HW2 SOLUTION

9-34. Show that if σ is a cycle of odd length, then σ^2 is a cycle.
Proof. $\sigma = (1, 2, 3, \cdots, m)$, ($m$은 홀수)라 하면,

$$\sigma^2 = (1, 2, 3, \cdots, m)(1, 2, 3, \cdots, m) = (1, 3, 5, \cdots, m-2, m, 2, 4, 6, \cdots, m-1).$$

따라서 σ^2은 cycle이다. $\square$

10-40. Show that if a group G with identity e has finite order n, then $a^n = e$ for all $a \in G$.
Proof. $H =< a >$의 order를 d라 하면, Theorem of Lagrange에 의해 $|H| \mid |G|$이므로 $n = dm$라 하자.

$$a^n = (a^d)^m = e^m = e. \square$$

10-45. Show that a finite cyclic group of order n has exactly one subgroup of each order d dividing n, and that these are all the subgroups it has.
Proof. $G = \mathbb{Z}_n$만 고려해도 충분하고, $n = dm$라 하자. $\langle m \rangle = \{m, 2m, 3m, \cdots, (d-1)m, 0\}$이고 $|\langle m \rangle| = d$이다.
한편 $H \leq G$, $|H| = d$, $a \in H$라 하면 $da \equiv 0 \pmod{n}$이므로 $a \equiv 0 \pmod{m}$, 즉 $a \in \langle m \rangle$. 따라서 $H \subset \langle m \rangle$이고 $|H| = |\langle m \rangle|$이므로 $H = \langle m \rangle$이다. 즉 order d인 subgroup은 $\langle n/d \rangle$로 유일하며, Theorem of Lagrange에 의해 이게 모든 subgroups 된다. $\square$

10-46. The Euler phi-function is defined for positive integers n by $\varphi(n) = s$, where s is the number of positive integers less than or equal to n that are relatively prime to n. Use Exercise 45 to show that

$$n = \sum_{d|n} \varphi(d),$$

the sum being taken over all positive integers d dividing n.
Proof. 45번 풀이의 notation을 그대로 따른다. Cor 6.16에 의해 $\varphi(d) = \sharp[\text{generators of } \langle n/d \rangle]$. $\mathbb{Z}_n$의 원소는 $\sum_{d|n} \varphi(d)$에 의해 정확히 한 번씩 세어지므로, $\sum_{d|n} \varphi(d) = n$이다. $\square$

11-6. Find the order of the given element of the direct product.

$$(3, 10, 9) \in \mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{15}$$

Proof. order n이라 하면, 다음을 만족하는 최소의 자연수이다.

$$(3n, 10n, 9n) \equiv (0, 0, 0) \pmod{4, 12, 15}$$
$$\Leftrightarrow \quad 4 \mid n,\ 6 \mid n,\ 5 \mid n.$$

그러므로 $n = \text{lcm}(4, 6, 5) = 60$. $\square$

11-11. Find all subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_4$ of order 4.

Proof. $G = \mathbb{Z}_2 \times \mathbb{Z}_4$, $H \le G$, $|H| = 4$라 하자. Fundamental Theorem of Finitely Generated Abelian Groups에 의해

$$H \cong \mathbb{Z}_4 \text{ or } \mathbb{Z}_2 \times \mathbb{Z}_2.$$

(1) $H \cong \mathbb{Z}_4$인 경우
$a \in G$, $|a| = 4$를 찾으면 $(0,1)$, $(0,3)$, $(1,1)$, $(1,3)$이므로

$$H = \{(0,1),(0,2),(0,3),(0,0)\},\ \ \{(1,1),(0,2),(1,3),(0,0)\}.$$

(2) $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$인 경우
$a \in G$, $|a| = 2$를 찾으면 $(0,2)$, $(1,0)$, $(1,2)$이므로

$$H = \{(0,0),(0,2),(1,0),(1,2)\}.$$

(1), (2)에 의해 subgroups of order $= 4$는 3개 있다. $\square$

11-50. Let H and K be groups and let $G = H \times K$. Recall that both H and K appear as subgroups of G in a natural way. Show that these subgroups H (actually $H \times \{e\}$) and K (actually $\{e\} \times K$) have the following properties.
a. Every element of G is of the form hk for some $h \in H$ and $k \in K$.
b. $hk = kh$ for all $h \in H$ and $k \in K$.
c. $H \cap K = \{e\}$.
Proof. $(h,k) = (h,e)(e,k)$.
$(h,e)(e,k) = (h,k) = (e,k)(h,e)$.
$H \cap K = \{(h,e) \mid h \in H\} \cap \{(e,k) \mid k \in K\} = \{(e,e)\}$. $\square$

11-52. Show that a finite abelian group is not cyclic if and only if it contains a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p.
Proof. 주어진 finite abelian group을 G라 하자.
($\Leftarrow$) cyclic group의 모든 subgroup은 cyclic인데, $\mathbb{Z}_p \times \mathbb{Z}_p$는 cyclic이 아니므로 G는 cyclic이 아니다.
($\Rightarrow$) Theorem 11.12에 의해 G는 subgroup isomorphic to $\mathbb{Z}_{p^r} \times \mathbb{Z}_{p^s}$를 포함해야 한다.

$$\langle p^{r-1} \rangle \times \langle p^{s-1} \rangle \le \mathbb{Z}_{p^r} \times \mathbb{Z}_{p^s}$$

인 subgroup을 잡으면 $\mathbb{Z}_p \times \mathbb{Z}_p$와 isomorphic이다. $\square$

13-44. Let $\phi : G \to G'$ be a group homomorphism. Show that if $|G|$ is finite, then $|\phi[G]|$ is finite and is a divisor of $|G|$.
Proof. Theorem 13.15에 의해 $|G|/|\mathrm{Ker}(\phi)| = |\phi[G]|$. 따라서 성립. $\square$

13-45. Let $\phi : G \to G'$ be a group homomorphism. Show that if $|G'|$ is finite, then $|\phi[G]|$ is finite and is a divisor of $|G'|$.
Proof. Theorem 13.12에 의해 $\phi[G] \le G'$. Theorem of Lagrange에 의해 성립. $\square$

13-47. Show that any group homomorphism $\phi : G \to G'$ where $|G|$ is a prime must either be the trivial homomorphism or a one-to-one map.
Proof. Theorem 13.12에 의해 $\mathrm{Ker}(\phi) \leq G$. Theorem of Lagrange에 의해 $|\mathrm{Ker}(\phi)| = p$ or 1. $\Rightarrow \phi$는 trivial homo 또는 one-to-one map. $\square$

14-8. Find the order of the given factor group.

$$G = (\mathbb{Z}_{11} \times \mathbb{Z}_{15})/\langle(1,1)\rangle.$$

Proof. $(1,1)$ generates $\mathbb{Z}_{11} \times \mathbb{Z}_{15}$이므로 $|G| = 1$. $\square$

14-34. Show that if a finite group G has exactly one subgroup H of a given order, then H is a normal subgroup of G.
Proof. $\forall g \in G$, inner automorphism $i_g : G \to G$를 생각하자. 만약 $i_g(a) = e$라 하면,

$$gag^{-1} = e \Rightarrow ga = g \Rightarrow a = e,$$

이므로 $i_g : G \to G$는 one-to-one. 따라서 $i_g[H] \leq G$의 order는 $|i_g[H]| = |H|$이고, 조건에 의해

$$i_g[H] = H.$$

$gHg^{-1} = H$ for all $g \in G$이므로 H는 normal subgroup. $\square$

15-12. Classify the given group according to the fundamental theorem of finitely generated abelian groups.

$$G = (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})/\langle(3,3,3)\rangle.$$

Proof.

$$\begin{aligned} \phi : \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} &\to \mathbb{Z}_3 \times \mathbb{Z} \times \mathbb{Z} \\ (l,m,n) &\mapsto (\bar{l}, m-l, n-l) \end{aligned}$$

라 하면,

$$\begin{aligned} \phi((l_1,m_1,n_1)+(l_2,m_2,n_2)) &= \phi((l_1+l_2, m_1+m_2, n_1+n_2)) \\ &= (\overline{l_1+l_2}, m_1+m_2-l_1-l_2, n_1+n_2-l_1-l_2) \\ &= (\overline{l_1}, m_1-l_1, n_1-l_1) + (\overline{l_2}, m_2-l_2, n_2-l_2) \\ &= \phi((l_1,m_1,n_1)) + \phi((l_2,m_2,n_2)) \end{aligned}$$

이므로 group homomorphism이다. $\forall(\overline{a}, b, c) \in \mathbb{Z}_3 \times \mathbb{Z} \times \mathbb{Z}$에 대해,

$$\phi((a, b+a, c+a)) = (\overline{a}, b, c)$$

이므로 ϕ는 surjective이다.
$\mathrm{Ker}(\phi) = \{(3t,3t,3t) \mid t \in \mathbb{Z}\} = \langle(3,3,3)\rangle$이므로, The Fundamental Homomorphism Theorem에 의해, $G \cong \mathbb{Z}_3 \times \mathbb{Z} \times \mathbb{Z}$이다. $\square$

15-14. Find both the center and the commutator subgroup of $\mathbb{Z}_3 \times S_3$.

Proof. Example 15.19와 Example 15.21에 의해 $Z(S_3) = \{\rho_0\}$이고 $S_3' = A_3$임을 알고 있다.
(G' = the commutator subgroup of G 의미)
연산이 componentwise이므로 ceter는

$$Z(\mathbb{Z}_3 \times S_3) = Z(\mathbb{Z}_3) \times Z(S_3) = \mathbb{Z}_3 \times \{\rho_0\}$$

이고, commutator subgroup은

$$(\mathbb{Z}_3 \times S_3)' = \mathbb{Z}_3' \times S_3' = \{0\} \times A_3$$

가 된다. □