MTL104 : Linear Algebra **Spring 2020-21**

Lecture 1

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1.1 Definitions

1.1.1 Cartesian Product

- Suppose G is a non-empty set, Then :

$$G \times G = \{(a, b); a \in G, b \in G\}$$

1.1.2 Binary Operation

- If $f : G \times G \to G$, then f is said to be a binary operation on the set G
- We often use the symbols $+, \times, \cdot, \circ$, etc to denote binary operations.
- For e.g., '+' is a binary operation in G only iff

$$\forall a, b \in G,\ a + b \in G \text{ and a+b is unique}$$

1.1.3 Conventions

- $\mathbb{N}$ - Set of Natural Numbers
- $\mathbb{Z}$ - Set of Integers
- $\mathbb{Q}$ - Set of Rational Numbers
- $\mathbb{R}$ - Set of Real Numbers
- $\mathbb{C}$ - Set of Complex Numbers

1.1.4 Algebraic Structure or Algebraic System

- A non-empty set G equipped with one or more binary operations is called an algebraic structure.
- Suppose * is a binary operation on G, then $(G, *)$ is an algebraic structure.
- E.g.
  - $(\mathbb{N}, +)$
  - $(\mathbb{R}, +, \cdot)$

1.1.5 Group

- Suppose S is a non-empty set and let * be a binary operation defined on S.
- i.e. $* : S \times S \to S$
- We say $(S, *)$ is a group if it satisfies the following properties :
  - $\forall a, b, c \in S,\ a * (b * c) = (a * b) * c$
  - $\exists z \in S$ such that, $\forall a \in S, a * z = z * a = a$ (Identity)
  - $\forall a \in S,\ \exists a^{-1} \in S$such that, $a * a^{-1} = a^{-1} * a = z$ (Inverse)

1.1.6 Abelian/Commutative Group

- If $(S, *)$ is a group such that $\forall a, b \in S,\ a * b = b * a$ (* is Commutative), then $(S, *)$ is called an Abelian group or Commutative Group.

1.1.7 Field

- Suppose F is a non-empty set equipped with two binary operations called addition and multiplication, denoted by '+' and '$\cdot$', respectively.
- That is, $\forall a, b \in F$, we have : $a + b \in F$ and $a \cdot b \in F$.
- Then the algebraic structure $(F, +, \cdot)$ is called a field, if the following properties are satisfied :
  1. Addition is commutative. i.e. $\forall a, b \in F,\ a + b = b + a$
  2. Addition is associative. i.e. $\forall a, b, c \in F,\ a + (b + c) = (a + b) + c$
  3. $\exists \mathbf{0} \in F$ (called zero), such that $\forall a \in F,\ a + \mathbf{0} = \mathbf{0} + a = a$
  4. $\forall a \in F,\ \exists(-a) \in F$, such that : $a + (-a) = \mathbf{0}$
  5. Multiplication is commutative. i.e. $\forall a, b \in F,\ a \cdot b = b \cdot a$
  6. Multiplication is associative. i.e. $\forall a, b, c \in F,\ a \cdot (b \cdot c) = (a \cdot b) \cdot c$
  7. $\exists \mathbf{1} \in F$ (called zero), such that $\forall a \in F,\ a \cdot \mathbf{1} = \mathbf{1} \cdot a = a$
  8. $\forall a \in F,\ \exists a^{-1} \in F$, such that : $a \cdot a^{-1} = \mathbf{1}$
  9. Multiplication Distributes over addition, i.e. $\forall a, b, c \in F,\ a \cdot (b + c) = a \cdot b + a \cdot c$ (left distribution) and $\forall a, b, c \in F,\ (a + b) \cdot c = a \cdot c + b \cdot c$ (right distribution)
- Notice that property 1-4 essentially states that $(F, +)$ is abelian. Similarly properties 5-8 states that $(F, *)$ is abelian.
- Note that $\mathbf{0}$ is called that Zero element of the field(F) and $\mathbf{1}$ is called the Unity element of the field(F).
- Equivalently, $(F, +, \cdot)$ is a field iff
  1. (F, $+$) is an abelian group.
  2. (F, $\cdot$) is an abelian group.
  3. Addition and Multiplication are linked by distributive property for both left and right distribution.
- Equivalently, A commutative division ring is a field.