고급수학 및 연습 1 기말고사

(2014년 6월 7일 오후 1:00-3:00)

학번:	이름:

모든 문제의 답에 풀이과정을 명시하시오. (총점 200점)

Problem 1. [20 pts] For the set $\mathbf{M} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$, let $L : \mathbf{M} \to \mathbf{M}$ be the function given by

$$L(P) = P + P^t \quad (P \in \mathbf{M}).$$

(a) (5 pts) By identifying the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{M}$ with the vector (a, b, c, d) in $\mathbb{R}^4$, show that L is a linear map.

(b) (10 pts) Find the matrix A which represents L, and compute $\det A$.

(c) (5 pts) Prove or disprove that if P_1, P_2, P_3, P_4 are linearly independent vectors in $\mathbf{M}$, then $L(P_1)$, $L(P_2)$, $L(P_3)$, $L(P_4)$ are also linearly independent.

Problem 2. [20 pts] Write T if the statement is true, or write F if the statement is false. (4 pts for each correct answer, -4 pts for each incorrect answer and 0 pts if you do not answer. **You don't need to explain your answer**.)

(a) (4 pts) For $n \times n$ matrices A and B, $AB = 0$ implies $BA = 0$.

(b) (4 pts) For an $n \times n$ matrix A, $\det(A^t A) = (\det A)^2$.

(c) (4 pts) For $n \times n$ matrices A and B, $\det(A + B) = \det A + \det B$.

(d) (4 pts) For $n \times n$ matrices A and B, $\det(AB) = (\det A)(\det B)$.

(e) (4 pts) For a 3×3 matrix A, $A^3 = 0$ implies $A^2 = 0$ or $A = 0$.

Problem 3. [20 pts] Consider the plane α in 3-dimensional space through a point P with a unit normal vector $\mathbf{n}$. For a vector X in 3-dimensional space, answer the following questions.

(a) (10 pts) Let $E(X)$ be the projection of X onto α. Show that $E(X) = X + ((P - X) \cdot \mathbf{n})\mathbf{n}$ and also show that $(E \circ E)(X) = E(X)$.

(b) (10 pts) Consider the linear transformation which projects a point in $\mathbb{R}^3$ onto the plane given by $x + 2y + 3z = 0$. Find the matrix A corresponding to this linear transformation, and compute $\det(A^{2014} - I)$.

Problem 4. [20 pts] Let $X(t)$ be a unit speed curve in $\mathbb{R}^3$. Fix a unit vector $\mathbf{v} \in \mathbb{R}^3$. Prove that if $Y(t) = X(t) \times \mathbf{v}$ is also a unit speed curve, then $X(t)$ is contained in a plane.

Problem 5. [20 pts] Consider the curve in 3-dimensional space determined by the following

$$x^2 + y^2 + z^2 = 4, \;\; (x - 1)^2 + y^2 = 1, \;\; z \geq 0.$$

(a) (10 pts) Parametrize this curve.

(b) (10 pts) Find the osculating plane at the point $(0, 0, 2)$.

Problem 6. [20 pts] Consider the plane curve

$$X(t) = \left(\arctan t, \frac{1}{2} \log(1 + t^2) \right).$$

(a) (10 pts) Find the length in the interval $0 \leq t \leq 1$.

(b) (10 pts) Find a parametric equation of the tangent line at $t = 1$.

Problem 7. [20 pts] For the function $f(x, y) = x^2$ defined on $\mathbb{R}^2$ and the curve $X(t) = \left(\frac{1-t^4}{1+t^4}, \frac{2t^2}{1+t^4}\right)$ $(-\infty < t < \infty)$, find the line integral $\int_X f ds$.

Problem 8. [20 pts] Consider the curve given by

$$r = z = e^\theta$$

in cylindrical coordinates. Parametrize this curve by arc length. (Use the arc length measured from $(r, \theta, z) = (1, 0, 1)$.)

Problem 9. [20 pts] Find the osculating circle of the curve $X(t) = (t, \sin t)$ at the point $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$.

Problem 10. [20 pts] Find the area of the region enclosed by the curve given by $r = 2\cos\theta - 1$ in polar coordinates, and containing the point $\left(\frac{1}{2}, 0\right)$.