Controlling Tail Risk Measures with Estimation Error*

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Abstract

Risk control plays an important role in economic decision making such as investment and banking regulation. In practice, controlling risks such as Value-at-Risk and expected shortfall comes with estimation error that can invalidate the risk guarantees put forward by the risk measures. This paper provides a novel way to hedge against estimation risk in general risk management problems under the assumption that a valid confidence interval is available for the risk measure. We call the class of risk measures that control the probability of risk events the *tail risk measure*. We then provide an *observable* risk estimate with which the true *unobservable* risk probability is bounded. We give a recommended choice of the tuning parameters that makes the risk estimate consistent. An empirical application to Value-at-Risk and expected shortfall demonstrates that the proposed risk estimates are not too conservative for practical use.

Keywords: risk control, estimation risk, risk measure, value-at-risk, expected shortfall.

JEL Codes: D81, G28, G32, C58.

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1 Introduction

Risk control plays an important role in economic decision making. In the context of portfolio management, Markowitz (1952) considered the variance of the return as the risk, and Roy (1952) considered the probability of a certain amount of loss to be the risk. This notion of minimizing the probability of a tail loss is associated with various concepts of risk measures (Goovaerts et al., 2003). In banking regulation, for example, Basel II imposed capital requirements based on *Value-at-Risk (VaR)*, and Basel 2.5 added requirements based on *expected shortfall (ES)* to reduce the chance of financial crisis (Chen, 2014). The literature proceeded to proposing various desiderata of risk measures (Artzner et al., 1999; Acerbi, 2002; Dowd and Blake, 2006; Ahmadi-Javid, 2012).

In practice, true risk measures are not observed, so estimates based on historical data must be employed. McNeil et al. (2005, pp. 40–41) warned the danger of a naive use of estimated risk, pointing out that it is subject to problems of estimation error, model risk, and market liquidity. The problem of estimation error is that the sampling error of the estimated risk may distort the desired characteristics of the risk measure. The problem of model risk arises when the postulated model is largely misspecified and its estimates are misleading or useless. The problem of market liquidity comes from the fact that an asset can change its prices for the sole reason of being liquidated; thus, the asset worth \$1 may not translate to \$1 in hand. For the latter two, the literature provides some methods to account for them. Stahl (1997) argued that multiplying 3 to VaR hedges against the model risk;[1] Leippold and Vanini (2002) showed that for ES, the multiplier for model risk can be lowered to 1.5; see also Hendricks and Hirtle (1997), Lopez (1998), and Novak (2010). Next, many institutions use "liquidity-adjusted" VaR that takes into account realistic holding periods; Gârleanu and Pedersen (2007) analyzed the feedback effect of such practice; Adrian and Brunnermeier (2016) proposed a measure of systemic risk that subsumes liquidity risk.

In contrast, despite the vast literature on statistical inference for risk measures, a practical way to account for estimation error in generic risk management problems has not been established. In the context of portfolio optimization, estimation error is known to be particularly problematic in Markowitz's (1952) mean-variance model

[1] Stahl's (1997) justification was retrospective, after Basel II employed the multiplier of 3.

(Michaud, 1989; Chopra and Ziemba, 1993), and many portfolio optimization methods are proposed to incorporate estimation error (Ceria and Stubbs, 2006; Michaud and Michaud, 2008). However, other risk management problems, such as banking regulation, are inherently different from portfolio optimization, and a generic method to uphold the desired risk guarantee is yet to be developed.

This paper proposes a general method to incorporate estimation error into risk control problems that comes with a guarantee. Under the assumption that a valid confidence interval is available for the risk measure, we provide a simple method to bound the true *unobservable* risk probability using an *observable* risk estimate. We characterize the class of risk measures to which our method applies and name them the *tail risk measure*, and show that VaR and ES—arguably the two most popular risk measures—are special cases of it.

The main idea of the paper is best illustrated in the case of VaR. Specifically, the VaR is defined as the maximum loss that can be incurred with a user-specified probability $1-p$. For example, if we let X be the random variable that represents the asset value of an institution, the VaR is given by the smallest value of C that satisfies[2]

$$P(-X > C) \leq p.$$

The idea of "controlling VaR" is to maintain the capital reserve of C so that the institution stays solvent with probability at least $1-p$ when X realizes. We denote such C as VaR_p and call it the *VaR of coverage* $1-p$.[3] We call the event $-X > C$ the *risk event*, and its probability the *risk probability*. The "risk guarantee" put forward by VaR is, therefore, that the risk probability is bound to be at most p.

The problem of estimation error is that, when the VaR is replaced by its estimate $\widehat{\mathrm{VaR}}_p$, we no longer have

$$P(-X > \widehat{\mathrm{VaR}}_p) \leq p$$

as a theoretical guarantee. In fact, we demonstrate in Section 2 that the actual coverage is distorted, that is, the left-hand side (LHS) is strictly greater than p when we use a naive VaR estimate.

[2]This C may be greater than the C' that attains the equality $P(-X > C') = p$ for various reasons. First, if the distribution of X is discrete, there may not exist such C'. Second, the distribution of X is not known in practice, so the practically feasible choice of C may be conservative.

[3]This $1-p$ is often called the "confidence level" in the literature, but because we deal with the "confidence interval" of the risk measure, we adopt a different term to avoid confusion.

Our idea is to "allot" the risk probability allowance p for two undesirable events separately: (1) the asset value exceeds the VaR and (2) the estimated VaR underestimates the true VaR. Let $p = q + r$ be a user-specified allocation of p, and suppose that a one-sided $(1 - r)$-confidence interval for VaR_q is available, that is,

$$P(\mathrm{VaR}_q > \widetilde{\mathrm{VaR}}_{q,r}) \leq r$$

holds for some observable $\widetilde{\mathrm{VaR}}_{q,r}$. Then, by the Bonferroni inequality, we have

$$P(-X > \widetilde{\mathrm{VaR}}_{q,r}) \leq P(-X > \mathrm{VaR}_q) + P(\mathrm{VaR}_q > \widetilde{\mathrm{VaR}}_{q,r}) \leq q + r = p.$$

This is because, when the event $-X > \widetilde{\mathrm{VaR}}_{q,r}$ takes place, we have that either (1) $-X > \mathrm{VaR}_q$ or (2) $\mathrm{VaR}_q > \widetilde{\mathrm{VaR}}_{q,r}$ must take place. Note that the LHS does not depend on the unknown, VaR_q, but only on the observable, $\widetilde{\mathrm{VaR}}_{q,r}$. Therefore, by holding the capital worth $\widetilde{\mathrm{VaR}}_{q,r}$, we can make sure that the actual risk probability is capped by p, even including the sampling variation of $\widetilde{\mathrm{VaR}}_{q,r}$. A legitimate concern here is that the Bonferroni inequality is known to be a "loose" bound. To address this point, we demonstrate in the empirical application in Section 4 that $\widetilde{\mathrm{VaR}}_{q,r}$ is only modestly larger than a naive estimate, roughly by 10–50%. Also, we give a practical recommendation of the allocation (q, r) that is not conservative asymptotically so that no allowance is wasted in the limit as the sample size grows.

This paper relates to the long history of concerns on estimation error in risk measures (Jorion, 1996; Hendricks, 1996; Pritsker, 1997, 2006; Barone-Adesi and Giannopoulos, 2001; Berkowitz and O'Brien, 2002; Aussenegg and Miazhynskaia, 2006; Thiele, 2019). Chen (2008) noted that the effective sample size for ES at confidence level $1 - p$ is the actual sample size times p^2, stating that the estimate's high volatility is a common challenge for statistical inference. Caccioli et al. (2018) noted that because of high dimensionality of institutional portfolios and the lack of long-run stationarity, portfolio optimization is plagued by (relatively) small sample sizes.

This paper is organized as follows. Section 2 presents a simulation exercise that demonstrates that estimation error acts adversally to risk control. Section 3 defines the class of tail risk measures and presents a simple method to control the true risk probability using an observable quantity. Section 4 applies our method to an empirical application of controlling VaR and ES in a stylized investment problem and assess how large our risk estimates can be relative to naive estimates. Section 5 concludes.

2 Coverage Distortion due to Estimation Error

The defining feature of VaR is that the probability of the loss exceeding the VaR is bounded at a desired level, $P(-X > \mathrm{VaR}_p) \leq p$. We show, however, that if we substitute the VaR with an estimator, the actual risk probability can indeed be larger than the proclaimed level p, building on a generalized autoregressive conditional heteroskedasticity (GARCH) model.[4]

The setup of the simulation is as follows. First, we draw $T = 200$ observations $\{X_1, \dots, X_T\}$ from a GARCH(1,1) model,

$$\begin{cases} X_t = \sigma_t z_t, \\ \sigma_t^2 = \omega + \alpha z_{t-1}^2 + \beta \sigma_{t-1}^2, \end{cases}$$

where the true parameters are $\omega = 0.001$, $\alpha = 0.05$, and $\beta = 0.9$, and z_t follows independent standard normal distribution. We first estimate the GARCH parameters (ω, α, β) by maximum likelihood estimation (MLE) and fit innovations $\{\hat{z}_1, \dots, \hat{z}_T\}$ and volatility $\hat{\sigma}_T$. We then estimate VaR_p for $p \in \{0.05, 0.01\}$ for the next observation X_{T+1} by $\widehat{\mathrm{VaR}}_p = -\hat{\sigma}_T \kappa_p$, where κ_p is given by one of the three canonical quantile estimators:

1. *Parametric.* The p-quantile of a standard normal distribution.
2. *Semiparametric.* Weissman's (1978) estimator of the p-quantile of $\{\hat{z}_1, \dots, \hat{z}_T\}$,

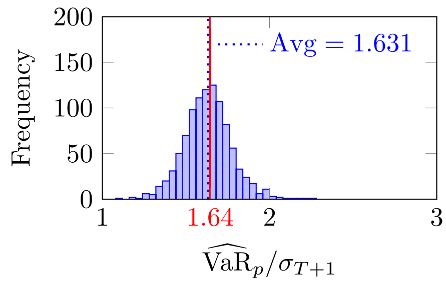
$$\kappa_p = \hat{z}_{(k)} - \left(\hat{z}_{(k)} - \frac{1}{k} \sum_{i=1}^{k} \hat{z}_{(i)} \right) \log\left(\frac{k}{pT} \right) \qquad \text{for} \quad k = 10.$$

3. *Nonparametric.* The empirical p-quantile of $\{\hat{z}_1, \dots, \hat{z}_T\}$.

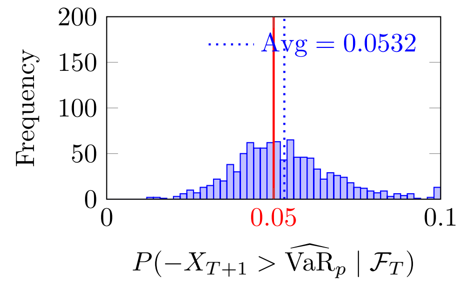
Since $z_T \sim N(0, 1)$, we can calculate the exact risk probability conditional on the history, $P(-X_{T+1} > \widehat{\mathrm{VaR}}_p \mid \mathcal{F}_T)$.[5] We repeat this exercise for $S = 1{,}000$ times with newly drawn observations and compute the unconditional risk probability $P(-X_{T+1} > \widehat{\mathrm{VaR}}_p)$. We see below that better estimation of VaR does not necessarily lead to less distortion of the risk probability.

[4]Simulation of i.i.d. returns exhibits the same problem. See Kaji (2018) for details.

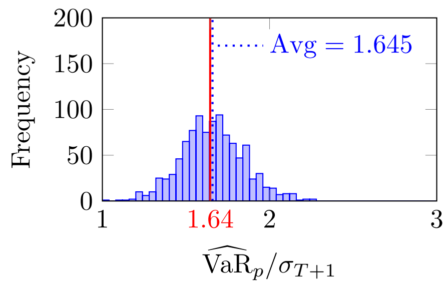
[5]Here, $\mathcal{F}_T$ is the σ-field that contains all information up to time T, including the VaR estimate.



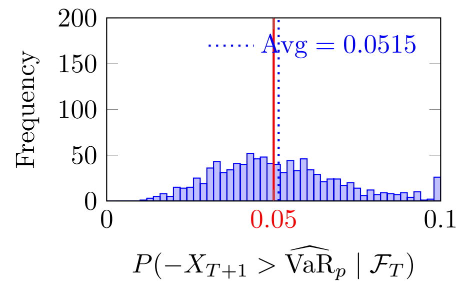
(a) Parametric estimator of $\mathrm{VaR}_{0.05}$. MSE $= 0.0207$.



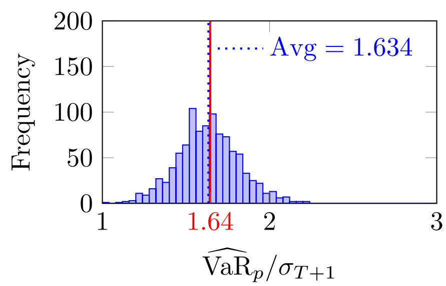
(b) Coverage of parametric estimator. p-value for $H_0 : \mathrm{Avg} = 0.05$ is 9.04×10^{-11}.



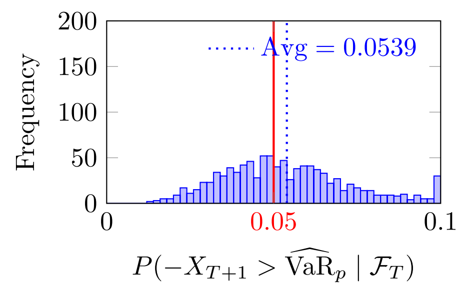
(c) Semiparametric estimator of $\mathrm{VaR}_{0.05}$. MSE $= 0.0355$.



(d) Coverage of semiparametric estimator. p-value for $H_0 : \mathrm{Avg} = 0.05$ is 0.017.



(e) Nonparametric estimator of $\mathrm{VaR}_{0.05}$. MSE $= 0.0339$.



(f) Coverage of nonparametric estimator. p-value for $H_0 : \mathrm{Avg} = 0.05$ is 6.48×10^{-10}.

Figure 1: Histograms of the estimated VaR and its coverage for $p = 0.05$. The simulated observations follow GARCH(1,1). The simulation size is $S =$ 1,000, and the sample size in each iteration is $T = 200$. The red lines on the left figures are the true VaR, and those on the right are the intended risk probability. The dotted blue lines on the right figures are the average risk probability that estimates the actual unconditional coverage, $P(-X_{T+1} > \widehat{\mathrm{VaR}}_p)$. The p-values in the right captions are for the hypothesis that that the actual coverage equals the intended level. We conclude that the actual coverage is worse than the intended level in all three cases.

Figure 1 presents the histograms of the estimated risk and the risk probability for VaR of coverage 95% ($p = 0.05$). Figure 1a shows the histogram of $\widehat{\mathrm{VaR}}_p$ for X_{T+1} normalized by the volatility σ_{T+1}. By the property of the GARCH model, the variance of X_{T+1} differs in each iteration. Therefore, normalization by the true volatility makes VaR estimates comparable across iterations. The red line indicates the true $\mathrm{VaR}_p/\sigma_{T+1}$, which is simply the absolute value of the p-quantile of the standard normal distribution.

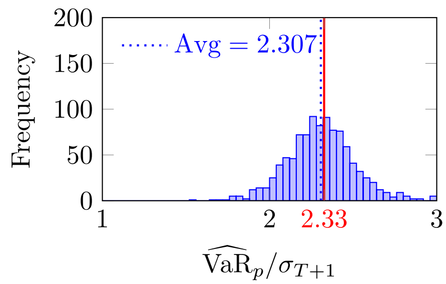
Figure 1b is the histogram of the risk probability conditional on the history, $P(-X_{T+1} > \widehat{\mathrm{VaR}}_p \mid \mathcal{F}_T)$. The red line indicates the intended risk probability $p = 0.05$, and the blue dotted line is the average of the conditional risk probability from each iteration, which estimates the actual unconditional risk probability, $P(-X_{T+1} > \widehat{\mathrm{VaR}}_p)$. These numbers suggest that if we maintain the liquid asset worth $\widehat{\mathrm{VaR}}_p$, the risk event $-X_{T+1} > \widehat{\mathrm{VaR}}_p$ takes place about 5.32% of the time, more often than the intended 5%. However, this "5.32%" is subject to the "sampling error" of the simulation draws. To take this into account, we conduct a t-test. By the nature of the simulation, each draw of the conditional risk probability is independent and identically distributed. With this, it is straightforward to compute the p-value for the hypothesis that the true unconditional risk probability equals the intended level,

$$H_0 : P(-X_{T+1} > \widehat{\mathrm{VaR}}_p) = p.$$

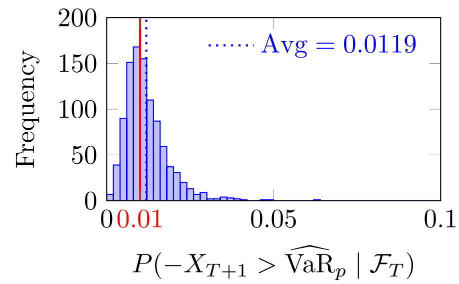
The p-value is 9.04×10^{-11} as given in the caption of Figure 1b. Combined with the fact that the estimate 5.32% is above 5%, we conclude that the unconditional risk probability exceeds the intended level with statistical significance.

Figures 1c and 1d are the same figures when the VaR is estimated by the semiparametric estimator. It is interesting to note that, while the semiparametric estimator *overestimates* the true VaR on average (that is, the blue dotted line is above the red line in Figure 1c), the risk probability is still *above* the intended level (that is, the blue dotted line is above the red line in Figure 1d). Figures 1e and 1f are the corresponding figures for the nonparametric estimator of VaR.

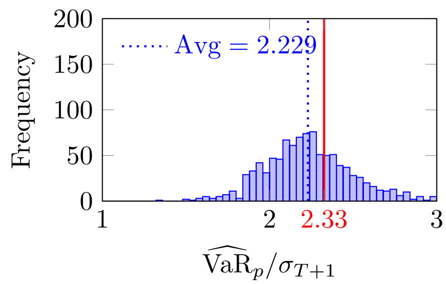
Being correctly specified, the parametric estimator achieves the smallest mean squared error (MSE). The MSE of the parametric estimator is 0.0207, while those of the semi- and nonparametric estimators are 0.0355 and 0.0339 (as given in the captions of Figures 1a, 1c, and 1e). Thus, in the context of this simulation, the parametric estimator is the "best" among the three. However, this relationship



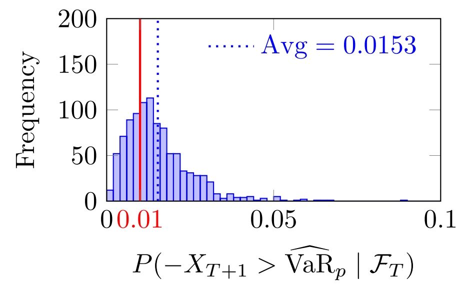
(a) Parametric estimator of $\mathrm{VaR}_{0.01}$. MSE = 0.0414.



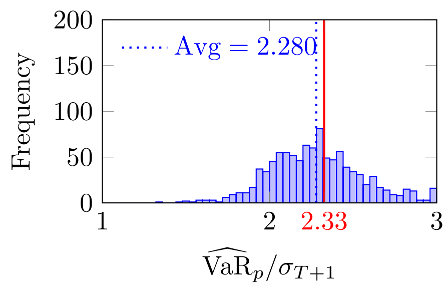
(b) Coverage of parametric estimator. p-value for H_0 : Avg = 0.01 is 5.02×10^{-19}.



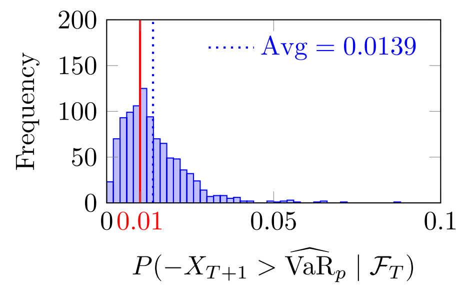
(c) Semiparametric estimator of $\mathrm{VaR}_{0.01}$. MSE = 0.0749.



(d) Coverage of semiparametric estimator. p-value for H_0 : Avg = 0.01 is 4.15×10^{-57}.



(e) Nonparametric estimator of $\mathrm{VaR}_{0.01}$. MSE = 0.0786.



(f) Coverage of nonparametric estimator. p-value for H_0 : Avg = 0.01 is 5.86×10^{-34}.

Figure 2: The same figures as Figure 1 but for $p = 0.01$. The same comments apply.

does not carry over to the risk probability. The distortion of the risk probability is $5.32\% - 5.00\% = 0.32\%$ for the parametric estimator, which is worse than that of the semiparametric estimator, 0.15%. In other words, higher precision of VaR estimation should not be taken as evidence of less distortion of the risk probability.

Figure 2 gives the corresponding histograms for the VaR of coverage 99% ($p = 0.01$). The rates of distortion are substantially worse for $p = 0.01$ than for $p = 0.05$. For example, the distortion rate of the risk probability for the parametric estimator for $p = 0.05$ is $(0.0532 - 0.05)/0.05 \approx 6.4\%$. Similarly, they are 3.0% and 7.8% for the semi- and nonparametric estimators for $p = 0.05$. Meanwhile, for $p = 0.01$, they are respectively 18.6%, 53.2%, and 38.8%. This is especially problematic when we want to minimize the probability of a catastrophic event such as financial crisis.[6]

3 Main Results

This section presents the main results of the paper. Section 3.1 defines the class of risk measures we consider. Section 3.2 provides a method to control the true risk probability and formally proves the guarantee. Section 3.3 gives a practical recommendation of the tuning parameters (q, r).

3.1 Definition

The risk measures to which our method applies are the ones that come with probability guarantees. We start with reformulating VaR and then generalize it. Let $(\Omega, \mathcal{F}, P)$ be the probability space we consider.

The key feature of VaR is that it provides a guarantee on the probability of a *risk event*, $\{-X > C\}$. Recall that VaR_p is formally defined as the smallest value C such that

$$P(-X > C) \leq p.$$

[6]For regulatory capital calculations within the Basel framework, 99.9% coverage ($p = 0.001$) is typically used for VaR.

Equivalently, it can be cast as the smallest value of C such that

$$\underbrace{\sup_{E\in\mathcal{F}}\Big\{P(E):}_{\substack{\text{the max probability}\\ \text{of a risk event}}}\underbrace{\inf\{-X(\omega):\omega\in E\}-C>0}_{\substack{\text{in which even the smallest}\\ \text{loss exceeds } C}}\Big\}\underbrace{\leq p}_{\substack{\text{is at}\\ \text{most } p}}.$$

The middle component,

$$\inf\{-X(\omega):\omega\in E\}-C,$$

is the key riskiness metric embedded in VaR. In other words, VaR defines the risk of an event E to be such that, even the best-case loss in the event is greater than the capital reserve C. Holding the capital worth C, therefore, corresponds to keeping the maximum probability of such an event at or below p.

Other risk measures can be obtained by replacing this riskiness component. Acerbi et al. (2001) defined the *ES of coverage* $1-p$ as

$$\mathrm{ES}_p(X) \coloneqq \frac{1}{p}\int_0^p \mathrm{VaR}_\alpha(X)d\alpha.$$

This can be cast as the smallest value of C such that

$$\underbrace{\sup_{E\in\mathcal{F}}\Big\{P(E):}_{\substack{\text{the max probability}\\ \text{of a risk event}}}\underbrace{\mathbb{E}[-X\mid E]-C>0}_{\substack{\text{in which the expected}\\ \text{loss exceeds } C}}\Big\}\underbrace{\leq p}_{\substack{\text{is at}\\ \text{most } p}}.$$

Thus, ES measures the riskiness of X by $\mathbb{E}[-X\mid E]-C$ when the event E takes place and the capital reserve is C.[7]

Let $\chi(X,C\mid E)$ be an arbitrary metric of *riskiness* of return X under event E that is nonincreasing in reserve C. The tail risk measure is formally defined as follows.

Definition (tail risk measure)**.** The *tail risk measure of coverage* $1-p$ is the infimum of C such that

$$\sup_{E\in\mathcal{F}}\Big\{P(E):\chi(X,C\mid E)>0\Big\}\leq p.$$

If there is no event E that attains $\chi(X,C\mid E)>0$ for a fixed C, we may understand the corresponding supremum to be 0.

[7] If we adopt $\mathbb{E}[-X\mid X\in X(E)]-C$, we obtain what is known as the tail conditional expectation (Acerbi, 2002).

Intuitively, the tail risk measure controls the probability of the undesirable risk event $\chi > 0$. The supremum with respect to E represents a search for the maximum probability that it can happen. The value C is the "capital reserve" with which we can avoid the risk event with probability at least $1 - p$. This is a direct control of the risk probability compared to ones relying on the Markov bound (Goovaerts et al., 2003).

The choice of χ defines a risk measure. It is straightforward to extend it to various tail risks, such as the "two-sided" expected shortfall,

$$\chi(X, C \mid E) = \mathbb{E}[|X| \mid E] - C,$$

which may be useful when we want to use a common risk measure for longing and shorting the asset. We may also obtain the "utility-based" shortfall risk,

$$\chi(X, C \mid E) = \mathbb{E}[\ell(-X - C) \mid E],$$

for a nondecreasing loss function $\ell : \mathbb{R} \to \mathbb{R}$. This allows one to, e.g., incorporate loss aversion into the risk measure.

Instead of the utility-based risk, we may also consider risk based on the dual theory of Yaari (1987). Such a risk measure is known as the *distortion risk measure* (Wang et al., 1997; Artzner et al., 1999; Tsukahara, 2014). Let $g : [0, 1] \to [0, 1]$ be the *distortion function* that is right-continuous and increasing with $g(0) = 0$ and $g(1) = 1$. The *distorted probability* is defined as $P^* = g \circ P$. Let $\mathbb{E}^*$ be the expectation with respect to P^*. Then, for example, the tail distortion expected shortfall can be obtained by setting

$$\chi(X, C \mid E) = \mathbb{E}^*[-X \mid E] - C.$$

If we want to control the distorted probability instead of the true probability, we may replace $P(E)$ with $P^*(E)$ in the definition.

3.2 Controlling the Tail Risk Measure

This section provides a method to control the actual risk probability with an observable risk estimate. The essential assumption is that a (one-sided) confidence interval for the tail risk measure c_q is available, that is, we have access to an observable quantity $\tilde{c}_{q,r}$ that satisfies

$$P(c_q > \tilde{c}_{q,r}) \leq r$$

for an arbitrary confidence level $1 - r$.

To lay out how the Bonferroni bound can be generalized to arbitrary tail risk measures, let us illustrate it for the case of ES. Note that the risk probability of ES can be written as

$$\sup_{E \in \mathcal{F}} P(E,\, \mathbb{E}[-X \mid E] > \mathrm{ES}_p).$$

Since "$\mathbb{E}[-X \mid E] > \mathrm{ES}_p$" is a nonrandom statement, the probability is either $P(E)$ when it holds or 0 otherwise. If we substitute a confidence interval into the ES, we can apply the Bonferroni bound as

$$P(E,\, \mathbb{E}[-X \mid E] > \widetilde{\mathrm{ES}}_{q,r}) \leq P(E,\, \mathbb{E}[-X \mid E] > \mathrm{ES}_q) + P(E,\, \mathrm{ES}_q > \widetilde{\mathrm{ES}}_{q,r}),$$

since violation of both inequalities on the right implies violation of the one on the left. By taking the supremum of both sides with respect to E, we find

$$\begin{aligned}\sup_{E \in \mathcal{F}} P(E,\, \mathbb{E}[-X \mid E] > \widetilde{\mathrm{ES}}_{q,r}) &\leq \sup_{E \in \mathcal{F}} \big[P(E,\, \mathbb{E}[-X \mid E] > \mathrm{ES}_q) + P(E,\, \mathrm{ES}_q > \widetilde{\mathrm{ES}}_{q,r})\big] \\ &\leq \sup_{E \in \mathcal{F}} P(E,\, \mathbb{E}[-X \mid E] > \mathrm{ES}_q) + P(\mathrm{ES}_q > \widetilde{\mathrm{ES}}_{q,r}) \\ &\leq q + r.\end{aligned}$$

Thus, the risk probability *including the sampling error of* $\widetilde{ES}_{q,r}$ is bounded by $q + r$. The general statement is as follows.

Theorem 1 (Risk control with estimation error)**.** *Let c_q be the tail risk measure of coverage $1 - q$, and $\tilde{c}_{q,r}$ the $(1 - r)$-confidence bound of c_q, that is, $P(c_q > \tilde{c}_{q,r}) \leq r$. Then, for $p = q + r$, we have*

$$\sup_{E \in \mathcal{F}} P\big(E,\, \chi(X, \tilde{c}_{q,r} \mid E) > 0\big) \leq p.$$

Proof. Fix an event $E \in \mathcal{F}$. By the Bonferroni inequality, we have

$$P\big(E,\, \chi(X, \tilde{c}_{q,r} \mid E) > 0\big) \leq P\big(E,\, \chi(X, c_q \mid E) > 0\big) + P(E,\, c_q > \tilde{c}_{q,r}).$$

Taking the supremum of both sides with respect to E, we find

$$\begin{aligned}
\sup_{E\in\mathcal{F}} P\big(E,\, \chi(X, \tilde{c}_{q,r} \mid E) > 0\big) &\le \sup_{E\in\mathcal{F}} \big\{ P\big(E,\, \chi(X, c_q \mid E) > 0\big) + P(E,\, c_q > \tilde{c}_{q,r}) \big\} \\
&\le \sup_{E\in\mathcal{F}} \big\{ P(E) : \chi(X, c_q \mid E) > 0 \big\} + P(c_q > \tilde{c}_{q,r}) \\
&\le q + r = p.
\end{aligned}$$

This completes the proof. ■

Note that Theorem 1 does not make use of asymptotic arguments. Therefore, it is valid in finite samples as long as the confidence interval is. It is however the case that many existing inference methods are justified as asymptotic approximations. In that case, the risk statement given in Theorem 1 should be understood as an asymptotic statement as well.

While the theorem is valid for an arbitrary choice of (q, r), we cannot "hunt" for the smallest $\tilde{c}_{q,r}$ after observing the data. This changes the distribution of $\tilde{c}_{q,r}$ for randomness introduced by the hunting, and the probability bound will no longer be valid. Such is an example of data snooping. In the next section, we discuss a practical choice of (q, r).

The theorem assumes a one-sided confidence interval, but the upper bound of any two-sided confidence interval is by construction a valid one-sided confidence interval at the same confidence level. It is, however, more conservative than necessary. Also, our results only make use of a confidence interval but not an estimator, so an inferential method that does not require an estimator can also be used.

Estimation and inference for risk measures are a widely studied area, ranging from parametric to nonparametric methods. Embrechts et al. (1997) considered estimation and inference of VaR and ES based on extreme value theory. Chen and Tang (2005) and Scaillet (2004) discussed nonparametric estimation and inference of VaR and ES. Linton and Xiao (2013) and Hill (2015) considered nonparametric estimation and inference of ES when X may not have a variance. Belomestny and Krätschmer (2012) established asymptotic normality of plug-in estimators of law-invariant coherent risk measures. Gao and Song (2008) derived asymptotic distribution of VaR and ES estimators in the GARCH model estimated by the filtered historical simulation method. There is also large literature on estimation and inference of VaR and ES conditional on covariates (Chernozhukov and Umantsev, 2001; Cai and Wang, 2008; Kato, 2012;

Chernozhukov and Fernández-Val, 2011; Chun et al., 2012; Chernozhukov et al., 2017; Martins-Filho et al., 2018).

3.3 Practical Choice of (q, r)

As the sample size increases, estimation of the risk measure is expected to be more precise. This provokes a thought that we can spend less allowance on r and let q be closer to p. In this section, we discuss the choice of (q, r) as a function of the sample size T, assuming that the confidence interval is constructed with a (sub)polynomially-tailed distribution.

Suppose, first, that $\hat{c}_q$ is a normally-distributed unbiased estimator with known variance σ_q^2, that is,

$$\sqrt{T}(\hat{c}_q - c_q) \sim N(0, \sigma_q^2).$$

The one-sided $(1-r)$-confidence bound for c_q is then given by

$$\hat{c}_q + \frac{\sigma_q}{\sqrt{T}}\kappa_r,$$

where κ_r is the $(1-r)$th quantile of the standard normal distribution. This suggests that the confidence bound shrinks at rate $1/\sqrt{T}$ for fixed r and blows up as $r \to 0$ for fixed T. In reasonable situations, we can expect that σ_q is continuous at $q = p$, so it is enough to consider a shrinking sequence $r \to 0$ such that $\kappa_r/\sqrt{T}$ goes to zero.

Let ϕ and Φ be the pdf and cdf of a standard normal distribution. Then, κ_r can be expressed as

$$\kappa_r = \Phi^{-1}(1-r).$$

We know, by the property of the Mills ratio, that for $x \geq 1/\sqrt{2\pi}$, we have

$$1 - \Phi(x) = \Phi(-x) < \frac{1}{x}\phi(x) \leq e^{-x^2/2}.$$

Therefore, for $0 < r \leq e^{-1/4\pi}$, we find

$$\kappa_r = \Phi^{-1}(1-r) = -\Phi^{-1}(r) < \sqrt{-2\log r}.$$

Thus, we can let $r \to 0$ at the speed at which $\kappa_r/\sqrt{T} < \sqrt{-2\log r}/\sqrt{T} \to 0$, i.e., $r \gg e^{-T/2}$. Therefore, r can indeed go very fast to zero; e.g., any rational function in

T converges slower than $e^{-T/2}$. For the sample sizes of order hundreds to thousands, $r = 1/T$ would be a good choice.

In other cases, the confidence interval may result from a distribution that is more heavy-tailed than the normal distribution, such as the t-distribution. Suppose that κ_r is bounded by a polynomial function,

$$\kappa_r \lesssim r^{-\frac{1}{\nu}},$$

for some $\nu > 0$ when r is close to zero. This is the case when, e.g., κ_r is the quantile of a t-distribution with ν degrees of freedom. Then, we need that $r^{-1/\nu}/\sqrt{T}$ go to zero, i.e., $r \gg T^{-\nu/2}$. In other words, if $\nu > 2$ (which corresponds to having a finite variance), the choice $r = 1/T$ is justified.

We summarize that our recommended choice is $r = T^{-1}$ for applications where the estimator is root-n consistent and the asymptotic distribution has a finite variance.[8] This covers most estimators in the literature. One exception is Linton and Xiao (2013), who considered estimation of ES when the underlying time series has an infinite variance. They observed that the convergence rate is strictly slower than $\sqrt{T}$ and the limiting distribution is a stable law. In that case, we will need to find our own choice of (q, r) by going through similar consideration. Or, we may use trimming proposed by Hill (2015) to bring it back to an asymptotically normal estimator and stick to our recommendation.

4 Empirical Application to VaR and ES

In this section, we apply our method to real data drawing on an investment problem. The purpose is to demonstrate that our proposed quantity is practically not too large compared to a naive estimator, despite relying on a Bonferroni bound.

Consider the problem of controlling VaR and ES of a portfolio with three assets: the stock of the Bank of America Corp. (BAC), the stock of Morgan Stanley (MS), and the index fund for the Dow Jones Industrial Average (DJI). The choice of assets is due to the ease of access to the data, but it can be any assets, e.g., on a bank's balance sheet. We use the daily adjusted close values from February 23, 1993 to

[8]The value of q is simply $p - r$ if we only concern one asset. When there are multiple assets, we may want to adjust q depending on our purpose. See Section 4.

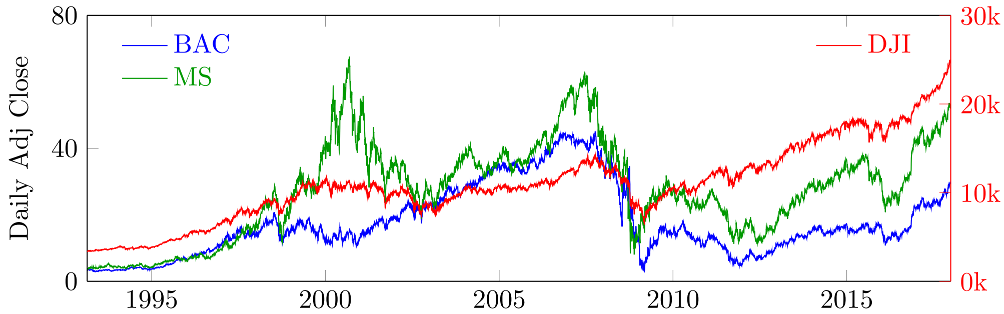


Figure 3: Daily adjusted close values of BAC, MS, and DJI from Feb. 23, 1993 to Dec. 31, 2017. BAC and MS are on the left y-axis and DJI on the right.

December 31, 2017. The price data are retrieved from Yahoo! Finance. Figure 3 shows the adjusted close values of these assets.

Denoting by Y_t the daily close value, the daily return is calculated as $X_t = (Y_t - Y_{t-1})/Y_{t-1}$. Figure 4 shows the daily returns of the three assets. The daily portfolio return is given by

$$X = w_1 X_{\mathrm{BAC}} + w_2 X_{\mathrm{MS}} + w_3 X_{\mathrm{DJI}},$$

where X_{TIC} is the daily return of the stock of ticker symbol TIC, and (w_1, w_2, w_3) are the weights. Our goal is to estimate VaR and ES of coverage $1-p$ of the return X of the entire portfolio without losing probability guarantees.

We model each daily return as a GARCH(1,1) process,

$$\begin{cases} X_t = \mu + \sigma_t z_t, \\ \sigma_t^2 = \omega + \alpha z_{t-1}^2 + \beta \sigma_{t-1}^2, \end{cases}$$

where $\{z_t\}$ are i.i.d. random variables and $\omega, \alpha, \beta \geq 0$ are GARCH parameters.[9] We use Gao and Song (2008) to estimate VaR and ES for individual assets.

Let $r = 1/T$ and $q = p - 3r$. The reason why q is more conservative than $p - r$ is that the asymptotic distribution given in Gao and Song (2008, Theorems 3.2 and 3.3) are of the marginal distribution of each estimator. To ignore the correlation of risk estimates, we need to allot $r = 1/T$ to the confidence bound of each asset separately. This is yet another application of the Bonferroni inequality. If there are many assets, this would lead to a very conservative bound. An alternative is to use the bootstrap procedure in Gao and Song (2008) and estimate the joint distribution of risk measures

[9] X_t is stationarity if $\alpha + \beta < 1$ and i.i.d. if $\alpha = \beta = 0$.

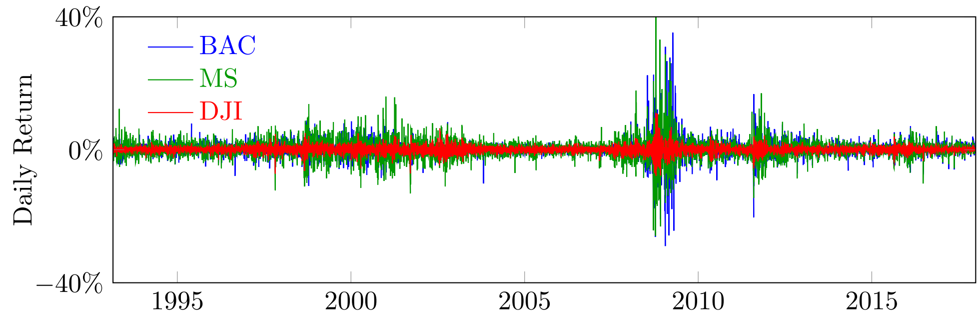


Figure 4: Daily returns of BAC, MS, and DJI from Feb. 23, 1993 to Dec. 31, 2017. We see stochastic trends in their volatility.

across assets. To the best of our knowledge, however, the validity of bootstrap in this context is not fully understood.

Our estimation procedure goes as follows.

1. Estimate the GARCH parameters $(\mu, \omega, \alpha, \beta)$ by quasi-MLE assuming normality of z_t, and fit $\hat{\sigma}_T$ and $\{\hat{z}_1, \dots, \hat{z}_{t-1}\}$.

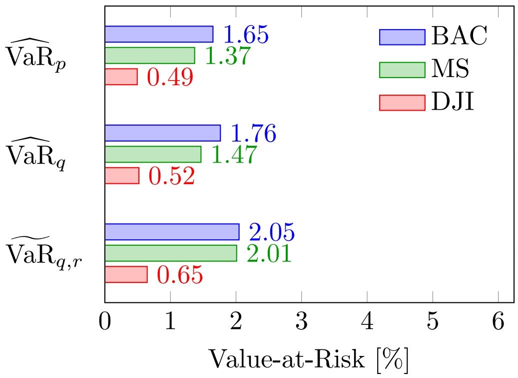
2. Let κ_q be the empirical q-quantile of $\hat{z}_t$ and compute the q-trimmed average of $\hat{z}_t$ by $\eta_q = \sum_{t=1}^{T} \hat{z}_t \mathbb{1}\{\hat{z}_t < \kappa_q\} / \sum_{t=1}^{T} \mathbb{1}\{\hat{z}_t < \kappa_q\}$.

3. Obtain the point estimates of VaR_q and ES_q by $\widehat{\mathrm{VaR}}_q = \hat{\sigma}_T \kappa_q$ and $\widehat{\mathrm{ES}}_q = \hat{\sigma}_T \eta_q$.

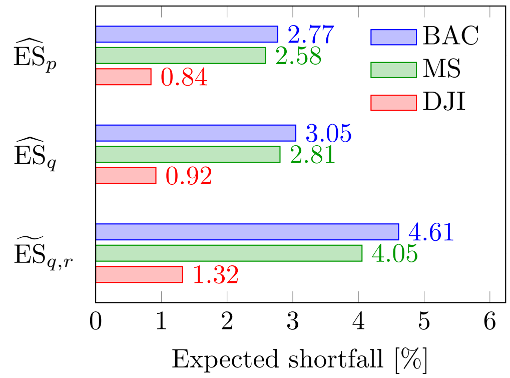
4. Estimate $\mathrm{Var}(\widehat{\mathrm{VaR}}_q)$ and $\mathrm{Var}(\widehat{\mathrm{ES}}_q)$ using analytical formulae provided in Gao and Song (2008, Theorems 3.2 and 3.3).

5. Construct the one-sided $(1-r)$-confidence bounds $\widetilde{\mathrm{VaR}}_{q,r}$ and $\widetilde{\mathrm{ES}}_{q,r}$, which are our proposed risk estimates.

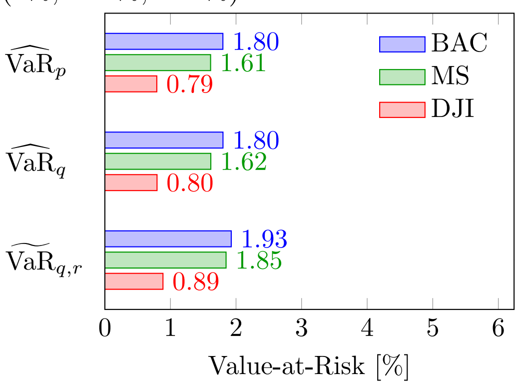
Figure 5 shows the estimates of VaR and ES, which are also summarized in Table 1. Figures 5a and 5b use the data from November 1, 2016 to December 31, 2017, consisting of $T = 293$ daily returns; Figures 5c and 5d from February 23, 1993 to December 31, 2017, totaling $T = 6{,}260$ observations. The top three bars in Figure 5a show the "naive" estimates of VaR of coverage $1-p$, which have no guarantee on the actual risk probability as discussed in Section 2. The next three bars are the interim estimates of VaR of coverage $1-q$. The last three bars are the upper bounds of the one-sided $(1-r)$-confidence intervals of VaR_q; they satisfy the risk guarantee that the probability of the next loss going above is less than p.



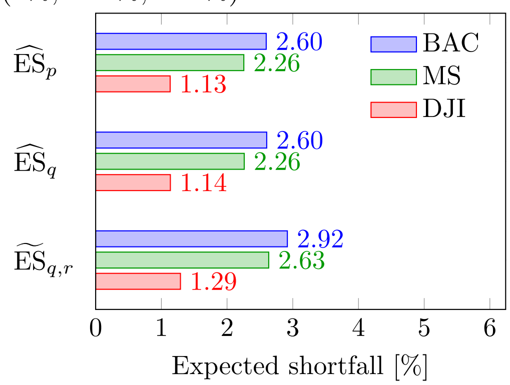
(a) Estimated VaR for $(p, q, r) = (5\%, 3.98\%, 0.34\%)$. Uses the short data.



(b) Estimated ES for $(p, q, r) = (5\%, 3.98\%, 0.34\%)$. Uses the short data.



(c) Estimated VaR for $(p, q, r) = (5\%, 4.95\%, 0.016\%)$. Uses the long data.



(d) Estimated ES for $(p, q, r) = (5\%, 4.95\%, 0.016\%)$. Uses the long data.

Figure 5: Estimated VaR and ES of daily returns for $(p, q, r) = (0.05, p - 3/T, 1/T)$. The top figures use the data from Nov. 1, 2016 to Dec. 31, 2017 ($T = 293$), and the bottom from Feb. 23, 1993 to Dec. 31, 2017 ($T = 6{,}260$).

The numbers in parentheses in Table 1 are the multipliers relative to the "naive" estimates; in columns (1–3) they are ratios relative to column (1), and in columns (4–6) ratios relative to column (4); for example, in column (3), $\widetilde{\mathrm{VaR}}_{q,r}/\widehat{\mathrm{VaR}}_p$, and in column (6), $\widetilde{\mathrm{ES}}_{q,r}/\widehat{\mathrm{ES}}_p$. The VaR has overall low multipliers, meaning that estimation error can be addressed without being too conservative. The ES has bigger multipliers, but the magnitudes are still comparable to the multipliers needed for model risk.[10] Whenever we can construct a confidence interval, we advocate the use of our method since the multipliers heavily depend on the sample size. When statistical inference is not available, using a fixed multiplier of 2 may be practically reasonable to address

[10] The recommended multiplier for model risk is 3 for VaR (Stahl, 1997) and 1.5 for ES (Leippold and Vanini, 2002).

Table 1: Estimates of 95% VaR and ES in the percentage units. The numbers in parentheses are the multipliers relative to $\widehat{\mathrm{VaR}}_p$ and $\widehat{\mathrm{ES}}_p$.

	(1) $\widehat{\mathrm{VaR}}_p$ [%]	(2) $\widehat{\mathrm{VaR}}_q$ [%]	(3) $\widetilde{\mathrm{VaR}}_{q,r}$ [%]	(4) $\widehat{\mathrm{ES}}_p$ [%]	(5) $\widehat{\mathrm{ES}}_q$ [%]	(6) $\widetilde{\mathrm{ES}}_{q,r}$ [%]
Nov. 1, 2016 to Dec. 31, 2017 ($T = 293$)						
BAC	1.649	1.762	2.047	2.772	3.046	4.612
	(1.000)	(1.069)	(1.241)	(1.000)	(1.099)	(1.664)
MS	1.369	1.466	2.009	2.584	2.806	4.054
	(1.000)	(1.071)	(1.468)	(1.000)	(1.086)	(1.569)
DJI	0.494	0.520	0.646	0.842	0.917	1.322
	(1.000)	(1.052)	(1.308)	(1.000)	(1.089)	(1.570)
Feb. 24, 1993 to Dec. 31, 2017 ($T = 6{,}260$)						
BAC	1.800	1.801	1.927	2.596	2.604	2.917
	(1.000)	(1.001)	(1.071)	(1.000)	(1.003)	(1.124)
MS	1.612	1.616	1.848	2.255	2.261	2.634
	(1.000)	(1.002)	(1.146)	(1.000)	(1.003)	(1.168)
DJI	0.792	0.796	0.886	1.134	1.137	1.291
	(1.000)	(1.004)	(1.119)	(1.000)	(1.003)	(1.138)

estimation error of VaR and ES.

With the above results, we may control the risk of the entire portfolio. Assuming that VaR is subadditive (Ibragimov, 2005, p. 25; McNeil et al., 2005, Theorem 6.8), we can bound the VaR of the entire portfolio of coverage $1 - p$ by

$$w_1 2.05\% + w_2 2.01\% + w_3 0.65\%,$$

where $(2.05\%, 2.01\%, 0.65\%)$ are the $\widetilde{\mathrm{VaR}}_{q,r}$ from Figure 5a and (w_1, w_2, w_3) are weights of the portfolio on BAC, MS, and DJI. Note that even without the assumption of subadditivity, we can directly compute the bound by applying the same exercise to the transformed historical data $\{w_1 X_{\mathrm{BAC}} + w_2 X_{\mathrm{MS}} + w_3 X_{\mathrm{DJI}}\}$. The advantage of this is that it will not be conservative. If we use subadditivity, it eliminates the need to re-estimate the risk as we consider other weights, but at the expense of being more conservative than necessary.

Since ES is subadditive by construction (McNeil et al., 2005, Proposition 6.9), we can bound the ES of our portfolio of coverage 95% by

$$w_1 4.61\% + w_2 4.05\% + w_3 1.32\%,$$

where the numbers come from the $\widetilde{\mathrm{ES}}_{q,r}$ in Figure 5b.

We may also allow for short positions (negative weights). Observe that the risk associated with shorting X_t is equal to the risk of longing $-X_t$. Then, we can apply the same method with the risk estimates on the other tail.

5 Conclusion

Many risk measures are motivated by a certain guarantee on the probability of a ruin. Estimation error involved in risk assessment is one of many factors that impair the intended risk guarantee (Section 2). We addressed this issue and proposed a method to recover the intended risk probability guarantee with an observable risk estimate, namely the confidence interval of the risk measure.

We characterized the class of risk measures to which our method can be applied, and named them the *tail risk measure* (Section 3.1). We showed that the class contains VaR and ES, and discussed that it can be extended to various other tail risks (Section 3.2). Our method of risk control is based on the Bonferroni inequality and hence is robust against arbitrary correlation between the risk variable X and the risk estimate. We also provided a recommendation of the tuning parameters (q, r) so that our risk estimate is consistent (Section 3.3).

In the empirical application, we applied our method to the VaR and ES estimation for an arbitrary portfolio of three assets (Section 4). We found in our setup that our proposed risk estimates on VaR are generally larger than the naive estimates by 10–50%, and those on ES by 20–70%. These are modest inflation compared to multipliers for model risk. This demonstrated that our method produces risk estimates that are practically not too conservative.

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