Midterm Exam I: April 13, 2004 .

1. (20 points) Assume that a typical raindrop is spherical in shape. Starting at time $t = 0$, the raindrop of radius r_0 falls from rest from a cloud and begins to evaporate. If it is assumed that a raindrop evaporates in such a manner that its shape remains spherical, then it also makes sense to assume that the rate of evaporation (that is, the rate at which it loses mass) is proportional to its surface area. [Note that a sphere of radius r has volume $4\pi r^3/3$ and surface area $4\pi r^2$.]

   (a) (10 points) Show that this latter assumption implies that the rate at which the radius r of the raindrop decreases is a constant. Find $r(t)$.

   (b) (10 points) From Newton's second law: $\frac{d}{dt}[mv] = mg$, derive a differential equation for the velocity $v(t)$ of the falling raindrop at time t. Ignore air resistance.

(a) Let ρ be the density of the raindrop.

mass : $m(t) = \rho \cdot \frac{4}{3}\pi r(t)^3$

surface area : $S(t) = 4\pi r(t)^2$ (+3)

Assumption : $m'(t) = k S(t)$, for some k

Conclusion : $r'(t)$ is a constant (+2)

Pf :
$$m'(t) = \rho \cdot 4\pi r(t)^2 \cdot r'(t)$$
$$k S(t) = k \cdot 4\pi r(t)^2$$
$$m'(t) = k S(t)$$
(+2)

$$\Rightarrow \rho \cdot 4\pi r(t)^2 \cdot r'(t) = k \cdot 4\pi r(t)^2$$
$$\therefore r'(t) = k/\rho$$
$$(r(t) = r_0 + (k/\rho)t)$$
(+3)

(b) $m'(t) \cdot v(t) + m(t) \cdot v'(t) = m(t) \cdot g$, where g is gravitational constant.
(+3)

$$\rho \cdot 4\pi r(t)^2 \cdot r'(t) \cdot v(t) + \rho \cdot \frac{4}{3}\pi r(t)^3 \cdot v'(t) = \rho \cdot \frac{4}{3}\pi r(t)^3 \cdot g$$
(+3)

$$3r'(t) \cdot v(t) + r(t)\, v'(t) = r(t) \cdot g$$
$$v'(t) + \frac{3r'(t)}{r(t)} v(t) = g$$
$$v'(t) + \frac{3\, v(t)}{t + r_0\rho/k} = g$$
(+4)

2. (10 points) Solve the following initial value problem

$$4x^2y'' + y = 0. \quad y(1) = 2. \ y'(1) = 4.$$

$4m(m-1)+1=0 \Rightarrow (2m-1)^2=0$ (+2)

$y = c_1\sqrt{x} + c_2\sqrt{x}\cdot\ln x$ (+3)

$y' = \frac{c_1}{2}x^{-\frac{1}{2}} + \frac{c_2}{2}x^{-\frac{1}{2}}\cdot\ln x + c_2 x^{-\frac{1}{2}}$ (+2)

$y(1)=2 \Rightarrow c_1=2$

$y'(1)=4 \Rightarrow 1+c_2=4 \Rightarrow c_2=3$ (+2)

$y = 2\sqrt{x} + 3\sqrt{x}\cdot\ln x$ (+1)

3. (10 points) Find the orthogonal trajectories of the following family of curves

$$y = \frac{1}{x+c}$$

$y' = -\frac{1}{(x+c)^2} = -y^2$ (+2)

For orthogonal trajectories:

$\frac{dy}{dx} = \frac{1}{y^2}$ (+3)

$dx - y^2dy = 0$ (+3)

$x - \frac{1}{3}y^3 = c$ (+2)

4. (15 points) Solve the following initial value problem

$$y'' - y = \cosh x = \frac{e^x + e^{-x}}{2}, \quad y(0) = 2, \; y'(0) = 12.$$

$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$ (+1)

$y_p = Kxe^x + Mxe^{-x}$ (+3)

$y_p' = Ke^x + Me^{-x} + (Kxe^x - Mxe^{-x})$

$y_p'' = 2(Ke^x - Me^{-x}) + (Kxe^x + Mxe^{-x})$ (+2)

$y_p'' - y_p = 2(Ke^x - Me^{-x}) = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

$\therefore K = \frac{1}{4}, \quad M = -\frac{1}{4}$ (+2)

$y_p = \frac{1}{4}xe^x - \frac{1}{4}xe^{-x}$

$y = y_h + y_p$

$= c_1 e^x + c_2 e^{-x} + \frac{1}{4}x(e^x - e^{-x})$ (+2)

$y' = c_1 e^x - c_2 e^{-x} + \frac{1}{4}(e^x - e^{-x}) + \frac{1}{4}x(e^x + e^{-x})$ (+2)

$y(0) = 2 \Rightarrow c_1 + c_2 = 2$

$y'(0) = 12 \Rightarrow c_1 - c_2 = 12$ (+2)

$\therefore c_1 = 7, \quad c_2 = -5$

$\therefore y(x) = 7e^x - 5e^{-x} + \frac{1}{4}x(e^x - e^{-x})$ (+1)

5. (10 points) Solve the following initial value problem

$$y'' + 4y = \begin{cases} 4t, & \text{if } 0 \le t < 1, \\ 8t - 4, & \text{if } 1 \le t < \infty; \end{cases} \qquad y(0) = 0,\ y'(0) = 0$$

$$y'' + 4y = 4t + 4(t-1)u(t-1) \quad +3$$

$$s^2Y + 4Y = \frac{4}{s^2} + \frac{4}{s^2}\cdot e^{-s} \quad +2$$

$$(s^2+4)Y = \frac{4}{s^2}\cdot(1+e^{-s})$$

$$Y = \frac{4}{s^2(s^2+4)}\cdot(1+e^{-s})$$

$$= \left(\frac{1}{s^2} - \frac{1}{s^2+4}\right)(1+e^{-s}) \quad +2$$

$$y(t) = t - \frac{1}{2}\sin 2t + \left[(t-1) - \frac{1}{2}\sin 2(t-1)\right]u(t-1)$$

$+3$

6. (15 points) Using Laplace transforms, solve the following integral equation:

$$y(t) + 2\int_0^t y(\tau)\cos(t-\tau)d\tau = 4e^{-t} + \sin t$$

$$Y + 2\cdot\frac{s}{s^2+1}\cdot Y = \frac{4}{s+1} + \frac{1}{s^2+1} \quad +3$$

$$\frac{(s+1)^2}{s^2+1}Y = \frac{4}{s+1} + \frac{1}{s^2+1}$$

$+3$

$$Y = \frac{4(s^2+1)}{(s+1)^3} + \frac{1}{(s+1)^2} \quad +3$$

$$= \frac{4(s+1)^2 - 8(s+1) + 8}{(s+1)^3} + \frac{1}{(s+1)^2} \quad +3$$

$$y(t) = 4\cdot e^{-t} - 8t\cdot e^{-t} + 4t^2\cdot e^{-t} + t\cdot e^{-t}$$

$$= (4 - 7t + 4t^2)\cdot e^{-t} \quad +3$$

7. **(20 points)** Using Laplace transforms and the relation

$$\frac{1}{s^n(s-1)} = \frac{-s^{n-1} - s^{n-2} - \cdots - 1}{s^n} + \frac{1}{s-1},$$

solve the following initial value problem:

$$y_1' = tu(t-1), \qquad y_1(0) = 1,$$
$$y_2' = 4y_1 + y_2, \qquad y_2(0) = -1.$$

+1

$$\boxed{\begin{aligned} sY_1 - 1 &= \left(\frac{1}{s^2}+\frac{1}{s}\right)\cdot e^{-s} \\ sY_2 + 1 &= 4Y_1 + Y_2 \end{aligned}}$$

+2 +1

$$\begin{bmatrix} s & 0 \\ -4 & s-1 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1+\left(\frac{1}{s^2}+\frac{1}{s}\right)e^{-s} \\ -1 \end{bmatrix} \to +1$$

+1

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \frac{1}{s(s-1)}\begin{bmatrix} s-1 & 0 \\ 4 & s \end{bmatrix}\begin{bmatrix} 1+\left(\frac{1}{s^2}+\frac{1}{s}\right)e^{-s} \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s} & 0 \\ \frac{4}{s(s-1)} & \frac{1}{s-1} \end{bmatrix}\begin{bmatrix} 1+\left(\frac{1}{s^2}+\frac{1}{s}\right)e^{-s} \\ -1 \end{bmatrix}$$

+2

$$Y_1 = \frac{1}{s} + \left(\frac{1}{s^3}+\frac{1}{s^2}\right)e^{-s} \quad +1$$

$$Y_2 = 4\left(\frac{1}{s-1}-\frac{1}{s}\right) + 4\left[\frac{1}{s^3(s-1)}+\frac{1}{s^2(s-1)}\right]e^{-s} - \frac{1}{s-1} \quad +1$$

$$= \frac{3}{s-1} - \frac{4}{s} + 4\left[\frac{-s^2-s-1}{s^3} + \frac{-s-1}{s^2} + \frac{2}{s-1}\right]e^{-s} \quad +1$$

$$y_1(t) = \underbrace{1}_{+1} + \underbrace{\left[\tfrac{1}{2}(t-1)^2 + (t-1)\right]}_{+1}\underbrace{u(t-1)}_{+1}$$

$$y_2(t) = \underbrace{3e^t}_{+1} \underbrace{-4}_{+1} + 4\left[\underbrace{-\tfrac{1}{2}(t-1)^2}_{+1} \underbrace{-2(t-1)}_{+1} \underbrace{-2}_{+1} + \underbrace{2e^{(t-1)}}_{+1}\right]u(t-1)$$

8. **(Extra Credit: 20 points)** Solve the following differential equation

$$\frac{dy}{dx} = -\frac{1}{x+y}.$$

$$\underbrace{1}_{=P} \cdot dx + \underbrace{(x+y)}_{=Q} dy = 0$$

$$\frac{1}{Q}(P_y - Q_x) = \frac{1}{x+y}(0-1) = -\frac{1}{x+y} : \text{not a function of } x$$

Let $\hat{x} = y$, $\hat{y} = x$, (+5)

$$(\hat{x} + \hat{y})\,d\hat{x} + d\hat{y} = 0 \quad (+2)$$

$$\frac{1}{\hat{Q}}(\hat{P}_{\hat{y}} - \hat{Q}_{\hat{x}}) = 1 - 0 = 1 = \hat{R}(\hat{x}) \quad (+3)$$

$$F(\hat{x}) = \exp\left(\int \hat{R}(\hat{x})\,d\hat{x}\right) = e^{\hat{x}}$$

$$F(y) = e^{y} : \text{integrating factor} \quad (+2)$$

$$e^{y} \cdot dx + e^{y}(x+y)\,dy = 0 \quad (+2)$$

$$u(x,y) = xe^{y} + k(y) \quad (+2)$$

$$\frac{\partial u}{\partial y} = xe^{y} + k'(y) = e^{y}(x+y)$$

$$\therefore k'(y) = y \cdot e^{y}$$

$$k(y) = (y-1)e^{y} + c^{*} \quad (+3)$$

$$\therefore xe^{y} + (y-1)e^{y} = c$$

$$(x+y-1)e^{y} = c \quad (+1)$$