2014년도 1학기 고급수학 및 연습 1 기말고사 모범답안

1. (a) $L(c_1P_1+c_2P_2) = (c_1P_1+c_2P_2)+(c_1P_1+c_2P_2)^t = c_1P_1+c_2P_2+c_1P_1^t+c_2P_2^t$

$(c_1, c_2 \in \mathbb{R},\ P_1, P_2 \in M)$

$= c_1(P_1+P_1^t)+c_2(P_2+P_2^t) = c_1L(P_1)+c_2L(P_2).$

Thus, L is a linear map. 5 pts

(b) $L((a,b,c,d)) = L\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix}$

$= (2a, b+c, b+c, 2d) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

Thus, $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ represents L 5 pts

Also, $\det(A) = \det\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = 0$ 5 pts

(c) Take $P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $P_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $P_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $P_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\rightsquigarrow$ Then $\{P_1, P_2, P_3, P_4\}$ is a linearly independent vectors in M.

However, $0 \cdot L(P_1) + 1 \cdot L(P_2) + (-1) \cdot L(P_3) + 0 \cdot L(P_4) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0.$

$\rightsquigarrow \{L(P_1), L(P_2), L(P_3), L(P_4)\}$ is linearly dependent. 5 pts

2. (a) F, consider $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ & $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

(b) T

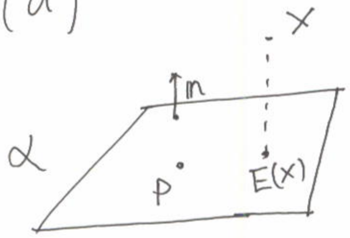
(c) F, consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ & $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

(d) T

(e) F, consider $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

3.

(a)



$$E(X) - X = P_n(P - X) = ((P-X)\cdot n)\, n.$$

$$\Rightarrow E(X) = X + ((P-X)\cdot n)\, n.$$

$\lrcorner$ +5pts

$E(X)$ is on the plane α $\Rightarrow (P - E(X))\cdot n = 0.$

$$\Rightarrow E\circ E(X) = E(X) + ((P - E(X))\cdot n)\, n = E(X).$$

$\lrcorner$ +5pts

(b) $P = (0,0,0).$ $n = \frac{1}{\sqrt{14}}(1,2,3).$

$$\Rightarrow E(X) = X - (X\cdot n)\, n = X - \frac{1}{14}(x+2y+3z)(1,2,3)$$

$$= \left(I - \frac{1}{14}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} \right) X.$$

Hence, $A = I - \frac{1}{14}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 13 & -2 & -3 \\ -2 & 10 & -6 \\ -3 & -6 & 5 \end{pmatrix}.$ $\lrcorner$ +5pts.

From (a), $E\circ E(X) = E(X)$ for all $X \in \mathbb{R}^3$, and we have $A^2 = A$.

$$\Rightarrow A^{2014} - I = A - I = -\frac{1}{14}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix},$$

and $\det(A^{2014} - I) = 0.$ $\left(\because \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} \right\} \text{ is linearly dependent} \right).$

$\lrcorner$ +5pts.

* (b) 에서 $n = (1,2,3)$ 으로 계산할 경우 −5pts.

\# 4. $|X'(t)| = 1$. $\mathbb{V} \in \mathbb{R}^3$ fixed.

$$Y(t) = X(t) \times \mathbb{V}. \quad \Rightarrow \quad Y'(t) = X'(t) \times \mathbb{V} \qquad \lrcorner \; +5 \text{ pts}.$$

$$1 = |Y'(t)|^2 = |X'(t) \times \mathbb{V}|^2 = |X'(t)|^2 \cdot |\mathbb{V}|^2 - (X'(t) \cdot \mathbb{V})^2 = 1 - (X'(t) \cdot \mathbb{V})^2,$$

and $X'(t) \cdot \mathbb{V} = 0$ for any t. $\lrcorner$ +5 pts.

Take and fix $t_0 \in \mathbb{R}$.

$$\Rightarrow \quad X(t) \cdot \mathbb{V} = \int_{t_0}^{t} X'(\tau) \cdot \mathbb{V} \, d\tau + X(t_0) \cdot \mathbb{V} = X(t_0) \cdot \mathbb{V} \quad ; \text{constant}.$$

Therefore, $X(t)$ is contained in the plane

$$X \cdot \mathbb{V} = c \quad \text{for} \quad c = X(t_0) \cdot \mathbb{V}. \qquad \lrcorner \; +10 \text{ pts}.$$

* $X'(t) \cdot \mathbb{V} = 0$ 증명 이후의 서술이 명확하지 않으면

그 부분에 대한 부분점수 없음.

#5.

a) $x^2+y^2+z^2=4$ — i)

$(x-1)^2+y^2=1$ — ii)

$z \geq 0$.

ii) $\Rightarrow$ $x = 1+\cos\theta$. $y=\sin\theta$ $(0 \leq \theta < 2\pi)$ 치환

$\Rightarrow$ i) 에서 $z^2 = 2-2\cos\theta = 4\sin^2\frac{\theta}{2}$

$z \geq 0$, 이고 $\sin\frac{\theta}{2} \geq 0$ for $0 \leq \theta < 2\pi$.

$\therefore$ $z = 2\sin\frac{\theta}{2}$

$(0 \leq \theta < 2\pi)$

$\therefore$ $(x, y, z) = (1+\cos\theta.\ \sin\theta.\ 2\sin\frac{\theta}{2}) = X(\theta)$

+10

b) $X(\theta) = (1+\cos\theta.\ \sin\theta.\ 2\sin\frac{\theta}{2})$

$X'(\theta) = (-\sin\theta.\ \cos\theta.\ \cos\frac{\theta}{2})$

$X''(\theta) = (-\cos\theta,\ -\sin\theta,\ -\frac{1}{2}\sin\frac{\theta}{2})$

$X(\theta) = (0,0,2) \Rightarrow \theta = \pi \Rightarrow \begin{cases} X'(\theta) = (0,-1,0) \\ X''(\theta) = (1, 0, -\frac{1}{2}) \end{cases}$

+5

$\therefore$ osculating plane

$(x.\ y.\ z-2)\cdot((0,-1,0)\times(1,0.\ -\frac{1}{2}))$

$= \frac{x}{2} + (z-2) = 0.$

+5

- #5-a)에서 범위 안 적을 시
  5점 감점.

6.

a) $X(t) = (\arctan t, \frac{1}{2}\log(1+t^2))$

$X'(t) = \left(\frac{1}{1+t^2}, \frac{t}{1+t^2}\right) \Rightarrow |X'(t)| = \frac{1}{\sqrt{1+t^2}}$

$\therefore$ length for $0 \le t \le 1$ $= \int_0^1 |X'(t)|\, dt$ +5

$= \int_0^1 \frac{1}{\sqrt{1+t^2}}\, dt = \int_0^{\frac{\pi}{4}} \sec s\, ds \quad (t = \tan s)$

$= \left[\log|\tan s + \sec s|\right]_0^{\frac{\pi}{4}} = \log(1+\sqrt{2})$ +5

b) $X(1) = (\frac{\pi}{4}, \frac{1}{2}\log 2) \quad X'(1) = (\frac{1}{2}, \frac{1}{2})$ +5

$\therefore$ parametric equation of the tangent line

: $(x, y) = (\frac{\pi}{4}, \frac{1}{2}\log 2) + t\cdot(\frac{1}{2}, \frac{1}{2})$

$(t \in \mathbb{R})$ +5

— #6 – a) 의 처음 5점은 $|X'(t)|$ 와

arc length 식을 정확히 서술해야 함.

#7. $X(t) = \left(\frac{1-t^4}{1+t^4}, \frac{2t^2}{1+t^4} \right), \quad -\infty < t < \infty$

$$\Rightarrow X'(t) = \left(\frac{-8t^3}{(1+t^4)^2}, \frac{4t(1-t^4)}{(1+t^4)^2} \right)$$

. $|X'(t)| = \dfrac{4|t|}{1+t^4}$

. $f(X(t)) = \left(\dfrac{1-t^4}{1+t^4} \right)^2$

따라서,

$$\int_X f\,ds = \int_{-\infty}^{\infty} \frac{4|t|(1-t^4)^2}{(1+t^4)^3}\,dt \quad +10$$

$$= 2\int_0^{\infty} \frac{4t(1-t^4)^2}{(1+t^4)^3}\,dt \quad (\because \text{우함수})$$

$t^2 = \tan\theta$

$2t\,dt = \sec^2\theta\,d\theta$

$$= 4\int_0^{\frac{\pi}{2}} \frac{(1-\tan^2\theta)^2}{(1+\tan^2\theta)^3} \sec^2\theta\,d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} (\cos^4\theta - 2\cos^2\theta\sin^2\theta + \sin^4\theta)\,d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \cos^2 2\theta\,d\theta = \pi \quad +10$$

$Y(S) = (\cos S, \sin S)$ 로 재매개화한 경우, S의 범위를 $-\pi < S < \pi$ 로 하면 원래 곡선과 다르므로 점수를 주지 않았습니다.

옳은 대응 관계는,

$t: -\infty \longrightarrow 0 \quad \longleftrightarrow \quad S: \pi \longrightarrow 0$

적분범위 등

$t: 0 \longrightarrow \infty \quad \longleftrightarrow \quad S: 0 \longrightarrow \pi$

앞부분에 틀린 점이 있으면 점수 없음.

#8. 주어진 곡선은 직교좌표계로 다음과 같다.

$$X(\theta) = (e^{\theta}\cos\theta, e^{\theta}\sin\theta, e^{\theta})$$

점 $(r, \theta, z) = (1, 0, 1)$은 $\theta = 0$일 때와 대응되므로,

길이함수 $s(\theta)$는 다음과 같다.

$$s(\theta) = \int_0^{\theta} |X'(\tilde{\theta})|\, d\tilde{\theta}$$

$$= \int_0^{\theta} \sqrt{3}e^{\tilde{\theta}}\, d\tilde{\theta} = \sqrt{3}(e^{\theta} - 1)$$ +10

$$\Rightarrow \theta = \log\left(1 + \frac{s}{\sqrt{3}}\right)$$

따라서, 호의 길이로 재매개화된 곡선은

$$Y(s) = \left(\left(1+\frac{s}{\sqrt{3}}\right)\cos\log\left(1+\frac{s}{\sqrt{3}}\right), \left(1+\frac{s}{\sqrt{3}}\right)\sin\log\left(1+\frac{s}{\sqrt{3}}\right), 1+\frac{s}{\sqrt{3}}\right)$$

$(s \geq 0)$ +10

#9 $X'(t) = (1, \cos t)$, $X''(t) = (0, -\sin t)$.

$$\kappa(t) = \frac{|-\sin t|}{(1+\cos^2 t)^{\frac{3}{2}}}, \quad \kappa\left(\frac{\pi}{4}\right) = \frac{2}{3\sqrt{3}} \Rightarrow \text{radius} = \frac{3}{2}\sqrt{3}$$

+10

$$\vec{\kappa}(t) = \frac{1}{\sqrt{1+\cos^2 t}} \left(\frac{(1, \cos t)}{\sqrt{1+\cos^2 t}} \right)', \quad \vec{\kappa}\left(\frac{\pi}{4}\right) = \left(\frac{2}{9}, -\frac{2\sqrt{2}}{9}\right)$$

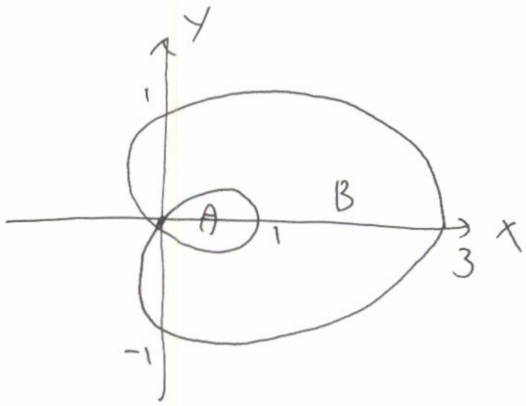
중심 : $X\left(\frac{\pi}{4}\right) + \dfrac{\vec{\kappa}\left(\frac{\pi}{4}\right)}{\kappa\left(\frac{\pi}{4}\right)^2} = \left(\frac{\pi}{4} + \frac{3}{2}, -\sqrt{2}\right)$

$\Rightarrow$ 중심이 $\left(\frac{\pi}{4} + \frac{3}{2}, -\sqrt{2}\right)$ 이고 반지름이 $\frac{3}{2}\sqrt{3}$ 인 원

+10

#10,

$r = 2\cos\theta - 1$ 을 그리면



$(\frac{1}{2}, 0)$을 포함하는 부분은 A이고 이때

θ의 범위는 $0 \le \theta \le \frac{\pi}{3}$

$\frac{5\pi}{3} \le \theta \le 2\pi$

+10

넓이 $= 2 \times \int_0^{\frac{\pi}{3}} \frac{1}{2} r^2 \, d\theta$

$= \pi - \frac{3}{2}\sqrt{3}$

+10.