4. Construct a derivation of type $((\alpha \to \beta) \to \gamma) \to \beta \to \gamma$ and the associated typed $\lambda$-term

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{\Gamma, y:\alpha \vdash z:\beta}\,\text{Var}
    }{\Gamma \vdash \lambda y^{\alpha}.z : \alpha \to \beta}\,\text{Abs}
    \qquad
    \dfrac{}{\Gamma \vdash x : (\alpha \to \beta) \to \gamma}\,\text{Var}
  }{\overbrace{x : (\alpha \to \beta) \to \gamma,\ z:\beta}^{\Gamma} \vdash x(\lambda y^{\alpha}.z) : \gamma}\,\text{App}
}{
  \dfrac{x : (\alpha \to \beta) \to \gamma \vdash \lambda z^{\beta}.x(\lambda y^{\alpha}.z) : \beta \to \gamma}{\vdash \lambda x^{(\alpha \to \beta) \to \gamma} z^{\beta}.x(\lambda y^{\alpha}.z) : ((\alpha \to \beta) \to \gamma) \to \beta \to \gamma}\,\text{Abs}
}\,\text{Abs}
$$

1. Prove that the following statements are derivable in STLC (provide type derivation)

   (a) $f : Bool \to Bool \vdash f\ (if\ false\ then\ true\ else\ false)\ :\ Bool$

   (b) $f : Bool \to Bool \vdash \lambda x : Bool.\ f\ (if\ x\ then\ false\ else\ x)\ :\ Bool \to Bool$

   assuming

$$
\dfrac{\Gamma \vdash e\ :\ Bool \quad \Gamma \vdash v\ :\ \tau \quad \Gamma \vdash u\ :\ \tau}{\Gamma \vdash if\ e\ then\ v\ else\ u\ :\ \tau}\ \text{T-If}
$$

a).

$$
\dfrac{
  \dfrac{
    \dfrac{}{f:Bool \to Bool \vdash false : Bool}\,\text{Var?}
    \quad
    \dfrac{}{f:Bool \to Bool \vdash true : Bool}\,\text{Var?}
    \quad
    \dfrac{}{f:Bool \to Bool \vdash false : Bool}\,\text{Var?}
  }{f:Bool \to Bool \vdash if\ false\ then\ true\ else\ false : Bool}
  \qquad
  \dfrac{}{f:Bool \to Bool \vdash f : Bool \to Bool}\,\text{Var}
}{f : Bool \to Bool \vdash f(if\ false\ then\ true\ else\ false) : Bool}\,\text{App}
$$

b).

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{\Gamma \vdash x : Bool}\,\text{Var}
      \quad
      \dfrac{}{\Gamma \vdash false : Bool}\,\text{Var}
      \quad
      \dfrac{}{\Gamma \vdash x : Bool}\,\text{Var}
    }{\Gamma \vdash if\ x\ then\ false\ else\ x : Bool}
    \qquad
    \dfrac{}{\Gamma \vdash f : Bool \to Bool}\,\text{Var}
  }{\overbrace{f : Bool \to Bool, x : Bool}^{\Gamma} \vdash f(if\ x\ then\ false\ else\ x) : Bool}\,\text{App}
}{f : Bool \to Bool \vdash \lambda x^{Bool}.f(if\ x\ then\ false\ else\ x) : Bool \to Bool}\,\text{Abs}
$$

3. Compute the most general (principal) type of the following terms

(a) $S = \lambda x\, y\, z.\ x\, z\, (y\, z)$
(b) $K = \lambda x\, y.\ x$
(c) $S\, K\, K$
(d) $I = \lambda x.\ x$

a). $z : \alpha$
$y : \alpha \to \beta$
$x : \alpha \to \beta \to \gamma$
$(\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma$

b). $\alpha \to \beta \to \alpha$

c). $\lambda z. (\lambda x y. x)\, z\, ((\lambda x y. x)\, z)$
$\lambda z. z$
$\alpha \to \alpha$

d). $\alpha \to \alpha$

2. Find all inhabitants (closed terms) of the following types (both in à la Curch and à la Curry):

(a) $(\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma$
(b) $\alpha \to \beta \to (\alpha \to \beta \to \gamma) \to \gamma$
(c) $((\alpha \to \beta \to \alpha) \to \alpha) \to \alpha$
(d) $\beta \to ((\alpha \to \beta) \to \gamma) \to \gamma$
(e) $\alpha \to (\alpha \to \alpha) \to \alpha$

a). $\lambda x y z. y (x z)$ $\qquad$ $\lambda x^{\alpha \to \beta} y^{\beta \to \gamma} z^{\alpha}. y (x z)$

b). $\lambda x y z. z x y$ $\qquad$ $\lambda x^{\alpha} y^{\beta} z^{\alpha \to \beta \to \gamma}. z x y$

c). $\lambda x. x (\lambda y z. y)$ $\qquad$ $\lambda x^{(\alpha \to \beta \to \alpha) \to \alpha}. x (\lambda y z. y)$

d). $\lambda x y. y (\lambda z. x)$ $\qquad$ $\lambda x^{\beta} y^{(\alpha \to \beta) \to \gamma}. y (\lambda z. x)$

e). $\lambda x y. x$ $\qquad$ $\lambda x^{\alpha} y^{\alpha \to \alpha}. x$
$\lambda x y. y x$ $\qquad$ $\lambda x^{\alpha} y^{\alpha \to \alpha}. y x$
$\lambda x y. y (y x)$ $\qquad$ $\lambda x^{\alpha} y^{\alpha \to \alpha}. y (y x)$
$\vdots$ $\qquad$ $\vdots$

7. Provide step-by-step evaluation of term $fact\ c_3$ with both call-by-name and call-by-value reduction strategies.

*Solution.*

This one is not done, it is too time consuming.

Call by name:

$f = \lambda f.\lambda n.\ \text{test}\ (iszero\ n)\ c_1\ (times\ n\ (f\ (pred\ n)))$

$\mathbb{Y}\ f\ c_2 = (\lambda f.\ (\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ f\ c_2$

$(\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x))\ c_2$

$f\ ((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ c_2$

$(\lambda f.\lambda n.\ \text{test}\ (iszero\ n)\ c_1\ (times\ n\ (f\ (pred\ n))))\ ((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ c_2$

$(\lambda n.\ \text{test}\ (iszero\ n)\ c_1\ (times\ n\ (((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ n))))\ c_2$

$\text{test}\ (iszero\ c_2)\ c_1\ (times\ c_2\ (((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ c_2)))$

$(\lambda l.\lambda m.\lambda n.\ l\ m\ n)\ (iszero\ c_2)\ c_1\ (times\ c_2\ (((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ c_2)))$

$(\lambda m.\lambda n.\ (iszero\ c_2)\ m\ n)\ c_1\ (times\ c_2\ (((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ c_2)))$

$(\lambda n.\ (iszero\ c_2)\ c_1\ n)\ (times\ c_2\ (((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ c_2)))$

$(iszero\ c_2)\ c_1\ (times\ c_2\ (((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ c_2)))$

$((\lambda m.\ m\ (\lambda x.\ false)\ true)\ c_2)\ c_1\ (times\ c_2\ (((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ c_2)))$

$(c_2\ (\lambda x.\ false)\ true)\ c_1\ (times\ c_2\ (((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ c_2)))$

$((\lambda s.\lambda z.\ s(s\ z))\ (\lambda x.\ false)\ true)\ c_1\ (times\ c_2\ (((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ c_2)))$

$(\lambda z.\ (\lambda x.\ false)\ ((\lambda x.\ false)\ ((\lambda x.\ false)\ z)))\ true)\ c_1\ (times\ c_2\ (((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ c_2)))$

$(\lambda x.\ false)\ ((\lambda x.\ false)\ ((\lambda x.\ false)\ true)))\ c_1\ (times\ c_2\ (((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ c_2)))$

$false\ c_1\ (times\ c_2\ (((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ c_2)))$

$(\lambda t.\lambda f.\ f)\ c_1\ (times\ c_2\ (((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ c_2)))$

$(\lambda f.\ f)\ (times\ c_2\ (((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ c_2)))$

$times\ c_2\ (((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ c_2))$

$(\lambda m.\lambda n.\lambda s.\lambda z.\ m\ (n\ s)\ z)\ c_2\ (((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ c_2))$

$(\lambda n.\lambda s.\lambda z.\ c_2\ (n\ s)\ z)\ (((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ c_2))$

$\lambda s.\lambda z.\ c_2\ ((((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ c_2))\ s)\ z$

$\lambda s.\lambda z.\ (\lambda s.\lambda z.\ s(s\ z))\ ((((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ c_2))\ s)\ z$

$\lambda s.\lambda z.\ (\lambda z.\ ((((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ c_2))\ s)\ (((((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ c_2))\ s)\ z))\ z$

$\lambda s.\lambda z.\ ((((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ c_2))\ s)\ (((((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ c_2))\ s)\ z)$

$\lambda s.\lambda z.\ (((f\ ((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x))))\ (pred\ c_2))\ s)\ (((((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ c_2))\ s)\ z)$

$\lambda s.\lambda z.\ ((((\lambda f.\lambda n.\ \text{test}\ (iszero\ n)\ c_1\ (times\ n\ (f\ (pred\ n))))\ ((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x))))\ (pred\ c_2))\ s)\ \ldots$

$\lambda s.\lambda z.\ (((\lambda n.\ \text{test}\ (iszero\ n)\ c_1\ (times\ n\ (((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ n))))\ (pred\ c_2))\ s)\ \ldots$

$\lambda s.\lambda z.\ ((\ \text{test}\ (iszero\ (pred\ c_2))\ c_1\ (times\ (pred\ c_2)\ (((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ (pred\ c_2)))))\ s)\ \ldots$

$\lambda s.\lambda z.\ (((\lambda l.\lambda m.\lambda n.\ l\ m\ n)\ (iszero\ (pred\ c_2))\ c_1\ (times\ (pred\ c_2)\ (((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ (pred\ c_2)))))\ s)\ \ldots$

$\lambda s.\lambda z.\ (((\lambda m.\lambda n.\ (iszero\ (pred\ c_2))\ m\ n)\ c_1\ (times\ (pred\ c_2)\ (((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ (pred\ c_2)))))\ s)\ \ldots$

$\lambda s.\lambda z.\ (((iszero\ (pred\ c_2))\ c_1\ (times\ (pred\ c_2)\ (((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ (pred\ c_2)))))\ s)\ \ldots$

$\lambda s.\lambda z.\ ((((\lambda m.\ m\ (\lambda x.\ false)\ true)\ (pred\ c_2))\ c_1\ (times\ (pred\ c_2)\ (((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ (pred\ c_2)))))\ s)\ \ldots$

$\lambda s.\lambda z.\ (((((pred\ c_2)\ (\lambda x.\ false)\ true))\ c_1\ (times\ (pred\ c_2)\ (((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ (pred\ c_2)))))\ s)\ \ldots$

$\lambda s.\lambda z.\ ((((((\lambda n.\lambda s.\lambda z.\ n\ (\lambda f.\lambda h.\ f\ (g\ f))\ (\lambda u.z)\ (\lambda v.v))\ c_2)\ (\lambda x.\ false)\ true))\ c_1\ (times\ (pred\ c_2)\ (((\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (pred\ (pred\ c_2)))))\ s)\ \ldots$

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