고급수학 및 연습 2 중간고사

(2009년 10월 17일 오후 1:00-3:00)

학번:	이름:

모든 문제의 답에 풀이과정을 명시하시오. (총점 200점)

1 (30 points). Let f be the function defined as follows:

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+|y|^3} & \text{if } (x,y) \neq (0,0). \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) (10 points) Is f continuous at $(0,0)$?

(b) (10 points) Find $D_{\mathbf{v}}f(0,0)$ when $\mathbf{v} = (1,1)$.

(c) (10 points) Is f differentiable at $(0,0)$?

2 (20 points). Let f be a differentiable function defined on an open set containing the sphere $S : x^2+y^2+z^2 = 14$ in $\mathbb{R}^3$. When f is restricted to S, f attains a maximum value at $P = (1,2,-3)$. Find the equation of the tangent plane at P of the level surface of f containing the point P. (Here $\text{grad} f(P) \neq (0,0,0)$.)

3 (20 points). Suppose that the Hessian of the $\mathcal{C}^2$ function $z = f(x,y)$ at P is $\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$. Find the second-order partial derivative $D_{\mathbf{v}}{}^2 f(P)$. And on the coordinate plane sketch the set of all vectors $\mathbf{v}$ such that $D_{\mathbf{v}}{}^2 f(P) > 0$.

4 (20 points). Find the second-degree Taylor polynomial of the function $f(x,y) = \log(2x+y+1)$ at the origin $(0,0)$.

5 (25 points). Explain that the maximum and minimum values of $x+y+z$ exists when $x^2+y^2+z^2+2x = 1$, $x \geq -1$. And find these values.

6 (25 points). In $\mathbb{R}^2$, Laplace's equation is

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

in rectangular coordinates. Show that we may express this using polar coordinates as follows:

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0\,.$$

7 (20 points). For the curve $X(t) = (\cos(2t), \sin(2t))\,(0 \leq t \leq 2\pi)$, evaluate the line integral $\displaystyle\int_X \mathbf{F}\cdot d\mathbf{s}$ of the vector field $\mathbf{F}(x,y) = \dfrac{(x-y, x+y)}{x^2+y^2}$ along the curve X.

8 (20 points). Answer the following problems.

(a) (10 points) Suppose that a differentiable function $g(r)$ of a single variable has the derivative which is never zero. Show that for a function $f(X) := g(|X|)$ $(X \in \mathbb{R}^3 - \{\mathbf{0}\})$, the gradient $\text{grad } f(X)$ is parallel to X.

(b) (10 points) Suppose that for a differentiable function $f(X)$, there exists a function $g(X)$ such that $\text{grad } f(X) = g(X)X$ for $X \in \mathbb{R}^n$. Show that $f(X)$ is constant on any sphere with the center at the origin.

(Hint: consider a spherical curve $X(t)$ connecting two points X_0, X_1 on the sphere.)

9 (20 points). Suppose the function $z = f(x,y)$ satisfies

$$x^2 y = z(z+y)(z-y),$$

where f is defined in some open set $U \subset \mathbb{R}^2$ containing $(\sqrt{6},1)$. Show that f is differentiable at $(\sqrt{6},1)$ and compute $\text{grad} f$ at the point $(\sqrt{6},1)$.