Exercise 1

a)

$$
\dfrac{\dfrac{\dfrac{}{\Gamma \vdash false : Bool}\,\text{T-False} \quad \dfrac{}{\Gamma \vdash true : Bool}\,\text{T-Tr} \quad \dfrac{}{\Gamma \vdash false : Bool}\,\text{T-False}}{\Gamma \vdash (\text{if } false \text{ then } true \text{ else } false) : Bool}\,\text{T-If} \qquad \dfrac{}{\Gamma \vdash \Gamma}\,\text{Var}}{\underbrace{f : Bool \to Bool}_{\Gamma} \vdash f(\text{if } false \text{ then } true \text{ else } false) : Bool}\,\text{App}
$$

b)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash false : Bool}\,\text{T-False} \quad \dfrac{}{\Gamma \vdash true : Bool}\,\text{T-True} \quad \dfrac{}{\Gamma \vdash false \cdot Bool}\,\text{T-False}}{\Gamma, x : Bool \vdash (\text{if } x \text{ then } false \text{ else } x) : Bool}\,\text{T-If} \qquad \dfrac{}{\Gamma, x : Bool \vdash f : Bool \to Bool}\,\text{Var}}{\Gamma, x : Bool \vdash M : Bool}\,\text{App}}{\Gamma \vdash \lambda x : Bool .\ M : Bool \to Bool}\,\text{Abs}/\to I}{\underbrace{f : Bool \to Bool}_{\Gamma} \vdash \lambda x : Bool .\ f\underbrace{(\text{if } x \text{ then } false \text{ else } x)}_{M} : Bool \to Bool}
$$

Exercise 2

a) $(\underset{f}{\alpha \to \beta}) \to (\underset{g}{\beta \to \gamma}) \to \alpha \to \gamma$

Let : $g : \beta \to \gamma$, $f : \alpha \to \beta$, $x : \alpha$

↳ **Church** : $\lambda f^{\alpha \to \beta}\, g^{\beta \to \gamma}\, x^{\alpha} .\ g(f(x))$

Curry : $\lambda f g x .\ g(f(x))$

* This will be the only inhabitant, since we cannot return "$\alpha \to \gamma$", and neither "$(\beta \to \gamma) \to \alpha \to \gamma$". There exist no inhabitants which would satisfy these 2 cases, and thus the only option is to return "γ" and take 3 inputs.

b) $\alpha \to \beta \to (\alpha \to \beta \to \gamma) \to \gamma$

Church: $\lambda x^{\alpha} y^{\beta} f^{\alpha \to \beta \to \gamma}. f\,x\,y$

Curry: $\lambda x y f . f\,x\,y$

* Similar to (a), the only possibility is to take 3 inputs and return "γ". This inhabitant is the only possibility.

c) $((\alpha \to \beta \to \alpha) \to \alpha) \to \alpha$

Curch: $\lambda f^{(\alpha \to \beta \to \alpha) \to \alpha} . f(\lambda g^{(\alpha \to \beta \to \alpha)} . g(\lambda x^{\alpha} y^{\beta} . x))$

Curry: $\lambda f . f(\lambda g . g(\lambda x y . x))$

* Since the inhabitant of this type can only take one input and return one output, this will be the only solution.

d) $\beta \to ((\alpha \to \beta) \to \gamma) \to \gamma$

Curch: $\lambda x^{\beta} f^{(\alpha - \beta) \to \gamma} . f(\lambda y^{\alpha} . x)$

Curry: $\lambda x f . f(\lambda y . x)$

* Similarly to (c), the only option here is to return γ and take 2 inputs, thus this being the only solution.

e) $\alpha \to (\alpha \to \alpha) \to \alpha$

Curch:
- $\lambda x^{\alpha} f^{\alpha \to \alpha} . x$
- $\lambda x^{\alpha} f^{\alpha \to \alpha} . f(x)$
- $\lambda x^{\alpha} f^{\alpha \to \alpha} . f(f(x))$
- $\vdots$

Curry:
- $\lambda x f . x$
- $\lambda x f . f(x)$
- $\lambda x f . f(f(x))$
- $\vdots$

Exercise 3:

a) $S = \lambda xyz.\, x z (y z)$

$x : \gamma \to \beta \to \alpha$

$z : \gamma$

$y : \gamma \to \beta$

$(\gamma \to \beta \to \alpha) \to (\gamma \to \beta) \to \gamma \to \alpha$

b) $K = \lambda x y.\, x$

$x : \alpha$

$y : \beta$

$\alpha \to \beta \to \alpha$

c) $S K K$

$SKK = (\lambda xyz.\, xz(yz))\ KK =$

$= \lambda z.\, Kz(Kz) =$ $\to$ $Kz = (\lambda xy.\, x) z = z$

$= \lambda z.\, K z z =$

$= \lambda z.\, z$

$\Rightarrow \alpha \to \alpha$

d) $I = \lambda x.\, x$

$\alpha \to \alpha$

Exercise 4

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{}{\Gamma \vdash (\alpha \to \beta) \to \gamma}\,Ax
        \qquad
        \dfrac{}{\Gamma \vdash \alpha \to \beta}\,Ax
      }{\Gamma \equiv (\alpha \to \beta) \to \gamma,\ \beta \vdash \gamma}\,{\to}E
    }{(\alpha \to \beta) \to \gamma,\ \beta \to \gamma}\,{\to}I
  }{(\alpha \to \beta) \to \gamma \vdash \beta \to \gamma}\,{\to}I
}{\vdash ((\alpha \to \beta) \to \gamma) \to \beta \to \gamma}\,{\to}I
$$

Associated lambda term:

$$\lambda f^{((\alpha \to \beta) \to \gamma)}\, x^{\beta} .\ f(\lambda y^{\alpha} . x)$$

Exercise 5

a) $M, N ::= x \mid \lambda x : \tau . N \mid MN \mid (M, N) \mid \langle N| \mid |N\rangle$

$\tau, \sigma ::= \alpha_i \mid \tau \to \sigma \mid \sigma \times \tau$

b) $\dfrac{\Gamma \vdash x : \sigma \quad \Gamma\ y : \tau}{\Gamma \vdash (x, y) : \sigma \times \tau}\,Prod$ $\qquad$ $\dfrac{\Gamma \vdash x : \sigma \times \tau}{\Gamma \vdash \langle x| : \sigma}\,Fst$

$\dfrac{\Gamma \vdash x : \sigma \times \tau}{\Gamma : |x\rangle : \tau}\,Snd$

Exercise 6

?

Exercise 7

CBV

$\text{fact } c_3 =$

$= \text{if } (\text{iszero } c_3) \text{ then } c_1 \text{ else times } c_3 \text{ (fact (pred } c_3))$

$= \text{if } (\text{iszero } c_3) \text{ then } c_1 \text{ else times } c_3 \text{ (fact } c_2)$

$= \text{if false then } c_1 \text{ else times } c_3 \text{ (fact } c_2)$

$= \text{times } c_3 \text{ (fact } c_2) =$

$= \text{times } c_3 \text{ (if (iszero } c_2) \text{ then } c_1 \text{ else times } c_2 \text{ (fact (pred } c_2))$

$= \text{times } c_3 \text{ (if (iszero } c_2) \text{ then } c_1 \text{ else times } c_2 \text{ (fact } c_1)$

$= \text{times } c_3 \text{ (if false then } c_1 \text{ else times } c_2 \text{ (fact } c_1)$

$= \text{times } c_3 \text{ (times } c_2 \text{ (fact } c_1)) =$

$= \text{times } c_3 \text{ (times } c_2 \text{ (if (iszero } c_1) \text{ then } c_1 \text{ else times } c_1 \text{ (fact (pred } c_1))$

$= \text{times } c_3 \text{ (times } c_2 \text{ (if (iszero } c_1) \text{ then } c_1 \text{ else times } c_1 \text{ (fact } c_0) =$

$= \text{times } c_3 \text{ (times } c_2 \text{ (if false then } c_1 \text{ else times } c_1 \text{ (fact } c_0) =$

$= \text{times } c_3 \text{ (times } c_2 \text{ (times } c_1 \text{ (fact } c_0)) =$

$= \text{times } c_3 \text{ (times } c_2 \text{ (times } c_1 \text{ (if (iszero } c_0) \text{ then } c_1 \text{ else times } c_0$
$\text{(fact (pred } c_0)) =$

$= \text{times } c_3 \text{ (times } c_2 \text{ (times } c_1 \text{ (if true then } c_1 \text{ else times } c_0$
$\text{(fact (pred } c_0)) =$

$= \text{times } c_3 \text{ (times } c_2 \text{ (times } c_1 \ c_1) =$

$= \text{times } c_3 \text{ (times } c_2 \ c_1) =$

$= \text{times } c_3 \ c_2 =$

$= c_6$