MTL104 : Linear Algebra	**Spring 2020-21**

Lecture 2

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2.1 Definitions

2.1.1 Subfield

- A subfield of a field K is a subset L of K that is a field with respect to the field operations inherited from K.
- Example : $\mathbb{R}$ is a subfield of $\mathbb{C}$

2.1.2 Characteristic of a field

- If F is a field, it may be possible to add the unit element($\mathbf{1}$) to itself a finite number of times to obtain $\mathbf{0}$. That is,
$$\mathbf{1} + \mathbf{1} + \mathbf{1} + ... + \mathbf{1} = \mathbf{0}$$
- This is not possible in the field of complex numbers.
- In such cases where it is not possible to obtain $\mathbf{0}$ by adding $\mathbf{1}$ a finite number of times then the field F is a field of *characteristic zero*.
- Otherwise, the least n such that adding $\mathbf{1}$ n times results in $\mathbf{0}$ is called the *characteristic* of the field.

2.1.3 Ring

- A ring is a set R with two binary operations $+$, $\cdot$ such that :
  1. Both operations are closed.
  2. R is *abelian* under addition.
  3. Multiplication is distributed over addition on both left and right.
  4. Multiplication is associative
- Note that $\mathbf{1}$ might not be an element of the ring.
- In case $\mathbf{1} \in R$, then R is called a ring with unity.
- A commutative division ring is called a field.

2.1.4 Finite Fields

- If the number of elements in a field is finite, then the field is called a finite field.
- Example : If $\mathbb{Z}_n = \{x; 0 \leq x < n\}$, then Z_p is a finite field if p is a prime number.
- Also note that Z_4 is not a field.

2.2 Vector Spaces

2.2.1 Definition

- Elements of a field are called scalars.
- $(V, +, \cdot)$ is called a vector space over a field K if :
- $\cdot : V \times V \to V$ and $+ : V \times V \to V$ exist, such that :
  1. $(V, +)$ is an abelian group.
  2. $\alpha \in F$ and $x, y \in V$, then : $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$
  3. $\alpha, \beta \in F$ and $x \in V$, then : $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$
  4. $\alpha, \beta \in F$ and $x \in V$, then : $(\alpha\beta) \cdot x = \alpha(\beta \cdot x)$
  5. $\forall x \in V$, $\mathbf{1} \cdot x = x$
- Example : n-tuple space

2.2.2 Example

- Take n-tuple space as an example.
- Let V be the set of all ordered n-tuples of elements of any field F for a fixed integer n. That is,

$$V = \{(a_1, a_2, ..., a_n) : a_i \in F\}$$

- Then V is a vector space over F, with the following $\cdot$ and $+$:
  1. Let $x = (a_1, a_2, ..., a_n)$ and $y = (b_1, b_2, ..., b_n)$
  2. $x + y = (a_1 + b_1, a_2 + b_2, ..., a_n + b_n)$ (Addition)
  3. $\alpha x = (\alpha a_1, \alpha a_2, ..., \alpha a_n)$ (Scalar Multiplication)
  4. $x = y$ iff $\forall i \in \{1, 2, ..., n\}, a_i = b_i$

2.2.3 Properties

1. A vector space over a field K can be regarded as a vector space over any of it's subfield(S) of F
2. F(F) is a vector space over any field F.
   - $\mathbb{R}$ is not a vector space over $\mathbb{C}$ as it is not closed under scalar multiplication.
3. Set f(x) of polynomials over a field F is a vector space. (With conventional addition and multiplication)

4. The set of all convergent sequences is a vector space over the field of real numbers.
5. The set of all finite matrices with real elements is a vector space over real numbers
6. Let K be an arbitrary field. Let X be any non-empty set. Consider the set V of all funcitons from X to K. The sum of any two functions $f, g \in V$ is the function $f + g \in V$ defined by :

   $$(f + g)(x) = f(x) + g(x)$$

   Where the scalar product with $\alpha \in K$, $f \in V$, $\alpha f \in V$ is defined by :

   $$(\alpha f)(x) = \alpha f(x)$$

   is a vector space over the field K.