Exercise 1

a)

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \text{false} : \text{Bool}}\,\text{T-False} \quad \dfrac{}{\Gamma \vdash \text{true} : \text{Bool}}\,\text{T-True} \quad \dfrac{}{\Gamma \vdash \text{false} : \text{Bool}}\,\text{T-False}}{\Gamma \vdash (\text{if false then true else false}) : \text{Bool}}\,\text{T-If} \quad \dfrac{}{\Gamma \vdash \Gamma}\,\text{Var}}{f : \text{Bool} \to \text{Bool} \vdash f\,(\text{if false then true else false}) : \text{Bool}}\,\text{App}$$

Γ

b)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \text{false} : \text{Bool}}\,\text{T-False} \quad \dfrac{}{\Gamma \vdash \text{true} : \text{Bool}}\,\text{T-True} \quad \dfrac{}{\Gamma \vdash \text{false} : \text{Bool}}\,\text{T-False}}{\Gamma, x : \text{Bool} \vdash (\text{if } x \text{ then false else } x) : \text{Bool}}\,\text{T-If} \quad \dfrac{}{\Gamma, x : \text{Bool} \vdash f : \text{Bool} \to \text{Bool}}\,\text{Var}}{\Gamma, x : \text{Bool} \vdash M : \text{Bool}}\,\text{App}}{\Gamma \vdash \lambda x : \text{Bool}.\ M : \text{Bool} \to \text{Bool}}\,\text{Abs}/{\to}\text{I}$$

$$f : \text{Bool} \to \text{Bool} \vdash \lambda x : \text{Bool}.\ f\,(\text{if } x \text{ then false else } x) : \text{Bool} \to \text{Bool}$$

Γ $\qquad$ M

Exercise 2

a) $(\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma$

($\alpha \to \beta$: f, $\beta \to \gamma$: g)

Let: $g : \beta \to \gamma$, $f : \alpha \to \beta$, $x : \alpha$

↳ Church: $\lambda f^{\alpha \to \beta}\, g^{\beta \to \gamma}\, x^{\alpha}.\ g(f(x))$

Curry: $\lambda f\, g\, x.\ g(f(x))$

*This will be the only inhabitant, since we cannot return "$\alpha \to \gamma$", and neither "$(\beta \to \gamma) \to \alpha \to \gamma$". There exist no inhabitants which would satisfy these 2 cases, and thus the only option is to return "γ" and take 3 inputs.

b) $\alpha \to \beta \to (\alpha \to \beta \to \gamma) \to \gamma$

Church: $\lambda x^{\alpha} y^{\beta} f^{\alpha \to \beta \to \gamma} . f\, x\, y$

Curry: $\lambda x y f . f\, x\, y$

* Similar to (a), the only possibility is to take 3 inputs and return "γ". This inhabitant is the only possibility.

c) $((\alpha \to \beta \to \alpha) \to \alpha) \to \alpha$

Curch: $\lambda f^{(\alpha \to \beta \to \alpha) \to \alpha} . f(\lambda g^{(\alpha \to \beta \to \alpha)} . g(\lambda x^{\alpha} y^{\beta} . x))$

Curry: $\lambda f . f(\lambda g . g(\lambda x y . x))$

* Since the inhabitant of this type can only take one input and return one output, this will be the only solution.

d) $\beta \to ((\alpha \to \beta) \to \gamma) \to \gamma$

Curch: $\lambda x^{\beta} f^{(\alpha \to \beta) \to \gamma} . f(\lambda y^{\alpha} . x)$

Curry: $\lambda x f . f(\lambda y . x)$

* Similarly to (c), the only option here is to return γ and take 2 inputs, thus this being the only solution.

e) $\alpha \to (\alpha \to \alpha) \to \alpha$

Curch
- $\lambda x^{\alpha} f^{\alpha \to \alpha} . x$
- $\lambda x^{\alpha} f^{\alpha \to \alpha} . f(x)$
- $\lambda x^{\alpha} f^{\alpha \to \alpha} . f(f(x))$
- $\vdots$

Curry
- $\lambda x f . x$
- $\lambda x f . f(x)$
- $\lambda x f . f(f(x))$
- $\vdots$

Exercise 3:

a) $S = \lambda x y z . x\ z\ (y\ z)$

$x : \gamma \to \beta \to \alpha$

$z : \gamma$

$y : \gamma \to \beta$

$(\gamma \to \beta \to \alpha) \to (\gamma \to \beta) \to \gamma \to \alpha$

b) $K = \lambda x\ y . x$

$x : \alpha$

$y : \beta$

$\alpha \to \beta \to \alpha$

c) $S\ K\ K$

$SKK = (\lambda x y z . x z (y z))\ K K =$

$= \lambda z . K z (K z) =$ → $Kz = (\lambda x y . x) z = z$

$= \lambda z . K z z =$

$= \lambda z . z$

$\Rightarrow \alpha \to \alpha$

d) $I = \lambda x . x$

$\alpha \to \alpha$

Exercise 4

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash (\alpha \to \beta) \to \gamma}\,Ax \qquad \dfrac{}{\Gamma \vdash \alpha \to \beta}\,Ax}{\Gamma \equiv (\alpha \to \beta) \to \gamma,\ \beta \vdash \gamma}\,{\to}E}{(\alpha \to \beta) \to \gamma,\ \beta \to \gamma}\,{\to}I}{(\alpha \to \beta) \to \gamma \vdash \beta \to \gamma}\,{\to}I}{\vdash ((\alpha \to \beta) \to \gamma) \to \beta \to \gamma}\,{\to}I$$

Associated lambda term:

$$\lambda f^{((\alpha \to \beta) \to \gamma)}\ x^{\beta}\ .\ f(\lambda y^{\alpha}.x)$$

Exercise 5

?

Exercise 6

?

Exercise 7

CBV

fact c_3 =

= if (iszero c_3) then c_1 else times c_3 (fact (pred c_3))

= if false then c_1 else times c_3 (fact c_2)

= times c_3 (fact c_2) =

= times c_3 (if (iszero c_2) then c_1 else times c_2 (fact (pred c_2))

= times c_3 (if false then c_1 else times c_2 (fact c_1)

= times c_3 (times c_2 (fact c_1)) =

= times c_3 (times c_2 (if (iszero c_1) then c_1 else times c_1 (fact (pred c_1))

= times c_3 (times c_2 (if false then c_1 else times c_1 (fact c_0)) =

= times c_3 (times c_2 (times c_1 (fact c_0)) =

= times c_3 (times c_2 (times c_1 (if (iszero c_0) then c_1 else times c_0

(fact (pred c_0)) =

= times c_3 (times c_2 (times c_1 (if true then c_1 else times c_0

(fact (pred c_0)) =

= times c_3 (times c_2 (times c_1 c_1) =

= times c_3 (times c_2 c_1) =

= times c_3 c_2 =

= c_6