MTL104 : Linear Algebra **Spring 2020-21**

Lecture 5

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5.1 Direct Sum

5.1.1 Direct sum of subspaces

Theorem : $V = W_1 \oplus W_2 \iff V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$

Proof($\implies$) :

- Suppose $V = W_1 \oplus W_2$
- $V = W_1 \oplus W_2 \implies V = W_1 + W_2$
- To prove : $W_1 \cap W_2 = \{0\}$
- Let $z = W_1 \cap W_2$ where $z \neq 0$
- Then : $z = \mathbf{0} + z = z + \mathbf{0}$ where blue elements belong to W_1 and green elements belong to W_2
- As z has been expressed in two ways, therefore the representation is not unique.
- Therefore z has to be $\mathbf{0}$. Hence proved.

Proof($\impliedby$):

- Suppose $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$
- To Prove : $V = W_1 \oplus W_2$
- let $z \in V$.
- Let blue represent the fact that the element belongs to W_1 and green elements belong to W_2
- Suppose the following two representations of z exist:
  - $z = x_1 + y_1$
  - $z = x_2 + y_2$
- Then $x_1 + y_1 = x_2 + y_2 = z$
- $x_1 - x_2 = y_1 - y_2$

- Clearly LHS = RHS
- But as $W_1 \cap W_2 = \{0\}$
- Therefore, $x_1 - x_2 = 0$ and $y_1 - y_2 = 0$
- Therefore $x_1 = x_2$ and $y_1 = y_2$
- Therefore each $z \in V$ is uniquely represented as $x + y$
- Therefore $V = W_1 \oplus W_2$

5.1.2 Extended direct sum

- A vector space V(F) is said to be direct sum of it's subspaces $W_1, W_2, ..., W_k$ i.e. $V = W_1 \oplus W_2 \oplus ... \oplus W_k$ if and only if a vector $z \in V$ is uniquely expressible as $z = x_1 + x_2 + ... + x_k$ with $x_i \in W_i$
- $V = W_1 \oplus W_2 \oplus W_3 \oplus ... \oplus W_k$ Iff and only if :
  1. $V = W_1 + W_2 + ... + W_k$
  2. $W_i \cap (W_1 + W_2 + ... + W_{i-1} + W_{i+1} + ... + W_k) = \{0\}$ for each $i = 1, 2, .., k$

5.1.3 Complementary Subspaces

- If $V = W_1 \oplus W_2 (\implies V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\})$
- In such a case W_1 and W_2 are said to be complementary subspaces.
- Complementary subspaces are **NOT** unique.
- For example : Consider the vector space $\mathbb{R}^3(R)$, then the following are it's subspaces :
  - $M_1 = \{(0, x_2, x_3)\}$ (yz-plane), $N_1 = \{(x_1, 0, 0)\}$ (x-axis)
  - $M_2 = \{(x_1, 0, x_3)\}$ (xz-plane), $N_2 = \{(0, x_2, 0)\}$ (y-axis)
  - $M_3 = \{(x_1, x_2, 0)\}$ (xy-plane), $N_3 = \{(0, 0, x_3)\}$ (z-axis)
  - $V = M_1 \oplus N_1 = M_2 \oplus N_2 = M_3 \oplus N_3$
- **Theorem:** Corresponding to a given sub-space, there will be a unique complementary subspace.
- Proof :
  - Let W_1 be a given subspace and W_2 and W_3 be it's complementary subspaces so that $V_1 = W_1 \oplus W_2 = W_1 \oplus W_3$
  - Now let $z = x_1 + x_2$ where $x_1 \in W_1$ and $x_2 \in W_2$
  - Also let $z = x_1 + x_3$ where $x_1 \in W_1$ and $x_3 \in W_3$
  - $x_1 + x_2 = x_1 + x_3$
  - $x_2 = x_3$
  - Therefore $W_2 = W_3$

5.1.4 Dimension of a subspace

- Each subspace W of a finitely generated vector space V(F) of dimension n is a finite dimensional subspace with dimension $\leq n$
$$dim(W) \leq dim(V)$$
- $dim(W) = dim(U) \iff W = V$
- If W_1 and W_2 are subspaces of a finite dimension vector space V(F), then :
$$dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2)$$
- $V = W_1 \oplus W_2 \implies dim(V) = dim(W_1) + dim(W_2)$

5.1.5 Quotient Space

- Let W be any subspace of a vector space V(F). Let x be any vector in V. Then the set
$$W + x = \{w + x : w \in W\}$$
is called a coset of W in V generated by x.
$$\frac{V}{W} = \{W + x : x \in V\} = \text{ set of all cosets of W in V}$$
- $\mathbf{0} \in V$ and $W + \mathbf{0} = W$, therefore W is itself a coset of W in V
- If $x \in W$, then $W + x = W$(why?)
- If $W + x$ and $W + y$ are two cosets of W in V, then
$$W + x = W + y \iff x - y \in W$$
- Proof ($\implies$)
  - Since $\mathbf{0} \in W \implies \mathbf{0} + x \in W + x$, thus $x \in W + x$
  - Now $W + x = W + y \implies x \in W + y$
  - $x - y \in W + (y - y)$
  - $x - y \in W + \mathbf{0}$
- Proof ($\impliedby$)
  - $x - y \in W \implies W + x - y = W$
  - $W + (x - y + y) = W + y$
  - $W + x = W + y$
- **Theorem :** If W is a subspace of a vector space V(F), then set $\frac{V}{W}$ of all cosets $W + x$ where x is any arbitrary element of V, is a vector space over F, under the operations defined by :
  - **Vector Addition :** $\forall x, y \in V$, $(W + x) + (W + y) = W + x + y$
  - **Scalar Multiplication :** $\alpha(W + x) = W + \alpha x$

- $$dim\left(\frac{V}{W}\right) = dim(V) - dim(W)$$
  - Let m be the dimension of the subspace W of the vector space V(F)
  - Let $S_1 = \{x_1, x_2, ..., x_m\}$ be a basis of W.
  - Since S_1 is a linearly independent subset of V, it can be extended to form a basis of V.
  - Let $S_2 = \{x_1, x_2, ..., x_m, y_1, y_2, ..., y_n\}$
  - $dim(V) = m + n$, $dim(W) = m$
  - $dim(V) - dim(W) = m + n - m = n$
  - Therefore We have $dim\left(\frac{V}{W}\right) = n$
  - Claim : $S_3 = \{W + y_1, W + y_2, ..., W + y_n\}$ is a basis of $\frac{V}{W}$.
  - Prove that $L(S_3) = \frac{V}{W}$ and that S_3 is linearly independent.
  - To prove linear independence of S_3,
    * $\alpha_1(W + y_1) + \alpha_2(W + y_2) + \alpha_3(W + y_3) + ... + \alpha_n(W + y_n) = \mathbf{0}_{\frac{V}{W}}$
    * $W + (\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 + ... + \alpha_n y_n) = W + \mathbf{0}$
    * $\alpha_1 y_1 + \alpha_2 y_2 + ... + \alpha_n y_n \in W$
    * $\sum_{i=1}^{n} \alpha_i y_i =$ L.C. of S_1 (Any vector in W can be expressed as a L.C. of basis vectors.)
    * $\sum_{i=1}^{n} \alpha_i y_i = \sum_{i=1}^{m} \beta_i x_i$
    * $\sum_{i=1}^{n} \alpha_i y_i - \sum_{i=1}^{m} \beta_i x_i = 0$
    * $\alpha_i = 0$ and $\beta_i = 0$
  - To prove $L(S_3) = \frac{V}{W}$
    * Let $W + x \in \frac{V}{W}$
    * $x \in V \implies x = \sum_{i=1}^{n} \alpha_i x_i + \sum_{i=1}^{m} \beta_i y_i$
    * $x = z + \sum_{i=1}^{m} \beta_i y_i$ where $z = \sum_{i=1}^{n} \alpha_i x_i$
    * $W + x = W + (z + \sum_{i=1}^{m} \beta_i y_i)$
    * $W + x = W + \sum_{i=1}^{m} \beta_i y_i$
    * $W + x = (W + \beta_1 y_1) + (W + \beta_2 y_2) + ... + (W + \beta_m y_m)$
    * $W + x =$ L.C. of S_3
    * Therefore $L(S_3) = \frac{V}{W}$
  - S_2 is a basis of V
  - To prove $L(S_3) = \frac{V}{W}$
- $x, y \in V \implies x + y \in V$
- Also $\alpha \in F$ and $x \in V \implies \alpha x \in V$
- Therefore $W + (x + y) \in \frac{V}{W}$ and also $W + \alpha x \in \frac{V}{W}$
- Thus $\frac{V}{W}$ is closed with respect to addition of cosets and scalar multiplication of cosets.
- Zero element of $\frac{V}{W}$ is $W + \mathbf{0} = W$
- $W - x$ is the inverse of $W + x$

H.W. Prove $\frac{V}{W}$ is a vector space.