# Bits, Bytes, and Integers

Lecture 2
September 7th, 2017

Jae W. Lee (jaewlee@snu.ac.kr)
Computer Science and Engineering
Seoul National University

***Slide credits**: [CS:APP3e] slides from CMU; [COD5e] slides from Elsevier Inc.*

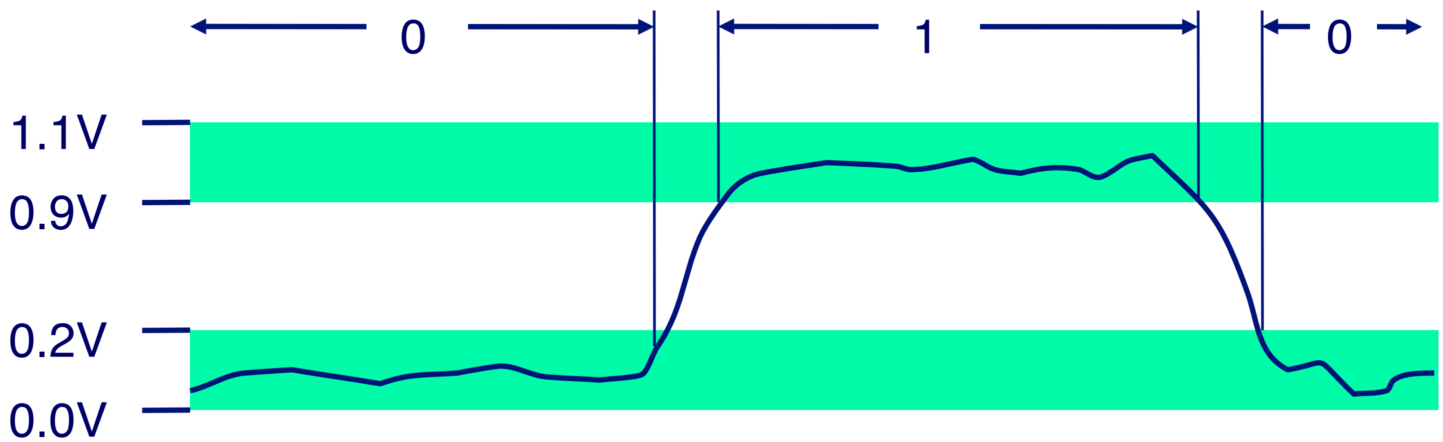
# Outline: Bits, Bytes, and Integers

**Textbook: [CS:APP3e] 2.1-2.3**

- **Representing information as bits**
- Bit-level manipulations
- Integers
  - Representation: unsigned and signed
  - Conversion, casting
  - Expanding, truncating
  - Addition, negation, multiplication, shifting
  - Summary
- Representations in memory, pointers, strings

# Everything is bits

- **Each bit is 0 or 1**
- **By encoding/interpreting sets of bits in various ways**
  - Computers determine what to do (instructions)
  - … and represent and manipulate numbers, sets, strings, etc…
- **Why bits? Electronic Implementation**
  - Easy to store with bistable elements
  - Reliably transmitted on noisy and inaccurate wires



# For example, can count in binary

- **Base 2 Number Representation**
  - Represent $15213_{10}$ as $11101101101101_2$
  - Represent $1.20_{10}$ as $1.0011001100110011[0011]..._2$
  - Represent $1.5213 \times 10^4$ as $1.1101101101101_2 \times 2^{13}$

# Encoding Byte Values

- **Byte = 8 bits**
  - Binary $00000000_2$ to $11111111_2$
  - Decimal: $0_{10}$ to $255_{10}$
  - Hexadecimal $00_{16}$ to $FF_{16}$
    - Base 16 number representation
    - Use characters ‘0’ to ‘9’ and ‘A’ to ‘F’
    - Write $FA1D37B_{16}$ in C as
      - 0xFA1D37B
      - 0xfa1d37b

| Hex | Decimal | Binary |
|---|---|---|
| 0 | 0 | 0000 |
| 1 | 1 | 0001 |
| 2 | 2 | 0010 |
| 3 | 3 | 0011 |
| 4 | 4 | 0100 |
| 5 | 5 | 0101 |
| 6 | 6 | 0110 |
| 7 | 7 | 0111 |
| 8 | 8 | 1000 |
| 9 | 9 | 1001 |
| A | 10 | 1010 |
| B | 11 | 1011 |
| C | 12 | 1100 |
| D | 13 | 1101 |
| E | 14 | 1110 |
| F | 15 | 1111 |

# Example Data Representations

| C Data Type | Typical 32-bit | Typical 64-bit | x86-64 |
|---|---|---|---|
| `char` | 1 | 1 | 1 |
| `short` | 2 | 2 | 2 |
| `int` | 4 | 4 | 4 |
| `long` | 4 | 8 | 8 |
| `float` | 4 | 4 | 4 |
| `double` | 8 | 8 | 8 |
| `long double` | − | − | 10/16 |
| pointer | 4 | 8 | 8 |

# Outline: Bits, Bytes, and Integers

**Textbook: [CS:APP3e] 2.1-2.3**

- Representing information as bits
- **Bit-level manipulations**
- Integers
  - Representation: unsigned and signed
  - Conversion, casting
  - Expanding, truncating
  - Addition, negation, multiplication, shifting
  - Summary
- Representations in memory, pointers, strings

# Boolean Algebra

- **Developed by George Boole in 19th Century**
  - Algebraic representation of logic
    - Encode "True" as 1 and "False" as 0

## And

- A&B = 1 when both A=1 and B=1

| & | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

## Or

- A|B = 1 when either A=1 or B=1

| \| | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 1 |

## Not

- ~A = 1 when A=0

| ~ | |
|---|---|
| 0 | 1 |
| 1 | 0 |

## Exclusive-Or (Xor)

- A^B = 1 when either A=1 or B=1, but not both

| ^ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |

# General Boolean Algebras

- **Operate on Bit Vectors**
  - Operations applied bitwise

```
  01101001     01101001     01101001
& 01010101   | 01010101   ^ 01010101   ~ 01010101
  01000001     01111101     00111100     10101010
```

- **All of the Properties of Boolean Algebra Apply**

# Example: Representing & Manipulating Sets

- **Representation**
  - Width w bit vector represents subsets of {0, …, w–1}
  - $a_j = 1$ if $j \in A$
    - 01101001 { 0, 3, 5, 6 }
    - *76543210*
    - 01010101 { 0, 2, 4, 6 }
    - *76543210*
- **Operations**

| | | | |
|---|---|---|---|
| & | Intersection | 01000001 | { 0, 6 } |
| \| | Union | 01111101 | { 0, 2, 3, 4, 5, 6 } |
| ^ | Symmetric difference | 00111100 | { 2, 3, 4, 5 } |
| ~ | Complement | 10101010 | { 1, 3, 5, 7 } |

# Bit-Level Operations in C

- **Operations &, |, ~, ^ Available in C**
  - Apply to any "integral" data type
    - `long, int, short, char, unsigned`
  - View arguments as bit vectors
  - Arguments applied bit-wise
- **Examples (Char data type)**
  - ~0x41 → 0xBE
    - ~$01000001_2$ → $10111110_2$
  - ~0x00 → 0xFF
    - ~$00000000_2$ → $11111111_2$
  - 0x69 & 0x55 → 0x41
    - $01101001_2$ & $01010101_2$ → $01000001_2$
  - 0x69 | 0x55 → 0x7D
    - $01101001_2$ | $01010101_2$ → $01111101_2$

# Contrast: Logic Operations in C

- **Contrast to Logical Operators**
  - `&&, ||, !`
    - View 0 as "False"
    - Anything nonzero as "True"
    - Always return 0 or 1
    - Early termination
- **Examples (char data type)**
  - `!0x41` → `0x00`
  - `!0x00` → `0x01`
  - `!!0x41` → `0x01`
  - `0x69 && 0x55` → `0x01`
  - `0x69 || 0x55` → `0x01`
  - `p && *p` (avoids null pointer access)

# Contrast: Logic Operations in C

- **Contrast to Logical Operators**
  - `&&, ||, !`
    - View 0 as "Fal
    - Anything nonze
    - Alway
    - Early
- **Example**
  - `!0x41`
  - `!0x00`
  - `!!0x41`
  - `0x69 && 0x55` → `0x01`
  - `0x69 || 0x55` → `0x01`
  - `p && *p` (avoids null pointer access)

**Watch out for && vs. & (and || vs. |)… one of the more common oopsies in C programming**

# Shift Operations

- **Left Shift: `x << y`**
  - Shift bit-vector **`x`** left **`y`** positions
    - – Throw away extra bits on left
    - Fill with 0's on right
- **Right Shift: `x >> y`**
  - Shift bit-vector **`x`** right **`y`** positions
    - Throw away extra bits on right
  - Logical shift
    - Fill with 0's on left
  - Arithmetic shift
    - Replicate most significant bit on left
- **Undefined Behavior**
  - Shift amount < 0 or ≥ word size

| | |
|---|---|
| **Argument** `x` | `01100010` |
| `<< 3` | `00010000` |
| **Log.** `>> 2` | `00011000` |
| **Arith.** `>> 2` | `00011000` |

| | |
|---|---|
| **Argument** `x` | `10100010` |
| `<< 3` | `00010000` |
| **Log.** `>> 2` | `00101000` |
| **Arith.** `>> 2` | `11101000` |

# Outline: Bits, Bytes, and Integers

**Textbook: [CS:APP3e] 2.1-2.3**

- Representing information as bits
- Bit-level manipulations
- **Integers**
  - **Representation: unsigned and signed**
  - Conversion, casting
  - Expanding, truncating
  - Addition, negation, multiplication, shifting
  - Summary
- Representations in memory, pointers, strings
- Summary

# Encoding Integers

**Unsigned**

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

**Two's Complement**

$$B2T(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

**Sign Bit**

```
short int x =  15213;
short int y = -15213;
```

- **C `short` 2 bytes long**

| | Decimal | Hex | Binary |
|---|---|---|---|
| `x` | 15213 | `3B 6D` | `00111011 01101101` |
| `y` | -15213 | `C4 93` | `11000100 10010011` |

- **Sign Bit**
  - For 2's complement, most significant bit indicates sign
    - 0 for nonnegative
    - 1 for negative

# Two-complement Encoding Example (Cont.)

```
x =      15213: 00111011 01101101
y =     -15213: 11000100 10010011
```

| Weight | 15213 | | -15213 | |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 0 | 1 | 2 |
| 4 | 1 | 4 | 0 | 0 |
| 8 | 1 | 8 | 0 | 0 |
| 16 | 0 | 0 | 1 | 16 |
| 32 | 1 | 32 | 0 | 0 |
| 64 | 1 | 64 | 0 | 0 |
| 128 | 0 | 0 | 1 | 128 |
| 256 | 1 | 256 | 0 | 0 |
| 512 | 1 | 512 | 0 | 0 |
| 1024 | 0 | 0 | 1 | 1024 |
| 2048 | 1 | 2048 | 0 | 0 |
| 4096 | 1 | 4096 | 0 | 0 |
| 8192 | 1 | 8192 | 0 | 0 |
| 16384 | 0 | 0 | 1 | 16384 |
| -32768 | 0 | 0 | 1 | -32768 |
| **Sum** | | **15213** | | **-15213** |

# Numeric Ranges

- **Unsigned Values**
  - *UMin* = 0
    000…0
  - *UMax* = $2^w - 1$
    111…1

- **Two's Complement Values**
  - *TMin* = $-2^{w-1}$
    100…0
  - *TMax* = $2^{w-1} - 1$
    011…1

- **Other Values**
  - Minus 1
    111…1

**Values for $W$ = 16**

| | Decimal | Hex | Binary |
|---|---|---|---|
| UMax | 65535 | FF FF | 11111111 11111111 |
| TMax | 32767 | 7F FF | 01111111 11111111 |
| TMin | -32768 | 80 00 | 10000000 00000000 |
| -1 | -1 | FF FF | 11111111 11111111 |
| 0 | 0 | 00 00 | 00000000 00000000 |

# Values for Different Word Sizes

| | W | | | |
|---|---|---|---|---|
| | **8** | **16** | **32** | **64** |
| **UMax** | 255 | 65,535 | 4,294,967,295 | 18,446,744,073,709,551,615 |
| **TMax** | 127 | 32,767 | 2,147,483,647 | 9,223,372,036,854,775,807 |
| **TMin** | -128 | -32,768 | -2,147,483,648 | -9,223,372,036,854,775,808 |

- **Observations**
  - $|TMin| = TMax + 1$
    - Asymmetric range
  - $UMax = 2 * TMax + 1$
- **C Programming**
  - #include <limits.h>
  - Declares constants, e.g.,
    - ULONG_MAX
    - LONG_MAX
    - LONG_MIN
  - Values platform specific

# Unsigned & Signed Numeric Values

| *X* | B2U(*X*) | B2T(*X*) |
|---|---|---|
| 0000 | 0 | 0 |
| 0001 | 1 | 1 |
| 0010 | 2 | 2 |
| 0011 | 3 | 3 |
| 0100 | 4 | 4 |
| 0101 | 5 | 5 |
| 0110 | 6 | 6 |
| 0111 | 7 | 7 |
| 1000 | 8 | –8 |
| 1001 | 9 | –7 |
| 1010 | 10 | –6 |
| 1011 | 11 | –5 |
| 1100 | 12 | –4 |
| 1101 | 13 | –3 |
| 1110 | 14 | –2 |
| 1111 | 15 | –1 |

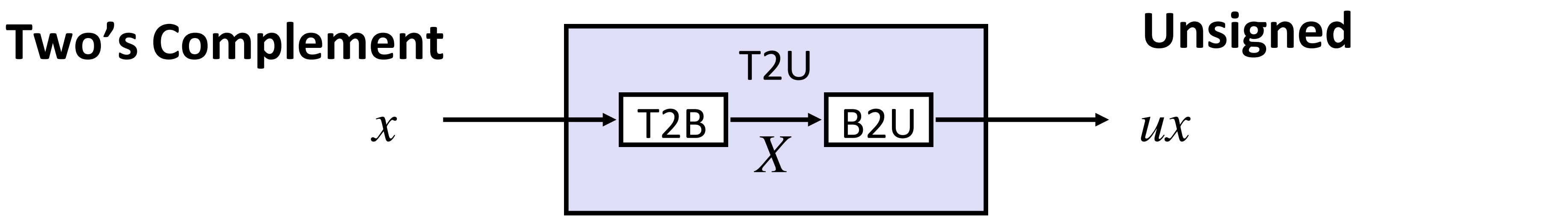
- **Equivalence**
  - Same encodings for nonnegative values
- **Uniqueness**
  - Every bit pattern represents unique integer value
  - Each representable integer has unique bit encoding
- **⇒ Can Invert Mappings**
  - U2B($x$) = B2U$^{-1}$($x$)
    - Bit pattern for unsigned integer
  - T2B($x$) = B2T$^{-1}$($x$)
    - Bit pattern for two's comp integer

# Outline: Bits, Bytes, and Integers

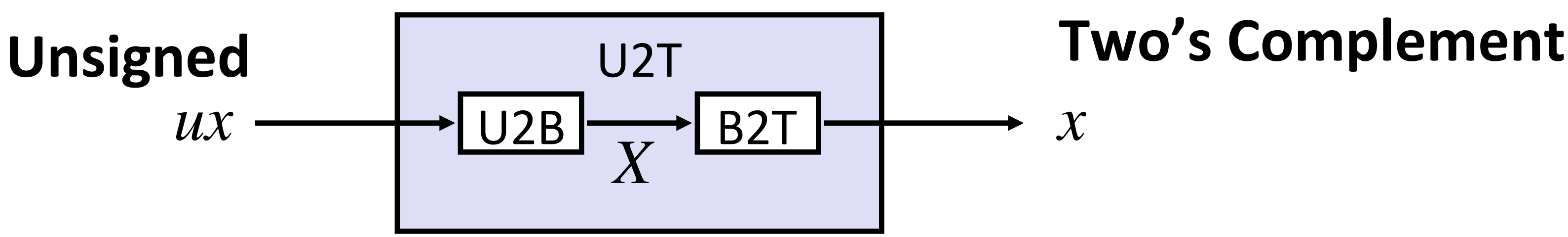
**Textbook: [CS:APP3e] 2.1-2.3**

- Representing information as bits
- Bit-level manipulations
- **Integers**
  - Representation: unsigned and signed
  - **Conversion, casting**
  - Expanding, truncating
  - Addition, negation, multiplication, shifting
  - Summary
- Representations in memory, pointers, strings

# Mapping Between Signed & Unsigned




Maintain Same Bit Pattern




Maintain Same Bit Pattern

- **Mappings between unsigned and two's complement numbers:**
  **Keep bit representations and reinterpret**

# Mapping Signed ↔ Unsigned

| Bits | Signed | Unsigned |
|---|---|---|
| 0000 | 0 | 0 |
| 0001 | 1 | 1 |
| 0010 | 2 | 2 |
| 0011 | 3 | 3 |
| 0100 | 4 | 4 |
| 0101 | 5 | 5 |
| 0110 | 6 | 6 |
| 0111 | 7 | 7 |
| 1000 | -8 | 8 |
| 1001 | -7 | 9 |
| 1010 | -6 | 10 |
| 1011 | -5 | 11 |
| 1100 | -4 | 12 |
| 1101 | -3 | 13 |
| 1110 | -2 | 14 |
| 1111 | -1 | 15 |

Signed $\xrightarrow{\text{T2U}}$ Unsigned

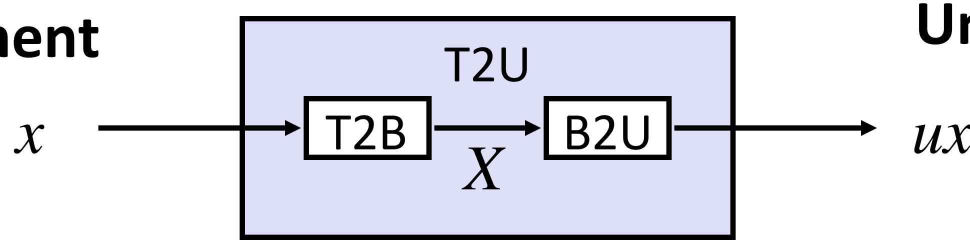
Signed $\xleftarrow{\text{U2T}}$ Unsigned

# Mapping Signed ↔ Unsigned

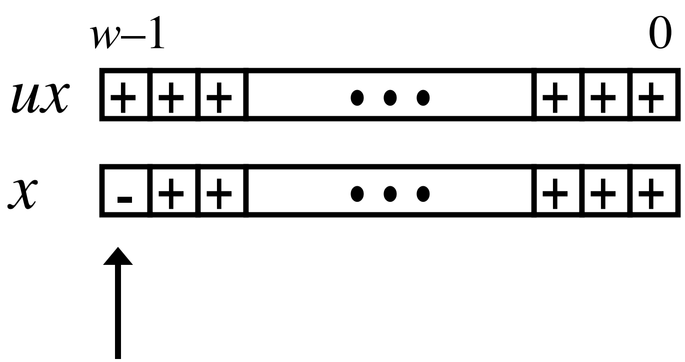
| Bits | Signed | | Unsigned |
|---|---|---|---|
| 0000 | 0 | | 0 |
| 0001 | 1 | | 1 |
| 0010 | 2 | | 2 |
| 0011 | 3 | | 3 |
| 0100 | 4 | = | 4 |
| 0101 | 5 | | 5 |
| 0110 | 6 | | 6 |
| 0111 | 7 | | 7 |
| 1000 | -8 | | 8 |
| 1001 | -7 | | 9 |
| 1010 | -6 | | 10 |
| 1011 | -5 | +/- 16 | 11 |
| 1100 | -4 | | 12 |
| 1101 | -3 | | 13 |
| 1110 | -2 | | 14 |
| 1111 | -1 | | 15 |

# Relation between Signed & Unsigned

**Two's Complement** **Unsigned**




Maintain Same Bit Pattern



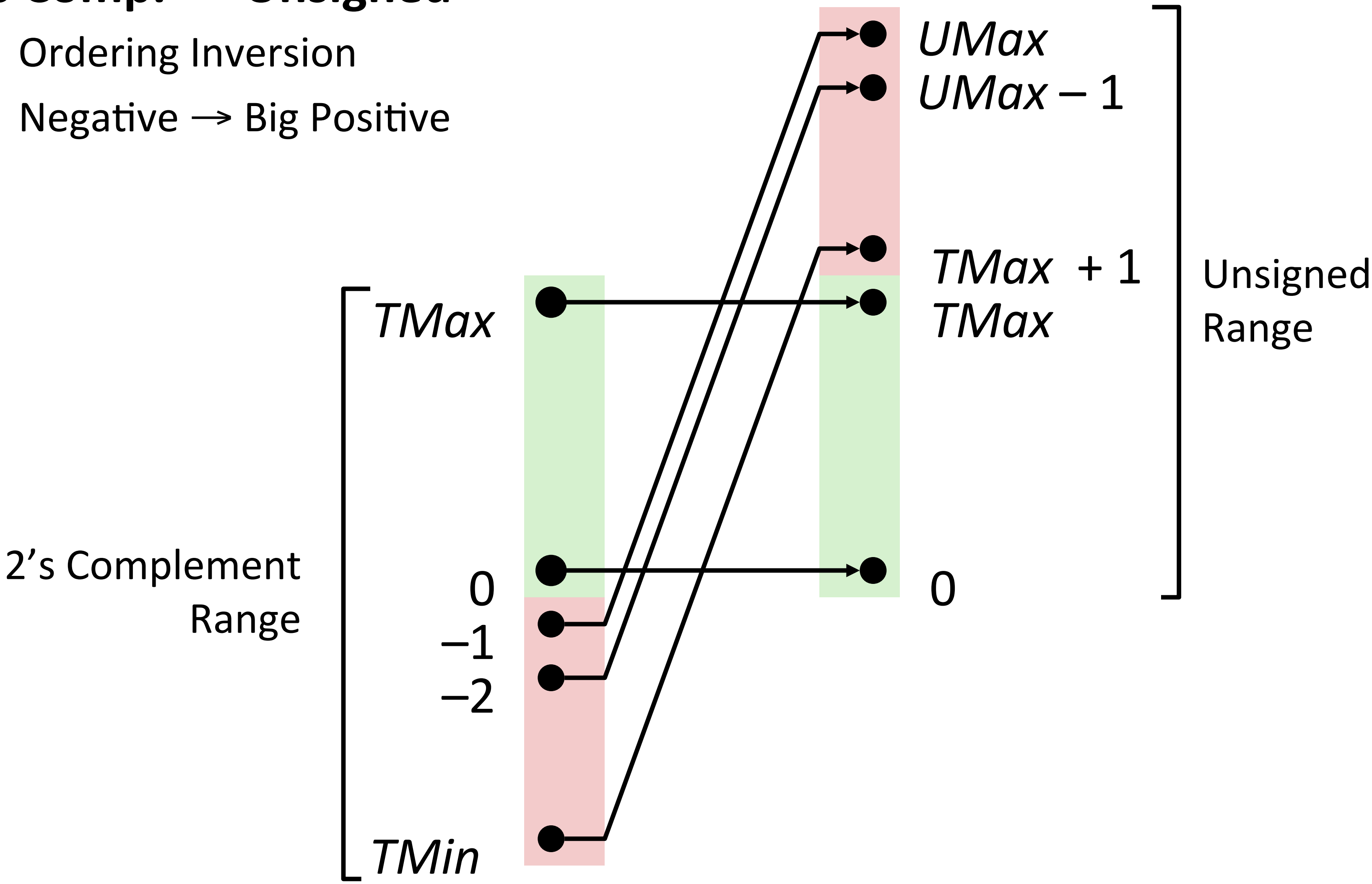

**Large negative weight**

*becomes*

**Large positive weight**

# Conversion Visualized

- **2's Comp. → Unsigned**
  - Ordering Inversion
  - Negative → Big Positive



# Signed vs. Unsigned in C

- **Constants**
  - By default are considered to be signed integers
  - Unsigned if have "U" as suffix

```
0U, 4294967259U
```

- **Casting**
  - Explicit casting between signed & unsigned same as U2T and T2U

```
int tx, ty;
unsigned ux, uy;
tx = (int) ux;
uy = (unsigned) ty;
```

  - Implicit casting also occurs via assignments and procedure calls

```
tx = ux;
uy = ty;
```

# Casting Surprises

- **Expression Evaluation**
  - If there is a mix of unsigned and signed in single expression, ***signed values implicitly cast to unsigned***
  - Including comparison operations **<, >, ==, <=, >=**
  - Examples for *W* = 32: **TMIN = -2,147,483,648 , TMAX = 2,147,483,647**

| Constant$_1$ | Constant$_2$ | Relation | Evaluation |
|---|---|---|---|
| 0 | 0U | **==** | **unsigned** |
| -1 | 0 | **<** | **signed** |
| -1 | 0U | **>** | **unsigned** |
| 2147483647 | -2147483647-1 | **>** | **signed** |
| 2147483647U | -2147483647-1 | **<** | **unsigned** |
| -1 | -2 | **>** | **signed** |
| (unsigned)-1 | -2 | **>** | **unsigned** |
| 2147483647 | 2147483648U | **<** | **unsigned** |
| 2147483647 | (int) 2147483648U | **>** | **signed** |

# Summary
# Casting Signed ↔ Unsigned: Basic Rules

- Bit pattern is maintained
- But reinterpreted
- Can have unexpected effects: adding or subtracting $2^w$

- Expression containing signed and unsigned int
  - `int` is cast to `unsigned`!!

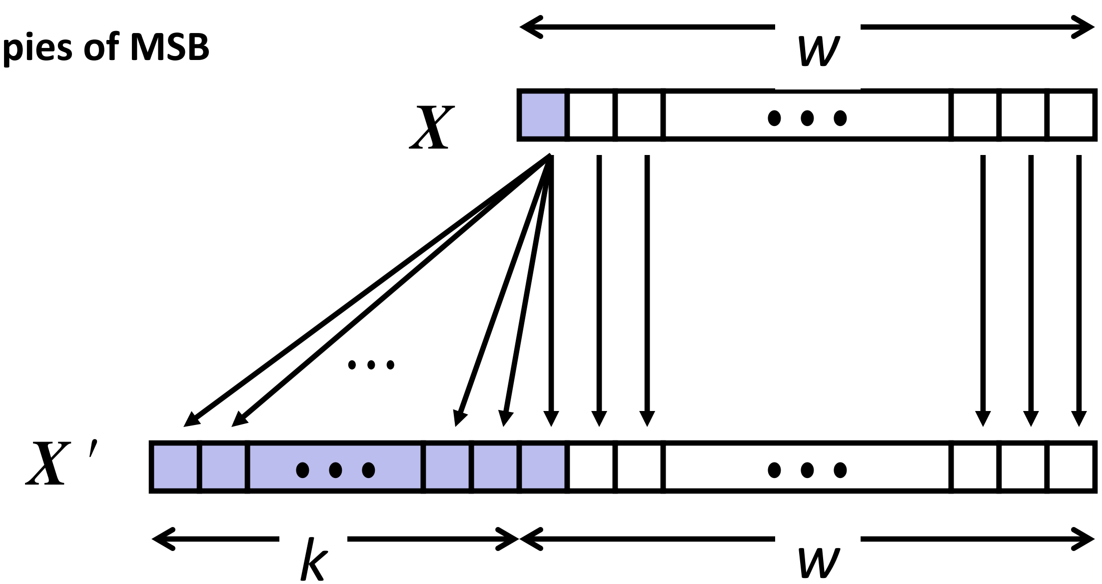
# Outline: Bits, Bytes, and Integers

**Textbook: [CS:APP3e] 2.1-2.3**

- Representing information as bits
- Bit-level manipulations
- **Integers**
  - Representation: unsigned and signed
  - Conversion, casting
  - **Expanding, truncating**
  - Addition, negation, multiplication, shifting
  - Summary
- Representations in memory, pointers, strings

# Sign Extension

- **Task:**
  - Given $w$-bit signed integer $x$
  - Convert it to $w$+$k$-bit integer with same value
- **Rule:**
  - Make $k$ copies of sign bit:
  - $X' = \underbrace{x_{w-1}, \ldots, x_{w-1}}_{k \text{ copies of MSB}}, x_{w-1}, x_{w-2}, \ldots, x_0$



# Sign Extension Example

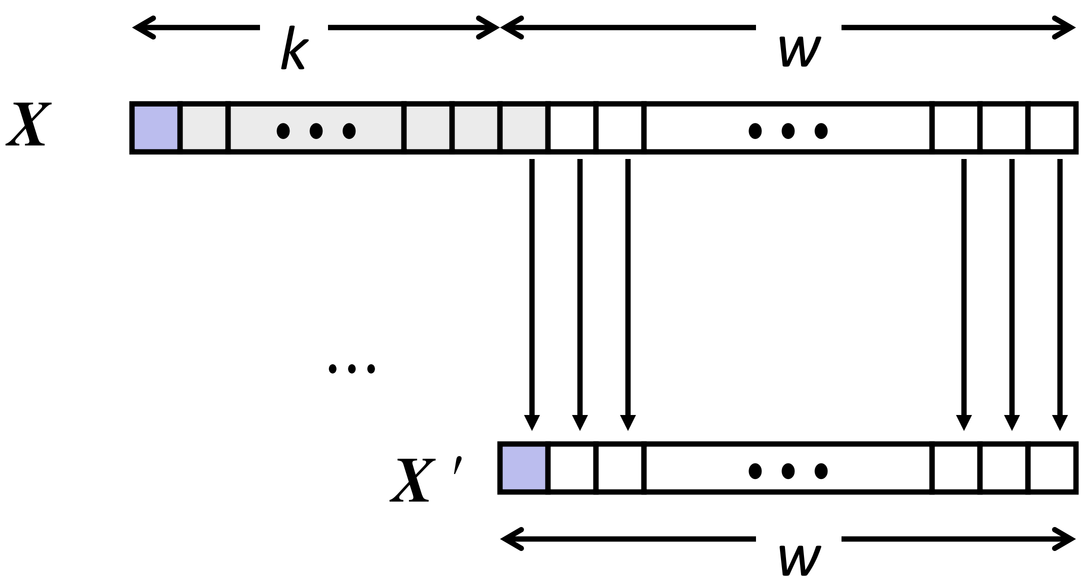
```
short int x =  15213;
int       ix = (int) x;
short int y = -15213;
int       iy = (int) y;
```

| | Decimal | Hex | Binary |
|---|---|---|---|
| x | 15213 | 3B 6D | 00111011 01101101 |
| ix | 15213 | 00 00 3B 6D | 00000000 00000000 00111011 01101101 |
| y | -15213 | C4 93 | 11000100 10010011 |
| iy | -15213 | FF FF C4 93 | 11111111 11111111 11000100 10010011 |

- Converting from smaller to larger integer data type
- C automatically performs sign extension

# Truncation

- **Task:**
  - Given k+$w$-bit signed or unsigned integer $X$
  - Convert it to $w$-bit integer X' with same value for "small enough" X
- **Rule:**
  - Drop top $k$ bits:
  - $X' = x_{w-1}, x_{w-2}, \ldots, x_0$



# Truncation: Simple Example

## No sign change

| | -16 | 8 | 4 | 2 | 1 |
|---|---|---|---|---|---|
| 2 = | 0 | 0 | 0 | 1 | 0 |

| | -8 | 4 | 2 | 1 |
|---|---|---|---|---|
| 2 = | 0 | 0 | 1 | 0 |

`2 mod 16 = 2`

| | -16 | 8 | 4 | 2 | 1 |
|---|---|---|---|---|---|
| -6 = | 1 | 1 | 0 | 1 | 0 |

| | -8 | 4 | 2 | 1 |
|---|---|---|---|---|
| -6 = | 1 | 0 | 1 | 0 |

`-6 mod 16 = 26U mod 16 = 10U = -6`

## Sign change

| | -16 | 8 | 4 | 2 | 1 |
|---|---|---|---|---|---|
| 10 = | 0 | 1 | 0 | 1 | 0 |

| | -8 | 4 | 2 | 1 |
|---|---|---|---|---|
| -6 = | 1 | 0 | 1 | 0 |

`10 mod 16 = 10U mod 16 = 10U = -6`

| | -16 | 8 | 4 | 2 | 1 |
|---|---|---|---|---|---|
| -10 = | 1 | 0 | 1 | 1 | 0 |

| | -8 | 4 | 2 | 1 |
|---|---|---|---|---|
| 6 = | 0 | 1 | 1 | 0 |

`-10 mod 16 = 22U mod 16 = 6U = 6`

# Summary: Expanding, Truncating: Basic Rules

- **Expanding (e.g., short int to int)**
  - Unsigned: zeros added
  - Signed: sign extension
  - Both yield expected result
- **Truncating (e.g., unsigned to unsigned short)**
  - Unsigned/signed: bits are truncated
  - Result reinterpreted
  - Unsigned: mod operation
  - Signed: similar to mod
  - For small numbers yields expected behavior

# Outline: Bits, Bytes, and Integers

**Textbook: [CS:APP3e] 2.1-2.3**

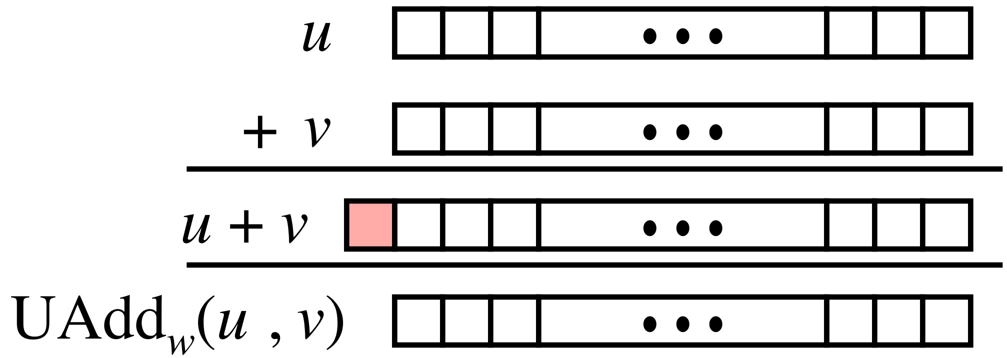
- Representing information as bits
- Bit-level manipulations
- **Integers**
  - Representation: unsigned and signed
  - Conversion, casting
  - Expanding, truncating
  - **Addition, negation, multiplication, shifting**
- Representations in memory, pointers, strings
- Summary

# Unsigned Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



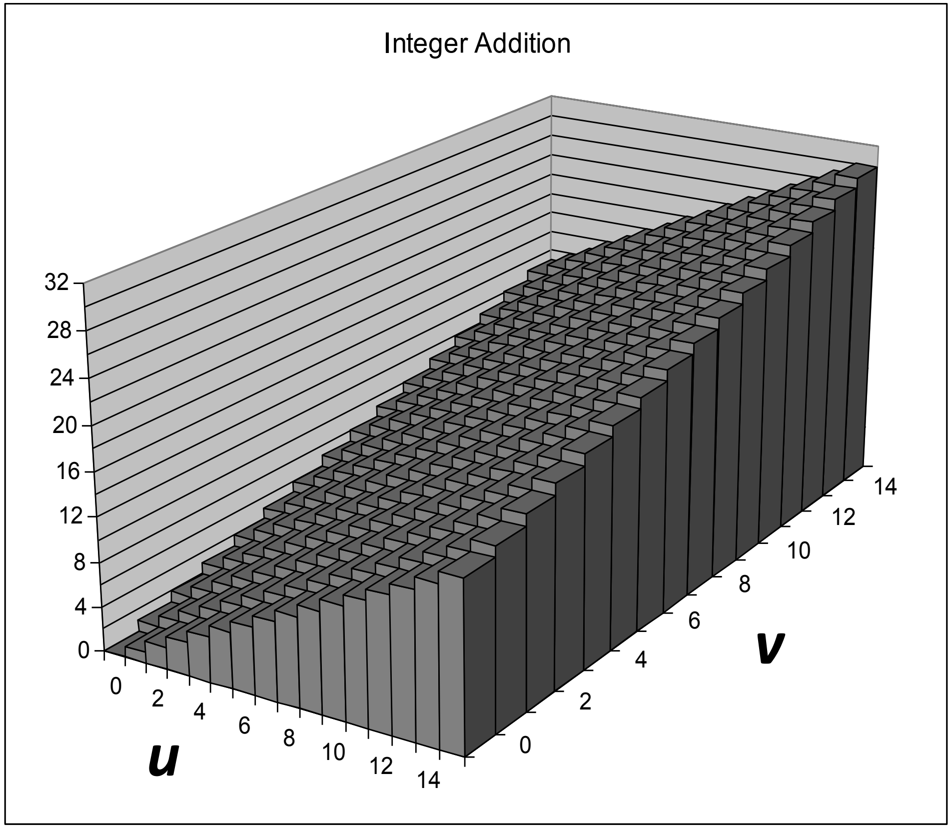

- **Standard Addition Function**
  - Ignores carry output
- **Implements Modular Arithmetic**

  $s \quad = \quad \mathrm{UAdd}_w(u\ ,\ v) \quad = \quad u + v \mod 2^w$

# Visualizing (Mathematical) Integer Addition

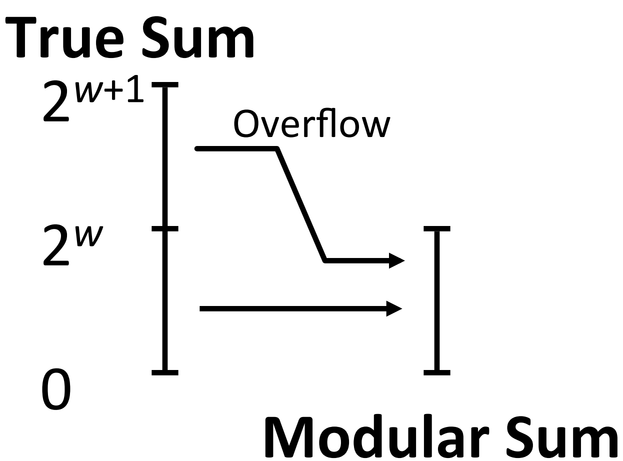
- **Integer Addition**
  - 4-bit integers *u*, *v*
  - Compute true sum $Add_4(u, v)$
  - Values increase linearly with *u* and *v*
  - Forms planar surface

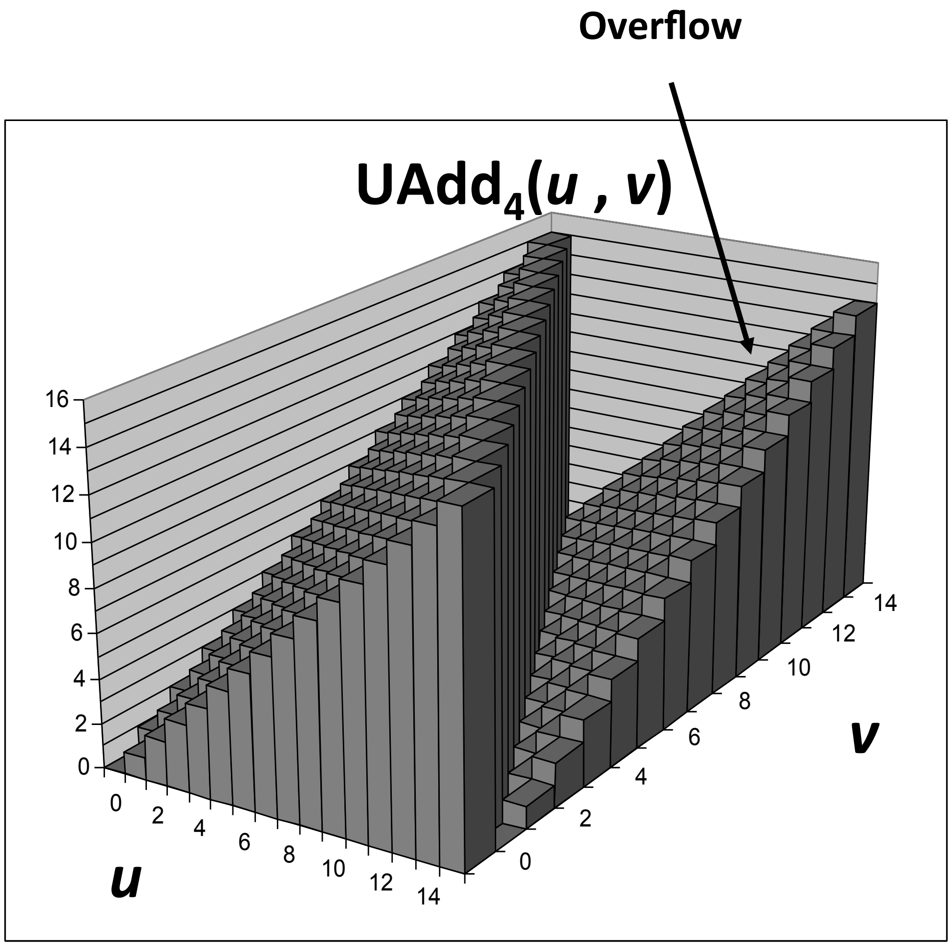
## $Add_4(u , v)$



# Visualizing Unsigned Addition

- **Wraps Around**
  - If true sum ≥ $2^w$
  - At most once



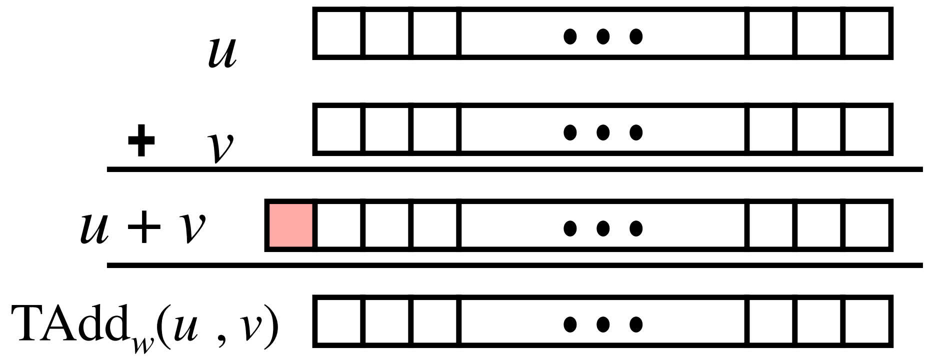



# Two's Complement Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



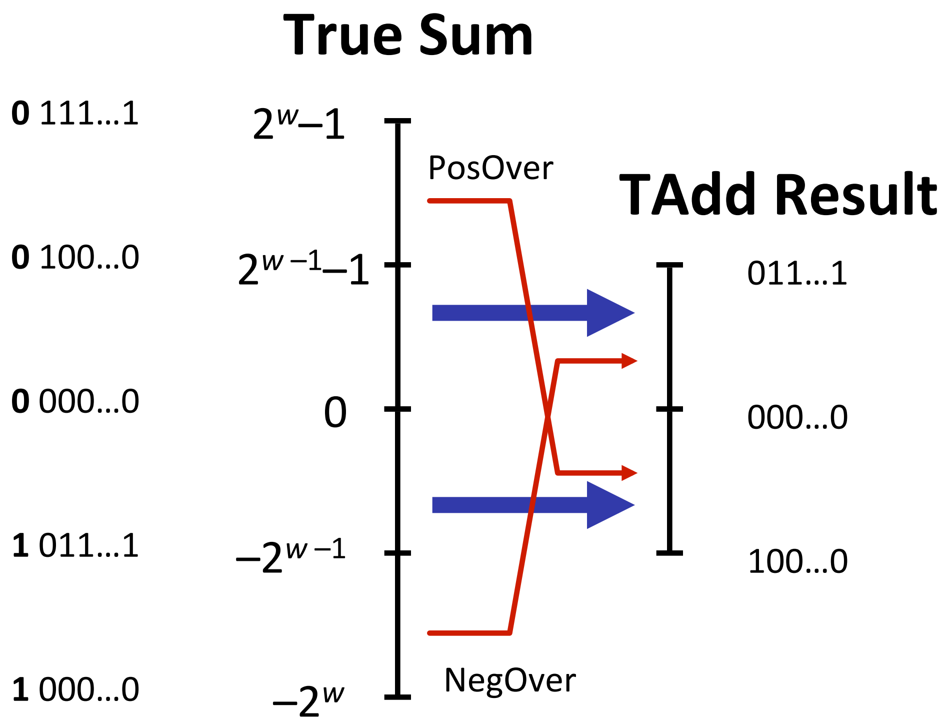

- **TAdd and UAdd have Identical Bit-Level Behavior**
  - Signed vs. unsigned addition in C:

```
int s, t, u, v;
s = (int) ((unsigned) u + (unsigned) v);
t = u + v
```

  - Will give `s == t`

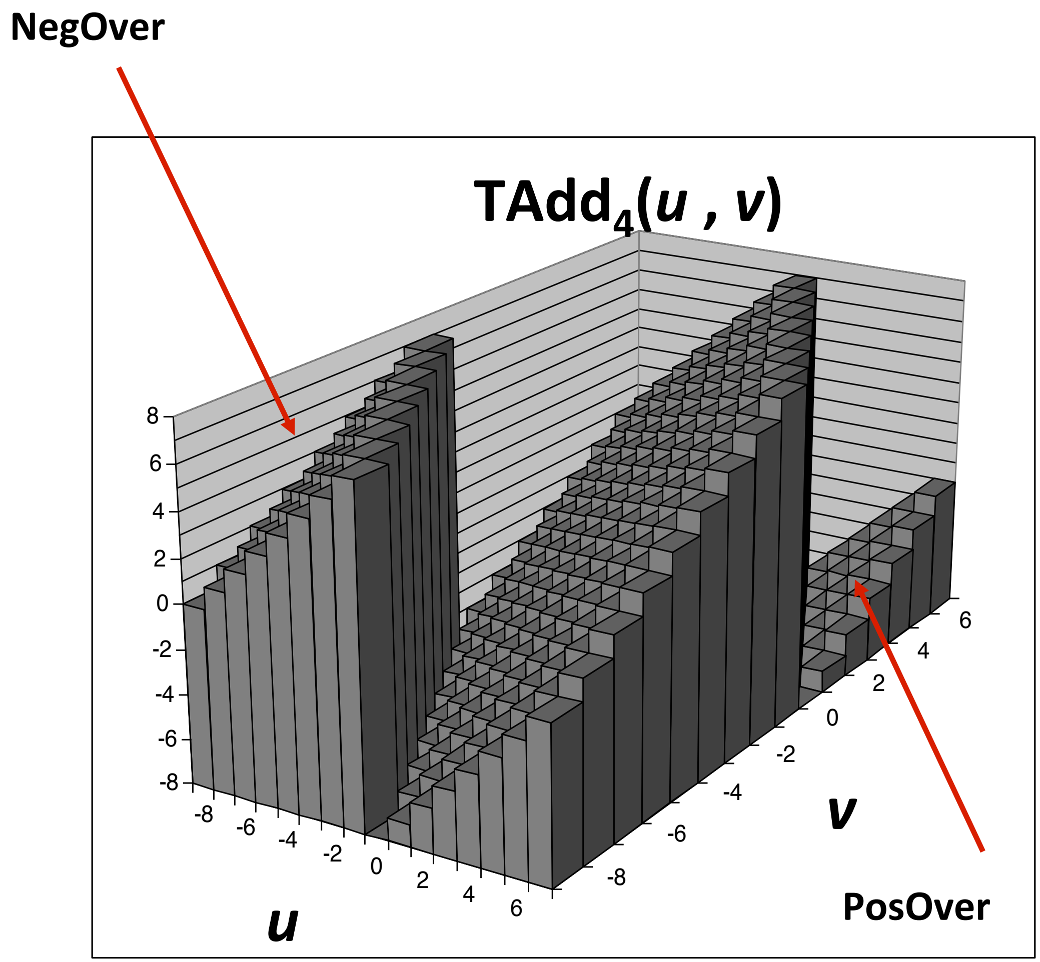
# TAdd Overflow

- **Functionality**
  - True sum requires $w$+1 bits
  - Drop off MSB
  - Treat remaining bits as 2's comp. integer



# Visualizing 2's Complement Addition

- **Values**
  - 4-bit two's comp.
  - Range from -8 to +7
- **Wraps Around**
  - If sum $\geq 2^{w-1}$
    - Becomes negative
    - At most once
  - If sum $< -2^{w-1}$
    - Becomes positive
    - At most once



# Multiplication

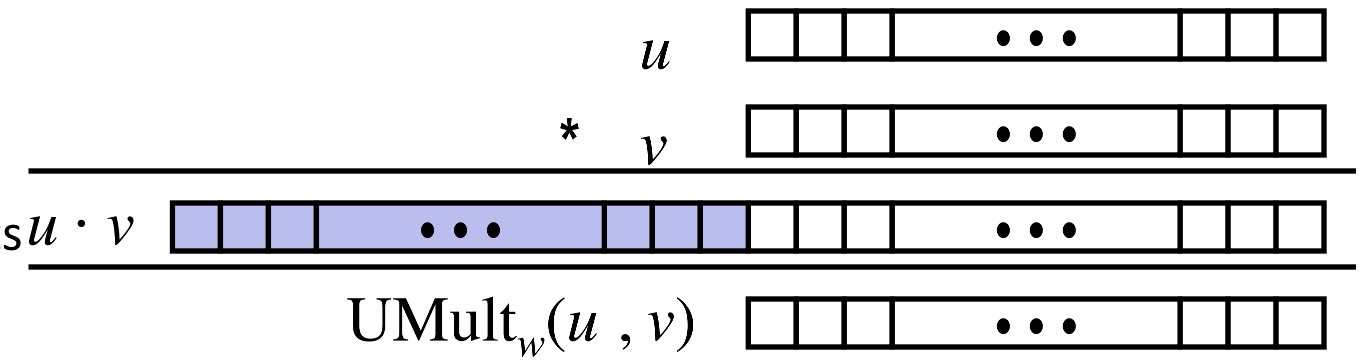
- **Goal: Computing Product of *w*-bit numbers *x*, *y***
  - Either signed or unsigned
- **But, exact results can be bigger than *w* bits**
  - Unsigned: up to 2*w* bits
    - Result range: $0 \le x * y \le (2^{w} – 1)^{2} = 2^{2w} – 2^{w+1} + 1$
  - Two's complement min (negative): Up to 2*w*-1 bits
    - Result range: $x * y \ge (–2^{w–1})*(2^{w–1}–1) = –2^{2w–2} + 2^{w–1}$
  - Two's complement max (positive): Up to 2*w* bits, but only for $(TMin_{w})^{2}$
    - Result range: $x * y \le (–2^{w–1})^{2} = 2^{2w–2}$
- **So, maintaining exact results…**
  - would need to keep expanding word size with each product computed
  - is done in software, if needed
    - e.g., by "arbitrary precision" arithmetic packages

# Unsigned Multiplication in C

Operands: *w* bits

True Product: 2**w* bits

Discard *w* bits: *w* bits




- **Standard Multiplication Function**
  - Ignores high order *w* bits
- **Implements Modular Arithmetic**

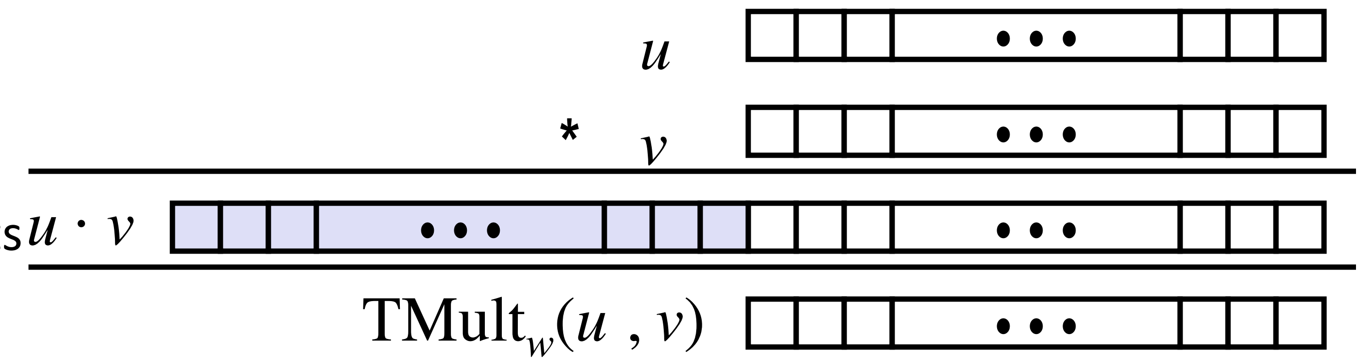
  $\mathrm{UMult}_w(u, v) = u \cdot v \bmod 2^w$

# Signed Multiplication in C

Operands: *w* bits

True Product: 2**w* bits

Discard *w* bits: *w* bits




- **Standard Multiplication Function**
  - Ignores high order *w* bits
  - Some of which are different for signed vs. unsigned multiplication
  - Lower bits are the same

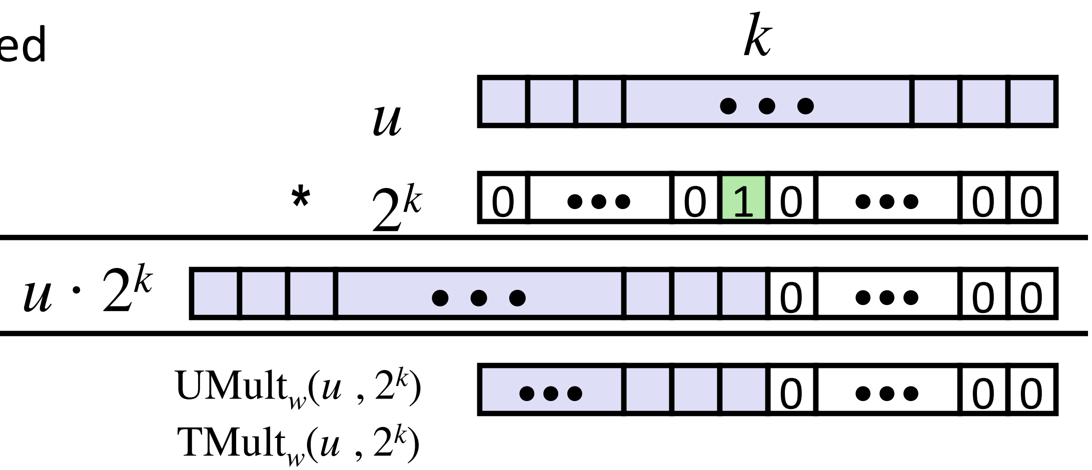
# Power-of-2 Multiply with Shift

- **Operation**
  - `u << k` gives `u * 2`$^k$
  - Both signed and unsigned

Operands: *w* bits

True Product: *w*+*k* bits

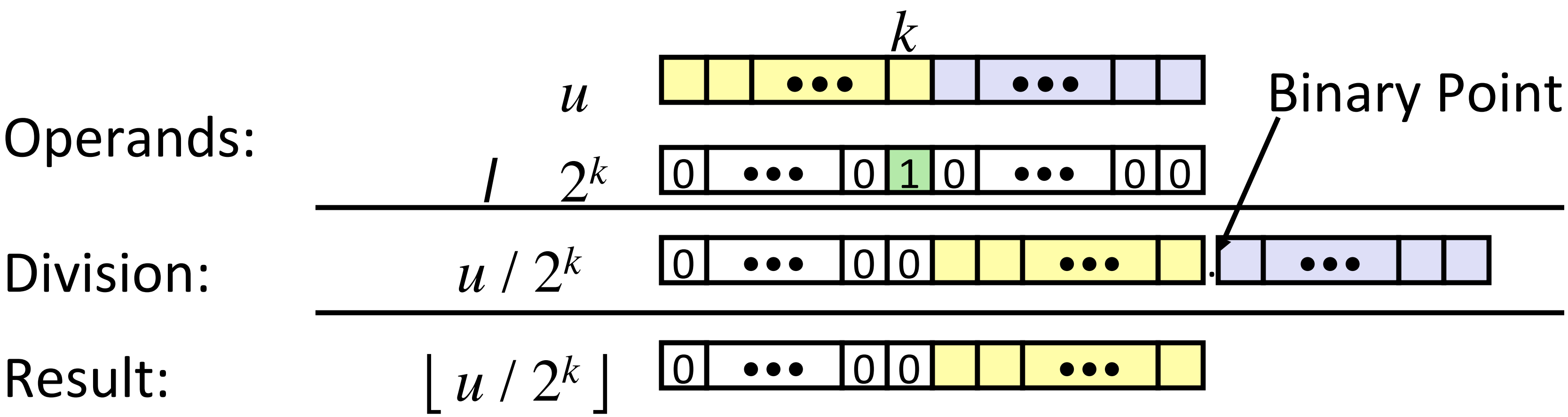
Discard *k* bits: *w* bits




- **Examples**
  - `u << 3        ==   u * 8`
  - `(u << 5) – (u << 3)==    u * 24`
  - Most machines shift and add faster than multiply
    - Compiler generates this code automatically

# Unsigned Power-of-2 Divide with Shift

- **Quotient of Unsigned by Power of 2**
  - `u >> k` gives $\lfloor$ `u / ` $2^k \rfloor$
  - Uses logical shift




| | Division | Computed | Hex | Binary |
|---|---|---|---|---|
| `x` | 15213 | 15213 | `3B 6D` | `00111011 01101101` |
| `x >> 1` | 7606.5 | 7606 | `1D B6` | `00011101 10110110` |
| `x >> 4` | 950.8125 | 950 | `03 B6` | `00000011 10110110` |
| `x >> 8` | 59.4257813 | 59 | `00 3B` | `00000000 00111011` |

# Outline: Bits, Bytes, and Integers

**Textbook: [CS:APP3e] 2.1-2.3**

- Representing information as bits
- Bit-level manipulations
- **Integers**
  - Representation: unsigned and signed
  - Conversion, casting
  - Expanding, truncating
  - Addition, negation, multiplication, shifting
  - **Summary**
- Representations in memory, pointers, strings

# Arithmetic: Basic Rules

- **Addition:**
  - Unsigned/signed: Normal addition followed by truncate, same operation on bit level
  - Unsigned: addition mod $2^w$
    - Mathematical addition + possible subtraction of $2^w$
  - Signed: modified addition mod $2^w$ (result in proper range)
    - Mathematical addition + possible addition or subtraction of $2^w$

- **Multiplication:**
  - Unsigned/signed: Normal multiplication followed by truncate, same operation on bit level
  - Unsigned: multiplication mod $2^w$
  - Signed: modified multiplication mod $2^w$ (result in proper range)

# Why Should I Use Unsigned?

- ***Don't*** **use without understanding implications**
  - Easy to make mistakes

```
unsigned i;
for (i = cnt-2; i >= 0; i--)
  a[i] += a[i+1];
```

  - Can be very subtle

```
#define DELTA sizeof(int)
int i;
for (i = CNT; i-DELTA >= 0; i-= DELTA)
  . . .
```

# Counting Down with Unsigned

- **Proper way to use unsigned as loop index**

```
unsigned i;
for (i = cnt-2; i < cnt; i--)
  a[i] += a[i+1];
```

- **See Robert Seacord, *Secure Coding in C and C++***
  - C Standard guarantees that unsigned addition will behave like modular arithmetic
    - 0 – 1 → *UMax*
- **Even better**

```
size_t i;
for (i = cnt-2; i < cnt; i--)
  a[i] += a[i+1];
```

  - Data type **`size_t`** defined as unsigned value with length = word size
  - Code will work even if **`cnt`** = *UMax*
  - What if **`cnt`** is signed and < 0?

# Why Should I Use Unsigned? (cont.)

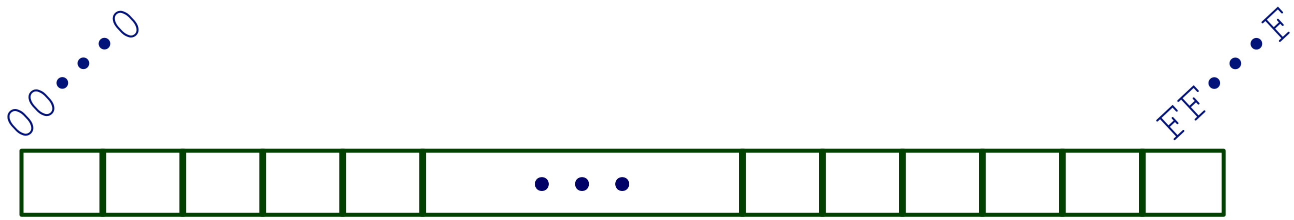
- ***Do*** **Use When Performing Modular Arithmetic**
  - Multiprecision arithmetic
- ***Do*** **Use When Using Bits to Represent Sets**
  - Logical right shift, no sign extension

# Outline: Bits, Bytes, and Integers

**Textbook: [CS:APP3e] 2.1-2.3**

- Representing information as bits
- Bit-level manipulations
- Integers
  - Representation: unsigned and signed
  - Conversion, casting
  - Expanding, truncating
  - Addition, negation, multiplication, shifting
  - Summary
- **Representations in memory, pointers, strings**

# Byte-Oriented Memory Organization




- **Programs refer to data by address**
  - Conceptually, envision it as a very large array of bytes
    - In reality, it's not, but can think of it that way
  - An address is like an index into that array
    - and, a pointer variable stores an address

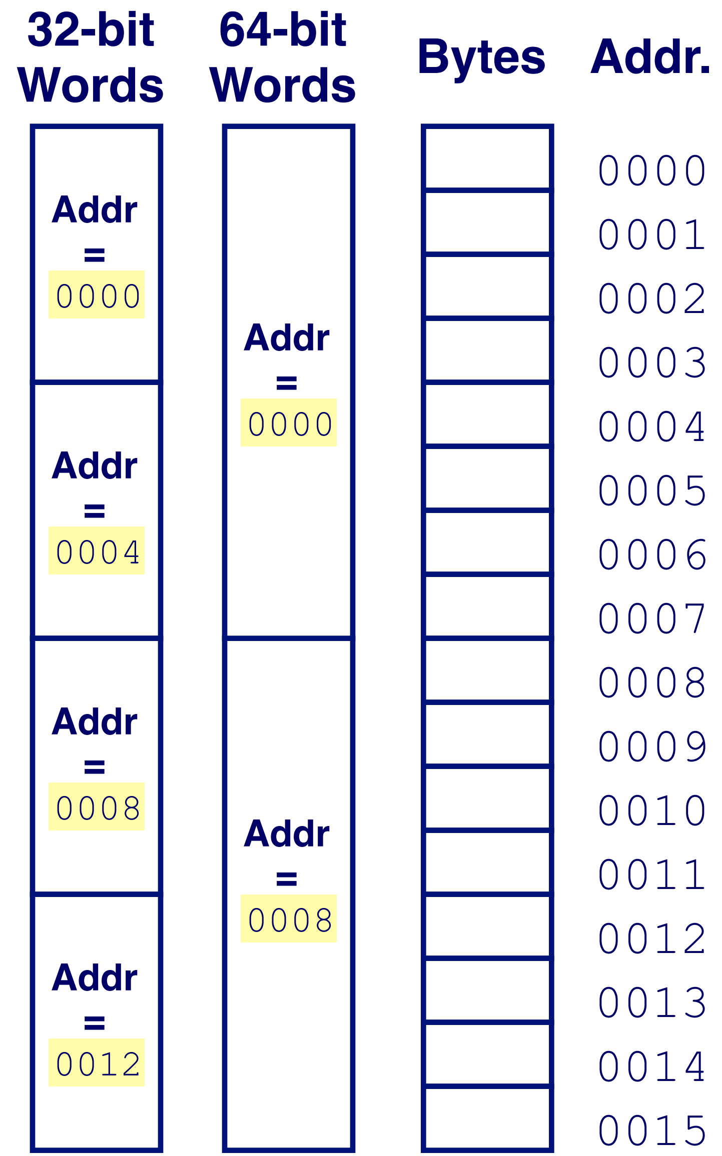
- **Note: system provides private address spaces to each "process"**
  - Think of a process as a program being executed
  - So, a program can clobber its own data, but not that of others

# Machine Words

- **Any given computer has a "Word Size"**
  - Nominal size of integer-valued data
    - and of addresses
  - Until recently, most machines used 32 bits (4 bytes) as word size
    - Limits addresses to 4GB ($2^{32}$ bytes)
  - Increasingly, machines have 64-bit word size
    - Potentially, could have 18 EB (exabytes) of addressable memory
    - That's 18.4 X $10^{18}$
  - Machines still support multiple data formats
    - Fractions or multiples of word size
    - Always integral number of bytes

# Word-Oriented Memory Organization

- **Addresses Specify Byte Locations**
  - Address of first byte in word
  - Addresses of successive words differ by 4 (32-bit) or 8 (64-bit)



# Example Data Representations

| C Data Type | Typical 32-bit | Typical 64-bit | x86-64 |
|---|---|---|---|
| **`char`** | 1 | 1 | 1 |
| **`short`** | 2 | 2 | 2 |
| **`int`** | 4 | 4 | 4 |
| **`long`** | 4 | 8 | 8 |
| **`float`** | 4 | 4 | 4 |
| **`double`** | 8 | 8 | 8 |
| **`long double`** | − | − | 10/16 |
| pointer | 4 | 8 | 8 |

# Byte Ordering

- **So, how are the bytes within a multi-byte word ordered in memory?**
- **Conventions**
  - Big Endian: Sun, PPC Mac, Internet
    - Least significant byte has highest address
  - Little Endian: x86, ARM processors running Android, iOS, and Windows
    - Least significant byte has lowest address

# Byte Ordering Example

- **Example**
  - Variable x has 4-byte value of 0x01234567
  - Address given by &x is 0x100

**Big Endian**

| | | 0x100 | 0x101 | 0x102 | 0x103 | | |
|---|---|---|---|---|---|---|---|
| | | 01 | 23 | 45 | 67 | | |

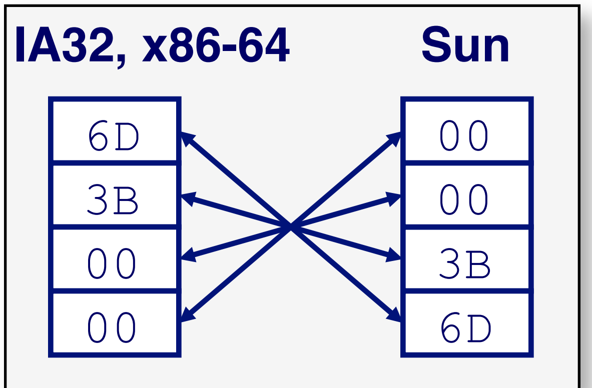
**Little Endian**

| | | 0x100 | 0x101 | 0x102 | 0x103 | | |
|---|---|---|---|---|---|---|---|
| | | 67 | 45 | 23 | 01 | | |

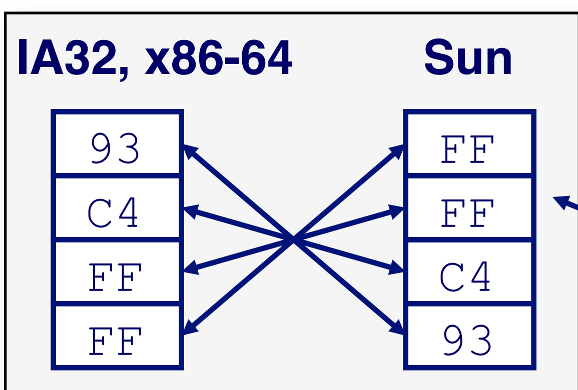
# Representing Integers

| | | | | |
|---|---|---|---|---|
| **Decimal:** | 15213 | | | |
| **Binary:** | 0011 | 1011 | 0110 | 1101 |
| **Hex:** | 3 | B | 6 | D |

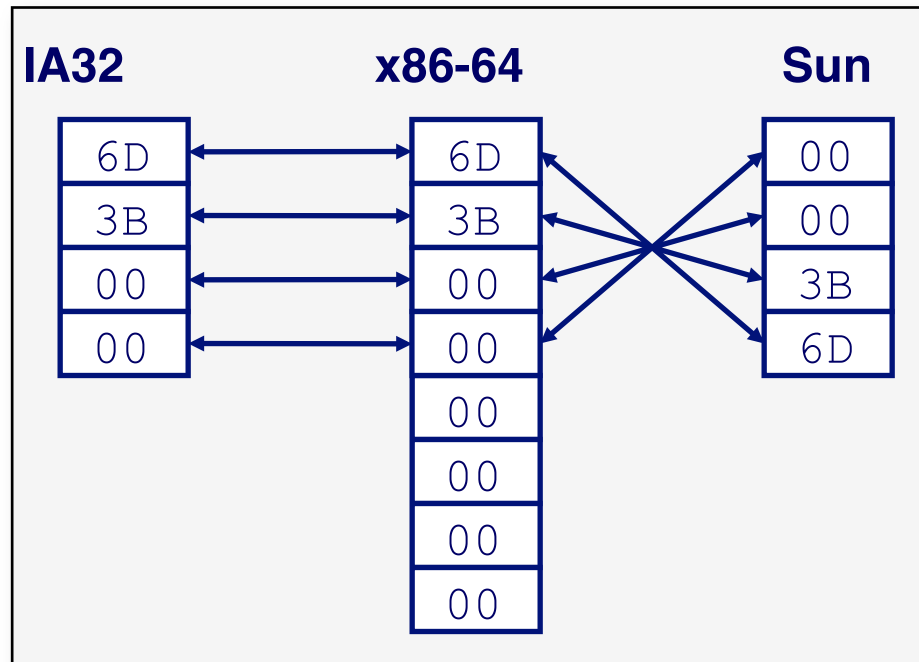
```
int A = 15213;
```




```
int B = -15213;
```




```
long int C = 15213;
```




**Two's complement representation**

# Examining Data Representations

- **Code to Print Byte Representation of Data**
  - Casting pointer to unsigned char * allows treatment as a byte array

```
typedef unsigned char *pointer;

void show_bytes(pointer start, size_t len){
  size_t i;
  for (i = 0; i < len; i++)
    printf("%p\t0x%.2x\n",start+i, start[i]);
  printf("\n");
}
```

**Printf directives:**
%p: Print pointer
%x: Print Hexadecimal

# `show_bytes` Execution Example

```
int a = 15213;
printf("int a = 15213;\n");
show_bytes((pointer) &a, sizeof(int));
```

**Result (Linux x86-64):**

```
int a = 15213;
0x7fffb7f71dbc    6d
0x7fffb7f71dbd    3b
0x7fffb7f71dbe    00
0x7fffb7f71dbf    00
```

# Representing Pointers

```
int B = -15213;
int *P = &B;
```

| Sun | IA32 | x86-64 |
|---|---|---|
| EF | AC | 3C |
| FF | 28 | 1B |
| FB | F5 | FE |
| 2C | FF | 82 |
| | | FD |
| | | 7F |
| | | 00 |
| | | 00 |

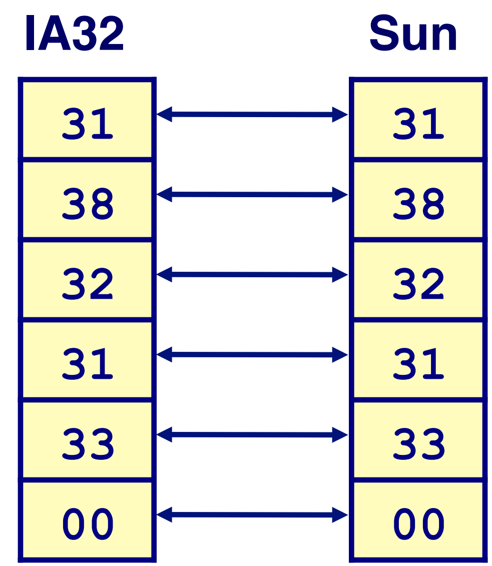
Different compilers & machines assign different locations to objects

Even get different results each time run program

# Representing Strings

```
char S[6] = "18213";
```

- **Strings in C**
  - Represented by array of characters
  - Each character encoded in ASCII format
    - Standard 7-bit encoding of character set
    - Character “0” has code 0x30
      - Digit *i* has code 0x30+*i*
  - String should be null-terminated
    - Final character = 0
- **Compatibility**
  - Byte ordering not an issue



# Integer C Puzzles

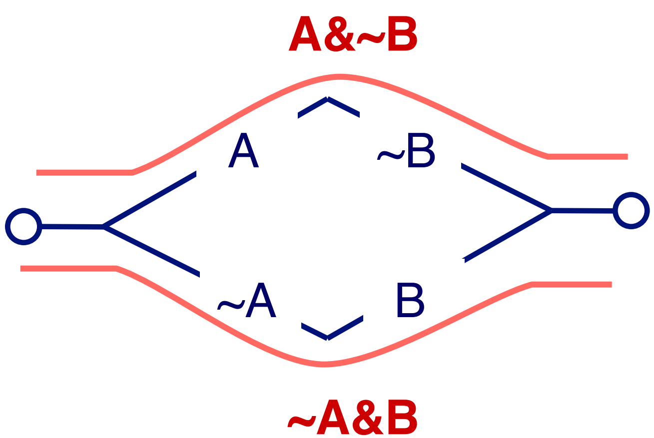
```
x < 0                    ⇒  ((x*2) < 0)    ✗
ux >= 0                                    ✓
x & 7 == 7               ⇒  (x<<30) < 0    ✓
ux > -1                                    ✗
x > y                    ⇒  -x < -y        ✗
x * x >= 0                                 ✗
x > 0 && y > 0           ⇒  x + y > 0      ✗
x >= 0                   ⇒  -x <= 0        ✓
x <= 0                   ⇒  -x >= 0        ✗
(x|-x)>>31 == -1                           ✗
ux >> 3 == ux/8                            ✓
x >> 3 == x/8                              ✗
x & (x-1) != 0                             ✗
```

## Initialization

```
int x = foo();
int y = bar();
unsigned ux = x;
unsigned uy = y;
```

# Application of Boolean Algebra

- **Applied to Digital Systems by Claude Shannon**
  - 1937 MIT Master's Thesis
  - Reason about networks of relay switches
    - Encode closed switch as 1, open switch as 0




**Connection when**

**A&~B | ~A&B**

**= A^B**

# Binary Number Property

## Claim

$$1 + 1 + 2 + 4 + 8 + \ldots + 2^{w\text{-}1} = 2^w$$

$$1 + \sum_{i=0}^{w-1} 2^i \quad = \quad 2^w$$

- **w = 0:**
  - $1 = 2^0$
- **Assume true for w-1:**
  - $\underbrace{1 + 1 + 2 + 4 + 8 + \ldots + 2^{w\text{-}1}}_{=\ 2^w} + 2^w \quad = \quad 2^w + 2^w \quad = \quad 2^{w+1}$

# Code Security Example

```
/* Kernel memory region holding user-accessible data */
#define KSIZE 1024
char kbuf[KSIZE];

/* Copy at most maxlen bytes from kernel region to user buffer */
int copy_from_kernel(void *user_dest, int maxlen) {
    /* Byte count len is minimum of buffer size and maxlen */
    int len = KSIZE < maxlen ? KSIZE : maxlen;
    memcpy(user_dest, kbuf, len);
    return len;
}
```

- **Similar to code found in FreeBSD's implementation of getpeername**
- **There are legions of smart people trying to find vulnerabilities in programs**

# Typical Usage

```
/* Kernel memory region holding user-accessible data */
#define KSIZE 1024
char kbuf[KSIZE];

/* Copy at most maxlen bytes from kernel region to user buffer */
int copy_from_kernel(void *user_dest, int maxlen) {
    /* Byte count len is minimum of buffer size and maxlen */
    int len = KSIZE < maxlen ? KSIZE : maxlen;
    memcpy(user_dest, kbuf, len);
    return len;
}
```

```
#define MSIZE 528

void getstuff() {
    char mybuf[MSIZE];
    copy_from_kernel(mybuf, MSIZE);
    printf("%s\n", mybuf);
}
```

# Malicious Usage

```
/* Declaration of library function memcpy */
void *memcpy(void *dest, void *src, size_t n);
```

```
/* Kernel memory region holding user-accessible data */
#define KSIZE 1024
char kbuf[KSIZE];

/* Copy at most maxlen bytes from kernel region to user buffer */
int copy_from_kernel(void *user_dest, int maxlen) {
    /* Byte count len is minimum of buffer size and maxlen */
    int len = KSIZE < maxlen ? KSIZE : maxlen;
    memcpy(user_dest, kbuf, len);
    return len;
}
```

```
#define MSIZE 528

void getstuff() {
    char mybuf[MSIZE];
    copy_from_kernel(mybuf, -MSIZE);
    . . .
}
```

# Mathematical Properties

- **Modular Addition Forms an *Abelian Group***
  - **Closed** under addition

    $0 \le \text{UAdd}_w(u, v) \le 2^w - 1$
  - **Commutative**

    $\text{UAdd}_w(u, v) = \text{UAdd}_w(v, u)$
  - **Associative**

    $\text{UAdd}_w(t, \text{UAdd}_w(u, v)) = \text{UAdd}_w(\text{UAdd}_w(t, u), v)$
  - **0** is additive identity

    $\text{UAdd}_w(u, 0) = u$
  - Every element has additive **inverse**
    - Let $\text{UComp}_w(u) = 2^w - u$

      $\text{UAdd}_w(u, \text{UComp}_w(u)) = 0$

# Mathematical Properties of TAdd

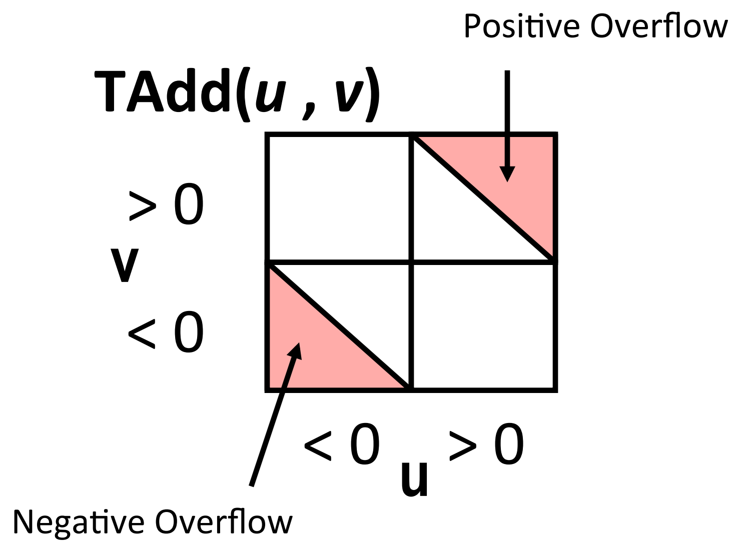
- **Isomorphic Group to unsigneds with UAdd**
  - $\mathrm{TAdd}_w(u, v) = \mathrm{U2T}(\mathrm{UAdd}_w(\mathrm{T2U}(u), \mathrm{T2U}(v)))$
    - Since both have identical bit patterns

- **Two's Complement Under TAdd Forms a Group**
  - Closed, Commutative, Associative, 0 is additive identity
  - Every element has additive inverse

$$TComp_w(u) = \begin{cases} -u & u \neq TMin_w \\ TMin_w & u = TMin_w \end{cases}$$

# Characterizing TAdd

- **Functionality**
  - True sum requires *w*+1 bits
  - Drop off MSB
  - Treat remaining bits as 2's comp. integer




$$TAdd_w(u,v) = \begin{cases} u + v + 2^w & u + v < TMin_w \quad \textbf{(NegOver)} \\ u + v & TMin_w \le u + v \le TMax_w \\ u + v - 2^w & TMax_w < u + v \quad \textbf{(PosOver)} \end{cases}$$

# Negation: Complement & Increment

- **Negate through complement and increase**

  `~x + 1 == -x`

- **Example**
  - Observation: `~x + x == 1111…111 == -1`

| | |
|---|---|
| x | 1 0 0 1 1 1 0 1 |
| + ~x | 0 1 1 0 0 0 1 0 |
| -1 | 1 1 1 1 1 1 1 1 |

**x = 15213**

| | Decimal | Hex | Binary |
|---|---|---|---|
| `x` | 15213 | `3B 6D` | `00111011 01101101` |
| `~x` | -15214 | `C4 92` | `11000100 10010010` |
| `~x+1` | -15213 | `C4 93` | `11000100 10010011` |
| `y` | -15213 | `C4 93` | `11000100 10010011` |

# Complement & Increment Examples

x = 0

| | Decimal | Hex | Binary |
|---|---:|---|---|
| 0 | 0 | 00 00 | 00000000 00000000 |
| ~0 | -1 | FF FF | 11111111 11111111 |
| ~0+1 | 0 | 00 00 | 00000000 00000000 |

x = TMin

| | Decimal | Hex | Binary |
|---|---:|---|---|
| x | -32768 | 80 00 | 10000000 00000000 |
| ~x | 32767 | 7F FF | 01111111 11111111 |
| ~x+1 | -32768 | 80 00 | 10000000 00000000 |

**Canonical counter example**

# Compiled Multiplication Code

**C Function**

```
long mul12(long x)
{
  return x*12;
}
```

**Compiled Arithmetic Operations**

```
leaq (%rax,%rax,2), %rax
salq $2, %rax
```

**Explanation**

```
t <- x+x*2
return t << 2;
```

- **C compiler automatically generates shift/add code when multiplying by constant**

# Compiled Unsigned Division Code

### C Function

```
unsigned long udiv8
      (unsigned long x)
{
  return x/8;
}
```

### Compiled Arithmetic Operations

```
  shrq $3, %rax
```

### Explanation

```
  # Logical shift
  return x >> 3;
```

- **Uses logical shift for unsigned**
- **For Java Users**
  - Logical shift written as >>>

# Compiled Signed Division Code

## C Function

```
long idiv8(long x)
{
  return x/8;
}
```

## Compiled Arithmetic Operations

```
  testq %rax, %rax
  js    L4
L3:
  sarq $3, %rax
  ret
L4:
  addq $7, %rax
  jmp  L3
```

## Explanation

```
  if x < 0
    x += 7;
  # Arithmetic shift
  return x >> 3;
```

- **Uses arithmetic shift for int**
- **For Java Users**
  - Arith. shift written as >>

# Arithmetic: Basic Rules

- **Unsigned ints, 2's complement ints are isomorphic rings: isomorphism = casting**

- **Left shift**
  - Unsigned/signed: multiplication by $2^k$
  - Always logical shift

- **Right shift**
  - Unsigned: logical shift, div (division + round to zero) by $2^k$
  - Signed: arithmetic shift
    - Positive numbers: div (division + round to zero) by $2^k$
    - Negative numbers: div (division + round away from zero) by $2^k$
      Use biasing to fix

# Properties of Unsigned Arithmetic

- **Unsigned Multiplication with Addition Forms Commutative Ring**
  - Addition is commutative group
  - Closed under multiplication

    $0 \leq \mathrm{UMult}_w(u\,, v) \leq 2^w - 1$
  - Multiplication Commutative

    $\mathrm{UMult}_w(u\,, v) = \mathrm{UMult}_w(v\,, u)$
  - Multiplication is Associative

    $\mathrm{UMult}_w(t, \mathrm{UMult}_w(u\,, v)) = \mathrm{UMult}_w(\mathrm{UMult}_w(t, u\,), v)$
  - 1 is multiplicative identity

    $\mathrm{UMult}_w(u\,, 1) = u$
  - Multiplication distributes over addtion

    $\mathrm{UMult}_w(t, \mathrm{UAdd}_w(u\,, v)) = \mathrm{UAdd}_w(\mathrm{UMult}_w(t, u\,), \mathrm{UMult}_w(t, v))$

# Properties of Two's Comp. Arithmetic

- **Isomorphic Algebras**
  - Unsigned multiplication and addition
    - Truncating to *w* bits
  - Two's complement multiplication and addition
    - Truncating to *w* bits
- **Both Form Rings**
  - Isomorphic to ring of integers mod $2^w$
- **Comparison to (Mathematical) Integer Arithmetic**
  - Both are rings
  - Integers obey ordering properties, e.g.,

$$u > 0 \quad \Rightarrow \quad u + v > v$$

$$u > 0, v > 0 \quad \Rightarrow \quad u \cdot v > 0$$

  - These properties are not obeyed by two's comp. arithmetic

*TMax* `+ 1` `==` *TMin*

`15213 * 30426 == -10030` (16-bit words)

# Reading Byte-Reversed Listings

- **Disassembly**
  - Text representation of binary machine code
  - Generated by program that reads the machine code
- **Example Fragment**

| Address | Instruction Code | Assembly Rendition |
|---|---|---|
| `8048365:` | `5b` | `pop %ebx` |
| `8048366:` | `81 c3 ab 12 00 00` | `add $0x12ab,%ebx` |
| `804836c:` | `83 bb 28 00 00 00 00` | `cmpl $0x0,0x28(%ebx)` |

- **Deciphering Numbers**
  - Value: **`0x12ab`**
  - Pad to 32 bits: **`0x000012ab`**
  - Split into bytes: **`00 00 12 ab`**
  - Reverse: **`ab 12 00 00`**