< 2013년 1학기 고급수학 및 연습 1 기말고사 모범답안 >

#1

(a) l의 방향벡터를 $\vec{v}$라 하면, $\vec{v} = (1, 2, 3)$이다

그러므로 $P_l(\vec{x}) = \dfrac{\vec{v}\cdot\vec{x}}{\vec{v}\cdot\vec{v}}\vec{v}$ $(\vec{x} = (x, y, z))$

$$= \frac{x+2y+3z}{14}(1, 2, 3)$$ ┘ 5점

이때 $P_l(e_1) = \frac{1}{14}(1, 2, 3)$

$P_l(e_2) = \frac{2}{14}(1, 2, 3)$

$P_l(e_3) = \frac{3}{14}(1, 2, 3)$ 이므로

$$A = \frac{1}{14}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$ 이다. ┘ 10점.

(b) 직선 m의 방정식은 $t\vec{v} + (0, 0, 4)$ 이므로

$$P_m(\vec{x}) = P_l(\vec{x} - (0, 0, 4)) + (0, 0, 4)$$

$$= A\vec{x} - A\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$$ ┘ +10점

$$= A\vec{x} + \frac{1}{7}\begin{pmatrix} -6 \\ -12 \\ 10 \end{pmatrix}.$$

$\therefore \vec{b} = \frac{1}{7}(-6, -12, 10)$ ┘ 15점

#2

(a) $2Ax\cdot Ay = |Ax+Ay|^2 - |Ax|^2 - |Ay|^2$

$= 2013^2|x+y|^2 - 2013^2|x|^2 - 2013^2|y|^2$

$= 2013^2\cdot(2\,x\cdot y)$

$\therefore\ Ax\cdot Ay = 2013^2\, x\cdot y$ for all $x, y \in \mathbb{R}^n$] 10점

(b)

$2013^2 x\cdot y = Ax\cdot Ay = x^t A^t A\, y$ for all $x\ y \in \mathbb{R}^n$

$x = e_i,\ y = e_j$ $1 \le i, j \le n$ 을 대입하면,

$2013^2\,\delta_{ij} = [A^tA]_{ij}$

$\therefore\ A^tA = 2013^2 I_n$] 5점

$\therefore\ |\det A| = |\det(2013^2 I_n)|^{1/2} = (2013^{2n})^{1/2} = 2013^n$] 10점

※ 행렬, 행렬식의 잘못된 성질을 이용한 경우 0점.

#3

Let X, Y be 2×2 matrices and t be a real number.

Then

$$T(X+Y) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}(X+Y) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}X + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}Y$$

$$= T(X) + T(Y)$$

$$T(tX) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}(tX) = t\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}(X)$$

$$= t\,T(X)$$

$\therefore$ T is a linear map ⌋ 10점

$$T\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} \quad T\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix}$$

$$T\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix} \quad T\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix}$$

The corresponding matrix is

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{pmatrix}$$

" ⌋ 20점

※ 2x2 행렬에 대응하는 $\mathbb{R}^4$ 벡터의 성분 순서를 잘못 쓴경우 5점 감점.

4

$$\det\begin{pmatrix} 3a_1+5b_1 & 3a_2+5b_2 & 3a_3+5b_3 \\ 4b_1+5c_1 & 4b_2+5c_2 & 4b_3+5c_3 \\ 8c_1+5a_1 & 8c_2+5a_2 & 8c_3+5a_3 \end{pmatrix}$$

$$= \det\left(\begin{pmatrix} 3 & 5 & 0 \\ 0 & 4 & 5 \\ 5 & 0 & 8 \end{pmatrix} \cdot A\right) \quad \left(A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}\right)$$

10pt

$$= \det\begin{pmatrix} 3 & 5 & 0 \\ 0 & 4 & 5 \\ 5 & 0 & 8 \end{pmatrix} \cdot \det(A)$$

5pt

$$= 221 \cdot \det(A),$$

$$\therefore t = 221$$

5pt

#4- 풀이 2

$\vec{a} = (a_1, a_2\ a_3)$ $\vec{b} = (b_1\ b_2\ b_3)$ $\vec{c} = (c_1\ c_2\ c_3)$ 라하면

$$\det\begin{pmatrix} 3a_1+5b_1 & 3a_2+5b_2 & 3a_3+5b_3 \\ 4b_1+5c_1 & 4b_2+5c_2 & 4b_3+5c_3 \\ 8c_1+5a_1 & 8c_2+5a_2 & 8c_3+5a_3 \end{pmatrix} = \det\begin{pmatrix} 3\vec{a}+5\vec{b} \\ 4\vec{b}+5\vec{c} \\ 8\vec{c}+5\vec{a} \end{pmatrix}$$

$$= \det\begin{pmatrix} 3\vec{a} \\ 4\vec{b} \\ 8\vec{c} \end{pmatrix} + \det\begin{pmatrix} 3\vec{a} \\ 4\vec{b} \\ 5\vec{a} \end{pmatrix} + \det\begin{pmatrix} 3\vec{a} \\ 5\vec{c} \\ 8\vec{c} \end{pmatrix} + \det\begin{pmatrix} 3\vec{a} \\ 5\vec{c} \\ 5\vec{a} \end{pmatrix}$$

$$+ \det\begin{pmatrix} 5\vec{b} \\ 4\vec{b} \\ 8\vec{c} \end{pmatrix} + \det\begin{pmatrix} 5\vec{b} \\ 4\vec{b} \\ 5\vec{a} \end{pmatrix} + \det\begin{pmatrix} 5\vec{b} \\ 5\vec{c} \\ 5\vec{c} \end{pmatrix} + \det\begin{pmatrix} 5\vec{b} \\ 5\vec{c} \\ 5\vec{a} \end{pmatrix}$$

10점.

$$= 3\times4\times8 \det\begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix} + 5^3\times(-1)^2 \det\begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix}$$

15점.

$$= (96+125)\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

$$= 221 \det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

20점

※ 행렬식의 bilinearity을 잘못 사용한 경우 0 점

#5

a) For $\vec{v} = \overrightarrow{AB} = (1, 2, -1)$, $\vec{\omega} = \overrightarrow{AC} = (-1, 1, 2)$,

the normal vector of the plane P

containing $L_1 = \overline{AB}$, $L_2 = \overline{AC}$ is

$$\vec{n} = \vec{v} \times \vec{\omega} = (5, -1, 3)$$ 5 pt

$\therefore$ Equation of P : $(\vec{X} - \overrightarrow{OA}) \cdot \vec{n} = 0$

$$\Rightarrow 5(x-1) - (y-1) + 3(z-3) = 0$$

$$\Rightarrow 5x - y + 3z - 13 = 0.$$ 5 pt

b) The area of S: the parallelogram with two sides L_1, L_2 is

$$\text{Area}(S) = |\vec{v} \times \vec{\omega}| = \sqrt{35}$$ 5 pt

Also, for the plane Q : $3x - 5y + z = 1$,

the cosine of the angle θ between P, Q is

$$\cos\theta = \frac{\vec{n} \cdot (3, -5, 1)}{|\vec{n}|\,|(3, -5, 1)|} = \frac{23}{35}$$

$\therefore$ The area of the parallelogram obtained by orthogonal projecting S onto Q is

$$\text{Area}(S) \times \cos\theta = \frac{23}{35}\sqrt{35}$$

5 pt.

6. Let $x, y, z, w > 0$ be distinct and

$X_1 := X(x)$, $X_2 := X(y)$, $X_3 := X(z)$, $X_4 = X(w)$

In order to show vectors $\overrightarrow{OX_i}$ $1 \le i \le 4$ are linearly independent, it suffices to show that

$$\det\begin{pmatrix} \overrightarrow{OX_1} \\ \overrightarrow{OX_2} \\ \overrightarrow{OX_3} \\ \overrightarrow{OX_4} \end{pmatrix} = \det\begin{pmatrix} x & x^2 & x^3 & x^4 \\ y & y^2 & y^3 & y^4 \\ z & z^2 & z^3 & z^4 \\ w & w^2 & w^3 & w^4 \end{pmatrix} \neq 0.$$

5 points

$$\det\begin{pmatrix} x & x^2 & x^3 & x^4 \\ y & y^2 & y^3 & y^4 \\ z & z^2 & z^3 & z^4 \\ w & w^2 & w^3 & w^4 \end{pmatrix} = xyzw \det\begin{pmatrix} 1 & x & x^2 & x^3 \\ 1 & y & y^2 & y^3 \\ 1 & z & z^2 & z^3 \\ 1 & w & w^2 & w^3 \end{pmatrix} = xyzw \det\begin{pmatrix} 1 & x & x^2 & x^3 \\ 0 & y-x & y^2-x^2 & y^3-x^3 \\ 0 & z-x & z^2-x^2 & z^3-x^3 \\ 0 & w-x & w^2-x^2 & w^3-x^3 \end{pmatrix}$$

$$= xyzw(y-x)(z-x)(w-x) \det\begin{pmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+xy+x^2 \\ 0 & 1 & z+x & z^2+zx+x^2 \\ 0 & 1 & w+x & w^2+wx+x^2 \end{pmatrix}$$

$$= xyzw(y-x)(z-x)(w-x) \det\begin{pmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+xy+x^2 \\ 0 & 0 & z-y & z^2-y^2+x(z-y) \\ 0 & 0 & w-y & w^2-y^2+x(w-y) \end{pmatrix}$$

$$= xyzw(y-x)(z-x)(w-x)(z-y)(w-y) \det\begin{pmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+xy+x^2 \\ 0 & 0 & 1 & z+y+x \\ 0 & 0 & 1 & w+y+x \end{pmatrix}$$

$$= xyzw(y-x)(z-x)(w-x)(z-y)(w-y) \det\begin{pmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+xy+x^2 \\ 0 & 0 & 1 & z+y+x \\ 0 & 0 & 0 & w-z \end{pmatrix}$$

Thus, $\det\begin{pmatrix} \overrightarrow{OX_1} \\ \overrightarrow{OX_2} \\ \overrightarrow{OX_3} \\ \overrightarrow{OX_4} \end{pmatrix} = xyzw(y-x)(z-x)(w-x)(z-y)(w-y)(w-z)$

$\neq 0$

since x, y, z, w are distinct. 20 points

* 행렬식 계산 값이 0이 아니라는 것을 명확히 보이지 않은 경우 (−10)

7. l: $r = 1 + \cos\theta \quad (0 \le \theta \le 2\pi)$

l': $r = 1 + \cos\theta \quad (0 \le \theta < \pi)$

(a) $X(\theta) = (r\cos\theta, r\sin\theta)$ 일 때.

$$|X'| = \sqrt{r^2 + (r')^2} = \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}$$

$$= \sqrt{2 + 2\cos\theta} = \sqrt{4\cos^2\frac{\theta}{2}} = 2\left|\cos\frac{\theta}{2}\right|$$

$$\frac{ds}{d\theta} = |X'| = 2\cos\frac{\theta}{2} \quad (\because 0 \le \theta < \pi)$$

$$s = \int 2\cos\frac{\theta}{2}\, d\theta = 4\sin\frac{\theta}{2}$$ 5점

$$\frac{1-\cos\theta}{2} = \sin^2\frac{\theta}{2} = \frac{s^2}{16} \Rightarrow \cos\theta = 1 - \frac{s^2}{8}$$

$$\sin\theta \underset{(\because \sin\theta \ge 0)}{=} \sqrt{1 - \cos^2\theta} = \sqrt{1 - 1 + \frac{s^2}{4} - \frac{s^4}{64}} = \frac{s}{2}\sqrt{1 - \frac{s^2}{16}}$$

$$X = \left((1+\cos\theta)\cos\theta, (1+\cos\theta)\sin\theta\right)$$

$$= \left(\left(2 - \frac{s^2}{8}\right)\left(1 - \frac{s^2}{8}\right), \left(2 - \frac{s^2}{8}\right)\frac{s}{2}\sqrt{1 - \frac{s^2}{16}}\right)$$

$$= \left(2 - \frac{3}{8}s^2 + \frac{s^4}{64}, s\left(1 - \frac{s^2}{16}\right)^{\frac{3}{2}}\right), \quad 0 \le s < 4$$

($0 \le \theta < \pi$ 이므로 $0 \le 4\sin\frac{\theta}{2} < 4$) 10점

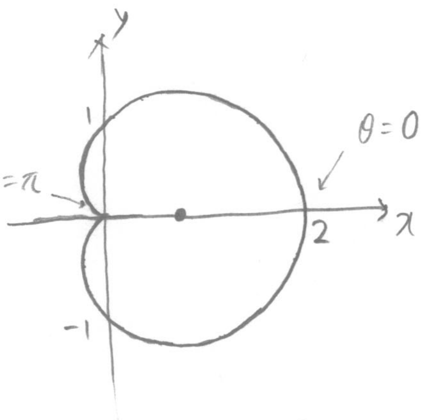
※ s의 범위를 안 쓰면 2점 감점

(b) l'의 길이가 4이므로 $s=2$인 점을 찾으면 된다.

$$s=2 \Rightarrow X=\left(2-\frac{3}{8}\cdot 4+\frac{1}{64}\cdot 16,\ 2\cdot\left(1-\frac{4}{16}\right)^{\frac{3}{2}}\right)$$

$$=\left(\frac{3}{4},\ \frac{3\sqrt{3}}{4}\right) \qquad \cdots \quad 10\text{점}$$

(c)



$r=1+\cos\theta$

곡선 l이 x축대칭이므로

$\bar{y}=0.$ $\cdots$ 5점

이고, $\bar{x}$는 곡선 l'의 중심의 x좌표와 같다.

$$\bar{x}=\frac{1}{4}\int_0^4 x\,ds$$

$$=\frac{1}{4}\int_0^4\left(2-\frac{3}{8}s^2+\frac{1}{64}s^4\right)ds$$

$$=\frac{1}{4}\left[2s-\frac{1}{8}s^3+\frac{1}{320}s^5\right]_0^4$$

$$=\frac{1}{4}\left(8-8+\frac{16}{5}\right)$$

$$=\frac{4}{5} \qquad \cdots \quad 5\text{점}$$

l의 중심은 $\left(\frac{4}{5},\ 0\right)$

#8.

평면곡선 $X(t) = (x(t), y(t))$ 에 대해,

$$\kappa(t) = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{((x'(t))^2 + (y'(t))^2)^{\frac{3}{2}}}$$ 임을 이용하자.

5점

$r = f(\theta)$ 이므로, $X(\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$ 로 둘수있다.

즉, $x(\theta) = f(\theta)\cos\theta$, $y(\theta) = f(\theta)\sin\theta$

$x'(\theta) = f'(\theta)\cos\theta - f(\theta)\sin\theta$

$x''(\theta) = f''(\theta)\cos\theta - 2f'(\theta)\sin\theta - f(\theta)\cos\theta$

$y'(\theta) = f'(\theta)\sin\theta + f(\theta)\cos\theta$

$y''(\theta) = f''(\theta)\sin\theta + 2f'(\theta)\cos\theta - f(\theta)\sin\theta$

$\Rightarrow (x'(\theta))^2 + (y'(\theta))^2 = (f'(\theta))^2 + (f(\theta))^2 = (r')^2 + r^2$

$$\begin{aligned}
x'(\theta)y''(\theta) - x''(\theta)y'(\theta) = &\{f'(\theta)f''(\theta)\sin\theta\cos\theta + 2(f'(\theta))^2\cos^2\theta - f(\theta)f'(\theta)\sin\theta\cos\theta \\
&- f(\theta)f''(\theta)\sin^2\theta - 2f(\theta)f'(\theta)\sin\theta\cos\theta + (f(\theta))^2\sin^2\theta\} \\
&- \{f'(\theta)f''(\theta)\sin\theta\cos\theta + f(\theta)f''(\theta)\cos^2\theta - 2(f'(\theta))^2\sin^2\theta \\
&- 2f(\theta)f'(\theta)\sin\theta\cos\theta - f(\theta)f'(\theta)\sin\theta\cos\theta - (f(\theta))^2\cos^2\theta \\
= &\ 2(f'(\theta))^2 - f(\theta)f''(\theta) + (f(\theta))^2 = 2(r')^2 - rr'' + r^2
\end{aligned}$$

$$\Rightarrow \kappa(\theta) = \frac{|2(r')^2 - rr'' + r^2|}{\{(r')^2 + r^2\}^{\frac{3}{2}}}$$

20점

9. $\vec{\kappa} = \frac{1}{|x'|}\left(\frac{x'}{|x'|}\right)' = \frac{1}{|x'|} \frac{x''|x'| - x' \frac{x' \cdot x''}{|x'|}}{|x'|^2}$

$$= \frac{|x'|^2 x'' - (x' \cdot x'')x'}{|x'|^4}$$

$y = e^x$의 그래프를 $x(t) = (t, e^t)$로 매개화하면

$x' = (1, e^t)$, $x'' = (0, e^t)$

$$\vec{\kappa} = \frac{(1+e^{2t})(0, e^t) - e^{2t}(1, e^t)}{(1+e^{2t})^2} = \frac{(-e^{2t}, e^t)}{(1+e^{2t})^2}$$

$$\kappa = |\vec{\kappa}| = \frac{e^t\sqrt{1+e^{2t}}}{(1+e^{2t})^2} = \frac{e^t}{(1+e^{2t})^{\frac{3}{2}}}$$ ┘ 10점

$$\kappa'(t) = \frac{e^t(1+e^{2t})^{\frac{3}{2}} - e^t \cdot \frac{3}{2}(1+e^{2t})^{\frac{1}{2}} \cdot 2e^{2t}}{(1+e^{2t})^3} = \frac{e^t(1-2e^{2t})}{(1+e^{2t})^{\frac{5}{2}}}$$

$\kappa'(t) = 0$ 인 경우는 $1 - 2e^{2t} = 0 \Leftrightarrow t = \frac{1}{2}\log\frac{1}{2}$ 뿐이다.

$t > \frac{1}{2}\log\frac{1}{2}$ 이면 $\kappa'(t) < 0$,

$t < \frac{1}{2}\log\frac{1}{2}$ 이면 $\kappa'(t) > 0$ 이므로

$t = \frac{1}{2}\log\frac{1}{2}$ 인 점이 유일한 극댓값이다.

따라서 이 경우 곡률이 최대가 되고, 곡률반경은 최소가 된다.

┘ 15점

$t = \frac{1}{2}\log\frac{1}{2}$ 일 때,

$$\vec{K}(t) = \frac{4}{9}\left(-\frac{1}{2}, \frac{1}{\sqrt{2}}\right) = \left(-\frac{2}{9}, \frac{2\sqrt{2}}{9}\right)$$

$$K(t) = \frac{2}{3\sqrt{3}}$$

접촉원의 중심은 $X(t) + \vec{K}(t)/K(t)^2$

$$= \left(\frac{1}{2}\log\frac{1}{2}, \frac{1}{\sqrt{2}}\right) + \frac{27}{4}\left(-\frac{2}{9}, \frac{2\sqrt{2}}{9}\right)$$

$$= \left(\frac{1}{2}\log\frac{1}{2} - \frac{3}{2}, 2\sqrt{2}\right)$$

25점