< 2013년 1학기 고급수학 및 연습1 중간고사 모범답안 >

Problem 1 :

(a) Note that $\tan x \leq \frac{4}{\pi}x$ for $x \ll 1$

$\Rightarrow 0 < n\tan\frac{\pi}{2^{n+1}} \leq n\cdot\frac{4}{\pi}\cdot\frac{\pi}{2^{n+1}} = \frac{n}{2^{n-1}}$ for $n \gg 1$

Since $\sum\frac{n}{2^{n-1}} < \infty$ by ratio test,

$\sum n\tan\frac{\pi}{2^{n+1}} < \infty$ by Comparison test | 10점

※ $\sum\frac{n}{2^{n-1}} < \infty$ 인 이유를 적지 않으면 5점감점.

(b) Let $a_n = n!\left(\frac{e^2}{n}\right)^n > 0$

$\Rightarrow \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty} e^2\cdot\left(\frac{n}{n+1}\right)^n = e > 1$

$\Rightarrow$ By ratio test $\sum n!\left(\frac{e^2}{n}\right)^n = \infty$ | 10점

※ 부분점수 없음.

(c) Since $\lim_{n\to\infty}\sqrt[n]{n} = 1$, $\lim_{n\to\infty}\frac{(-1)^n}{\sqrt[n]{n}} \neq 0$

$\Rightarrow$ By the test of divergence, $\sum\frac{(-1)^n}{\sqrt[n]{n}} = \infty$ | 10점

※ 부분점수 없음

(d) Let $a_n = \dfrac{1}{n-\log n} > 0$

Observe that $f(x) = x - \log x$ is increasing, since $f'(x) = 1 - \frac{1}{x} > 0$

So $a_n \geq a_{n+1}$.

$$\lim_{n\to\infty} \frac{1}{n-\log n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1 - \frac{\log n}{n}} = 0.$$

Therefore $\sum (-1)^n a_n = \sum \dfrac{(-1)^n}{n-\log n} < \infty$ by Alternating series test | 10점

※ '$a_n \geq a_{n+1}$' 이나 '$\lim_{n\to\infty} a_n = 0$' 임을 언급하지 않은 경우 각 5점 감점.

(e) Note that $\sin x \leq x$ and $\tan x \leq \frac{4}{\pi} \cdot x$ for $x \ll 1$

$$\Rightarrow \frac{1}{\sqrt{n}} \sin\left(\tan\frac{1}{n}\right) \leq \frac{1}{\sqrt{n}} \cdot \frac{4}{\pi} \cdot \frac{1}{n} = \frac{8}{\pi^2} \cdot \frac{1}{n^{\frac{3}{2}}} \quad \text{for } n \gg 1$$

Since $\sum \frac{1}{n^p} < \infty \Leftrightarrow p > 1$, $\sum \frac{8}{\pi^2} \cdot \frac{1}{n^{\frac{3}{2}}} < \infty$

By Comparison test, $\sum \frac{1}{\sqrt{n}} \sin\left(\tan\frac{1}{n}\right) < \infty$ | 10점

※ $\sum \frac{1}{n^{\frac{3}{2}}} < \infty$ 임을 아무 근거 없이 사용하면 5점감점.

Problem 2.

① Let $a_n := \log(1+\frac{1}{n})$, $n \geq 1$.

Find the radius of convergence of $\sum_{n\geq 1} a_n x^n$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\log(1+\frac{1}{n+1})}{\log(1+\frac{1}{n})} \underset{\text{L'Hôpital}}{=} \lim_{n\to\infty}\frac{\frac{1}{n+2}-\frac{1}{n+1}}{\frac{1}{n+1}-\frac{1}{n}} = \lim_{n\to\infty}\frac{\frac{-1}{(n+2)(n+1)}}{\frac{-1}{(n+1)(n)}} = 1.$$

$\therefore$ 1 is the radius of convergence of $\sum_{n\geq 1} a_n x^n$. 10 points

② If we let $x=1$,

$$\sum_{n\geq 1} a_n = \sum_{n\geq 1}\log(1+\tfrac{1}{n}) = \lim_{m\to\infty}\sum_{n=1}^{m}\log(1+\tfrac{1}{n}) = -\lim_{m\to\infty}(\log 1 - \cancel{\log 2} + \cancel{\log 2} - \cancel{\log 3} + \cdots \cancel{\log m} - \log(m+1))$$

$$= \infty.$$

$\therefore \sum_{n\geq 1} a_n x^n$ diverges at $x=1$. 5 points

③ If we let $x=-1$,

$$\sum_{n\geq 1} a_n(-1)^n = \sum_{n\geq 1}\log(1+\tfrac{1}{n})(-1)^n$$

a) $\lim_{n\to\infty} a_n = 0$

b) $0 \leq a_n$ and $\frac{a_{n+1}}{a_n} = \frac{\log(1+\frac{1}{n+1})}{\log(1+\frac{1}{n})} \leq 1$ ie. decreasing.

$\therefore \sum_{n\geq 1} a_n(-1)^n$ converges by Alternating series test. 5 points.

Thus $-1 \leq x < 1$ is the interval of convergence of $\sum_{n\geq 1}\log(1+\frac{1}{n})x^n$.

Problem 3.

Let $f(x) = \sum_{n \geq 0} a_n x^n$

Then $a_0 = 1$, $a_1 = 1$ and $\sum_{n \geq 2} n(n-1) a_n x^{n-2} + \sum_{n \geq 0} a_n x^n = 0$.

$\Rightarrow \sum_{n \geq 0} \left((n+2)(n+1) a_{n+2} + a_n \right) x^n = 0.$

$\therefore \quad a_{n+2} = - \dfrac{a_n}{(n+2)(n+1)}$

So, we have

$$a_{2n} = - \frac{a_{2n-2}}{2n(2n-1)} = (-1)^2 \frac{a_{2n-4}}{2n(2n-1)(2n-2)(2n-3)}$$

$$\cdots = (-1)^n \frac{a_0}{(2n)!} = (-1)^n \frac{1}{(2n)!}.$$

$$a_{2n+1} = - \frac{a_{2n-1}}{(2n+1)(2n)} = (-1)^2 \frac{a_{2n-3}}{(2n+1)(2n)(2n-1)(2n-2)}.$$

$$\cdots = (-1)^n \frac{a_1}{(2n+1)!} = (-1)^n \frac{1}{(2n+1)!}$$

$$\therefore f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

10 points

$$= \cos x + \sin x.$$

$$f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}.$$

10 points.

#4. (a)

$x = \tanh^{-1} y$, $y = \tanh x$

$$\frac{dy}{dx} = \operatorname{sech}^2 x = 1 - \tanh^2 x$$

$$\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1} = \frac{1}{1-\tanh^2 x} = \frac{1}{1-y^2}$$

(b) $$\sum_{n=0}^{\infty} y^{2n} = \frac{1}{1-y^2}$$

$$\sum_{n=0}^{\infty} \frac{1}{2n+1} y^{2n+1} = \int_0^y \frac{dt}{1-t^2} = \frac{1}{2}\log\frac{1+y}{1-y}$$

$(|y|<1)$

$$\sum_{n=0}^{\infty} \frac{1}{2n+1}\left(\frac{1}{2}\right)^{2n+1} = \frac{1}{2}\log\frac{1+\frac{1}{2}}{1-\frac{1}{2}}$$

$$= \frac{1}{2}\log 3$$

※ (a)를 이용해서 $\tanh^{-1}\frac{1}{2}$ 이라고 쓰면 5점

$\tanh^{-1}\frac{1}{2} = \frac{1}{2}\log 3$ 까지 계산하면 10점

#5.

(풀이 1) $g(x) = \arcsin x$ 라 하면,

$$g'(x) = \frac{1}{\sqrt{1-x^2}}$$ ⋯ 5점

$$g''(x) = \frac{x}{(1-x^2)^{\frac{3}{2}}}, \quad g^{(3)}(x) = \frac{1+2x^2}{(1-x^2)^{\frac{5}{2}}}$$ ⋯ 10점

$$g(0) = 0, \quad g'(0) = 1, \quad g''(0) = 0, \quad g^{(3)}(0) = 1$$ ⋯ 15점

$$g(x) = x + \frac{1}{3!}x^3 + \cdots$$

$$f(x) = x^2 g(x) = x^3 + \frac{1}{3!}x^5 + \cdots$$

$$T_5 f(x) = x^3 + \frac{1}{6}x^5$$ ⋯ 20점

(풀이 2) 이항정리 $(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n$ 을 이용하면

$$(1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots$$

$$(1-x^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \cdots$$

$$\arcsin x = \int_0^x (1-t^2)^{-\frac{1}{2}} = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots$$

$$x^2 \arcsin x = x^3 + \frac{1}{6}x^5 + \cdots$$

$$T_5 f(x) = x^3 + \frac{1}{6}x^5.$$

#6

Let $f(x) := \int_0^x \arctan t \, dt$.

$f'(x) = \arctan x$

$f''(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n (x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ $(|x|<1)$

$\Rightarrow f'(x) - \underbrace{f'(0)}_{0} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} x^{2n+1}$ $(|x|<1)$

$\Rightarrow f(x) - \underbrace{f(0)}_{0} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)} x^{2n+2}$ $(|x|<1)$

$\Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n(2n-1)} x^{2n}$ $(|x|<1)$ +10

$\Rightarrow \left| f\left(\frac{1}{10}\right) - T_4 f\left(\frac{1}{10}\right) \right| = \left| \sum_{n=3}^{\infty} (-1)^n \frac{\left(\frac{1}{10}\right)^{2n}}{2n(2n-1)} \right|$ +5

$\leq \frac{1}{6 \cdot 5} \left(\frac{1}{10}\right)^6 < \left(\frac{1}{10}\right)^7$

$\therefore f\left(\frac{1}{10}\right) \approx 0 + \frac{1}{2}\left(\frac{1}{10}\right)^2 - \frac{1}{12}\left(\frac{1}{10}\right)^4 \pm \left(\frac{1}{10}\right)^7$

$= \frac{1}{200} - \frac{1}{12}\left(\frac{1}{10}\right)^4 \pm \left(\frac{1}{10}\right)^7$ +5

※ · 오차 계산에 설명이 없으면 (−5)

· 멱급수의 수렴 범위 $(|x|<1)$ 언급이 없으면 (−5)

7

a) $f(x) = (1+x)^r$

$$\Rightarrow f^{(n)}(x) = r(r-1)\cdots(r-n+1)(1+x)^{r-n}$$

(In any case, $r \in \mathbb{N}$ or $r \notin \mathbb{N}$)

so $$\frac{f^{(n)}(0)}{n!} = \frac{r(r-1)\cdots(r-n+1)}{n!} = \binom{r}{n}$$

$$\therefore Tf(x) = \sum_{n=0}^{\infty} \binom{r}{n} x^n$$

5pt

— 부분 점수 없음.

b) $f(0) = Tf(0) = 1$.

For $0 < x < 1$, There exist $x^* \in (0, x)$ such that

$$R_n f(x) = \frac{f^{(n+1)}(x^*)}{(n+1)!} x^{n+1} \quad \text{(Taylor's theorem)}$$

$$\Rightarrow |R_n f(x)| = \left| \frac{f^{(n+1)}(x^*)}{(n+1)!} x^{n+1} \right| = \left| \binom{r}{n+1} \right| (1+x^*)^{r-n-1} \cdot x^{n+1}$$

5pt

For $n \geq r-1$. $r-n-1 \leq 0 \Rightarrow (1+x^*)^{r-n-1} \leq 1$. ($x^* \in (0,1)$)

so, $n \geq r-1 \Rightarrow |R_n f(x)| \leq \left| \binom{r}{n+1} \right| \cdot x^{n+1} = a_n$

Let.

Now. $\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{r-n}{n+1} x \right| \longrightarrow x < 1$ as $n \to \infty$

so $a_n \longrightarrow 0$ as $n \to \infty$ ($\because \sum a_n < \infty$ by ratio test)

$\therefore |R_n f(x)| = a_n \longrightarrow 0$ as $n \to \infty$, so $f(x) = Tf(x)$ 10pt

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- $R_n f(x)$ 를 정확히 적으면 5점
- $R_n f(x)$ 가 0으로 수렴함을 정확히 보이면 10점

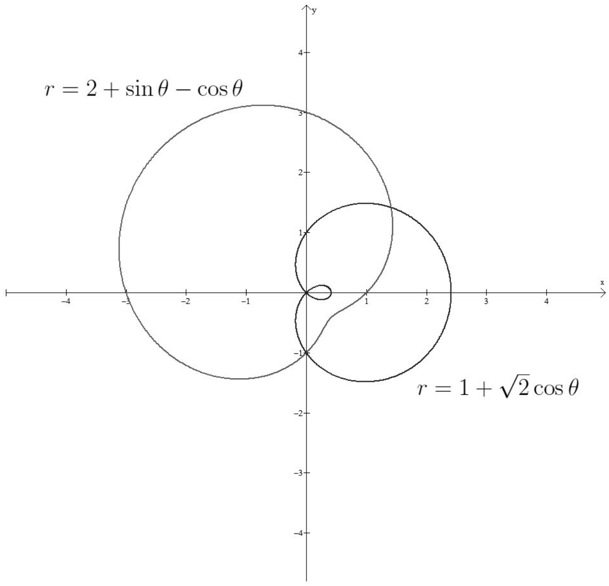
- 미분방정식을 이용한경우

  $Tf(x) = \sum \binom{r}{n} x^n$ 의 수렴반경에 관한

  설명이 없으면 $\boxed{-5}$

#8

a)



— 개형이 비슷하며 최대 최소가 분명히 나타나져 있어야 함. 각 그래프 당 5점.

b) If (r, θ): intersection point,

$r = 2 + \sin\theta - \cos\theta = 1 + \sqrt{2}\cos\theta$.

$\Rightarrow 1 + \sin\theta = (\sqrt{2}+1)\cos\theta$.

$\Rightarrow 1 + 2\sin\theta + \sin^2\theta = (3+2\sqrt{2})\cos^2\theta = (3+2\sqrt{2})(1-\sin^2\theta)$

$\Rightarrow \sin\theta = -1$ or $\frac{\sqrt{2}}{2}$.

Then for $\begin{cases} \sin\theta = -1, \quad \theta = \frac{3}{2}\pi \Rightarrow r = 1 \\ \sin\theta = \frac{\sqrt{2}}{2} \quad \begin{cases} \cos\theta = -\frac{\sqrt{2}}{2}, \Rightarrow 2 + \sin\theta - \cos\theta \neq 1 + \sqrt{2}\cos\theta \quad \therefore X \\ \cos\theta = \frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4} \Rightarrow r = 2 \end{cases} \end{cases}$

$\therefore$ Intersection points are

$(1, \frac{3}{2}\pi)$, $(2, \frac{\pi}{4})$ in polar coordinate.

4pt 4pt

$\Rightarrow (0, -1)$, $(\sqrt{2}, \sqrt{2})$ in Cartesian coordinate.

so square of the distance is

$(\sqrt{2})^2 + (\sqrt{2}+1)^2 = 5 + 2\sqrt{2}$

2pt.

- Intersection point 하나당 4점.
  이때, 그래프에 점을 표기하는 것만으로는 부족하며
  정확히 계산해 모든 교점을 구해야함.

- 거리. 2점.

#9

$$\lim_{R\to\infty} \int_1^R \cos(x^3)\,dx = \lim_{R\to\infty} \int_1^{R^3} \frac{\cos(x)}{3x^{2/3}}\,dx$$

Let $a_n = \displaystyle\int_{\frac{\pi}{2}+(n-1)\pi}^{\frac{\pi}{2}+n\pi} \frac{\cos(x)}{3x^{2/3}}\,dx$

Note that $a_n/|a_n| = (-1)^n$ (Alterating)

Since $\displaystyle\sum_{n=1}^{N} a_n \le \int_{\frac{\pi}{2}}^{R^3} \frac{\cos x}{3x^{2/3}} \le \sum_{n=1}^{N+1} a_n$, for some N;

(or $\displaystyle\sum_{n=1}^{N+1} a_n \le \int_{\frac{\pi}{2}}^{R^3} \frac{\cos x}{3x^{2/3}} \le \sum_{n=1}^{N} a_n$)

If $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent, then $\displaystyle\lim_{R\to\infty}\int_1^R \cos(x^3)\,dx$ is also convergent

Let $b_n = |a_n|$

$$\Rightarrow b_{n+1} = \int_{\frac{\pi}{2}+n\pi}^{\frac{\pi}{2}+(n+1)\pi} \frac{|\cos x|}{3x^{2/3}}\,dx = \int_{\frac{\pi}{2}+(n+1)\pi}^{\frac{\pi}{2}+(n)\pi} \frac{|\cos x|}{3(x+\pi)^{2/3}}\,dx$$

$$\le \int_{\frac{\pi}{2}+(n-1)\pi}^{\frac{\pi}{2}+n\pi} \frac{|\cos x|}{3x^{2/3}}\,dx = b_n$$

$\therefore b_n$: decreasing $\lrcorner + 5$

On the otherhand

$$b_n \le \int_{\frac{\pi}{2}+(n-1)\pi}^{\frac{\pi}{2}+n\pi} \frac{1}{3x^{2/3}}\,dx = \left(\frac{\pi}{2}+n\pi\right)^{1/3} - \left(\frac{\pi}{2}+(n-1)\pi\right)^{\frac{1}{3}}$$

$$= \frac{\pi}{\left(\frac{\pi}{2}+n\pi\right)^{\frac{2}{3}} + \left(\frac{\pi}{2}+n\pi\right)^{\frac{1}{3}}\left(\frac{\pi}{2}+(n-1)\pi\right)^{1/3} + \left(\frac{\pi}{2}+(n-1)\pi\right)^{\frac{2}{3}}} \xrightarrow[n\to\infty]{} 0$$

Therefore. b_n : decrasing and $\lim_{n\to\infty} b_n = 0$

By alternating series test,

Σa_n : convergent.

$\therefore \lim_{R\to\infty} \int_1^R \cos x^3 \, dx$: exists

+5

※ $\left.\begin{array}{l} b_n \geq b_{n+1} \\ \lim_{n\to\infty} b_n = 0 \end{array}\right\}$ 을 명확히 설명하여야 배점.