Engineering Mathematics I
(Comp 400.001)

Midterm Exam I: April 15, 2000

Solution Set

Problem	Score	Problem	Score
1		8	
2		9	
3		10	
4		11	
5		12	
6		13	
7		14	
		Total	

Name: ____________________

ID No: ____________________

1. (20 points) A student borrows \$8000 to buy a car. The lender charges interest at an annual rate of 10%. Assuming that interest is compounded continuously and that the borrower makes payments continuously at a constant annual rate k, determine the payment rate k that is required to pay off the loan in three years. Also determine how much interest is paid during the three-year period. **You need to set up a differential equation and solve it. Don't reply on your high school math.**

Ⓐ $y'(t) = 0.1\, y(t) - k$ +10

$\left[\; y'(t) = (0.1 - k)\, y(t) \quad +5 \;\right]$

이렇게하면 partial credit

$$\frac{dy}{y - 10k} = \frac{dt}{10}$$

$$\ln(y - 10k) = \frac{1}{10}t + c$$

$$y = 10k + \alpha \cdot e^{\frac{1}{10}t}$$

$$y(0) = 10k + \alpha = 8000$$

$$\therefore \alpha = 8000 - 10k$$

$$y(t) = 10k + (8000 - 10k)\, e^{\frac{1}{10}t}$$

$$10k + (8000 - 10k)\, e^{\frac{3}{10}} = 0$$

$$\therefore k = 800 \cdot \frac{e^{0.3}}{e^{0.3} - 1}.$$

+5

Ⓑ
$$\int_0^3 0.1\, y(t)\, dt = \int_0^3 \frac{1}{10} \cdot \left[8000 \cdot \frac{e^{0.3}}{e^{0.3}-1} + 8000 \cdot \frac{-1}{e^{0.3}-1} e^{\frac{1}{10}t} \right] dt$$

$$= 800 \cdot \frac{10 - 7 \cdot e^{0.3}}{e^{0.3} - 1}$$

+5

or

$$3 \cdot k - 8000 = 800 \cdot \frac{3e^{0.3}}{e^{0.3}-1} - 800 \cdot \frac{10(e^{0.3}-1)}{e^{0.3}-1}$$

$$= 800 \cdot \frac{10 - 7 \cdot e^{0.3}}{e^{0.3} - 1}$$

2. (10 points) Solve the following initial value problem

$$xy' = (y-x)^3 + y, \quad y(1) = \frac{3}{2}.$$

$u = y - x$
$u' = y' - 1$
$u'' = y''$
$\Rightarrow$

$x \cdot (u'+1) = u^3 + u + x, \quad u(1) = \frac{3}{2} - 1 = \frac{1}{2}$

$x \cdot u' = u(u^2+1)$

$\frac{du}{u(u^2+1)} = \frac{dx}{x}$

+5

$\left[\frac{1}{u} - \frac{1}{2} \cdot \frac{2u}{u^2+1}\right] du = \frac{1}{x} \cdot dx$

$\ln|u| - \frac{1}{2}\ln(u^2+1) = \ln|x| + c$

$\frac{u^2}{u^2+1} = \alpha \cdot x^2$

$\therefore \alpha = \frac{1}{5}$

+3

$u = \frac{x}{\sqrt{5-x^2}}, \quad y = u + x = \frac{x + x\sqrt{5-x^2}}{\sqrt{5-x^2}}$

+2

↳ 사소한 실수는 (-1)

3. (10 points) Solve the following equation

$$(2\cos y + 4x^2)dx = x \sin y dy.$$

$P = 2\cos y + 4x^2, \quad Q = -x\sin y$

$\frac{1}{Q}(P_y - Q_x) = \frac{1}{-x\sin y}(-2\sin y + \sin y) = \frac{1}{x}$

$F(x) = \exp\left(\int \frac{1}{x} dx\right) = \exp(\ln x) = x$

$(2x\cos y + 4x^3)dx - x^2 \sin y\, dy = 0$

+5

$u(x,y) = \int(-x^2 \sin y)dy = x^2\cos y + f(x)$

$\frac{\partial u}{\partial x} = 2x\cos y + f'(x) = 2x\cos y + 4x^3$

$\therefore f(x) = x^4 + c$

$\therefore u(x,y) = x^2 \cos y + x^4 + c = 0$

+5

4. (~~10~~ 5 points) Solve the following equation

$$xy' + 4y = 8x^4, \ \ y(1) = 2.$$

$$y' + \frac{4}{x}y = 8x^3$$

$$y = e^{-\int \frac{4}{x}dx}\left[\int e^{\int \frac{4}{x}dx} \cdot 8x^3 dx + c\right] \quad (+3)$$

$$= e^{-4\ln x}\left[\int e^{4\ln x} \cdot 8x^3 dx + c\right]$$

$$= x^{-4}\left[\int 8x^7 dx + c\right]$$

$$= x^{-4}\left[x^8 + c\right] = x^4 + c \cdot x^{-4} \quad (+1)$$

$$y(1) = 1 + c = 2, \quad \therefore c = 1$$

$$\therefore y(x) = x^4 + x^{-4} \quad (+1)$$

5. (5 points) Find the general solution of the following equation

$$x^2y'' + 2xy' + 2y = 0, \quad t > 0.$$

$$m(m-1) + 2m + 2 = 0$$

$$m^2 + m + 2 = 0$$

$$m = -\frac{1}{2} \pm \frac{\sqrt{7}}{2}i \quad (+2)$$

$$y = x^{-\frac{1}{2}}\left[A\cos\left(\frac{\sqrt{7}}{2}\ln x\right) + B\sin\left(\frac{\sqrt{7}}{2}\ln x\right)\right] \quad (+3)$$

중간과정에서 약간의 실수 (-1)

많은 실수 (-2)

6. (10 points) Find the general solution of the following equation

$$y'' - 5y' + 4y = 3 + 2e^x.$$

$$\lambda^2 - 5\lambda + 4 = 0$$

$$(\lambda - 1)(\lambda - 4) = 0$$

$$\therefore y_h = c_1 e^x + c_2 e^{4x}$$

+3

$$y_p = c_3 + c_4 x e^x$$

+5

$$y_p' = c_4 e^x + c_4 x e^x$$

$$y_p'' = 2c_4 e^x + c_4 x e^x$$

$$\Rightarrow \quad c_3 = \frac{3}{4}, \quad c_4 = -\frac{2}{3}$$

$$\therefore y_p = \frac{3}{4} - \frac{2}{3} x e^x$$

+2

$$\therefore y = y_h + y_p = c_1 e^x + c_2 e^{4x} + \frac{3}{4} - \frac{2}{3} x e^x.$$

7. (10 points) Find the general solution of the following equation

$$x^2y'' - 4xy' + 6y = 7x^4 \sin x.$$

$m(m-1) - 4m + 6 = 0$

$m^2 - 5m + 6 = 0$ (+3)

$m = 2$ or 3

$y_1 = x^2, \quad y_2 = x^3, \quad W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^4$

$y_p = -x^2 \int \frac{x^3}{x^4} \cdot 7x^2 \sin x + x^3 \int \frac{x^2}{x^4} \cdot 7x^2 \sin x \, dx$ (+3)

$= \cdots\cdots$

$= -7x^2 \sin x$ (+2)

$y = y_h + y_p$ (+2)

$= c_1 x^2 + c_2 x^3 - 7x^2 \sin x$

8. (15 points) Three solutions of a second-order nonhomogeneous linear equation

$$L[y] = g(x)$$

are

$$\psi_1(x) = 3e^x, \; \psi_2(x) = 7e^x + e^{x^2}, \text{ and } \; \psi_3(x) = 5e^x + e^{-x^3} + e^{x^2}.$$

Find the solution of the following initial-value problem

$$L[y] = g; \; y(0) = 1, y'(0) = 2.$$

$\psi_2(x) - \psi_1(x) = 4e^x + e^{x^2}$

$\psi_3(x) - \psi_2(x) = -2e^x + e^{-x^3}$

+10

$y_h(x) = c_1(4e^x + e^{x^2}) + c_2(-2e^x + e^{-x^3})$

$y_p(x) = 3e^x$

$y(x) = y_h(x) + y_p(x)$

$= c_1(4e^x + e^{x^2}) + c_2(-2e^x + e^{-x^3}) + 3e^x$

+3

$y(0) = 1$: $\quad 5c_1 - c_2 = -2$

$y'(0) = 2$: $\quad 4c_1 - 2c_2 = -1$

$c_1 + c_2 = -1$

$\therefore \; c_1 = -\frac{1}{2}, \; c_2 = -\frac{1}{2}.$

+2

9. (15 points) Solve the following equation

$$y'' + 2y' - 3y = 8e^{-t} + \delta(t - \frac{1}{2}), \quad y(0) = 3, y'(0) = -5.$$

$$s^2Y - 3s + 5 + 2sY - 6 - 3Y = 8 \cdot \frac{1}{s+1} + e^{-\frac{1}{2}s}$$

$$(s^2 + 2s - 3)Y = 3s + 1 + \frac{8}{s+1} + e^{-\frac{1}{2}s}$$

+5

$$Y = \frac{3s+1}{(s+3)(s-1)} + \frac{8}{(s+3)(s+1)(s-1)} + \frac{1}{(s+3)(s-1)} e^{-\frac{1}{2}s}$$

$$= \frac{2}{s+3} + \frac{1}{s-1} + \frac{1}{s+3} + \frac{1}{s-1} + \frac{-2}{s+1}$$

$$+ \frac{1}{4}\left[\frac{-1}{s+3} + \frac{1}{s-1}\right] e^{-\frac{1}{2}s}$$

$$= \frac{3}{s+3} + \frac{2}{s-1} + \frac{-2}{s+1} + \frac{1}{4}\left[\frac{-1}{s+3} + \frac{1}{s-1}\right] e^{-\frac{1}{2}s}$$

+5

$$y(t) = 3 \cdot e^{-3t} - 2e^{-t} + 2e^{t} + \frac{1}{4}\left[-e^{-3(t-\frac{1}{2})} + e^{t-\frac{1}{2}}\right] u(t-\tfrac{1}{2})$$

10. (15 points) Applying convolution, find the solution of the following equation

$$y'' + 3y' + 2y = r(t);\ y(0) = 0, y'(0) = 0,$$

where

$$r(t) = \begin{cases} 4t & \text{if } 0 < t < 1, \\ 8 & \text{if } t > 1. \end{cases}$$

$$s^2Y + 3sY + 2Y = R(s)$$

$$Y = Q(s) \cdot R(s) = \frac{1}{s^2+3s+2} \cdot R(s)$$

$$Q(s) = \frac{1}{s+1} - \frac{1}{s+2},\ q(t) = e^{-t} - e^{-2t}$$

+5

$$y(t) = q(t) * r(t)$$

Case I: $0 < t < 1$:

$$y(t) = q(t) * r(t) = r(t) * q(t)$$

$$= \int_0^t r(\tau)\, q(t-\tau)\, d\tau = \int_0^t 4\tau \cdot \left[e^{-(t-\tau)} - e^{-2(t-\tau)}\right] d\tau$$

$$= 4e^{-t} \int_0^t \tau \cdot e^{\tau} d\tau - 4e^{-2t} \int_0^t \tau \cdot e^{2\tau} d\tau$$

$$= 4e^{-t}\left[\tau e^{\tau} - e^{\tau} + 1\right]_0^t - 4e^{-2t}\left[\frac{1}{2}\tau \cdot e^{2\tau} - \frac{1}{4}e^{2\tau} + \frac{1}{4}\right]_0^t$$

$$= 2t - 3 + 4e^{-t} - e^{-2t}$$

+5

Case II: $t > 1$:

$$y(t) = r(t) * q(t)$$

$$= \int_0^1 r(\tau) \cdot q(t-\tau)\, d\tau + \int_1^t r(\tau) \cdot q(t-\tau)\, d\tau$$

$$= 4e^{-t}\left[\tau e^{\tau} - e^{\tau} + 1\right]_0^1 - 4e^{-2t}\left[\frac{1}{2}\tau \cdot e^{2\tau} - \frac{1}{4} e^{2\tau} + \frac{1}{4}\right]_0^1$$

$$+ \int_1^t 8 \cdot \left[e^{-(t-\tau)} - e^{-2(t-\tau)}\right] d\tau$$

$$= 4e^{-t} - (1+e^2)e^{-2t} + 8e^{-t} \cdot \int_1^t e^{\tau} d\tau - 8e^{-2t} \int_1^t e^{2\tau} d\tau$$

$$= 4 + (4 - 8e)e^{-t} + (3e^2 - 1)e^{-2t}$$

+5

11. (5 points) Find the Laplace transform of the following function

$$f(t) = \int_0^t \sin(t-\tau)\cos\tau \, d\tau$$

$f(t) = \sin t * \cos t$ (+2)

$F(s) = \frac{1}{s^2+1} \cdot \frac{s}{s^2+1}$

$= \frac{s}{(s^2+1)^2}$ (+3)

12. (5 points) Applying convolution, find the inverse Laplace transform of the following function

$$F(s) = \frac{s}{(s+1)(s^2+4)}$$

$F(s) = \frac{1}{s+1} \cdot \frac{s}{s^2+4}$ (+2)

$f(t) = e^{-t} * \cos 2t$

$= \cos 2t * e^{-t}$

$= \int_0^t \cos 2\tau \cdot e^{-(t-\tau)} d\tau$

$= e^{-t} \int_0^t \cos 2\tau \cdot e^{\tau} d\tau$

$= \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t - \frac{1}{5}e^{-t}$ (+3)

13. (15 points) Solve the following initial value problem

$$y_1'' = -5y_1 + 2y_2,\ \ y_1(0) = 3, y_1'(0) = 0,$$
$$y_2'' = 2y_1 - 2y_2,\ \ \ \ y_2(0) = 1, y_2'(0) = 0.$$

$$\begin{cases} s^2 Y_1 - 3s = -5Y_1 + 2Y_2 \\ s^2 Y_2 - s = 2Y_1 - 2Y_2 \end{cases}$$

$$(s^2+5)Y_1 - 2Y_2 = 3s$$
$$-2Y_1 + (s^2+2)Y_2 = s$$

$$\begin{bmatrix} s^2+5 & -2 \\ -2 & s^2+2 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 3s \\ s \end{bmatrix}$$

+5

$$Y_1 = \frac{3s^3+8s}{(s^2+1)(s^2+6)} \leftarrow +2$$

$$= \frac{s}{s^2+1} + \frac{2s}{s^2+6} \leftarrow +2$$

$$Y_2 = \frac{s^3+11s}{(s^2+1)(s^2+6)} \leftarrow +2$$

$$= \frac{2s}{s^2+1} - \frac{s}{s^2+6} \leftarrow +2$$

$$\begin{cases} y_1 = \cos t + 2\cos\sqrt{6}t \\ y_2 = 2\cos t - \cos\sqrt{6}t \end{cases}$$

+2

14. (10 points) Find the inverse Laplace transform of the following function

$$\ln \frac{s+a}{s+b}$$

$$-\frac{d}{ds}\ln\left(\frac{s+a}{s+b}\right) = \frac{a-b}{(s+a)(s+b)}$$

$$= \frac{-1}{s+a} + \frac{1}{s+b}$$

+5

$$\mathcal{L}^{-1}\left(\ln\frac{s+a}{s+b}\right) = \mathcal{L}^{-1}\left(\int_s^\infty -\frac{d}{d\hat{s}}\ln\left(\frac{\hat{s}+a}{\hat{s}+b}\right)d\hat{s}\right)$$

$$= \frac{1}{t}\cdot\left[-e^{-at}+e^{-bt}\right]$$

+5