

Lecture 3

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3.1 Linear Combinations and Independence

3.1.1 Linear Independence

- A finite set $\{x_1, x_2, \dots, x_n\}$ of vectors in V is called linearly independent iff :

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \dots + \alpha_n x_n = \mathbf{0} \implies \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$$

- Let $X \subseteq V$, X is said to be linearly independent if every finite subset of X is linearly independent.

3.1.2 Properties of Linear Independence

- A set of vectors which contains $\mathbf{0}$ as one of its elements is linearly dependent.
- The set $\{x\}$ is linearly independent iff $x \neq 0$
- Any subset of a linearly independent set is linearly independent.
- Any superset of a linearly dependent set is linearly dependent.
- If $\{x_1, x_2, \dots, x_n\}$ is linearly independent and if $\{a_1, a_2, \dots, a_n\} \in F$ and $\{b_1, b_2, \dots, b_n\} \in F$, then

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + \dots + a_n x_n = b_1 x_1 + b_2 x_2 + b_3 x_3 + \dots + b_n x_n \implies a_1 = b_1, a_2 = b_2, \dots, a_n = b_n$$

- **Lemma :** The set of non-zero vectors $\{x_1, x_2, \dots, x_n\} \in V$ is linearly dependent if and only if any one of them, say x_j can be expressed as a *linear combination* of the other vectors.

3.1.3 Linear Combination of vectors

- If $V(F)$ is a vector space and $\{x_1, x_2, \dots, x_n\} \in V$ and $\{\alpha_1, \alpha_2, \dots, \alpha_n\} \in F$, then the vector $x = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = \sum_{i=1}^n (\alpha_i x_i)$ is called the linear combination of vectors $\{x_1, x_2, \dots, x_n\}$

3.1.4 Linear Span of a set

- The linear span of a non-empty subset S of $V(F)$ is the set of all linear combinations of any finite number of elements of S and is denoted by $L(S)$ (Some textbooks denote it as $\langle S \rangle$)

3.1.5 Basis of a set

- If S is a subset of a vector space $V(F)$ such that :
 1. S is a linearly independent set of vectors of $V(F)$
 2. Every vector v in $V(F)$ is a linear combination of elements of S or in other words $L(S) = V$, then S is called the basis of $V(F)$.
- Zero vector ($\mathbf{0}$) cannot be an element of any basis of $V(F)$

3.1.6 Finitely Generated Vector Spaces

- A vector space $V(F)$ is said to be finitely generated if there exists a finite subset S of V such that $V = L(S)$ or if $V(S)$ has a finite spanning set.
- Equivalently a vector space is finitely generated if it's basis is finite.
- In a finitely generate vector space $V(F)$, whose basis set is $B = x_1, x_2, \dots, x_n$, every vector $x \in V$ is uniquely expressible as a linear combination of vectors in B .
- **Existence Theorem :** There exists a basis for each finite dimensional vector space.

3.1.7 Dimension of a vector space

- The dimension of a finitely generated vector space $V(F)$ is the cardinality of any basis of $V(F)$ and is represented as $\dim(V)$.
- **Extension Theorem :** If $V(F)$ is a finitely generated vector space, and $A = x_1, x_2, \dots, x_m$ is any linearly independent set of vectors in V , then unless the x_i s already form a basis, we can find the vectors $\{y_1, y_2, \dots, y_{n-m}\}$ so that the extended set of n vectors $\{x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_{n-m}\}$ is a basis.
- In a finitely generated vector space, the cardinality of every basis is the same and is equal to the dimension of the vector space.

3.1.8 Subspace of a vector space

- A subset W of a vector space V over a field F is called a subspace of V if W is a vector space over F with the same operations of addition and multiplication as defined for V .
- **Theorem :** Let V be a vector space and W be a subset of V . Then W is a subspace of V if and only if the following three conditions hold for operations defined in V .
 1. $\mathbf{0} \in W$
 2. $x + y \in W$ whenever $x \in W$ and $y \in W$
 3. $\alpha x \in W$ whenever $x \in W$ and $\alpha \in F$