2014년 1학기 고급수학 및 연습 1 중간고사 모범답안.

# #1.

a) For $x \geq 0$, $\sin x \leq x$

so $\sum_{n=1}^{\infty} \sin^2 \frac{1}{n} \leq \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2 < \infty$ (Comparison test) 5점

Also, $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1-\frac{1}{2}}$ : converge 5점

$$\therefore \sum_{n=1}^{\infty} \left(\sin^2 \frac{1}{n} - \frac{1}{2^n}\right) = \sum_{n=1}^{\infty} \sin^2 \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$$

b) $\frac{n}{n-1} \geq 1$ for $n \geq 2$.

$$\therefore \sum_{n=2}^{\infty} \left(\frac{n}{n-1}\right)^{n^2} \geq \sum_{n=2}^{\infty} 1 = \infty \quad \text{(Comparison)}$$ 10점

c) $$0 \leq \lim_{n\to\infty} \frac{3^n}{n!} = \lim_{n\to\infty} 3 \cdot \frac{3}{2} \cdot 1 \cdot \left(\frac{3}{4} \cdots \cdot \frac{3}{n}\right)$$

$$\leq \lim_{n\to\infty} \frac{9}{2} \cdot \left(\frac{3}{4}\right)^{n-3} = 0. \quad \Rightarrow \lim_{n\to\infty} \frac{3^n}{n!} = 0.$$

$$\therefore \lim_{n\to\infty} \frac{e^n}{n! - 3^n + 2014} \cdot \frac{n!}{e^n} = \lim_{n\to\infty} \frac{1}{1 - \frac{3^n}{n!} + \frac{2014}{n!}} = 1$$

$\Rightarrow$ By limit comparison test,

$$\sum_{n=1}^{\infty} \frac{e^n}{n! - 3^n + 2014} < \infty \Leftrightarrow \sum_{n=1}^{\infty} \frac{e^n}{n!} < \infty. \quad \text{—} (*)$$ 5점

Now, let $a_n = \frac{e^n}{n!}$

#1.

Then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{e}{n+1} = 0$.

so by ratio test,

$$\sum_{n=1}^{\infty} \frac{e^n}{n!} < \infty. \Rightarrow \sum_{n=1}^{\infty} \frac{e^n}{n! - 3^n + 2014} < \infty. \text{ (by } (*)\text{)}$$

5점

d) $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ for $|x|<1$.

$$\therefore \quad x - \arctan x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n+1}}{2n+1}$$

Now, if $0 < x < 1$, $\Rightarrow \frac{x^{2n+1}}{2n+1} \searrow 0$. so

$$0 \leq \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n+1}}{2n+1} \leq \frac{x^3}{3}$$

5점

$\therefore n \geq 2 \Rightarrow 0 \leq \frac{1}{n} - \arctan \frac{1}{n} \leq \frac{1}{3n^3}$

$\therefore \sum_{n=2}^{\infty} \left( \frac{1}{n} - \arctan \frac{1}{n} \right) \leq \sum_{n=2}^{\infty} \frac{1}{3n^3} < \infty$ (Comparison)

$\Rightarrow \sum_{n=1}^{\infty} \left( \frac{1}{n} - \arctan \frac{1}{n} \right) < \infty$.

5점

#2.

$s \leq 1$ 인 경우, $\lim_{n\to\infty} \frac{n-\sqrt{n-1}}{2n^s+1} \neq 0$ 이므로, 주어진 급수가 발산한다.

$s > 1$ 인 경우, $a_n = \frac{n-\sqrt{n-1}}{2n^s+1}$, $b_n = \frac{1}{2n^{s-1}}$ 이라고 하면 $\lim_{n\to\infty} \frac{b_n}{a_n} = 1$ 이므로
$(a_n, b_n > 0)$

극한 비율판정법에 의해 급수 $\sum_{n=1}^{\infty} a_n$ 과 급수 $\sum_{n=1}^{\infty} b_n$ 의 수렴성이 동치이다.

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ 에 대한 적분 판정법에 의해 $1 < s \leq 2$ 인 경우 발산,

$s > 2$ 인 경우 수렴함을 알 수 있다.

※ $s > 2$ 인 경우의 수렴성을 확인 - 10점

$s \leq 2$ 인 경우에 대해 발산성을 확인 - 10점.

# #3

(a) $a_n := \sqrt{n+1} - \sqrt{n} > 0 \ (n \geq 0)$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} a_{n+1}}{(-1)^n a_n} \right| = \lim_{n\to\infty} \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n+1}} = 1.$$

$\Rightarrow$ The radius of convergence $= 1$. $+5$ pts.

$x = 1. \Rightarrow \sum_{n=0}^{\infty} (-1)^n a_n$ converges by the alternating series test.

$\left( a_n > 0, \quad a_n > a_{n+1} \text{ for } n \geq 0, \quad \lim_{n\to\infty} a_n = 0 \right)$

$x = -1 \Rightarrow \sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} \sqrt{n+1} = \infty$ diverges.

$\therefore$ The series converges on $(-1, 1]$ $+5$pts

(b) $b_n := 1 - \cos\frac{1}{n} = 2\sin^2\left(\frac{1}{2n}\right). \quad (n \geq 1)$

$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = 1$, The radius of convergence $= 1$ $+5$ pts.

$x = \pm 1 \Rightarrow \sum_{n=1}^{\infty} |b_n x^n| = \sum_{n=1}^{\infty} 2\sin^2\left(\frac{1}{2n}\right) \leq \sum_{n=1}^{\infty} \frac{1}{2n^2} < +\infty$

and the series converges absolutely.

$\therefore$ The series converges on $[-1, 1]$ $+5$pts

# 4.

(a) $a_n := \sqrt{n} \quad (n \geq 0)$

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, The radius of convergence $= 1$.

$x = \pm 1 \Rightarrow \lim_{n\to\infty} |\sqrt{n}\, x^n| = \infty$, and the series diverges

by the $n$-th term test.

$\therefore$ The series converges on $(-1, 1)$ $\lrcorner$ +5 pts

(b) If $|x| < 1$, then

$$\sum_{n=0}^{\infty} |n x^{n^2}| = \sum_{\substack{m \geq 0 \\ m:\text{Square}}} \sqrt{m}\, |x|^m \leq \sum_{m=0}^{\infty} m |x|^m < +\infty \text{ by (a).}$$

Otherwise, $|x| \geq 1$ ;

$\lim_{n\to\infty} |n x^{n^2}| = \infty$, and the series diverges by the $n$-th term test.

$\therefore$ The series converges on $(-1, 1)$ $\lrcorner$ + 10 pts

※ Root, Ratio test 등을 사용할 때,

절대값을 취해 정확히 쓰지 않으면 −5점.

#5.

$x = \tanh^{-1} y.$

$$\frac{dx}{dy} = \frac{1}{dy/dx} \text{ (Inverse function thm)} = \frac{1}{\operatorname{sech}^2 x}$$

$$= \frac{1}{1-\tanh^2 x} = \frac{1}{1-y^2} \quad \text{for } |y|<1. \quad +5$$

Also. $$\frac{1}{1-y^2} = \sum_{n=0}^{\infty} (y^2)^n \quad \text{for } |y|<1. \quad +5$$

$$\therefore \tanh^{-1} y = \int_0^y \frac{1}{1-t^2}\,dt \quad (\tanh^{-1} 0 = 0)$$

$$= \int_0^y \sum_{n=0}^{\infty} (t^2)^n\,dt = \sum_{n=0}^{\infty} \frac{y^{2n+1}}{2n+1} \quad \text{for } |y|<1. \quad +5$$

(Fundamental thm of Power series)

– 수렴반경에 대한 언급이 없으면 –5점

#6.

$f(x) = e^x + e^{2x}$

i) $f'(x) = e^x + 2e^{2x} > 0 \Rightarrow f$: strictly increasing $+5$

ii) $f$: conti. $\underline{\lim_{x \to -\infty} f(x) = 0.\ \lim_{x \to \infty} f(x) = \infty}$ (*)

$\Rightarrow$ range of $f$ is $(0, \infty)$ $+5$.

$\because$ for any $c \in (0, \infty)$. (*) implies that there exist $0 < x_1 < x_2$ such that $f(x_1) < c < f(x_2)$. $\therefore$ By Intermediate value thm. $c \in$ range $f$ ($f$: conti.)

$\therefore$ i) ii) $\Rightarrow$ $f$ has inverse $x = g(y)$ for $y > 0$.

Now, for $x = g(y)$.

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{e^x + 2e^{2x}}$$

$$g''(y) = \frac{d}{dy}\left(\frac{1}{e^x + 2e^{2x}}\right) = -\frac{e^x + 4e^{2x}}{(e^x + 2e^{2x})^2} \cdot \frac{1}{e^x + 2e^{2x}}$$

$$= -\frac{e^x + 4e^{2x}}{(e^x + 2e^{2x})^3}$$

#6.

$\therefore f(0)=2 \Rightarrow g(2)=0$,

$g'(2) = \frac{1}{1+2} = \frac{1}{3}$. $g''(2) = -\frac{1+4}{(1+2)^3} = -\frac{5}{27}$ +5

$$\therefore T_2^2 g(y) = g(2) + g'(2)(y-2) + \frac{g''(2)}{2}(y-2)^2$$

$$= \frac{1}{3}(y-2) - \frac{5}{54}(y-2)^2$$ +5.

7. $f(x) = \int_0^x \frac{dt}{1+t^4}$

$$f(x) = \int_0^x 1 - t^4 + t^8 - t^{12} + \cdots \, dt \qquad |x|, |t| < 1.$$

$$= \left[ t - \frac{1}{5}t^5 + \frac{1}{9}t^9 - \frac{1}{13}t^{13} + \cdots \right]_0^x$$

$$= x - \frac{1}{5}x^5 + \frac{1}{9}x^9 - \frac{1}{13}x^{13} + \cdots \qquad |x| < 1$$

] +10점

$$\left| f\left(\frac{1}{10}\right) - \frac{1}{10} + \frac{1}{5}\left(\frac{1}{10}\right)^5 - \frac{1}{9}\left(\frac{1}{10}\right)^9 \right|$$

$$\leq \left| -\frac{1}{13}\left(\frac{1}{10}\right)^{13} \right| \qquad (\because \text{교대급수의 성질})$$

$$\leq 10^{-10}$$

] +5점

멱급수의 수렴반경을 쓰지 않으면 −2.

8

1) $X \geq 0$ 일때는

$1 + \frac{x}{2!} + \frac{x^2}{4!} + \frac{x^3}{6!} + \cdots \geq 1$ 이므로 해가 없다 ⌋ +5점

2) $X < 0$ 일때

$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots \qquad t \in \mathbb{R}$

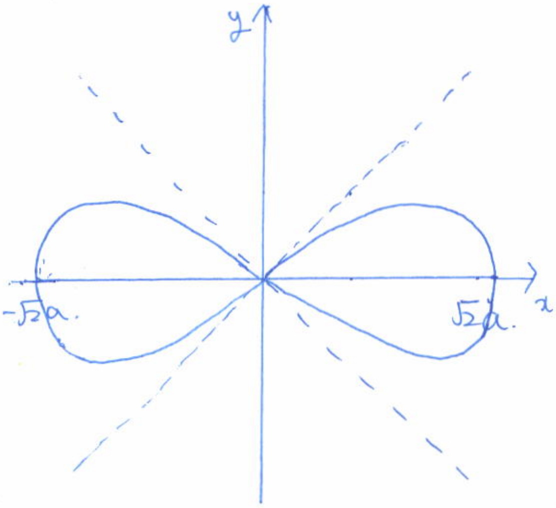
$t = \sqrt{-x}$ 로 하면

$\cos\sqrt{-x} = 1 + \frac{(-x)}{2!} + \frac{(-x)^2}{4!} + \frac{(-x)^3}{6!} + \cdots = 0.$

$\Rightarrow \quad X = -\left(\frac{(2n+1)}{2}\pi\right)^2, \quad n = 0, 1, 2, \cdots$ ⌋ +10점.

1) 2)에서 $X = -\left(\frac{(2n+1)}{2}\pi\right)^2$

9 (a)




+5

$\theta = \frac{5}{6}\pi \Rightarrow r^2 = 2a^2\cos\frac{5}{3}\pi = a^2$

$\Rightarrow r = \pm a$

$\theta = \frac{5}{6}\pi$ 인 반직선 과의 교점을 찾으면,

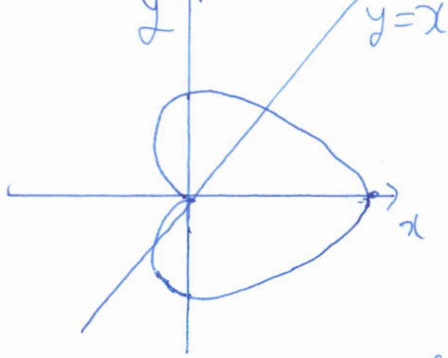
$A = (a\cos\frac{5}{6}\pi,\ a\sin\frac{5}{6}\pi) = (-\frac{\sqrt{3}}{2}a,\ \frac{1}{2}a)$.

그런데, 점 A, B, C 는 반지름이 $a$인 원위에 있고

점 B, C 는 원위의 반대편 두점이므로, $\angle BAC = \frac{\pi}{2}$ +5

☆ 두점을 찾으면 2점 감점.

(b).




∴ 교점 : $\theta = \frac{\pi}{4}$ 일때, $\left(\frac{\sqrt{2}+1}{2}, \frac{\sqrt{2}+1}{2}\right)$

$\theta = \frac{5\pi}{4}$ 일때, $\left(\frac{1-\sqrt{2}}{2}, \frac{1-\sqrt{2}}{2}\right)$

원점, $(0, 0)$

☆ 한개 틀리면 7점, 두개 틀리면 4점.

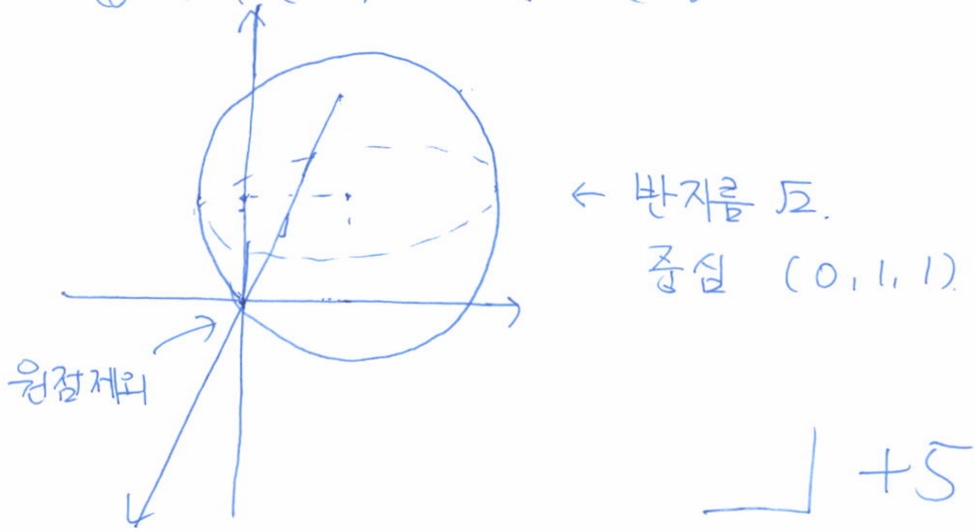
10.

$\rho = 2\cos\varphi + 2\sin\varphi\sin\theta , \rho>0$

$\Rightarrow \rho^2 = 2\rho\cos\varphi + 2\rho\sin\varphi\sin\theta , \rho>0$

$\Longleftrightarrow x^2+y^2+z^2 = 2z+2y , (x,y,z)\neq(0,0,0)$

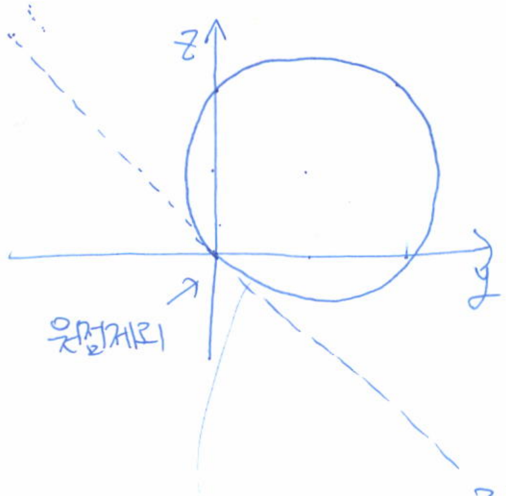
$\Longleftrightarrow x^2+(y-1)^2+(z-1)^2 = 2 \quad (x,y,z)\neq(0,0,0)$




+5

$0<\rho\le 2\sqrt{2}$. +5

yz 평면으로 정사영시켜보면,




$0\le\varphi<\frac{3}{4}\pi$ +5

$0\le\theta<2\pi$ +5

※ 부등호, 등호 틀리면 2점 감점.