Exercise 1

a)

$$\dfrac{\dfrac{}{\Gamma \vdash false : Bool}\text{T-False} \qquad \dfrac{}{\Gamma \vdash true : Bool}\text{T-True}}{\underbrace{f : Bool \to Bool}_{\Gamma} \vdash f(\text{if } false \text{ then } true \text{ else } false) : Bool}\text{T-If}$$

b)

$$\dfrac{\dfrac{\dfrac{x : Bool \in \{x : Bool\}}{\Gamma, x : Bool \vdash x : Bool}\text{Var} \qquad \dfrac{}{x : Bool \vdash false : Bool}\text{T-False}}{\Gamma, x : Bool \vdash M : Bool}\text{T-If}}{}$$

$$\dfrac{\dfrac{\Gamma, x : Bool \vdash M : Bool}{\Gamma \vdash \lambda x : Bool .\ M : Bool \to Bool}\text{Abs}/\!\to\! I}{\underbrace{f : Bool \to Bool}_{\Gamma} \vdash \lambda x : Bool.\ f(\underbrace{\text{if } x \text{ then } false \text{ else } x}_{M}) : Bool \to Bool}$$

Exercise 2

a) $(\underbrace{\alpha \to \beta}_{f}) \to (\underbrace{\beta \to \gamma}_{g}) \to \alpha \to \gamma$

Let : $g : \beta \to \gamma$, $f : \alpha \to \beta$, $x : \alpha$

↳ **Church** : $\lambda f^{\alpha \to \beta}\, g^{\beta \to \gamma}\, x^{\alpha}.\ g(f(x))$

Curry : $\lambda f\, g\, x.\ g(f(x))$

*This will be the only inhabitant, since we cannot return "$\alpha \to \gamma$", and neither "$(\beta \to \gamma) \to \alpha \to \gamma$". There exist no inhabitants which would satisfy these 2 cases, and thus the only option is to return "γ" and take 3 inputs.

b) $\alpha \to \beta \to (\alpha \to \beta \to \gamma) \to \gamma$

Church : $\lambda x^{\alpha} y^{\beta} f^{\alpha \to \beta \to \gamma} . f\, x\, y$

Curry : $\lambda x y f . f\, x\, y$

* Similar to (a), the only possibility is to take 3 inputs and return "γ". This inhabitant is the only possibility.

c) $((\alpha \to \beta \to \alpha) \to \alpha) \to \alpha$

Curch: $\lambda f^{((\alpha \to \beta \to \alpha) \to \alpha) \to \alpha} . f(\lambda g^{(\alpha \to \beta \to \alpha) \to \alpha} . g(\lambda x^{\alpha} y^{\beta} . x))$

Curry : $\lambda f . f(\lambda g . g(\lambda x y . x))$

* Since the inhabitant of this type can only take one input and return one output, this will be the only solution.

d) $\beta \to ((\alpha \to \beta) \to \gamma) \to \gamma$

Curch: $\lambda x^{\beta} f^{(\alpha - \beta) \to \gamma} . f(\lambda y^{\alpha} . x)$

Curry: $\lambda x f . f(\lambda y . x)$

* Similarly to (c), the only option here is to return γ and take 2 inputs, thus this being the only solution.

e) $\alpha \to (\alpha \to \alpha) \to \alpha$

Curch $\begin{cases} \lambda x^{\alpha} f^{\alpha \to \alpha} . x \\ \lambda x^{\alpha} f^{\alpha \to \alpha} . f(x) \end{cases}$

Curry $\begin{cases} \lambda x f . x \\ \lambda x f . f(x) \end{cases}$

* These 2 are the only solutions, since we wouldn't be able to take just 1 input of type "α". The only possibility remains to take 2 inputs and return one output, which can be done in the 2 ways mentioned.

Exercise 3:

a) $S = \lambda xyz.\ x z (y\ z)$

$x : \gamma \to \beta \to \alpha$

$z : \gamma$

$y : \gamma \to \beta$

$(\gamma \to \beta \to \alpha) \to (\gamma \to \beta) \to \gamma \to \alpha$

b) $K = \lambda x\ y.\ x$

$x : \alpha$

$y : \beta$

$\alpha \to \beta \to \alpha$

c) $S\ K\ K$

$SKK = (\lambda xyz.\ xz(yz))\ KK =$

$= \lambda z.\ Kz(Kz) =$ $\to$ $Kz = (\lambda xy.x)z = z$

$= \lambda z.\ Kzz =$

$= \lambda z.\ z$

$\Rightarrow \alpha \to \alpha$

d) $I = \lambda x.\ x$

$\alpha \to \alpha$

Exercise 4

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash (\alpha \to \beta) \to \gamma}\,Ax \qquad \dfrac{}{\Gamma \vdash \alpha \to \beta}\,Ax}{\Gamma \equiv (\alpha \to \beta) \to \gamma, \beta \vdash \gamma}\,{\to}E}{(\alpha \to \beta) \to \gamma,\ \beta \to \gamma}\,{\to}I}{(\alpha \to \beta) \to \gamma \vdash \beta \to \gamma}\,{\to}I}{\vdash ((\alpha \to \beta) \to \gamma) \to \beta \to \gamma}\,{\to}I
$$

Exercise 5

?

Exercise 6

?

Exercise 7

CBV

$\text{fact } c_3 =$

$= \text{if } (\text{iszero } c_3) \text{ then } c_1 \text{ else times } c_3 \,(\text{fact } (\text{pred } c_3))$

$= \text{if false then } c_1 \text{ else times } c_3 \,(\text{fact } c_2)$

$= \text{times } c_3 \,(\text{fact } c_2) =$

$= \text{times } c_3 \,(\text{if } (\text{iszero } c_2) \text{ then } c_1 \text{ else times } c_2 \,(\text{fact } (\text{pred } c_2)))$

$= \text{times } c_3 \,(\text{if false then } c_1 \text{ else times } c_2 \,(\text{fact } c_1))$

$= \text{times } c_3 \,(\text{times } c_2 \,(\text{fact } c_1)) =$

$= \text{times } c_3 \,(\text{times } c_2 \,(\text{if } (\text{iszero } c_1) \text{ then } c_1 \text{ else times } c_1 \,(\text{fact } (\text{pred } c_1)))$

$= \text{times } c_3 \,(\text{times } c_2 \,(\text{if false then } c_1 \text{ else times } c_1 \,(\text{fact } c_0) =$

$= \text{times } c_3 \,(\text{times } c_2 \,(\text{times } c_1 \,(\text{fact } c_0)) =$

$= \text{times } c_3 \,(\text{times } c_2 \,(\text{times } c_1 \,(\text{if } (\text{iszero } c_0) \text{ then } c_1 \text{ else times } c_0$
$(\text{fact } (\text{pred } c_0)) =$

$= \text{times } c_3 \,(\text{times } c_2 \,(\text{times } c_1 \,(\text{if true then } c_1 \text{ else times } c_0$
$(\text{fact } (\text{pred } c_0)) =$

$= \text{times } c_3 \,(\text{times } c_2 \,(\text{times } c_1 \; c_1) =$

$= \text{times } c_3 \,(\text{times } c_2 \; c_1) =$

$= \text{times } c_3 \; c_2 =$

$= c_6$