Chap 5. Series Solutions of ODEs

5.1 Power Series Method

Power Series

a power series is an infinite series of the form

$$\sum_{m=0}^{\infty} a_m (x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

a_m: coefficient; x_0: center

If $x_0 = 0$, we obtain a power series in powers of x

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots$$

Examples

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots \quad \left(|x|<1, \text{ geometric series}\right)$$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{ix} = \cos x + i \sin x = 1 + ix - \frac{x^2}{2!} - i \cdot \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m \cdot x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots$$

∘ The term "power series" does not include series of negative or fractional powers of x.

Idea of the Power Series Method

$y'' + p(x)y' + q(x)y = 0$, $p(x), q(x)$: polynomials or in power series

$y = \sum_{m=0}^{\infty} a_m x^m$, $y' = \sum_{m=1}^{\infty} m a_m x^{m-1}$, $y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$

$\Rightarrow$ The ODE gives a recurrence relation among a_m's.

Ex 1

$y' = 2xy$

$a_1 + 2a_2 x + 3a_3 x^2 + \cdots = 2x(a_0 + a_1 x + a_2 x^2 + \cdots)$

$a_1 = 0$, $2a_2 = 2a_0$, $3a_3 = 2a_1$, $4a_4 = 2a_2$, $\cdots$

$a_1 = 0$, $a_3 = 0$, $a_5 = 0$

$a_2 = a_0$, $a_4 = \frac{1}{2}a_2 = \frac{1}{2}a_0$, $a_6 = \frac{1}{3}a_4 = \frac{1}{3!}a_0$, $\cdots$

$y = a_0\left(1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \frac{1}{4!}x^8 + \cdots\right) = a_0 \cdot e^{x^2}$

Ex 2

$y'' + y = 0$

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} + \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum_{s=0}^{\infty} (s+2)(s+1) a_{s+2} x^s = -\sum_{s=0}^{\infty} a_s x^s$$

$$a_{s+2} = \frac{-a_s}{(s+2)(s+1)} \qquad (s = 0, 1, 2, \cdots)$$

$a_2 = -\frac{a_0}{2\cdot 1} = -\frac{1}{2!}a_0$, $a_3 = -\frac{a_1}{3\cdot 2} = -\frac{1}{3!}a_1$

$a_4 = -\frac{a_2}{4\cdot 3} = +\frac{1}{4!}a_0$, $a_5 = -\frac{a_3}{5\cdot 4} = +\frac{1}{5!}a_1$

$$\therefore y = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots\right)$$

$$= a_0 \cos x + a_1 \sin x$$

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5.2 Theory of the Power Series Method

Basic Concepts

$$\sum_{m=0}^{\infty} a_m (x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

$S_n(x) = a_0 + a_1(x-x_0) + \cdots + a_n(x-x_0)^n$: the nth partial sum

$R_n(x) = a_{n+1}(x-x_0)^{n+1} + \cdots$: the remainder

If for some x_1 the sequence $\{S_n(x_1)\}$ converges to $s(x_1)$,

$s(x_1) = \sum_{m=0}^{\infty} a_m (x_1 - x_0)^m$ and $s(x_1) = S_n(x_1) + R_n(x_1)$, for all n.

Convergence Interval, Radius of Convergence

Case 1 :

The series $\sum a_m (x-x_0)^m$ converges only at $x = x_0$

Case 2 :

The series converges for all x s.t $|x-x_0| < R$ and diverges for all x s.t $|x-x_0| > R$

Radius of convergence

R can be obtained from

(a) $R = 1 \Big/ \lim_{m\to\infty} \sqrt[m]{|a_m|}$, (b) $R = 1 \Big/ \lim_{m\to\infty} \left| \dfrac{a_{m+1}}{a_m} \right|$

Case 3 :

The series converges for all x and $R = \infty$.

Ex1 (The Useless Case 1)

$$\sum_{m=0}^{\infty} m!\, x^m = 1 + x + 2x^2 + 6x^3 + \cdots$$

$$\frac{a_{m+1}}{a_m} = m+1 \to \infty \text{ as } m \to \infty$$

This series converges only at $x=0$ and $R=0$.

Ex2 (The Usual Case 2)

For the geometric series,

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots$$

$a_m = 1$ for all m and $R=1$

Thus, the geometric series converges when $|x|<1$.

Ex3 (The Best Case 3)

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \cdots$$

$$\frac{a_{m+1}}{a_m} = \frac{1}{m+1} \to 0 \text{ as } m \to \infty$$

This series converges for all x and $R=\infty$

Ex4 (Radius of Convergence)

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{8^m} x^{3m} = 1 - \frac{x^3}{8} + \frac{x^6}{64} - \frac{x^9}{512} + \cdots$$

This is a series in powers of $t = x^3$ with $a_m = (-1)^m/8^m$

$$\left|\frac{a_{m+1}}{a_m}\right| = \frac{1}{8} \text{ and } R = 8$$

The series converges for $|t| = |x^3| < 8$ and $|x| < 2$

Operations on Power Series

Termwise Differentiation

$$y(x)=\sum_{m=0}^{\infty} a_m (x-x_0)^m, \quad y'(x)=\sum_{m=1}^{\infty} m a_m (x-x_0)^{m-1}, \quad \cdots$$

$\Rightarrow y(x), y'(x), y''(x), \cdots$ have the same radius of convergence R

Termwise Addition and Multiplication

$$f(x)=\sum_{m=0}^{\infty} a_m (x-x_0)^m, \quad g(x)=\sum_{m=0}^{\infty} b_m (x-x_0)^m$$

$$f(x)+g(x)=\sum_{m=0}^{\infty} (a_m+b_m)(x-x_0)^m \quad \text{and}$$

$$f(x)g(x)=\sum_{m=0}^{\infty} (a_0 b_m + a_1 b_{m-1} + \cdots + a_m b_0)(x-x_0)^m$$

converge for all $x \in I_f \cap I_g$.

convergence intervals of f and g

Existence of Power Series Solutions of ODE

Def (Real Analytic Function)

$f(x)$: analytic at $x=x_0$ if $f(x)=\sum_{m=0}^{\infty} a_m (x-x_0)^m$ for $|x-x_0|<R$ and $R>0$

Th 1 (Existence of Power Series Solutions)

(a) $y''+p(x)y'+q(x)y=r(x)$.

$p(x), q(x), r(x)$: analytic at $x=x_0$

$\Rightarrow$ every solution is analytic at $x=x_0$

(b) $\tilde{h}(x)y''+\tilde{p}(x)y'+\tilde{q}(x)y=\tilde{r}(x)$,

$\tilde{h}(x), \tilde{p}(x), \tilde{q}(x), \tilde{r}(x)$: analytic at $x=x_0$

and $\tilde{h}(x_0)\neq 0 \Rightarrow$ every solution is analytic at $x=x_0$.

5.3 Legendre's Equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad (n: \text{real constant})$$

Any solution of this equation is a Legendre function.
By Theorem 1 of Sec 5.2, there are power series solutions

$$y = \sum_{m=0}^{\infty} a_m x^m .$$

Substituting this function and its derivatives, we get

$$a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)} a_s \qquad (s=0,1,\cdots)$$

Using this recurrence relation, we obtain

$$y(x) = a_0 y_1(x) + a_1 y_2(x),$$

where

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - + \cdots$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - + \cdots$$

$y_1(x)$ and $y_2(x)$ converge for $|x|<1$ and linearly independent.
Hence, $y(x) = a_0 y_1(x) + a_1 y_2(x)$ is a general solution on $-1<x<1$.

Legendre Polynomials $P_n(x)$

If n is an even nonnegative integer, $y_1(x)$ is a polynomial of degree n. ($\because$ for $s=n$, $a_{n+2}=0$, $a_{n+4}=0$, $a_{n+6}=0$, $\cdots$)
If n is an odd nonnegative integer, $y_2(x)$ is a polynomial of degree n.
$P_n(x)$ are these polynomials multiplied by some constants so that $P_n(1)=1$
↳ Legendre polynomials.

The first few of these functions are

$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$

$P_3(x) = \frac{1}{2}(5x^3-3x)$, $P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$

$P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$, $\cdots$

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