Chapter 23

Gauss' Law

Lecture 4

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 → Modified by S.H. Seo

Johann Carl Friedrich Gauss

Carl Friedrich Gauß (1777–1855), painted by Christian Albrecht Jensen

Born	Johann Carl Friedrich Gauss 30 April 1777 Brunswick, Duchy of Brunswick-Wolfenbüttel, Holy Roman Empire
Died	23 February 1855 (aged 77) Göttingen, Kingdom of Hanover
Residence	Kingdom of Hanover
Nationality	German
Fields	Mathematics and physics
Institutions	University of Göttingen
Alma mater	University of Helmstedt
Thesis	*Demonstratio nova...* (1799)
Doctoral advisor	Johann Friedrich Pfaff
Other academic advisors	Johann Christian Martin Bartels
Doctoral students	Johann Listing Christian Ludwig Gerling Richard Dedekind Bernhard Riemann Christian Peters Moritz Cantor
Other notable students	Johann Encke Christoph Gudermann Peter Gustav Lejeune Dirichlet Gotthold Eisenstein Carl Wolfgang Benjamin Goldschmidt Gustav Kirchhoff Ernst Kummer August Ferdinand Möbius L. C. Schnürlein Julius Weisbach
Known for	See full list
Influenced	Friedrich Bessel Sophie Germain Ferdinand Minding
Notable awards	Lalande Prize (1810) Copley Medal (1838)

Signature

From Wikipedia

German 10-Deutsche Mark Banknote (1993; discontinued) featuring Gauss

Gauss (aged about 26) on East German stamp produced in 1977. Next to him: heptadecagon, compass and straightedge.

23 Summary

Gauss' Law

- Gauss' law is

$$\boxed{\varepsilon_0 \Phi = q_{enc}}$$ **Eq. 23-6**

- the net flux of the electric field through the surface:

$$\boxed{\Phi = \oint \vec{E} \cdot d\vec{A}}$$ **Eq. 23-6**

Applications of Gauss' Law

- surface of a charged conductor

$$\boxed{E = \frac{\sigma}{\varepsilon_0}}$$ **Eq. 23-11**

- Within the surface *E=0.*
- line of charge

$$\boxed{E = \frac{\lambda}{2\pi\varepsilon_0 r}}$$ **Eq. 23-12**

- Infinite non-conducting sheet

$$\boxed{E = \frac{\sigma}{2\varepsilon_0}}$$ **Eq. 23-13**

- Outside a spherical shell of charge

$$\boxed{E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}}$$ **Eq. 23-15**

- Inside a uniform spherical shell

$$\boxed{E = 0}$$ **Eq. 23-16**

- Inside a uniform sphere of charge

$$\boxed{E = \left(\frac{q}{4\pi\varepsilon_0 R^3}\right) r.}$$ **Eq. 23-20**

23-1 Electric Flux

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Learning Objectives

23.01 Identify that Gauss' law relates the electric field at points on a closed surface (real or imaginary, said to be a Gaussian surface) to the net charge enclosed by that surface.

23.02 Identify that the amount of electric field piercing a surface (not skimming along parallel to the surface) is the electric flux Φ through the surface.

23.03 Identify that an area vector for a flat surface is a vector that is perpendicular to the surface and that has a magnitude equal to the area of the surface.

23.04 Identify that any surface can be divided into area elements (patch elements) that are each small enough and flat enough for an area vector $d\vec{A}$ to be assigned to it, with the vector perpendicular to the element and having a magnitude equal to the area of the element.

23-1 Electric Flux

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Learning Objectives (Contd.)

23.05 Calculate the flux Φ through a surface by integrating the dot product of the electric field vector ***E*** and the area vector *d**A*** (for patch elements) over the surface, in magnitude- angle notation and unit-vector notation.

23.06 For a closed surface, explain the algebraic signs associated with inward flux and outward flux.

23.07 Calculate the net flux ϕ through a closed surface, algebraic sign included, by integrating the dot product of the electric field vector ***E*** and the area vector *d**A*** (for patch elements) over the full surface.

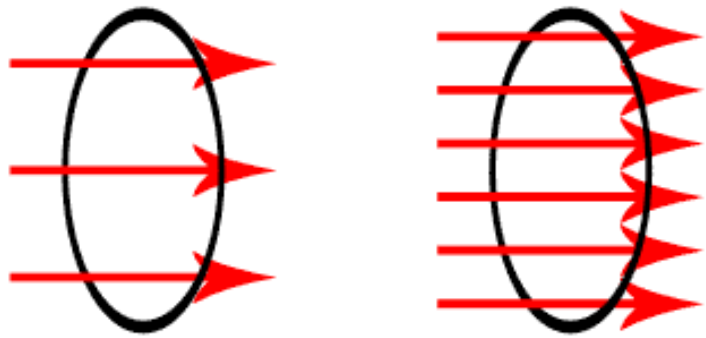
23.08 Determine whether a closed surface can be broken up into parts (such as the sides of a cube) to simplify the integration that yields the net flux through the surface.

What is Flux ?

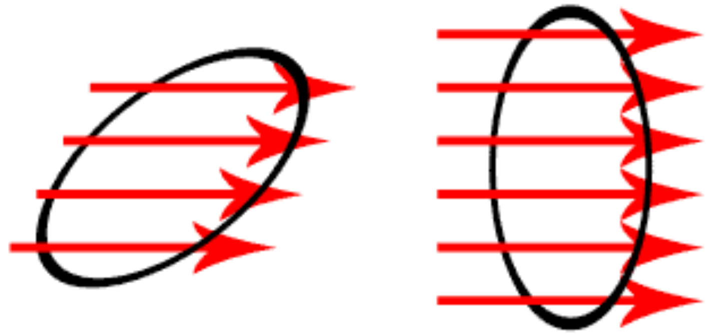
The word flux comes from Latin: fluxus means "flow”.

Flux: flow rate per unit area

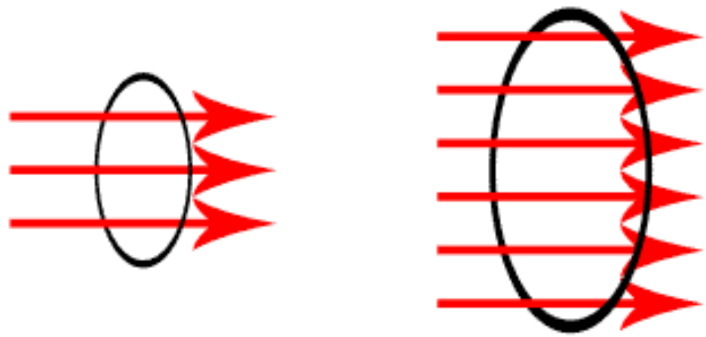
→ dimension: [quantity][time]$^{-1}$[area]$^{-1}$



Flux is proportional to the density of flow.



Flux varies by how the boundary faces the direction of flow.



Flux is proportional to the area within the boundary.

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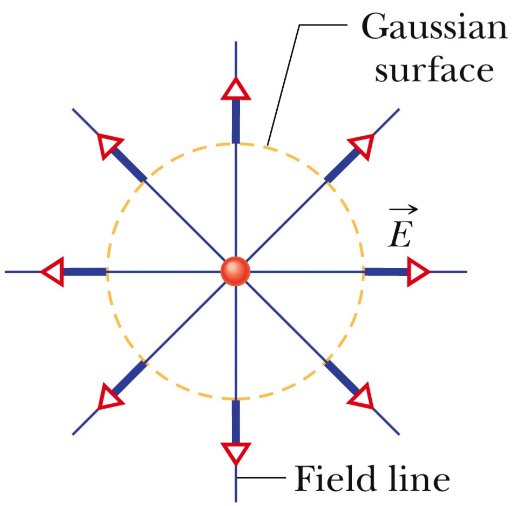
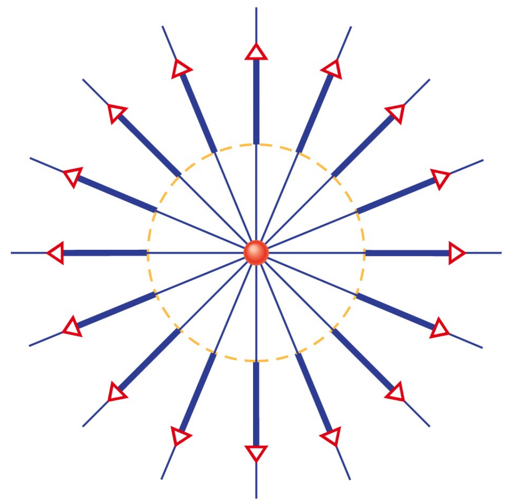
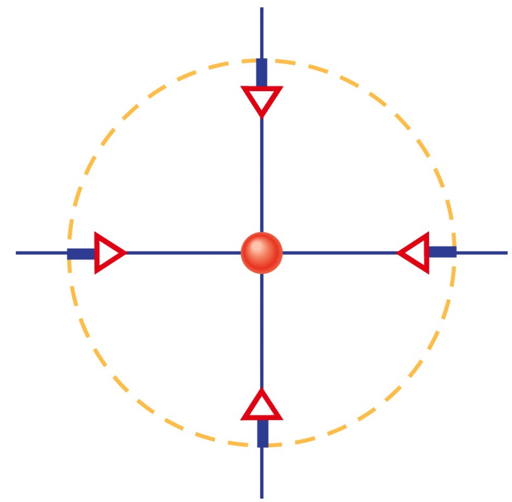
23-1 Electric Flux

Guass' law relates the electric field at points on a (closed) Gaussian surface to the net charge enclosed by that surface.







Electric field vectors and field lines pierce an imaginary, spherical **Gaussian surface** that encloses a particle with charge +Q.

Now the enclosed particle has charge +2Q.

Can you tell what the enclosed charge is now?
Answer: -0.5Q

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23-1 Electric Flux

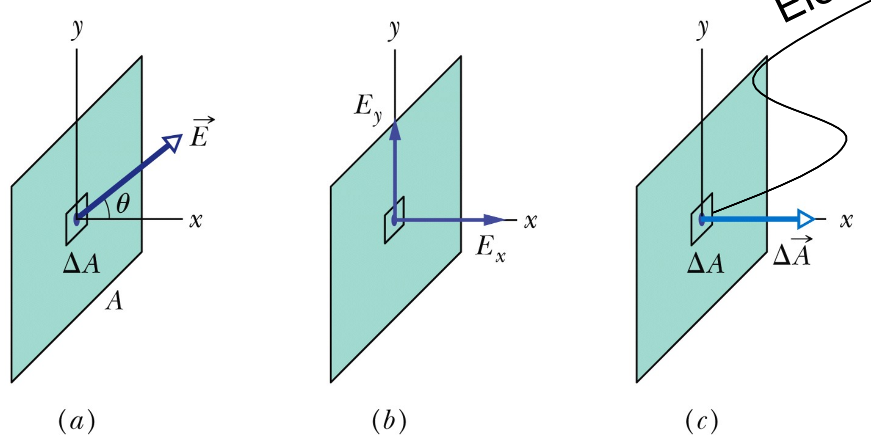
The **area vector** *dA* for an area element (patch element) on a surface is a vector that is perpendicular to the element and has a magnitude equal to the area *dA* of the element.

The **electric flux** *d*ϕ through a patch element with area vector ***dA*** is given by a dot product:

$$\boxed{d\Phi = \vec{E} \cdot d\vec{A}.}$$

Electric Flux

$$\Delta\Phi = (E\cos\theta)\,\Delta A.$$



(a) An electric field vector pierces a small square patch on a flat surface.

(b) Only the *x* component actually pierces the patch; the *y* component skims across it.

(c) The area vector of the patch is perpendicular to the patch, with a magnitude equal to the patch's area.

23-1 Electric Flux

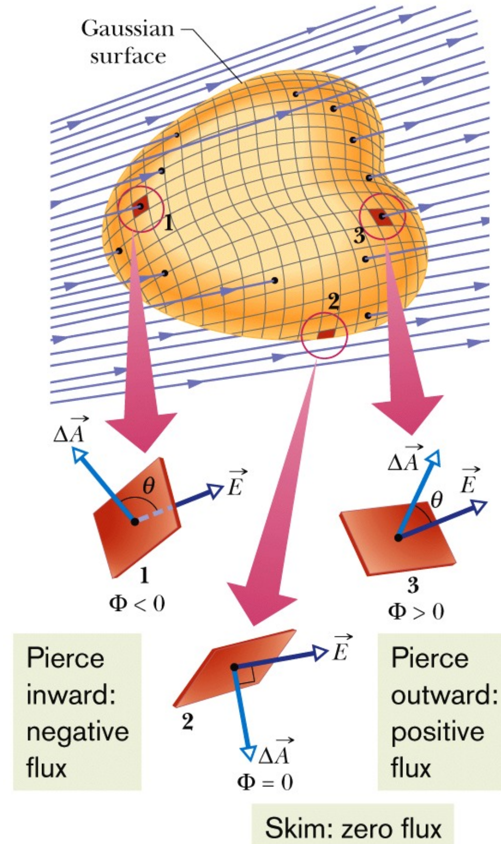
Now we can find the total flux by integrating the dot product over the full surface.
The **total flux** **through a surface** is given by

$$\boxed{\Phi = \int \vec{E} \cdot d\vec{A} \qquad \text{(total flux).}}$$

The **net flux** **through a closed surface** (which is used in Gauss' law) is given by

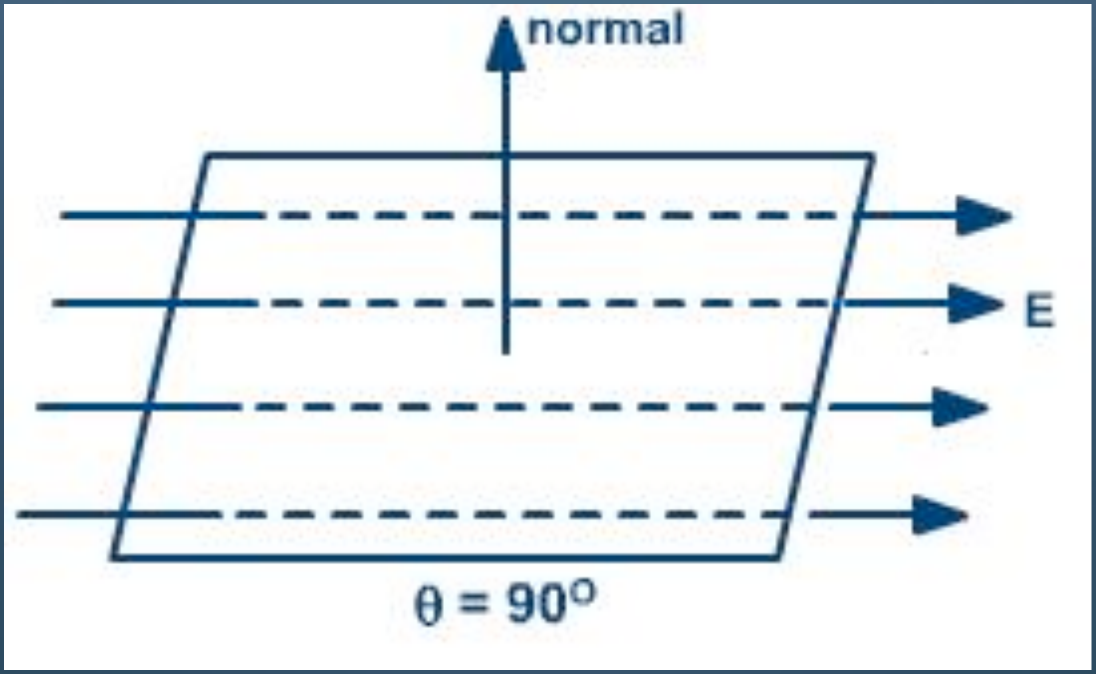
$$\boxed{\Phi = \oint \vec{E} \cdot d\vec{A} \qquad \text{(net flux).}}$$

where the integration is carried out over the entire surface.



An inward piercing field is negative flux. An outward piercing field is positive flux. A skimming field is zero flux.

Electric flux = ?



23-1 Electric Flux

Flux through a closed cylinder, uniform field

Figure 23-6 shows a Gaussian surface in the form of a closed cylinder (a Gaussian cylinder or G-cylinder) of radius R. It lies in a uniform electric field $\vec{E}$ with the cylinder's central axis (along the length of the cylinder) parallel to the field. What is the net flux Φ of the electric field through the cylinder?

KEY IDEAS

We can find the net flux Φ with Eq. 23-4 by integrating the dot product $\vec{E} \cdot d\vec{A}$ over the cylinder's surface. However, we cannot write out functions so that we can do that with one integral. Instead, we need to be a bit clever: We break up the surface into sections with which we can actually evaluate an integral.

Calculations: We break the integral of Eq. 23-4 into three terms: integrals over the left cylinder cap a, the curved cylindrical surface b, and the right cap c:

$$\Phi = \oint \vec{E} \cdot d\vec{A}$$

$$= \int_a \vec{E} \cdot d\vec{A} + \int_b \vec{E} \cdot d\vec{A} + \int_c \vec{E} \cdot d\vec{A}. \qquad (23\text{-}5)$$

Pick a patch element on the left cap. Its area vector $d\vec{A}$ must be perpendicular to the patch and pointing away from the interior of the cylinder. In Fig. 23-6, that means the angle between it and the field piercing the patch is 180°. Also, note that the electric field through the end cap is uniform and thus E can be pulled out of the integration. So, we can write the flux through the left cap as

$$\int_a \vec{E} \cdot d\vec{A} = \int E(\cos 180°)\, dA = -E \int dA = -EA,$$

where $\int dA$ gives the cap's area A $(= \pi R^2)$. Similarly, for the right cap, where $\theta = 0$ for all points,

$$\int_c \vec{E} \cdot d\vec{A} = \int E(\cos 0)\, dA = EA.$$

Finally, for the cylindrical surface, where the angle θ is 90° at all points,

$$\int_b \vec{E} \cdot d\vec{A} = \int E(\cos 90°)\, dA = 0.$$

Substituting these results into Eq. 23-5 leads us to

$$\Phi = -EA + 0 + EA = 0. \qquad \text{(Answer)}$$

The net flux is zero because the field lines that represent the electric field all pass entirely through the Gaussian surface, from the left to the right.

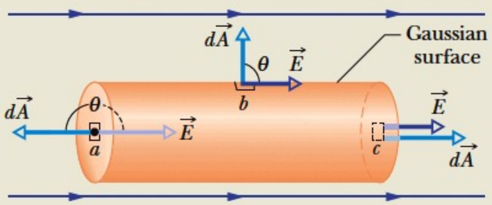


Figure 23-6 A cylindrical Gaussian surface, closed by end caps, is immersed in a uniform electric field. The cylinder axis is parallel to the field direction.

23-2 Gauss' Law

Learning Objectives

23.09 Apply Gauss' law to relate the net flux ϕ through a closed surface to the net enclosed charge q_{enc}.

23.10 Identify how the algebraic sign of the net enclosed charge corresponds to the direction (inward or outward) of the net flux through a Gaussian surface.

23.11 Identify that charge outside a Gaussian surface makes no contribution to the net flux through the closed surface.

23.12 Derive the expression for the magnitude of the electric field of a charged particle by using Gauss' law.

23.13 Identify that for a charged particle or uniformly charged sphere, Gauss' law is applied with a Gaussian surface that is a concentric sphere.

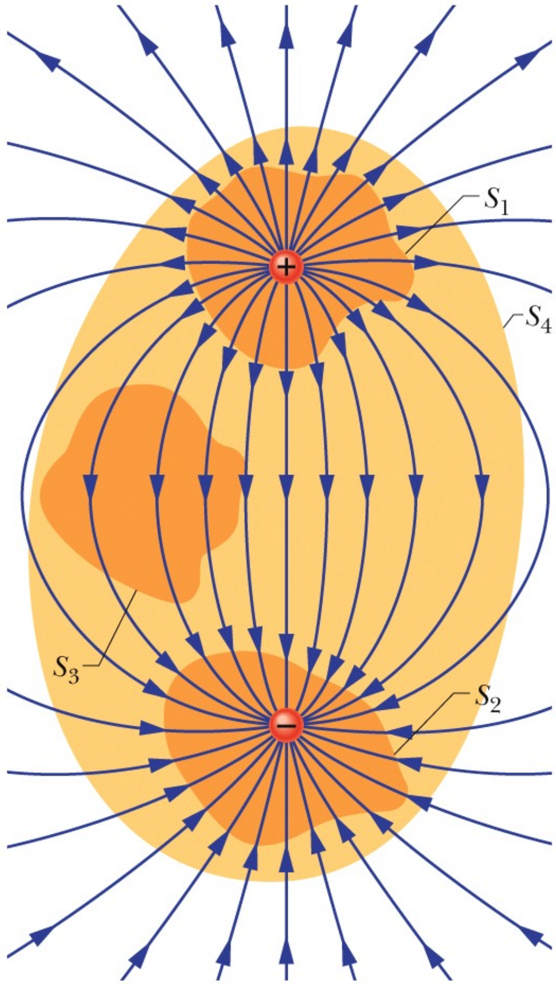
23-2 Gauss' Law

Gauss' law relates the net flux ϕ of an electric field through a closed surface (a Gaussian surface) to the *net* charge q_{enc} that is enclosed by that surface. It tells us that

$$\varepsilon_0 \Phi = q_{\text{enc}} \quad \text{(Gauss' law).}$$

we can also write Gauss' law as

$$\boxed{\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{\text{enc}} \quad \text{(Gauss' law).}}$$

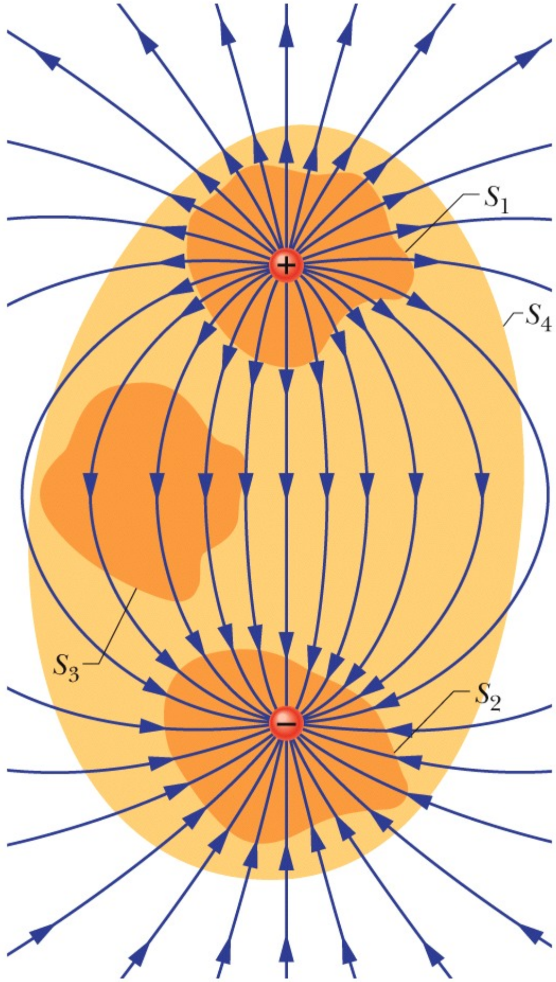


Two charges, equal in magnitude but opposite in sign, and the field lines that represent their net electric field. Four Gaussian surfaces are shown in cross section.

23-2 Gauss' Law

Surface S1.The electric field is outward for all points on this surface. Thus, the flux of the electric field through this surface is positive, and so is the net charge within the surface, as Gauss' law requires

Surface S2.The electric field is inward for all points on this surface. Thus, the flux of the electric field through this surface is negative and so is the enclosed charge, as Gauss' law requires.

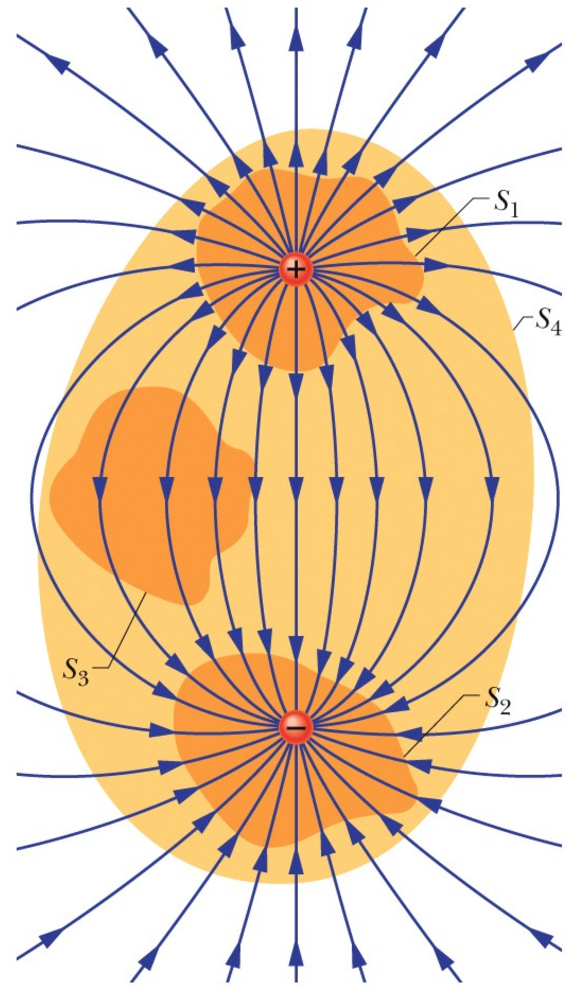


Two charges, equal in magnitude but opposite in sign, and the field lines that represent their net electric field. Four Gaussian surfaces are shown in cross section.

23-2 Gauss' Law

Surface S3.This surface encloses no charge, and thus q_{enc} = 0. Gauss' law requires that the net flux of the electric field through this surface be zero. That is reasonable because all the field lines pass entirely through the surface, entering it at the top and leaving at the bottom.

Surface S4.This surface encloses no net charge, because the enclosed positive and negative charges have equal magnitudes. Gauss' law requires that the net flux of the electric field through this surface be zero. That is reasonable because there are as many field lines leaving surface S4 as entering it.



Two charges, equal in magnitude but opposite in sign, and the field lines that represent their net electric field. Four Gaussian surfaces are shown in cross section.

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23-3 A Charged Isolated Conductor

Learning Objectives

23.14 Apply the relationship between surface charge density σ and the area over which the charge is uniformly spread.

23.15 Identify that if excess charge (positive or negative) is placed on an isolated conductor, that charge moves to the surface and none is in the interior.

23.16 Identify the value of the electric field inside an isolated conductor.

23.17 For a conductor with a cavity that contains a charged object, determine the charge on the cavity wall and on the external surface.

23.18 Explain how Gauss' law is used to find the electric field magnitude E near an isolated conducting surface with a uniform surface charge density σ.

23.19 For a uniformly charged conducting surface, apply the relationship between the charge density σ and the electric field magnitude E at points near the conductor, and identify the direction of the field vectors.

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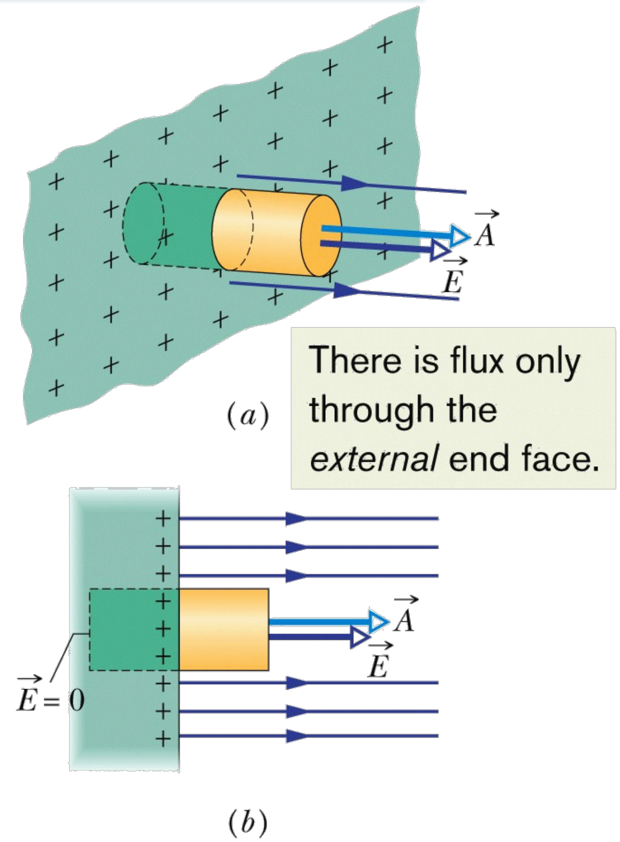
23-3 A Charged Isolated Conductor

If an excess charge is placed on an isolated conductor, that amount of charge will move entirely to the surface of the conductor. None of the excess charge will be found within the body of the conductor.

$$\boxed{E = \frac{\sigma}{\varepsilon_0} \quad \text{(conducting surface).}}$$

(a) Perspective view

(b) Side view of a tiny portion of a large, isolated conductor with excess positive charge on its surface. A (closed) cylindrical Gaussian surface, embedded perpendicularly in the conductor, encloses some of the charge. Electric field lines pierce the external end cap of the cylinder, but not the internal end cap. The external end cap has area A and area vector $\boldsymbol{A}$.



23-4 Applying Gauss' Law: Cylindrical Symmetry

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Learning Objectives

23.20 Explain how Gauss' law is used to derive the electric field magnitude outside a line of charge or a cylindrical surface (such as a plastic rod) with a uniform linear charge density λ.

23.21 Apply the relationship between linear charge density λ on a cylindrical surface and the electric field magnitude E at radial distance r from the central axis.

23.22 Explain how Gauss' law can be used to find the electric field magnitude inside a cylindrical non-conducting surface (such as a plastic rod) with a uniform volume charge density ρ.

23-4 Applying Gauss' Law: Cylindrical Symmetry

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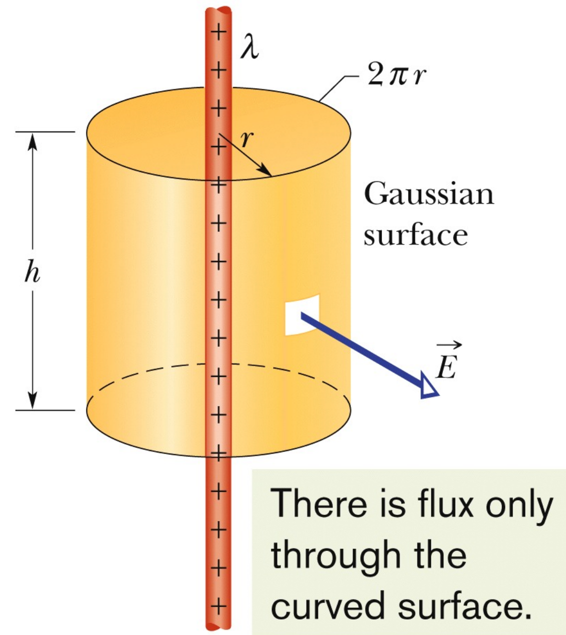
Figure shows a section of an infinitely long cylindrical plastic rod with a uniform charge density λ. The charge distribution and the field have cylindrical symmetry. To find the field at radius r, we enclose a section of the rod with a concentric Gaussian cylinder of radius r and height h.
The net flux through the cylinder from Gauss' Law reduces to

$$\Phi = EA\cos\theta = E(2\pi rh)\cos 0 = E(2\pi rh).$$

yielding $$\varepsilon_0\Phi = q_{enc},$$

$$\varepsilon_0 E(2\pi rh) = \lambda h,$$

$$\boxed{E = \frac{\lambda}{2\pi\varepsilon_0 r} \quad \text{(line of charge).}}$$



A Gaussian surface in the form of a closed cylinder surrounds a section of a very long, uniformly charged, cylindrical plastic rod.

23-5 Applying Gauss' Law: Planar Symmetry

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Learning Objectives

23.23 Apply Gauss' law to derive the electric field magnitude ***E*** near a large, flat, non-conducting surface with a uniform surface charge density σ.

23.24 For points near a large, flat non-conducting surface with a uniform charge density σ, apply the relationship between the charge density and the electric field magnitude ***E*** and also specify the direction of the field.

23.25 For points near two large, flat, parallel, conducting surfaces with a uniform charge density σ, apply the relationship between the charge density and the electric field magnitude ***E*** and also specify the direction of the field.

23-5 Applying Gauss' Law: Planar Symmetry

Non-conducting Sheet

Figure (a-b) shows a portion of a thin, infinite, non-conducting sheet with a uniform (positive) surface charge density σ. A sheet of thin plastic wrap, uniformly charged on one side, can serve as a simple model. Here, $\vec{E} \cdot d\vec{A}$

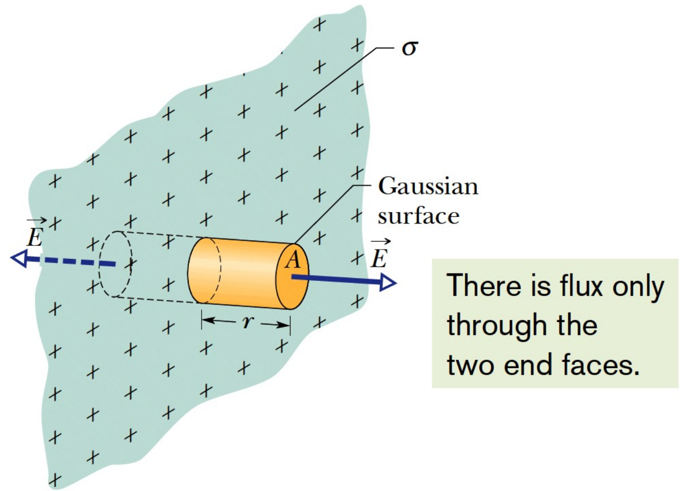
Is simply EdA and thus Gauss' Law,

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{enc},$$

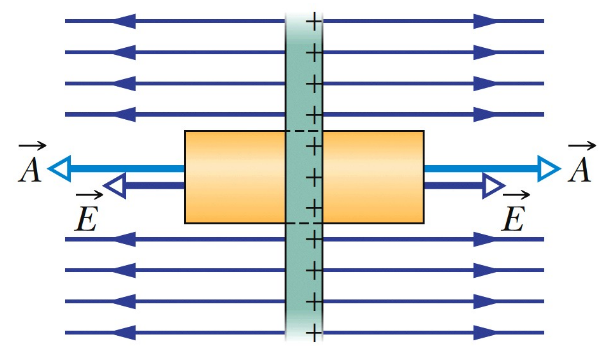
becomes $$\varepsilon_0(EA + EA) = \sigma A,$$

where σA is the charge enclosed by the Gaussian surface. This gives

$$\boxed{E = \frac{\sigma}{2\varepsilon_0} \quad \text{(sheet of charge).}}$$



(a)



(b)

23-5 Applying Gauss' Law: Planar Symmetry

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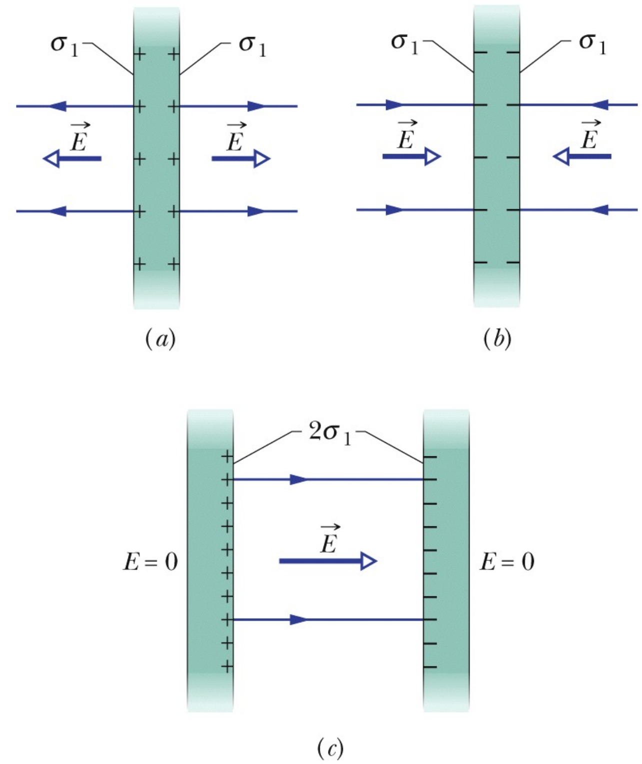
Two conducting Plates

Figure (a) shows a cross section of a thin, infinite conducting plate with excess positive charge**. Figure (b)** shows an identical plate with excess negative charge having the same magnitude of surface charge density *σ1*.
Suppose we arrange for the plates of **Figs. a** and **b** to be close to each other and parallel **(c)**. Since the plates are conductors, when we bring them into this arrangement, the excess charge on one plate attracts the excess charge



on the other plate, and all the excess charge moves onto the inner faces of the plates as in Fig.c. With twice as much charge now on each inner face, the electric field at any point between the plates has the magnitude

$$\boxed{E = \frac{2\sigma_1}{\varepsilon_0} = \frac{\sigma}{\varepsilon_0}.}$$

23-6 Applying Gauss' Law: Spherical Symmetry

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Learning Objectives

23.26 Identify that a shell of uniform charge attracts or repels a charged particle that is outside the shell as if all the shell's charge is concentrated at the center of the shell.

23.27 Identify that if a charged particle is enclosed by a shell of uniform charge, there is no electrostatic force on the particle from the shell.

23.28 For a point outside a spherical shell with uniform charge, apply the relationship between the electric field magnitude E, the charge q on the shell, and the distance r from the shell's center.

23.29 Identify the magnitude of the electric field for points enclosed by a spherical shell with uniform charge.

23.30 For a uniform spherical charge distribution (a uniform ball of charge), determine the magnitude and direction of the electric field at interior and exterior points.

23-6 Applying Gauss' Law: Spherical Symmetry

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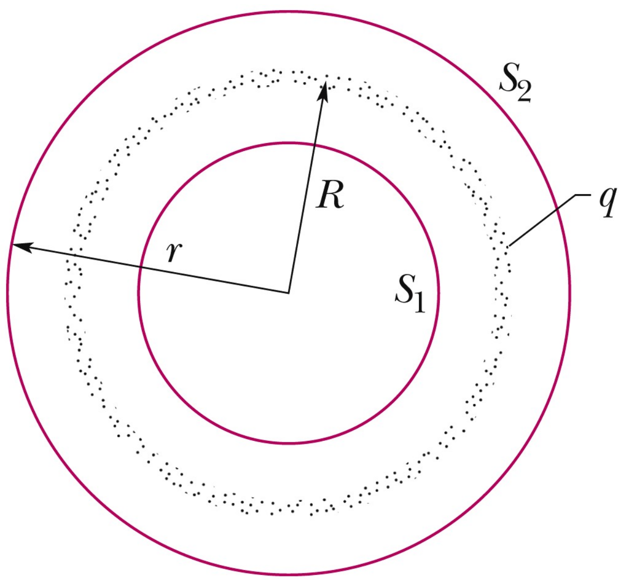
A shell of uniform charge attracts or repels a charged particle that is outside the shell as if all the shell's charge were concentrated at the center of the shell.



A thin, uniformly charged, spherical shell with total charge q, in cross section. Two Gaussian surfaces S1 and S2 are also shown in cross section. Surface S2 encloses the shell, and S1 encloses only the empty interior of the shell.

In the figure, applying Gauss' law to surface S_2, for which $r \geq R$, we would find that

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \quad \text{(spherical shell, field at } r \geq R\text{).}$$

And, applying Gauss' law to surface S_1, for which $r < R$,

$$E = 0 \quad \text{(spherical shell, field at } r < R\text{),}$$

If a charged particle is located inside a shell of uniform charge, there is no electrostatic force on the particle from the shell.

23-6 Applying Gauss' Law: Spherical Symmetry

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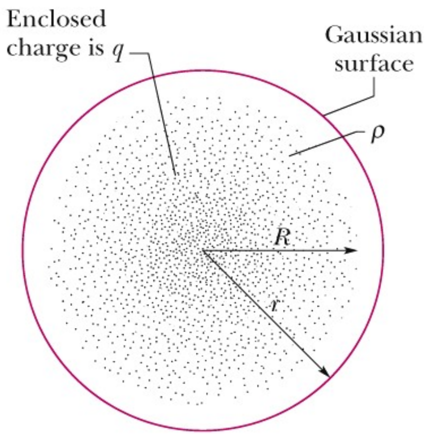
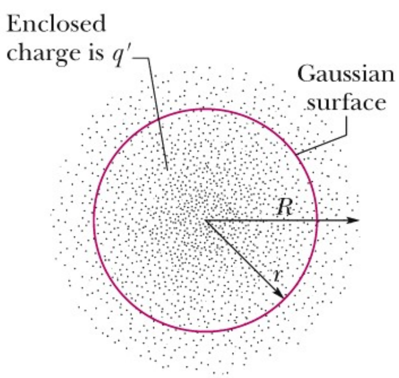


(a)



(b)

The flux through the surface depends on only the *enclosed* charge.

Inside a sphere with a uniform volume charge density, the field is radial and has the magnitude

$$\boxed{E = \left(\frac{q}{4\pi\varepsilon_0 R^3}\right) r \quad \text{(uniform charge, field at } r \le R\text{).}}$$

where q is the total charge, R is the sphere's radius, and r is the radial distance from the center of the sphere to the point of measurement as shown in figure.

$$\varepsilon_0 \underbrace{E4\pi r^2}_{\Phi} = \underbrace{\frac{q}{\frac{4\pi}{3}\cdot R^3}}_{\rho} \bullet \underbrace{\frac{4\pi}{3} r^3}_{V}$$

A concentric spherical Gaussian surface with $r > R$ is shown in (a).
A similar Gaussian surface with $r < R$ is shown in (b).

23 Summary

Gauss' Law

- Gauss' law is

$$\varepsilon_0 \Phi = q_{enc}$$ **Eq. 23-6**

- the net flux of the electric field through the surface:

$$\Phi = \oint \vec{E} \cdot d\vec{A}$$ **Eq. 23-6**

Applications of Gauss' Law

- surface of a charged conductor

$$E = \frac{\sigma}{\varepsilon_0}$$ **Eq. 23-11**

- Within the surface *E=0.*
- line of charge

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}$$ **Eq. 23-12**

- Infinite non-conducting sheet

$$E = \frac{\sigma}{2\varepsilon_0}$$ **Eq. 23-13**

- Outside a spherical shell of charge

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$ **Eq. 23-15**

- Inside a uniform spherical shell

$$E = 0$$ **Eq. 23-16**

- Inside a uniform sphere of charge

$$E = \left(\frac{q}{4\pi\varepsilon_0 R^3}\right)r.$$ **Eq. 23-20**