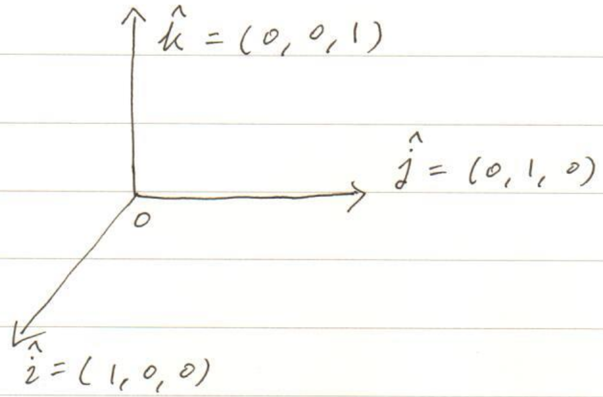
2016 Spring Physics 1 H.W.#1 Solution

$\boxed{1}$

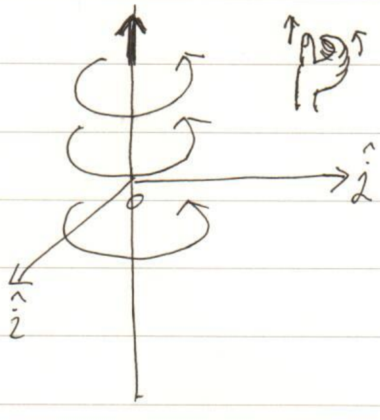
(a) $\hat{i}, \hat{j}, \hat{k}$ are unit vectors for x, y, z axis



The magnitude of $(\hat{i} \times \hat{j})$ is $|\hat{i}||\hat{j}|\sin\phi$, where ϕ is an angle between $\hat{i}$ and $\hat{j}$

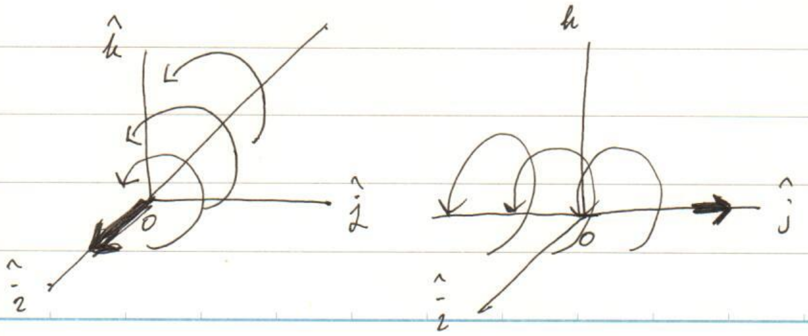
And $|\hat{i}| = 1$, $|\hat{j}| = 1$, $\sin(\frac{\pi}{2}) = 1$, magnitude of $(\hat{i} \times \hat{j}) = 1$

Then, The direction of $(\hat{i} \times \hat{j})$ is determined by a right-hand rule.



Using your right hand, Sweep $\hat{i}$ into $\hat{j}$
Then your thumb shows the direction of $(\hat{i} \times \hat{j})$

so, $\underline{\hat{i} \times \hat{j} = \hat{k}}$



For the same reason,

$\underline{\hat{j} \times \hat{k} = \hat{i}}$

$\underline{\hat{k} \times \hat{i} = \hat{j}}$

$\boxed{1}$ (b) $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{a}\cdot\left((b_y c_z - b_z c_y)\hat{i} + (b_z c_x - b_x c_z)\hat{j} + (b_x c_y - b_y c_x)\hat{k}\right)$

$= a_x b_y c_z - a_x b_z c_y + a_y b_z c_x - a_y b_x c_z + a_z b_x c_y - a_z b_y c_x$

$= b_x(a_z c_y - a_y c_z) + b_y(a_x c_z - a_z c_x) + b_z(a_y c_x - a_x c_y)$

$= \underline{\vec{b}\cdot(\vec{c}\times\vec{a})}$

$\overset{\text{also}}{=} c_x(a_y b_z - a_z b_y) + c_y(a_z b_x - a_x b_z) + c_z(a_x b_y - a_y b_x)$

$= \underline{\vec{c}\cdot(\vec{a}\times\vec{b})}$

(c) $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{a}\times\left((b_y c_z - b_z c_y)\hat{i} + (b_z c_x - b_x c_z)\hat{j} + (b_x c_y - b_y c_x)\hat{k}\right)$

$$\begin{aligned}
&= [a_y(b_x c_y - b_y c_x) - a_z(b_z c_x - b_x c_z)]\hat{i}\\
&+ [a_z(b_y c_z - b_z c_y) - a_x(b_x c_y - b_y c_x)]\hat{j}\\
&+ [a_x(b_z c_x - b_x c_z) - a_y(b_y c_z - b_z c_y)]\hat{k}
\end{aligned}$$

$$\begin{aligned}
&= [b_x(a_y c_y + a_z c_z) - c_x(a_y b_y + a_z b_z)]\hat{i}\\
&+ [b_y(a_x c_x + a_z c_z) - c_y(a_x b_x + a_z b_z)]\hat{j}\\
&+ [b_z(a_x c_x + a_y c_y) - c_z(a_x b_x + a_y b_y)]\hat{k}
\end{aligned}$$

Add and Subtract $a_x b_x c_x$ to 1st row $\Rightarrow b_x(\vec{a}\cdot\vec{c}) - c_x(\vec{a}\cdot\vec{b})$

$a_y b_y c_y$ to 2nd row $\Rightarrow b_y(\vec{a}\cdot\vec{c}) - c_y(\vec{a}\cdot\vec{b})$

$a_z b_z c_z$ to 3rd row $\Rightarrow b_z(\vec{a}\cdot\vec{c}) - c_z(\vec{a}\cdot\vec{b})$

$= (b_x\hat{i} + b_y\hat{j} + b_z\hat{k})(\vec{a}\cdot\vec{c}) - (c_x\hat{i} + c_y\hat{j} + c_z\hat{k})(\vec{a}\cdot\vec{b})$

$= \underline{\vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})}$

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$\boxed{1}$ (d) $\frac{d}{dt}(\vec{a}\cdot\vec{b}) = \frac{d}{dt}(a_x b_x + a_y b_y + a_z b_z)$

$$= \frac{da_x}{dt} b_x + \frac{da_y}{dt} b_y + \frac{da_z}{dt} b_z + a_x \frac{db_x}{dt} + a_y \frac{db_y}{dt} + a_z \frac{db_z}{dt}$$

$$= \underline{\frac{d\vec{a}}{dt}\cdot\vec{b} + \vec{a}\cdot\frac{d\vec{b}}{dt}}$$

(e) $\frac{d}{dt}(\vec{a}\times\vec{b}) = \frac{d}{dt}\left[(a_y b_z - a_z b_y)\hat{i} + (a_z b_x - a_x b_z)\hat{j} + (a_x b_y - a_y b_x)\hat{k}\right]$

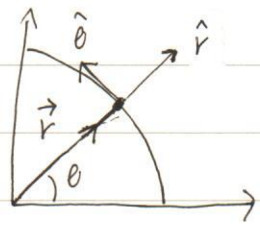
$\frac{d\hat{i}}{dt} = \frac{d\hat{j}}{dt} = \frac{d\hat{k}}{dt} = 0$

$$= \left[\frac{da_y}{dt} b_z + a_y \frac{db_z}{dt} - \frac{da_z}{dt} b_y - a_z \frac{db_y}{dt}\right]\hat{i}$$
$$+ \left[\frac{da_z}{dt} b_x + a_z \frac{db_x}{dt} - \frac{da_x}{dt} b_z - a_x \frac{db_z}{dt}\right]\hat{j}$$
$$+ \left[\frac{da_x}{dt} b_y + a_x \frac{db_y}{dt} - \frac{da_y}{dt} b_x - a_y \frac{db_x}{dt}\right]\hat{k}$$

$$= \left(\frac{da_y}{dt} b_z - \frac{da_z}{dt} b_y\right)\hat{i} + \left(\frac{da_z}{dt} b_x - \frac{da_x}{dt} b_z\right)\hat{j} + \left(\frac{da_x}{dt} b_y - \frac{da_y}{dt} b_x\right)\hat{k}$$
$$+ \left(a_y \frac{db_z}{dt} - a_z \frac{db_y}{dt}\right)\hat{i} + \left(a_z \frac{db_x}{dt} - a_x \frac{db_z}{dt}\right)\hat{j} + \left(a_x \frac{db_y}{dt} - a_y \frac{db_x}{dt}\right)\hat{k}$$

$$= \underline{\frac{d\vec{a}}{dt}\times\vec{b} + \vec{a}\times\frac{d\vec{b}}{dt}}$$

$\boxed{2}$ (i) polar coordinates



First, find $\hat{r}$ and $\hat{\theta}$

$$\vec{r} = x\hat{i} + y\hat{j}, \quad \hat{r} = \frac{x\hat{i} + y\hat{j}}{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

And, $\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

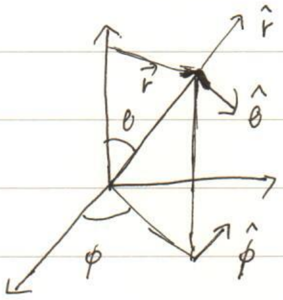
Then, $\vec{v} = \dfrac{d\vec{r}}{dt} = \dfrac{dx}{dt}\hat{i} + \dfrac{dy}{dt}\hat{j}$

$$\frac{dx}{dt} = \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}, \quad \frac{dy}{dt} = \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}$$

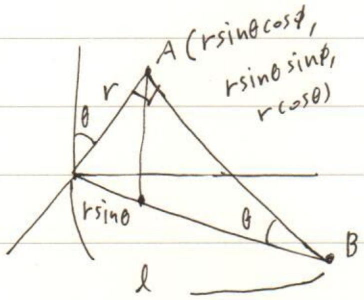
$$\vec{v} = \frac{dr}{dt}(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) + r\frac{d\theta}{dt}(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$

$$= \underline{\hat{r}\frac{dr}{dt} + \hat{\theta}\left(r\frac{d\theta}{dt}\right)}$$

(ii) spherical coordinates



$$\hat{r} = \frac{1}{r}(x\hat{i} + y\hat{j} + z\hat{k}) = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$



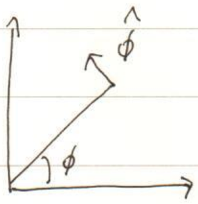
$l\sin\theta = r, \quad l = \dfrac{r}{\sin\theta}$

Then, $B = \left(\dfrac{r}{\sin\theta}\cos\phi,\ \dfrac{r}{\sin\theta}\sin\phi,\ 0\right)$

$\hookrightarrow \vec{\theta} = r\dfrac{\cos^2\theta}{\sin\theta}\cos\phi\,\hat{i} + r\dfrac{\cos^2\theta}{\sin\theta}\sin\phi\,\hat{j} - r\cos\theta\,\hat{k}$

make it a unit vector

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$



$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

Then, $\vec{v} = \dfrac{d\vec{r}}{dt} = \left(\dfrac{dr}{dt} \sin\theta \cos\phi + r\cos\theta \dfrac{d\theta}{dt} \cos\phi - r\sin\theta \sin\phi \dfrac{d\phi}{dt} \right) \hat{i}$

$$+ \left(\frac{dr}{dt} \sin\theta \sin\phi + r\cos\theta \frac{d\theta}{dt} \sin\phi + r\sin\theta \cos\phi \frac{d\phi}{dt} \right) \hat{j}$$

$$+ \left(\frac{dr}{dt} \cos\theta - r\sin\theta \frac{d\theta}{dt} \right) \hat{k}$$

$$= \frac{dr}{dt} \left(\sin\theta\cos\phi \, \hat{i} + \sin\theta\sin\phi \, \hat{j} + \cos\theta \, \hat{k} \right)$$

$$+ r\sin\theta \frac{d\phi}{dt} \left(-\sin\phi \, \hat{i} + \cos\phi \, \hat{j} \right)$$

$$+ r\frac{d\theta}{dt} \left(\cos\theta\cos\phi \, \hat{i} + \cos\theta\sin\phi \, \hat{j} - \sin\theta \, \hat{k} \right)$$

$$= \underline{\hat{r} \frac{dr}{dt} + \hat{\phi} \left(r\frac{d\phi}{dt} \sin\theta \right) + \hat{\theta} \left(r\frac{d\theta}{dt} \right)} \,//$$

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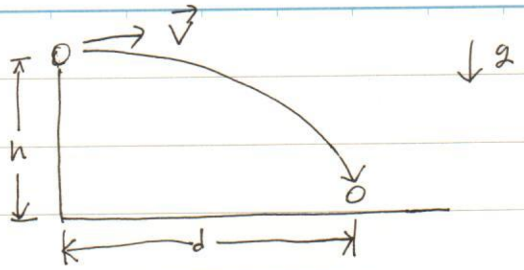


(i) y-component of an object : free-fall

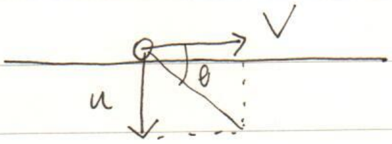
When an object hits the ground, let time becomes T (start : 0)

Then, $h = \frac{1}{2} g T^2$, $T = \sqrt{\frac{2h}{g}}$

While it falls, its x-component has constant speed.

$d = VT$, $\underline{V = d\sqrt{\frac{g}{2h}}}$

(ii)



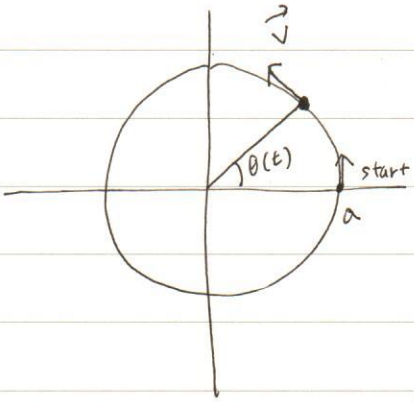
When it hits the ground,

$u = gT = \sqrt{2gh}$

Then, $\tan\theta = \frac{u}{V} = \frac{\sqrt{2gh}}{d\sqrt{\frac{g}{2h}}} = \frac{2h}{d}$

$\Rightarrow \theta = \arctan\left(\frac{2h}{d}\right)$

$\boxed{4}$



At time t, $|\vec{V}(t)| = V_0(1+\sin\omega t)$

$$\vec{V}(t) = \left[-V_0(1+\sin\omega t)\sin\theta(t)\right]\hat{i} + \left[V_0(1+\sin\omega t)\cos\theta(t)\right]\hat{j}$$

Here, need to find $\theta(t)$

The distance (from 0 to t) $= a\theta(t)$

$$s(t) = \int_0^t V_0(1+\sin\omega t)dt = V_0 t + \frac{V_0}{\omega}(1-\cos\omega t) = \frac{V_0}{\omega}(1+\omega t-\cos\omega t) = a\theta(t)$$

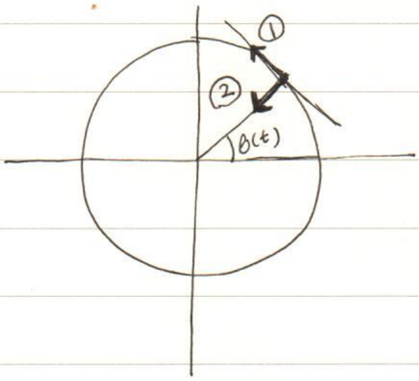
$$\theta(t) = \frac{V_0}{a\omega}(1+\omega t-\cos\omega t)$$

Then, the acceleration $\vec{a}(t) = \frac{d\vec{V}(t)}{dt} = \Big[-V_0\omega\cos\omega t\sin\left[\frac{V_0}{a\omega}(1+\omega t-\cos\omega t)\right]$

$$-V_0(1+\sin\omega t)\cos\left[\frac{V_0}{a\omega}(1+\omega t-\cos\omega t)\right]\frac{V_0}{a}(1+\sin\omega t)\Big]\hat{i}$$

$$+\Big[V_0\omega\cos\omega t\cos\left[\frac{V_0}{a\omega}(1+\omega t-\cos\omega t)\right]$$

$$-V_0(1+\sin\omega t)\sin\left[\frac{V_0}{a\omega}(1+\omega t-\cos\omega t)\right]\frac{V_0}{a}(1+\sin\omega t)\Big]\hat{j}$$



$$= V_0\omega\cos\omega t\left(-\sin\theta(t)\,\hat{i} + \cos\theta(t)\,\hat{j}\right)$$

$$- V_0(1+\sin\omega t)\frac{V_0}{a}(1+\sin\omega t)\left(\cos\theta(t)\,\hat{i} + \sin\theta(t)\,\hat{j}\right)$$

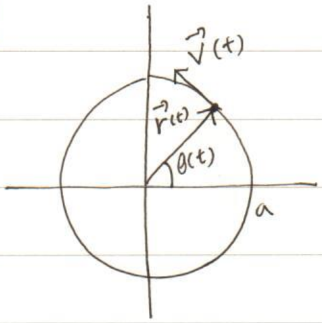
$$= \underbrace{\left(V_0\omega\cos\omega t\right)\hat{\theta}(t)}_{①} + \underbrace{\left(\frac{V_0^2}{a}(1+\sin\omega t)^2\right)\left(-\hat{r}(t)\right)}_{②}$$

$$= \frac{d|\vec{V}(t)|}{dt}\hat{\theta}(t) + \frac{|\vec{V}(t)|^2}{a}\left(-\hat{r}(t)\right)$$

Period is $\frac{8\pi}{\omega}$, $\theta\left(\frac{8\pi}{\omega}\right) = \frac{V_0}{a\omega}(1+8\pi-1) = \frac{8\pi V_0}{a\omega} = 2\pi$

$$\Rightarrow \underline{V_0 = \frac{1}{4}a\omega}$$

5



(a) When the motion is circular,

position vector $\vec{r}(t) = a\cos\theta(t)\,\hat{i} + a\sin\theta(t)\,\hat{j}$

velocity vector $\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = a\frac{d\theta}{dt}\left(-\sin\theta(t)\,\hat{i} + \cos\theta\,\hat{j}\right)$

Then, $\underline{\vec{r}(t)\cdot\vec{v}(t) = a^2\frac{d\theta}{dt}\left[-\sin\theta(t)\cos\theta(t) + \sin\theta(t)\cos\theta(t)\right] = 0}$ //

the position vector and velocity vector are perpendicular

(b) constant speed $\Rightarrow |\vec{v}(t)| = \sqrt{\vec{v}(t)\cdot\vec{v}(t)} = \sqrt{a^2\left|\frac{d\theta}{dt}\right|^2(\sin^2\theta(t)+\cos^2\theta(t))}$

$= a\left|\frac{d\theta}{dt}\right|$

so, $\frac{d\theta}{dt} = w$, w is constant

Then, $\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = -aw^2\left(\cos\theta(t)\,\hat{i} + \sin\theta(t)\,\hat{j}\right)$

so, $\underline{\vec{v}(t)\cdot\vec{a}(t) = -a^2w^3\left(-\sin\theta(t)\cos\theta(t) + \sin\theta(t)\cos\theta(t)\right) = 0}$ //

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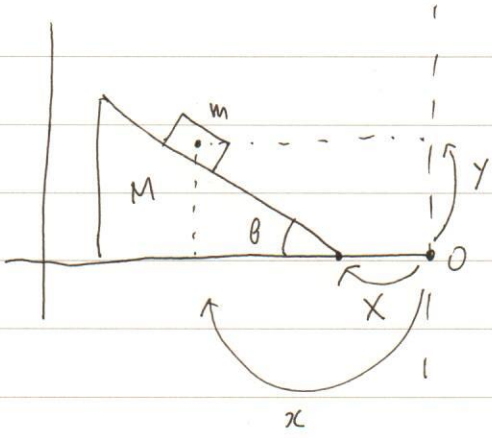


set the coordinates of M, m

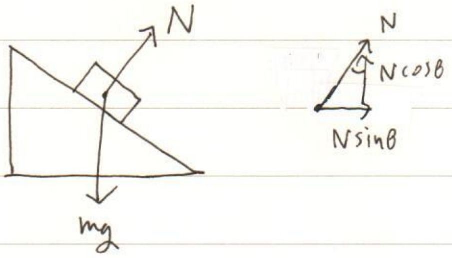
$M \Rightarrow X\hat{i}$ (M moves along a horizontal plane)

$m \Rightarrow x\hat{i} + y\hat{j}$

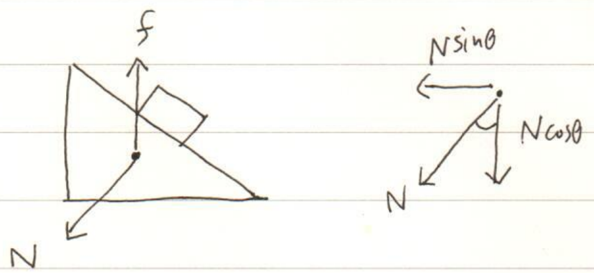
First, m slides on M, so use the angle θ which is constant

$\hookrightarrow \tan\theta = \dfrac{y}{X-x}$, $y = (X-x)\tan\theta$

Then, draw the forces



force on m



force on M

N is a normal force on inclined plane

f is a normal force by the ground

Then, their equation of motion

$N\cos\theta - mg = m\ddot{y} \cdots ①$

$N\sin\theta = m\ddot{x} \cdots ②$

$-N\sin\theta = M\ddot{X} \cdots ③$

From ② and ③, $\ddot{x} = -\frac{M}{m}\ddot{X}$

Then, ① is

$$N\cos\theta - mg = m(\ddot{X} - \ddot{x})\tan\theta$$

$$\rightarrow -M\frac{\cos\theta}{\sin\theta}\ddot{X} - mg = m\left(\ddot{X} + \frac{M}{m}\ddot{X}\right)\tan\theta$$

$$\rightarrow \left(M\frac{\cos\theta}{\sin\theta} + (m+M)\frac{\sin\theta}{\cos\theta}\right)\ddot{X} = -mg$$

$$\rightarrow \ddot{X} = \frac{-mg\cos\theta\sin\theta}{M\cos^2\theta + (m+M)\sin^2\theta} = \frac{-mg\cos\theta\sin\theta}{M + m\sin^2\theta}$$

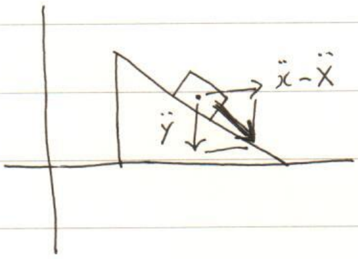
So, acceleration of M : $-\frac{mg\cos\theta\sin\theta}{M + m\sin^2\theta}\,\hat{i}$

And, $\ddot{x} = \frac{Mg\cos\theta\sin\theta}{M + m\sin^2\theta}$, $\ddot{y} = (\ddot{X} - \ddot{x})\tan\theta = -\frac{(M+m)g\sin^2\theta}{M + m\sin^2\theta}$

So, acceleration of m : $\frac{Mg\cos\theta\sin\theta}{M + m\sin^2\theta}\,\hat{i} - \frac{(M+m)g\sin^2\theta}{M + m\sin^2\theta}\,\hat{j}$

※ relative acceleration of m (to M) : $(\ddot{x} - \ddot{X})\,\hat{i} + \ddot{y}\,\hat{j}$

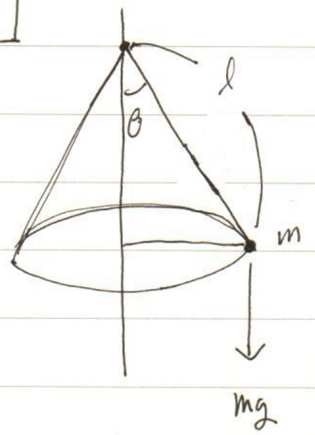
$$\Rightarrow \frac{(M+m)g\cos\theta\sin\theta}{M + m\sin^2\theta}\,\hat{i} - \frac{(M+m)g\sin^2\theta}{M + m\sin^2\theta}\,\hat{j}$$



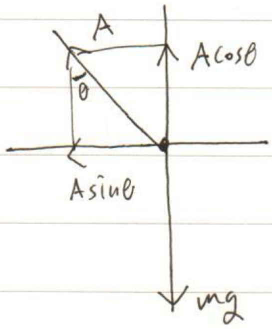
$\Rightarrow$ magnitude : $\frac{(M+m)g\sin\theta}{M + m\sin^2\theta}$

direction : downside along the inclined plane

$\boxed{7}$



Let the tension of a string A



$\Rightarrow mg = A\cos\theta$ (vertical net Force $= 0$)

$\Rightarrow rm\omega^2 = A\sin\theta$ (uniform circular motion)

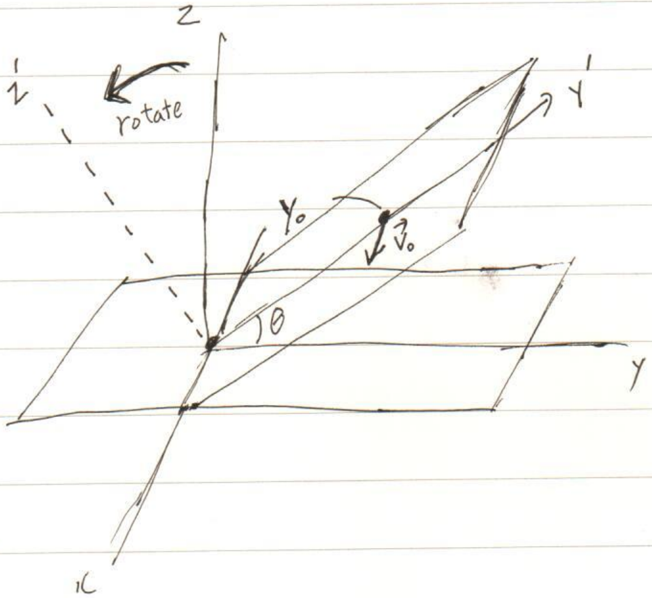
Here, $r = l\sin\theta$, $\omega = \frac{2\pi}{T}$

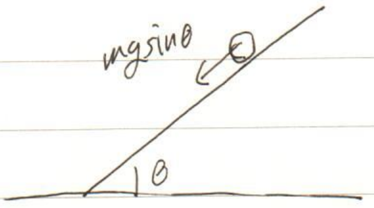
Then, $ml\sin\theta\left(\frac{2\pi}{T}\right)^2 = A\sin\theta$, $\quad A = \frac{4\pi^2 ml}{T^2}$

$$\Rightarrow \cos\theta = \frac{mg}{4\pi^2 ml/T^2} = \frac{gT^2}{4l\pi^2}$$

$\boxed{8}$ mass m is on the inclined plane $\Rightarrow$ $a = g\sin\theta$

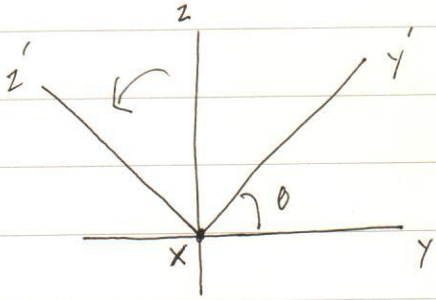
And direction is not vertical down





To solve this problem simply, use a linear transform - <u>rotate</u>

$(x, y, z) \to (x', y', z')$



$$\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} y' \\ z' \end{pmatrix}$$

And $x = x'$ (rotation pivot)

Then, bead moves on $x'y'$ plane, where $z' = 0$

The only acceleration is $\vec{a}' = -g\sin\theta\,\hat{j}'$

And initial velocity $\vec{v}'(0) = v_0\,\hat{i}'$, initial position $\vec{r}'(0) = y_0\,\hat{j}'$

Then, position vector $\vec{r}'(t) = v_0 t\,\hat{i}' + \left(y_0 - \frac{1}{2} g\sin\theta\, t^2\right)\hat{j}'$

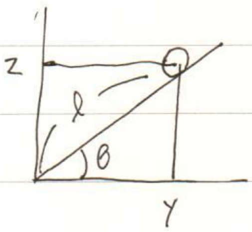
Then, use a linear transform

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix}$$

And, $z'(t) = 0$ here.

$$\Rightarrow \vec{r}(t) = (V_0 t)\,\hat{i} + \left(Y_0 - \frac{1}{2} g \sin\theta\, t^2\right)\cos\theta\,\hat{j} + \left(Y_0 - \frac{1}{2} g\sin\theta\, t^2\right)\sin\theta\,\hat{k}$$

This can be induced more simple by geometry



bead moves on the inclined plane

so $\frac{Z}{Y} = \tan\theta$

$\Rightarrow$ get the $l(t)$, and $Y = l(t)\cos\theta$, $Z = l(t)\sin\theta$

$l(t)$ comes from $-g\sin\theta$, $l(t) = Y_0 - \frac{1}{2} g \sin\theta\, t^2$

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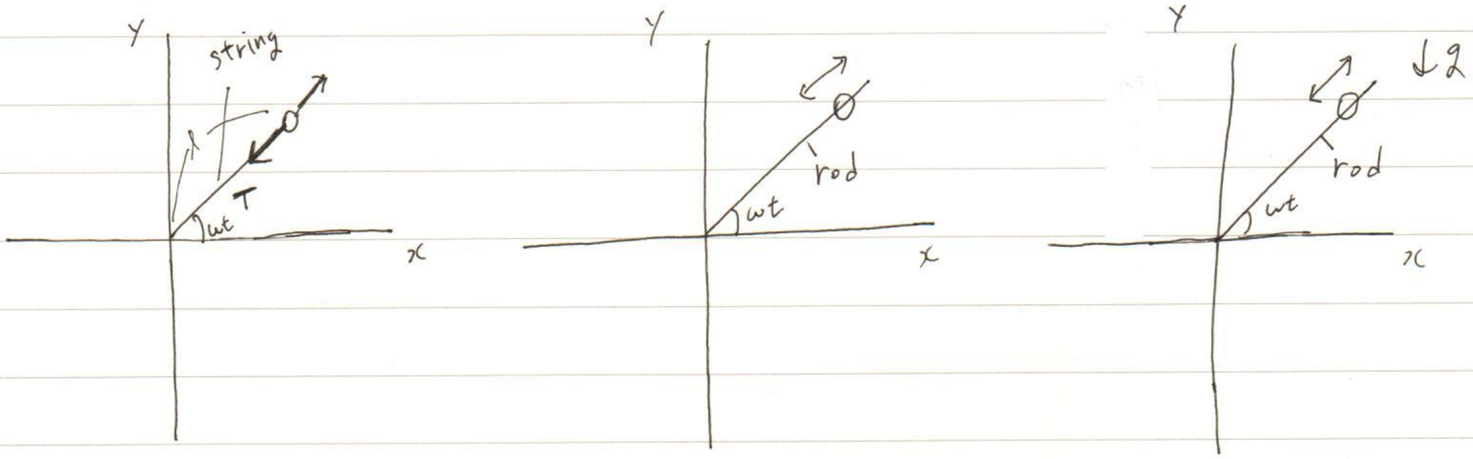

(1) (2) (3)

case (1) bead is hung on string, so its radial acceleration is $a = l\omega^2$

However, in the bead's frame, net centrifigual acceleration is 0

$\Rightarrow \; T = ml\omega^2 \;, \; \vec{F}_{net} = -T + ml\omega^2 = 0$

case (2) Now, l can vary by time, and there is no tension

$\Rightarrow \; \vec{F}_{net} = m\dfrac{d^2 l}{dt^2} = ml\omega^2$
(centrifigual)

case (3) Now, consider gravity. its centrifigual component is $-mg\sin\omega t$

$\Rightarrow \; m\dfrac{d^2 l}{dt^2} = ml\omega^2 - mg\sin\omega t$

$\Rightarrow \; \dfrac{d^2 l}{dt^2} = l\omega^2 - g\sin\omega t \;, \; \underline{\dfrac{d^2 l(t)}{dt^2} - \omega^2 l(t) + g\sin\omega t = 0}$ //

Now, Assume $l(t) = A\cosh\omega t + B\sinh\omega t + (g\sin\omega t)/2\omega^2$

$$\Rightarrow \frac{d}{dt}(\cosh\omega t) = \frac{d}{dt}\left(\frac{e^{\omega t}+e^{-\omega t}}{2}\right) = \omega\frac{e^{\omega t}-e^{-\omega t}}{2} = \omega\sinh\omega t$$

$$\frac{d}{dt}(\sinh\omega t) = \frac{d}{dt}\left(\frac{e^{\omega t}-e^{-\omega t}}{2}\right) = \omega\frac{e^{\omega t}+e^{-\omega t}}{2} = \omega\cosh\omega t$$

$$\Rightarrow \frac{d^2 l(t)}{dt^2} = \omega^2 A\cosh\omega t + \omega^2 B\sinh\omega t - g\sin\omega t/2$$

$$\Rightarrow \frac{d^2 l(t)}{dt^2} - \omega^2 l(t) + g\sin\omega t$$

$$= \omega^2(A\cosh\omega t + B\sinh\omega t) - g\sin\omega t/2$$
$$- \omega^2(A\cosh\omega t + B\sinh\omega t) - g\sin\omega t/2$$
$$+ g\sin\omega t = 0$$

//