Chap 4. Systems of ODEs

4.0 Basics of Matrices and Vectors

Eigenvalues, Eigenvectors

$$\underbrace{A\mathbf{x}}_{\text{eigenvector}} = \underbrace{\lambda}_{\text{eigenvalue}} \mathbf{x} \quad \text{for some } \mathbf{x} \neq \mathbf{0}$$

$\Leftrightarrow (A-\lambda I)\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$

$\Leftrightarrow \det(A-\lambda I) = 0$: characteristic equation of A

Ex 1 (Eigenvalue Problem)

$$A = \begin{bmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{bmatrix}, \quad A-\lambda I = \begin{bmatrix} -4.0-\lambda & 4.0 \\ -1.6 & 1.2-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = \lambda^2 + 2.8\lambda + 1.6 = 0$$

$$\lambda_1 = -2, \qquad \lambda_2 = -0.8$$

$$\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix}$$

∵ For $\lambda_1 = -2$,

$$\begin{bmatrix} -4.0+2.0 & 4.0 \\ -1.6 & 1.2+2.0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2.0 & 4.0 \\ -1.6 & 3.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x_1 + 2x_2 = 0$$

$$\therefore \mathbf{x}^{(1)} = c \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{for } c \neq 0$$

4.1 Systems of ODEs as Models

Ex 1 (Mixing Problem Involving Two Tanks)

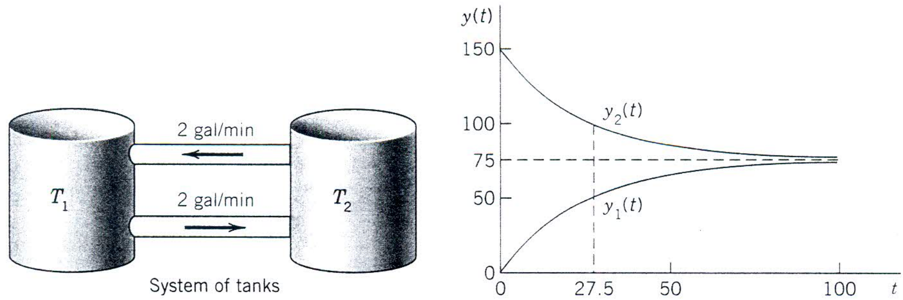


Fig. 77. Fertilizer content in Tanks T_1 (lower curve) and T_2

$$\begin{cases} y_1' = -0.02\,y_1 + 0.02\,y_2\,, & y_1(0) = 0 \\ y_2' = 0.02\,y_1 - 0.02\,y_2\,, & y_2(0) = 150 \end{cases}$$

$$\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A\mathbf{y}$$

Let $\mathbf{y} = \mathbf{x}e^{\lambda t}$,

$$\mathbf{y}' = \lambda \mathbf{x} e^{\lambda t}, \quad A\mathbf{x}e^{\lambda t} = \lambda \mathbf{x} e^{\lambda t}$$

$$\therefore A\mathbf{x} = \lambda \mathbf{x}$$

$$\det(A - \lambda I) = \lambda(\lambda + 0.04) = 0$$

$$\lambda_1 = 0 : \quad \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad \lambda_2 = -0.04 : \quad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{y} = c_1 \mathbf{x}^{(1)} e^{\lambda_1 t} + c_2 \mathbf{x}^{(2)} e^{\lambda_2 t}$$

$$= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.04t}$$

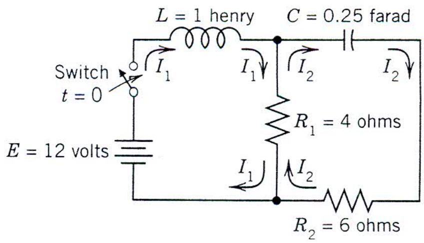
$$\mathbf{y}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 150 \end{bmatrix}$$

$$\therefore c_1 = 75, \quad c_2 = -75$$

$$\mathbf{y}(t) = 75 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 75 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.04t}$$

Ex2 (Electrical Network)

$$\begin{cases} I_1' + 4(I_1 - I_2) = 12 \\ 6I_2 + 4(I_2 - I_1) + 4\int I_2\,dt = 0 \end{cases}$$



$$\Rightarrow \begin{cases} I_1' = -4I_1 + 4I_2 + 12 \\ I_2' - 0.4I_1' + 0.4I_2 = 0 \end{cases}$$

$$\hookrightarrow I_2' = -1.6I_1 + 1.2I_2 + 4.8$$

$$\mathbf{J}' = A\mathbf{J} + \mathbf{g}, \quad \begin{bmatrix} I_1' \\ I_2' \end{bmatrix} = \begin{bmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} + \begin{bmatrix} 12 \\ 4.8 \end{bmatrix}$$

① Consider $\mathbf{J}' = A\mathbf{J}$ first, $\det(A - \lambda I) = 0$

$$\lambda_1 = -2: \quad \mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}; \quad \lambda_2 = -0.8: \quad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix}$$

$$\mathbf{J}_h = c_1\mathbf{x}^{(1)}e^{-2t} + c_2\mathbf{x}^{(2)}e^{-0.8t}$$

② For a particular solution, try a constant vector

$$\mathbf{J}_p = \mathbf{a} = [a_1\ a_2]^T \Rightarrow \mathbf{J}_p' = [0, 0]^T$$

$$A\mathbf{a} + \mathbf{g} = \mathbf{0} \quad \text{or} \quad A\mathbf{a} = -\mathbf{g}$$

$$\therefore \mathbf{a} = [3, 0]^T$$

$$\Rightarrow \mathbf{J} = \mathbf{J}_h + \mathbf{J}_p = c_1\mathbf{x}^{(1)}e^{-2t} + c_2\mathbf{x}^{(2)}e^{-0.8t} + \mathbf{a}$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 2c_1e^{-2t} + c_2e^{-0.8t} + 3 \\ c_1e^{-2t} + 0.8c_2e^{-0.8t} \end{bmatrix}$$

$$\begin{bmatrix} I_1(0) \\ I_2(0) \end{bmatrix} = \begin{bmatrix} 2c_1 + c_2 + 3 \\ c_1 + 0.8c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore c_1 = -4, \quad c_2 = 5$$

$$\mathbf{J} = -4\mathbf{x}^{(1)}e^{-2t} + 5\mathbf{x}^{(2)}e^{-0.8t} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\begin{cases} I_1 = -8e^{-2t} + 5e^{-0.8t} + 3 \\ I_2 = -4e^{-2t} + 4e^{-0.8t} \end{cases}$$

<u>Conversion of an nth-Order ODE to a System</u>

$$y^{(n)} = F(t, y, y', \cdots, y^{(n-1)})$$

Let $y_1 = y,\ y_2 = y',\ y_3 = y'',\ \cdots,\ y_n = y^{(n-1)}$.

$$\Rightarrow \begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ \cdots \\ y_{n-1}' = y_n \\ y_n' = F(t, y_1, \cdots, y_n) \end{cases}$$

<u>Ex3</u> (Mass on a Spring)

$$my'' + cy' + ky = 0 \quad \text{or} \quad y'' + \frac{c}{m}y' + \frac{k}{m}y = 0$$

$$\begin{cases} y_1' = y_2 \\ y_2' = -\frac{k}{m}y_1 - \frac{c}{m}y_2 \end{cases}$$

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \mathbf{y}, \quad \text{where} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m} - \lambda \end{vmatrix} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

When $m = 1,\ c = 2,\ k = 0.75$,

$$\lambda^2 + 2\lambda + 0.75 = (\lambda + 0.5)(\lambda + 1.5) = 0$$

$$\lambda_1 = -0.5 : \ \mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}; \quad \lambda_2 = -1.5 : \ \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix}$$

$$\mathbf{y} = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-0.5t} + c_2 \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} e^{-1.5t}$$

$$\therefore \underline{y = y_1 = 2c_1 e^{-0.5t} + c_2 e^{-1.5t}}$$

4.2 Basic Theory of Systems of ODEs

$$\begin{cases} y_1' = f_1(t, y_1, \cdots, y_n) \\ y_2' = f_2(t, y_1, \cdots, y_n) \\ \cdots \\ y_n' = f_n(t, y_1, \cdots, y_n) \end{cases} \quad \text{or } \mathbf{y}' = \mathbf{f}(t, \mathbf{y})$$

IVP:

$$y_1(t_0) = K_1,\ y_2(t_0) = K_2,\ \cdots,\ y_n(t_0) = K_n \quad \text{or } \mathbf{y}(t_0) = \mathbb{K}$$

Linear Systems

$$\mathbf{y}' = A\mathbf{y} + \mathbf{g}, \text{ where } A = [a_{ij}(t)]_{n\times n},\ \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \mathbf{g} = \begin{bmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix}$$

Homogeneous: $\mathbf{y}' = A\mathbf{y}$

Nonhomogeneous: $\mathbf{y}' = A\mathbf{y} + \mathbf{g}$, where $\mathbf{g} \neq \mathbf{0}$

Basis: ($\mathbf{y}^{(1)}, \cdots, \mathbf{y}^{(n)}$: basis
$\Leftrightarrow$ a linearly indep. set of n sols of $\mathbf{y}' = A\mathbf{y}$)

General Solution:

$$\mathbf{y} = c_1 \mathbf{y}^{(1)} + \cdots + c_n \mathbf{y}^{(n)}$$

Wronskian

$$W(\mathbf{y}^{(1)}, \cdots, \mathbf{y}^{(n)}) = \begin{vmatrix} y_1^{(1)} & y_1^{(2)} & \cdots & y_1^{(n)} \\ y_2^{(1)} & y_2^{(2)} & & y_2^{(n)} \\ \vdots & \vdots & & \vdots \\ y_n^{(1)} & y_n^{(2)} & \cdots & y_n^{(n)} \end{vmatrix}$$

$\mathbf{y}^{(1)}, \cdots, \mathbf{y}^{(n)}$: basis $\Leftrightarrow W(\mathbf{y}^{(1)}, \cdots, \mathbf{y}^{(n)}) \neq 0$

Remark

This definition is related to Section 2.6 with two indep. solutions y, z.

$$W(y, z) = \begin{vmatrix} y & z \\ y' & z' \end{vmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, \quad \text{where } y = y_1,\ y' = y_1' = y_2,\ z = z_1,\ z' = z_1' = z_2.$$

4.3 Constant-Coefficient Systems

$\underline{y' = Ay}$, where $A = [a_{ij}]$ is constant

Try $y = x e^{\lambda t} \Rightarrow y' = \lambda x e^{\lambda t} = Ay = Axe^{\lambda t}$

$$\therefore Ax = \lambda x.$$

$x^{(1)}, \cdots, x^{(n)}$: n eigenvectors of A, forming a basis

$\lambda_1, \cdots, \lambda_n$: eigenvalues of A

$\Rightarrow y^{(1)} = x^{(1)} e^{\lambda_1 t}, \cdots, y^{(n)} = x^{(n)} e^{\lambda_n t}$: solutions of $y' = Ay$

$$W(y^{(1)}, \cdots, y^{(n)}) = \begin{vmatrix} x_1^{(1)} e^{\lambda_1 t} & \cdots & x_1^{(n)} e^{\lambda_n t} \\ x_2^{(1)} e^{\lambda_1 t} & & x_2^{(n)} e^{\lambda_n t} \\ \vdots & & \vdots \\ x_n^{(1)} e^{\lambda_1 t} & \cdots & x_n^{(n)} e^{\lambda_n t} \end{vmatrix}$$

$$= e^{(\lambda_1 + \cdots + \lambda_n)t} \underbrace{\begin{vmatrix} x_1^{(1)} & \cdots & x_1^{(n)} \\ x_2^{(1)} & & x_2^{(n)} \\ \vdots & & \vdots \\ x_n^{(1)} & \cdots & x_n^{(n)} \end{vmatrix}}_{\neq 0}$$

(since $x^{(1)}, \cdots, x^{(n)}$ form a basis)

Th1 (General Solution)

If A has a linearly independent set of n eigenvectors

$\Rightarrow y^{(1)}, \cdots, y^{(n)}$ form a basis of solutions

$\underline{y = c_1 x^{(1)} e^{\lambda_1 t} + \cdots + c_n x^{(n)} e^{\lambda_n t}}$: general solution

4.6 Nonhomogeneous Linear Systems of ODEs

$$y' = Ay + g, \quad g(t) \neq 0$$

$$y = y^{(h)} + y^{(p)}$$

Method of Undetermined Coefficients

Ex1 (Modification Rule)

$$y' = Ay + g = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} y + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

$$\Rightarrow y^{(h)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

$$y^{(p)} = u\, te^{-2t} + v e^{-2t}$$

$$y^{(p)\prime} = u e^{-2t} - 2u\, te^{-2t} - 2v e^{-2t}$$

$$= Au\, te^{-2t} + Av e^{-2t} + g$$

$$Au = -2u \Rightarrow u = \begin{bmatrix} a \\ a \end{bmatrix}, \quad a \neq 0$$

$$u - 2v = Av + \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

$$\Rightarrow u = \begin{bmatrix} -2 \\ -2 \end{bmatrix} \text{ and } v = \begin{bmatrix} 0 \\ 4 \end{bmatrix} + k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $k = 0$,

$$y = y^{(h)} + y^{(p)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

$$\underbrace{-2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-2t} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} e^{-2t}}_{y^{(p)}}$$

For $k = -2$, $y^{(p)} = -2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-2t} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{-2t}$

Method of Variation of Parameters

$$y' = Ay + g$$

$$y^{(h)} = c_1 y^{(1)} + \cdots + c_n y^{(n)} = \underbrace{\left[y^{(1)} \cdots y^{(n)} \right]}_{\mathbb{Y}(t)} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$y^{(p)} = \mathbb{Y}(t)\,\mathbf{u}(t)$$

$$\underbrace{\mathbb{Y}'}_{A\mathbb{Y}}\mathbf{u} + \mathbb{Y}\mathbf{u}' = A\mathbb{Y}\mathbf{u} + g$$

$$\mathbb{Y}\mathbf{u}' = g$$

$$\mathbf{u}' = \mathbb{Y}^{-1} g$$

$$\mathbf{u}(t) = \int_{t_0}^{t} \mathbb{Y}^{-1}(\hat{t})\, g(\hat{t})\, d\hat{t} + \mathbf{c}$$

$$y = \mathbb{Y}\mathbf{u} = \mathbb{Y}\mathbf{c} + \mathbb{Y}\int_{t_0}^{t} \mathbb{Y}^{-1}(\hat{t})\, g(\hat{t})\, d\hat{t}$$

Ex2 (For the problem of Ex1)

$$\mathbb{Y} = \left[y^{(1)}\ y^{(2)} \right] = \begin{bmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix}, \quad g = \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

$$\mathbb{Y}^{-1} = \frac{1}{-2e^{-6t}} \begin{bmatrix} -e^{-4t} & -e^{-4t} \\ -e^{-2t} & e^{-2t} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{2t} & e^{2t} \\ e^{4t} & -e^{4t} \end{bmatrix}$$

$$\mathbf{u}' = \mathbb{Y}^{-1} g = \frac{1}{2} \begin{bmatrix} e^{2t} & e^{2t} \\ e^{4t} & -e^{4t} \end{bmatrix} \begin{bmatrix} -6e^{-2t} \\ 2e^{-2t} \end{bmatrix} = \begin{bmatrix} -2 \\ -4e^{2t} \end{bmatrix}$$

$$\mathbf{u}(t) = \int_0^t \begin{bmatrix} -2 \\ -4e^{2\hat{t}} \end{bmatrix} d\hat{t} = \begin{bmatrix} -2t \\ -2e^{2t} + 2 \end{bmatrix}$$

$$\mathbb{Y}\mathbf{u} = \begin{bmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix} \begin{bmatrix} -2t \\ -2e^{2t} + 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2t - 2 \\ -2t + 2 \end{bmatrix} e^{-2t} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} e^{-4t}$$

a solution of the homogeneous system

$$\therefore\ y = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} - 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-2t} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{-2t}$$