MTL104 : Linear Algebra **Spring 2020-21**

Lecture 4

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4.1 Subspaces

4.1.1 Examples of subspaces

- V(F) is also a subspace, so is $\{0\}$.
- Let V be the vector space $\mathbb{R}^3$. Then the set W consisting of those vectors whose third component is zero, i.e. $w = \{a, b, 0 : a, b \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$
- Let V be the vector space of all $n \times n$ matrices. Then the set W consisting of these matrices $A = [a_{ij}]$ for which $a_{ji} = a_{ij}$ (Symmetric matrices) is a subspace of V.
- Let V be the vector space of polynomials. Then the set W consisting of polynomials with degree $\leq n$, for a fixed n, is a subspace of V.
- Let V be the vector space of all functions for a non-empty set X into the real field $\mathbb{R}$. Then the set consisting of all bounded functions in V is a subspace of V.
  - A function $f \in V$ is bounded iff $\exists M \in R$ such that $|f(x)| \leq M$
- Let S be a non-empty subset of V(F), The set of all linear comination of vectors in S, denoted by $L(S)$, is a subspace of V containing S.
- Furthermore, if W is any other subspace of V containing S, then $L(S) \subseteq W$.
- The solution space of a system of linear equations $\subseteq R^{n\times 1}$
- In F^n, the set of all n-tuples $(x_1, x_2, ..., x_n)$ with $x_1 = 0$ is a subspace.
- The set of all hermitian matrices is **NOT** a subspace of the space of all $n \times n$ matrices over $\mathbb{C}$. The set of all $n \times n$ complex hermitian matrices is a vector space over the field of real numbers.

4.1.2 Few Properties of subspaces

1. Suppose W_1 and W_2 are subspaces of a vector space V(F), Then
   - $W_1 \cup W_2$ need not be a subspace of V
     - For example. Consider the vector space of $\mathbb{R}^2$, with $W_1 = \{(a, 0); a \in \mathbb{R}\}$ and $W_2 = \{(0, b); b \in (R)\}$. Then $(1, 0) \in W_1$ and $(0, 1) \in W_2$, but their addition $(1, 0) + (0, 1) = (1, 1) \notin W_1 \cup W_2$.

    - Hence addition is not closed, hence $W_1 \cup W_2$ is not a subspace of V
  - $W_1 \cap W_2$ is a subspace of V. Proof :
    - $\mathbf{0} \in W_1$, $\mathbf{0} \in W_2 \implies \mathbf{0} \in W_1 \cap W_2$
    - Let $x, y \in W_1 \cap W_2$, $\alpha \in F$,
    - Now $x \in W_1 \cap W_2 \implies x \in W_1$ and $x \in W_2$
    - Similarly $y \in W_1 \cap W_2 \implies y \in W_1$ and $y \in W_2$
    - Since W_1 is a subspace, $\alpha(x+y) \in W_1$
    - Similarly W_2 is a subspace, so $\alpha(x+y) \in W_2$
    - Therefor $\alpha(x+y) \in W_1 \cap W_2$

2. Let V be a vector space over the field F, then intersection of any collection of subspaces (i.e. Arbitrary intersection of subspaces) is a subspace in V.

4.1.3 Linear Sum of Subspaces

- Let W_1 and W_2 be two subspaces of a vector space V(F), Then the linear sum of subspaces, denoted by $W_1 + W_2$ is defined as :

$$W_1 + W_2 = \{x_1 + x_2 : x_1 \in W_1, x_2 \in W_2\}$$

- **Theorem :** If W_1 and W_2 are subspaces of a vector space V(F), then $W_1 + W_2$ is a subspace of V(F)
- Proof :
  - $\mathbf{0} = \mathbf{0} + \mathbf{0}$ where $\mathbf{0} \in W_1$ and $\mathbf{0} \in W_2$
  - let $x, y \in W_1 + W_2$, $\alpha \in F$, then :
    - $x = x_1 + x_2$ where $x_1 \in W_1$ and $x_2 \in W_2$
    - $y = y_1 + y_2$ where $y_1 \in W_1$ and $y_2 \in W_2$
  - Consider $\alpha x + y = \alpha(x_1 + x_2) + (y_1 + y_2) = (\alpha x_1 + y_1) + (\alpha x_2 + y_2)$
    - Now $(\alpha x_1 + y_1) \in W_1$ as W_1 is a subspace
    - Similarly $(\alpha x_2 + y_2) \in W_2$ as W_2 is a subspace
    - Therefore $\alpha x + y \in W_1 + W_2$
  - Therefore $W_1 + W_2$ is a subspace of V
- If $W_1, W_2, ..., W_k$ are subspaces of the vector space V(F). Then their linear sum, denote by $W_1 + W_2 + ... + W_k$ is defined as :

$$W_1 + W_2 + ... + W_k = x_1 + x_2 + ... + x_k : x_i \in W_i, 1 \leq i \leq k$$

- $W_1 + W_2 + ... + W_k$ is a subspace of V(F)

4.1.4 Direct Sum of Subspaces

- The vector space V over the field F is the direct sum of two vector spaces W_1 and W_2 if
  1. $V = W_1 + W_2$
  2. Every vector $z\ inV$ can be uniquely expressed as the sum of $x_1 + x_2$ where $x_1 \in W_1$ and $x_2 \in W_2$
- If the vector space V(F) is the direct sum of two subspaces, W_1 and W_2, we write,

$$V = W_1 \oplus W_2$$

- **Theorem :** $V = W_1 \oplus W_2 \iff V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$