + 2 \times 1.0000] = 1.2000$

$k_2 = hf(x_1, y_0 + k_1) = 0.2[4 - 0.2 + 2 \times 2.2000]$
$= 1.6400$ +2

$\therefore y_1 = y_0 + \frac{1}{2}[k_1 + k_2] = 2.4200$ +2

(C) $k_1 = hf(x_0, y_0) = 1.2000$ +2

$k_2 = hf(x_0 + 0.1, y_0 + 0.5k_1)$ +2
$= 0.2[4 - 0.1 + 2 \times 1.6] = 1.4200$

$k_3 = hf(x_0 + 0.1, y_0 + 0.5k_2]$ +2
$= 0.2[4 - 0.1 + 2 \times 1.71] = 1.4640$

$k_4 = hf(x_1, y_0 + k_3) = 0.2[4 - 0.2 + 2 \times 2.4640]$
$= 1.7456$ +2

$\therefore y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 2.4523$ +2