

# Chap 1. Matrices and Gauss Elimination.

## 1.1 Introduction

### Solving Linear Equations

$$\begin{cases} 1x + 2y = 3 \\ 4x + 5y = 6 \end{cases}$$

① Elimination:

$$\begin{cases} 1x + 2y = 3 \\ -3y = -6 \end{cases} \Rightarrow y = 2 \Rightarrow x + 4 = 3 \quad \therefore x = -1$$

② Determinants (Cramer's Rule)

$$y = \frac{\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}} = \frac{-6}{-3} = 2.$$

$$x = \frac{\begin{vmatrix} 3 & 2 \\ 6 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}} = \frac{3}{-3} = -1$$

Remark:

- o. Cramer's Rule is inefficient for a large system of linear equations.
- o. Gauss Elimination is a good method that is constantly used to solve large systems of equations.

## Four aspects of linear equations

### ① Geometry of planes:

two lines meeting at the point  $(x, y) = (-1, 2)$

### ② Matrix notation:

$$\text{Factorization } A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$$

### ③ Singular case:

Two parallel lines

$$\begin{cases} x + 2y = 3 \\ 4x + 8y = 6 \end{cases} \Rightarrow 0 = -6$$

No solution!

Two overlapping lines:  $\begin{cases} x + 2y = 3 \\ 4x + 8y = 12 \end{cases} \Rightarrow 0 = 0$

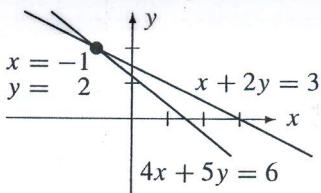
Infinitely many solutions!

### ④ The number of elimination steps:

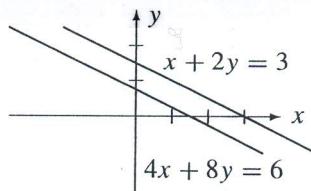
o. The computing cost often dominates the accuracy.

100 equations requires  $\approx \frac{1}{3} \times 10^6$  steps,

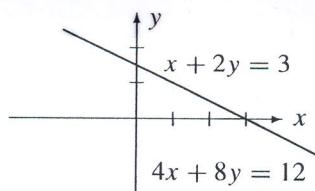
$\Rightarrow$  the round off error could be quite significant.



One solution  $(x, y) = (-1, 2)$



Parallel: No solution



Whole line of solutions

Figure 1.1 The example has one solution. Singular cases have none or too many.

## 1.2 The Geometry of Linear Equations

$$\begin{cases} 2x - y = 1 \\ x + y = 5 \end{cases}$$

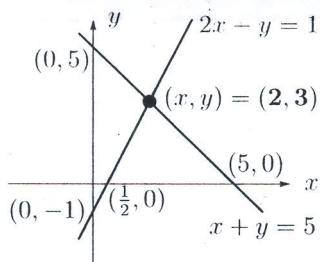
By Rows:

Two straight lines intersecting at  $(x, y) = (2, 3)$

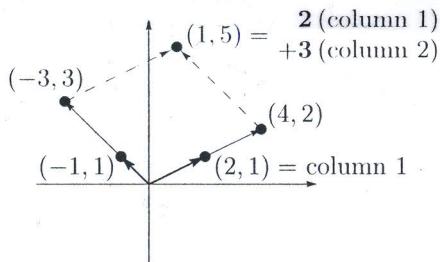
By Columns:

One vector equation  
 Column form:  $x \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

To find the combination of the column vectors  
 on the left that produces the vector on the right.



(a) Lines meet at  $x = 2, y = 3$



(b) Columns combine with 2 and 3

**Figure 1.2** Row picture (two lines) and column picture (combine columns).

[ Row Picture : Intersection of planes  
 Column Picture : Combination of columns ]

### Three Planes

$$\begin{cases} 2u + v + w = 5 \\ 4u - 6v = -2 \\ -2u + 7v + 2w = 9 \end{cases}$$

three planes  
 intersecting at  
 (1, 1, 2)

### Column Vectors and Linear Combinations

Column form:  $u \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + v \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} + w \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = b$

Linear combination:  $1 \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$

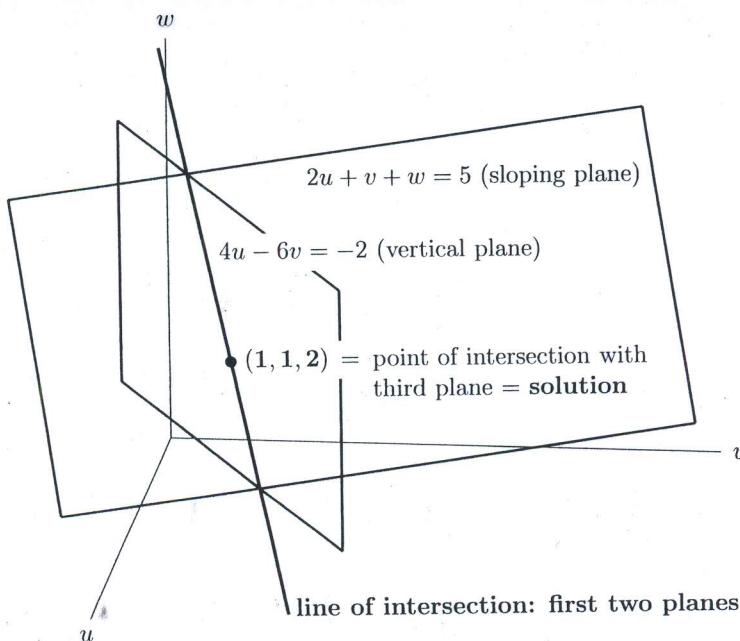


Figure 1.3 The row picture: three intersecting planes from three linear equations.

## The Singular Case

(a) Two planes may be parallel

$$\begin{cases} 2u + v + w = 5 \\ 4u + 2v + 2w = 11 \end{cases} \quad ] \text{ inconsistent}$$

parallel planes give no solution

(b) Every pair of planes intersects in a line, and those lines are parallel.

$$\begin{cases} u + v + w = 2 \\ 2u + 3v = 5 \\ 3u + v + 4w = 6 \end{cases} \quad \begin{matrix} \text{add up} \\ \text{to} \end{matrix} \quad \begin{cases} 3u + v + 4w = 7 \\ \downarrow \\ \text{inconsistent} \end{cases}$$

The third plane is not parallel to the other planes, but it is parallel to their line of intersection.

(c) Another singular system has an infinity of solutions

$$\begin{cases} u + v + w = 2 \\ 2u + 3v = 5 \\ 3u + v + 4w = 7 \end{cases} \quad \begin{matrix} \text{add up} \\ \text{to} \end{matrix} \quad \begin{cases} 3u + v + 4w = 7 \\ \downarrow \\ \text{the same} \end{cases}$$

too many solutions

(d) Three parallel planes:

No solution or a whole plane of solutions.

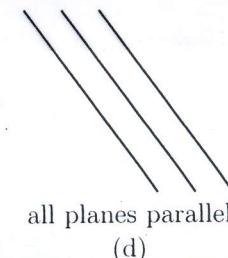
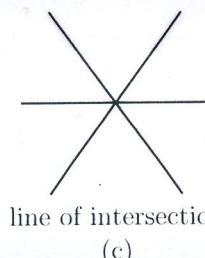
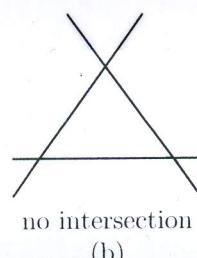
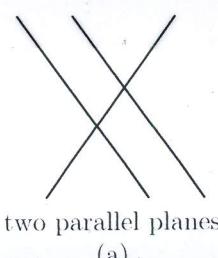


Figure 1.5 Singular cases: no solution for (a), (b), or (d), an infinity of solutions for (c).

## The Singular Case : Column Picture.

Three columns in the same plane

Solvable only for  $b$  in that plane

$$u \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + v \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = b$$

o. For  $b = (2, 5, 6)$ , it is non-solvable.

o. For  $b = (2, 5, 7)$ , it is possible and there are  
too many solutions.

The three columns can be combined in infinitely many ways to produce  $b$ .

When  $u=3, v=-1, w=-2$ ,

$$3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (-2) \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(u, v, w) = (1, 0, 1)$  is a solution, and we can add any multiple of  $(3, -1, -2)$  to  $(u, v, w) = (1, 0, 1)$  to produce a solution.

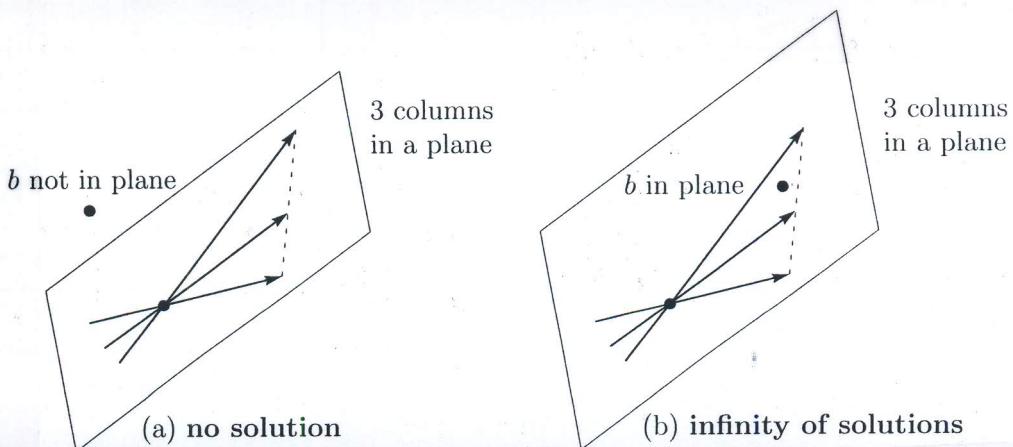


Figure 1.6 Singular cases:  $b$  outside or inside the plane with all three columns.

### 1.3 Example of Gaussian Elimination $\rightarrow$ pivots

$$\begin{cases} 2u + v + w = 5 \\ 4u - 6v = -2 \\ -2u + 7v + 2w = 9 \end{cases} \Rightarrow \begin{cases} (2u + v + w) = 5 \\ (-8v - 2w) = -12 \\ 8v + 3w = 14 \end{cases}$$

Forward  
Elimination

Backward  
Substitution

$$\begin{cases} (2u + v + w) = 5 \\ (-8v - 2w) = -12 \\ 1w = 2 \end{cases}$$

Triangular System

$$w=2, v=1, u=1$$

#### Remark

$$\left[ \begin{array}{cccc|c} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{array} \right] \xrightarrow{\text{Row 2} \rightarrow 2\text{Row 2}} \left[ \begin{array}{cccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & 2 & -12 \\ 0 & 8 & 3 & 14 \end{array} \right] \xrightarrow{\text{Row 3} \rightarrow \text{Row 3} + \text{Row 2}} \left[ \begin{array}{cccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & 2 & -12 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

The right-hand side as an extra column

### The Breakdown of Elimination

- o Pivots cannot be zero
- o If a zero appears in a pivot position,
  - i non-Singular case, exchange rows (Sec 1.5)
  - ii Singular case, either no solution or infinitely many solutions  
(Chap 2 deals with the singular cases.)

### Ex1 (Non-Singular Case)

$$\begin{cases} u + v + w = - \\ 2u + 2v + 5w = - \\ 4u + 6v + 8w = - \end{cases} \Rightarrow \begin{cases} u + v + w = - \\ 2u + 2v + 5w = - \\ 4u + 6v + 8w = - \end{cases}$$

↓  
exchange  
equations  
2 & 3

$$\begin{cases} u + v + w = - \\ 2u + 2v + 5w = - \\ 3w = - \end{cases}$$

### Ex2 (Singular Case)

$$\begin{cases} u + v + w = - \\ 2u + 2v + 5w = - \\ 4u + 4v + 8w = - \end{cases} \Rightarrow \begin{cases} u + v + w = - \\ 3w = - \\ 4w = - \end{cases}$$

No exchange of equations  
can avoid zero  
in the second pivot position.

#### Case1: (No Solution)

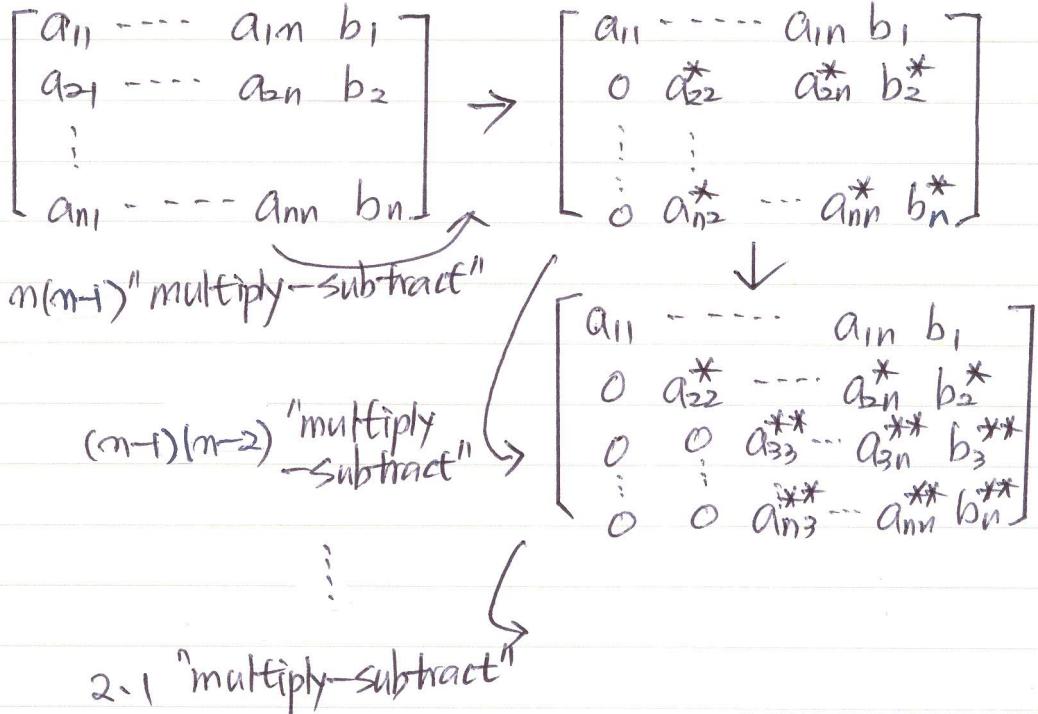
$$\begin{cases} u + v + w = - \\ 3w = b \\ 4w = \gamma \end{cases} \quad ) \text{ unsolvable}$$

#### Case2: (Infinitely many solutions)

$$\begin{cases} u + v + w = - \\ 3w = b \\ 4w = \gamma \end{cases} \quad ) \text{ consistent } (w=2)$$

But, we cannot decide  
 $u$  and  $v$ .

# The Cost of Elimination



Total "multiply-subtract":

$$\sum_{k=1}^{m-1} (k+1) \frac{1}{k} = \frac{1}{3} (m-1)m(m+1) \approx \underline{\underline{\frac{1}{3} \cdot n^3}}$$

## 1.4 Matrix Notation and Matrix Multiplication

$$\begin{cases} 2u + v + w = 5 \\ 4u - 6v = -2 \\ -2u + 7v + 2w = 9 \end{cases}$$

Nine coefficients

Three unknowns

Three right-hand sides

$$\mathbf{x} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} : \text{The unknown}; \quad \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} : \text{The solution}$$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} : \text{Coefficient matrix}$$

$$\text{Matrix form} \quad A\mathbf{x} = \mathbf{b} \quad : \quad \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

Two ways to multiply a matrix  $A$  by a vector  $\mathbf{x}$

①  $A\mathbf{x}$  by rows:

$$\begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + 1 \cdot 5 + 6 \cdot 0 \\ 3 \cdot 2 + 0 \cdot 5 + 1 \cdot 0 \\ 1 \cdot 2 + 1 \cdot 5 + 4 \cdot 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 7 \end{bmatrix}$$

②  $A\mathbf{x}$  by columns:

$$2 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 6 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 7 \end{bmatrix}$$

1A  $A\mathbf{x}$  is a combination of the columns of  $A$ .

The coefficients are the components of  $\mathbf{x}$ .

o. The  $i$ th component of  $A\mathbf{x}$  is  $\sum_{j=1}^n a_{ij}x_j$ ,  $\mathbf{x} = (x_j)_{j=1}^m$ .

## The Matrix Form of One Elimination Step.

$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  : Elementary matrix that subtracts  $2 \times (\text{the first equation})$  from the second

$$E\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 9 \end{bmatrix}$$

**[B]** The elementary matrix  $E_{ij}$  subtracts  $l$  times row  $j$  from row  $i$ , where  $E_{ij}$  includes  $-l$  in row  $i$ , column  $j$ .

Ex

$$E_{31}\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 - lb_1 \end{bmatrix}$$

## Matrix Multiplication

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix}$$

"Associative Law"

Matrix multiplication:  $(EA)\mathbf{x} = E(A\mathbf{x}) = EA\mathbf{x}$   
 $\Rightarrow$  We may write  $E(A\mathbf{x}) = E\mathbf{b}$  or  $(EA)\mathbf{x} = E\mathbf{b}$ .

Multiplication by columns:

$$AB = A[b_1 \ b_2 \ b_3] = [Ab_1 \ Ab_2 \ Ab_3]$$

**[IC]**  $A = (a_{ik})$ ,  $B = (b_{kj})$   
 $\Rightarrow (AB)_{ij} = a_{i1}b_{1j} + \dots + a_{in}b_{nj}$ .

**[ID]**

(i) Each entry of  $AB$  is the product of a row and a column:

$$(AB)_{ij} = (\text{row } i \text{ of } A) \text{ times } (\text{column } j \text{ of } B)$$

(ii) Each column of  $AB$  is the product of a matrix and a column:

$$\text{column } j \text{ of } AB = A \text{ times } (\text{column } j \text{ of } B)$$

(iii) Each row of  $AB$  is the product of a row and a matrix:

$$\text{row } i \text{ of } AB = (\text{row } i \text{ of } A) \text{ times } B.$$

**[IE]** Matrix multiplication is associative:

$$(AB)C = A(BC) = ABC$$

**[IF]** Matrix operations are distributive:

$$A(B+C) = AB+AC \text{ and}$$

$$(B+C)D = BD+CD$$

**[IG]** Matrix Multiplication is not commutative:

Usually  $FE \neq EF$

Ex4: But  $EF=FE$ , for these two  $\Rightarrow E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ,  $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$$\text{Ex1: } AB = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 5 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 11 & 10 \\ 4 & 8 & 0 \end{bmatrix}$$

$$\text{Ex2: } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 8 \end{bmatrix} = \begin{bmatrix} ? & ? \\ 2 & 3 \end{bmatrix} : \text{row exchange matrix}$$

$$\text{Ex3: } IA = A, \quad BI = B \quad \text{for } I: \text{identity matrix}$$

$$\text{Ex4: } E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$E\bar{E} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \bar{E}E$$

$$\text{Ex5: } G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow GE \neq EG$$

$$GE = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = EG$$

$$GFE = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$



The true order of elimination  
 $GFEA = U$  ← upper triangular

$$EFG = \begin{bmatrix} 1 & 0 & 0 \\ \cancel{-2} & 1 & 0 \\ \cancel{0} & \cancel{0} & 1 \end{bmatrix}$$

↳ these numbers are from  $E, \bar{E}, G$

↳ The right order for reversing the elimination steps.

## 1.5 Triangular Factors and Row Exchanges.

$$A\mathbf{x} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = \mathbf{b}$$

Elementary matrices:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow GFEA\mathbf{x} = U\mathbf{x} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix} = \mathbf{c}$$

(upper triangular)

$U\mathbf{x} = \mathbf{c}$  is solved by back-substitution

$$GFE = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

$$E^T F^T G^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = L$$

multipliers  $\ell = 2, -1, -1$

Triangular factorization  $A = LU$

with no exchanges of rows.

$L$ : lower triangular with 1s on the diagonal and multipliers  $l_{ij}$  below the diagonal

$U$ : the upper triangular matrix which appears after forward elimination.

The diagonal entries of  $U$  are the pivots.

$$\underline{\text{Ex1}} \quad A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = LU$$

$$\underline{\text{Ex2}} \quad A = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} \text{ cannot be factored into } A = LU$$

$$\underline{\text{Ex3}} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = LU$$

Ex4

$$A = \begin{bmatrix} 1 & 0 & 0 \\ b_{21} & 1 & 0 \\ b_{31} & b_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ b_{21} & 1 & 0 \\ b_{31} & b_{32} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = LU$$

$$A = LU : \begin{bmatrix} 1 & 0 & 0 \\ b_{21} & 1 & 0 \\ b_{31} & b_{32} & 1 \end{bmatrix} \begin{bmatrix} \text{row 1 of } U \\ \text{row 2 of } U \\ \text{row 3 of } U \end{bmatrix} = A$$

Apply the steps of elimination to both sides.  
Left side = the result of forward elimination  
to the right-hand side.]

Ex5

$$A = \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ -1 & 2 & -1 & \\ -1 & 2 & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

L                            U

One Linear System = Two Triangular Systems

① Factor:  $A = LU$

② Solve:  $L\mathbf{c} = \mathbf{b}$  and  $U\mathbf{x} = \mathbf{c}$  ( $\because A\mathbf{x} = L\cancel{U}\mathbf{x} = \mathbf{b}$ )

⇒ The solution for any new right-hand side  $\mathbf{b}$  can be found in only  $n^2$  operations.

(Far below the  $\frac{1}{3}n^3$  steps needed to factor  $A$ .)

Ex 6:

$$A\mathbf{x} = \mathbf{b} : \left\{ \begin{array}{l} x_1 - x_2 = 1 \\ -x_1 + 2x_2 - x_3 = 1 \\ -x_2 + 2x_3 - x_4 = 1 \\ -x_3 + 2x_4 = 1 \end{array} \right\} \Rightarrow \begin{array}{l} \text{splits into} \\ L\mathbf{c} = \mathbf{b} \text{ and} \\ U\mathbf{x} = \mathbf{c} \end{array}$$

$$L\mathbf{c} = \mathbf{b} : \left\{ \begin{array}{l} c_1 = 1 \\ -c_1 + c_2 = 1 \\ -c_2 + c_3 = 1 \\ -c_3 + c_4 = 1 \end{array} \right\} \Rightarrow \mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$U\mathbf{x} = \mathbf{c} : \left\{ \begin{array}{l} x_1 - x_2 = 1 \\ x_2 - x_3 = 2 \\ x_3 - x_4 = 3 \\ x_4 = 4 \end{array} \right\} \Rightarrow \mathbf{x} = \begin{bmatrix} 10 \\ 9 \\ 7 \\ 4 \end{bmatrix}$$

For these special tridiagonal matrices,  
the operation count drops from  $n^2$  to  $2n$ .

→ unique ( $L_1 D_1 U_1 = L_2 D_2 U_2 \Rightarrow L = L_2, R = D_2, U = U_2$ )

Remark:  $(A = LDU)$ , where  $L, U$ : Is on the diagonal  
 $D$ : the diagonal matrix of pivots

$$\begin{aligned} \text{Ex: } A &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \\ &= LDU. \end{aligned}$$

## Row Exchanges and Permutation Matrices

$$A\mathbf{x} = \mathbf{b} : \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Permutation zero pivot

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow PA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$$

(The unknowns  $u$  and  $v$  are not reversed!)

i In the nonsingular case, there is a permutation matrix  $P$  that reorders the rows of  $A$  to avoid zeros in the pivot positions. Then  $A\mathbf{x} = \mathbf{b}$  has a unique solution.

With the rows reordered in advance,

$PA$  can be factored into  $LU$ .

② In the singular case, no  $P$  can produce a full set of pivots: elimination fails.

\*  $P^T$  is always the same as  $P^T$

\* In practice, we exchange rows (and also columns) when the pivot is near zero to reduce the roundoff error.

Ex 1

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix} = U$$

row exchange, now  $b_{31} = 1, b_{21} = 2$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \underline{PA = LU}$$

## 1.6 Inverses and Transposes

Inverse matrix.  $\mathbf{b} = \mathbf{A}\mathbf{x} \Rightarrow \mathbf{A}^T\mathbf{b} = \mathbf{x}$   
 Thus  $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{x}$

Not all matrices have inverses.

An inverse is impossible when  $A\mathbf{x} = \mathbf{0}$  and  $\mathbf{x} \neq \mathbf{0}$ .

The inverse of A is a matrix B s.t.  $BA = I = AB$ .

There is at most one such  $B$ , denoted by  $A^+$ .

$$A^T A = I \quad , \quad A A^T = I.$$

## Notes

① The inverse exists if and only if

allowed

elimination produces n pivots (row exchanges).  
 Elimination solves  $Ax = b$  without explicitly finding  $x$ .

$$\textcircled{2} \quad BA = I \text{ and } AC = I \Rightarrow B = C$$

$$\text{For } B(AC) = (BA)C \Rightarrow \underline{\underline{B=C}}$$

left-inverse  $\rightarrow$  right-inverse

③ If  $A$  is invertible, the one and only one solution to  $Ax = b$  is  $x = A^{-1}b$ .

## ④ (Important!)

Suppose there is a nonzero vector  $\underline{x} \neq \underline{0}$  s.t.  $A\underline{x} = \underline{0}$ .

Then  $A$  cannot have an inverse.  $\uparrow$  Contradiction  $\downarrow$

( $\diamond$  If  $A$  is invertible, then  $A\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$   $\#$ )

$$\textcircled{5} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is invertible} \Leftrightarrow \textcircled{\text{ad}-bc \neq 0}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{determinant of } A$$

o. A matrix is invertible if its determinant is non-zero.

⑥ If  $A = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$ , then  $A^{-1} = \begin{bmatrix} 1/d_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1/d_n \end{bmatrix}$

and  $A^{-1}A = I$

[1L] For invertible matrices A and B,

the inverse of  $AB$ :  $(AB)^{-1} = B^{-1}A^{-1}$ .

$\therefore (CAB)(B^{-1}A^{-1}) = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$   
 $(B^{-1}A^{-1})(AB) = B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$

The Calculation of  $A^{-1}$ : The Gauss-Jordan Method

$\ast_i$ : columns of  $A^{-1} \Rightarrow A\ast_i = e_i \leftarrow$  Solve for  $i=1, \dots, n$

$$A\ast_i = e_i: \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} \ast_1 & \ast_2 & \ast_3 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ex1: (Gauss-Jordan)

$$\left[ A \ e_1 \ e_2 \ e_3 \right] = \left[ \begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 1 & 2 & 0 & 0 & 1 \end{array} \right] \quad \text{The First Half Forward Elimination}$$

pivot = 2  $\rightarrow$   $\left[ \begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{array} \right]$  GFE

pivot = -8  $\rightarrow$   $\left[ \begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array} \right] = \left[ \begin{array}{c|cc} U & & \\ \hline L^{-1} & 1 & 1 & 1 \end{array} \right]$

to be continued  $\rightsquigarrow$

→ Second Half that takes  $U$  to  $I$  &  $L^T \xrightarrow{\text{to}} U L^T = A^T$ ,  
 Creating zeros above the pivots, we reach  $A^T$

$$\begin{bmatrix} U & L^T \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

zeros above  
the pivots

$$\rightarrow \begin{bmatrix} 2 & 0 & 0 & \frac{12}{8} & -\frac{5}{8} & -\frac{6}{8} \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{12}{16} & -\frac{5}{16} & -\frac{6}{16} \\ 0 & 1 & 0 & \frac{4}{8} & -\frac{3}{8} & -\frac{2}{8} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} I & A^T \end{bmatrix}$$

The determinant is the product of pivots (2)(-8)(1).

Remark 1 (Don't Recommend it for solving  $Ax = b$ !)

Two triangular steps are better!

$x = A^T b$  separates into  $Lc = b$  and  $Ux = c$ .

⇒  $\frac{1}{2}n^2$  steps for each.

o It is a waste of time to form  $L^T$  and  $U^T$

o The multiplication  $A^T b$  would still take  $n^2$  steps

It is the solution that we want, and  
not all the entries on the inverse.

Remark 2 (Operation count for  $A^{-1}$ )

$$\frac{n^3}{6} + \frac{n^3}{3} + n\left(\frac{n^2}{2}\right) = n^3$$

↳ LU factorization      ↳ back-substitutions

↳ Forward Eliminations:  $\frac{1}{2}(n-j)^2$  for column  $e_j$ .

Invertible = Nonsingular (n pivots)

Suppose  $A$  has a full set of  $n$  pivots

$AA^{-1} = I$  gives  $A\mathbf{x}_i = \mathbf{e}_i$ ,  $\mathbf{x}_i$ : columns of  $A^{-1}$   
solved by elimination or Gauss-Jordan

$A^{-1} = [\mathbf{x}_1 \dots \mathbf{x}_n]$  is also a left-inverse.

(i.e. Solving  $AA^{-1} = I$  also solved  $A^{-1}A = I$ ).

∴ A 1-sided inverse of a squared matrix

$\Leftrightarrow$  is automatically a 2-sided inverse.

Every Gauss-Jordan step is a multiplication  
on the left by an elementary matrix:

①  $E_{ij}$ , ②  $P_{ij}$ , or ③  $D$  (or  $D^{-1}$ ) to divide by pivots.

$$\underbrace{(D^{-1} \dots E \dots P \dots E)}_{\hookrightarrow \text{The left inverse of } A} A = I$$

$\hookrightarrow$  The left inverse of  $A$

If exists, it equals the right inverse by Note 2.

∴ Every nonsingular matrix is invertible.

$\Rightarrow$  A cannot have a whole column of zeros.

Suppose elimination breaks down at some column

$$A \rightarrow \dots \rightarrow A' = \begin{bmatrix} d_1 & * & * & * \\ 0 & d_2 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{bmatrix} \rightarrow \boxed{\text{This matrix cannot have an inverse.}}$$

Using column operations, i.e. reparametrization,  
the whole third column can be made zero.

Therefore, the original  $A$  was not invertible.]

## The Transpose Matrix

Entries of  $A^T$ :  $(A^T)_{ij} = A_{ji}$

①  $(AB)^T = B^T A^T$

②  $(A^{-1})^T = (A^T)^{-1}$

∴ ① The first row of  $(AB)^T$  is the first column of  $AB$ .

The columns of  $A$  are weighted by the first column  
Equivalently, of  $B$ .  
the rows of  $A^T$  are weighted by

↳ } the first rows of  $B^T$   
exactly the first row of  $B^T A^T$ .

The other rows of  $(AB)^T$  and  $B^T A^T$  also agree.

②

$$AA^T = I \text{ and } A^T A = I$$

$$(A^{-1})^T A^T = I^T = I \text{ and } A^T (A^{-1})^T = I^T = I$$

$$\therefore (A^T)^{-1} = (A^{-1})^T \quad \boxed{\quad}$$

## Symmetric Matrices

$A$ : symmetric  $\Leftrightarrow A^T = A \Rightarrow A$  is a square matrix

A symmetric matrix need not be invertible. (ex: zero matrix)

But, if  $A^T$  exists, it is also symmetric.

$$\therefore (A^{-1})^T = (A^T)^{-1} = A^T \Leftarrow A^T = A$$

$$\therefore A^T \text{ is symmetric} \quad \boxed{\quad}$$

## Symmetric Products $R^T R$ , $R R^T$ , $L D L^T$

$$(R^T R)^T = R^T (R^T)^T = R^T R$$

$R R^T$  is also symmetric, but it is different from  $R^T R$  even if  $R$  and  $R^T$  are square matrices.

[IN] Suppose  $A = A^T$  can be factored into  $A = LDU$  without row exchanges. Then  $U = L^T$ .

$$\text{From } A = LDU, \quad A^T = U^T D^T L^T,$$

$$LDU = U^T D^T L^T$$

By the uniqueness of the  $LDU$  factorization  
(Problem 20 of page 53),  $L = U^T$  ]

Ex:  $\begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = L D L^T$

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \rightarrow \begin{bmatrix} a & b & c \\ 0 & d - \frac{b^2}{a} & e - \frac{bc}{a} \\ 0 & e - \frac{bc}{a} & f - \frac{c^2}{a} \end{bmatrix}$$

The smaller matrices stay symmetric as elimination proceeds, and we can work with half the matrix.  
⇒ The work of elimination is reduced from  $\frac{1}{3}n^3$  to  $\frac{1}{8}n^3$ .