MTL104 : Linear Algebra **Spring 2020-21**

Lecture 3

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3.1 Linear Combinations and Independence

3.1.1 Linear Independence

- A finite set $\{x_1, x_2, ..., x_n\}$ of vectors in V is called linearly independent iff :

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 ... + \alpha_n x_n = \mathbf{0} \implies \alpha_1 = \alpha_2 = ... = \alpha_n = 0$$

- Let $X \subseteq V$, X is said to be linearly independent if every finite subset of X is linearly independent.

3.1.2 Properties of Linear Independence

- A set of vectors with contains $\mathbf{0}$ as one of it's element is linearly dependent.
- The set $\{x\}$ is linearly independent iff $x \neq 0$
- Any subset of a linearly independent set is linearly independent.
- Any superset of a linearly dependent set is linearly dependent.
- If $\{x_1, x_2, ..., x_n\}$ is linearly independent and if $\{a_1, a_2, ..., a_n\} \in F$ and $\{b_1, b_2, ..., b_n\} \in F$, then

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + ...a_n x_n = b_1 x_1 + b_2 x_2 + b_3 x_3 + ... + b_n x_n \implies a_1 = b_1, a_2 = b_2, ..., a_n = b_n$$

- **Lemma :** The set of non-zero vectors $\{x_1, x_2, ..., x_n\} \in V$ is linearly dependent if and only if any one of them, say x_j can be expressed as a *linear combination* of the other vectors.

3.1.3 Linear Combination of vectors

- If V(F) is a vector space and $\{x_1, x_2, ..., x_n\} \in V$ and $\{\alpha_1, \alpha_2, ..., \alpha_n\} \in F$, then the vector $x = \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n = \sum_{i=1}^{n}(\alpha_i x_i)$ is called the linear combination of vectors $\{x_1, x_2, ..., x_n\}$

3.1.4 Linear Span of a set

- The linear span of a non-empty subset S of V(F) is the set of all linear combinations of any finite number of elements of S and is denoted by $L(S)$ (Some textbooks denote it as $< S >$)

3.1.5 Basis of a set

- If S is a subset of a vector space V(F) such that :
    1. S is a linearly independent set of vectors of V(F)
    2. Every vector $v\ inV(F)$ is a linear combination of elements of S or in other words $L(S) = V$, then S is called the basis of V(F).
- Zero vector ($\mathbf{0}$) cannot be an element of any basis of V(F)

3.1.6 Finitely Generated Vector Spaces

- A vector space V(F) is said to be finitely generated if there exists a finite subset S of V such that $V = L(S)$ or if $V(S)$ has a finite spanning set.
- Equivalently a vector space is finitely generated if it's basis is finite.
- In a finitely generate vector space V(F), whose basis set is $B = x_1, x_2, ..., x_n$, every vector $x \in V$ is uniquely expressible as a linear combination of vectors in B.
- **Existence Theorem :** There exists a basis for each finite dimensional vector space.

3.1.7 Dimension of a vector space

- The dimension of a finitely generated vector space V(F) is the cardinality of any basis of V(F) and is represented as $dim(V)$.
- **Extension Theorem :** If V(F) is a finitely generated vector space, and $A = x_1, x_2, ..., x_m$ is any linearly independent set of vectors in V, then unless the $x_i s$ already form a basis, we can find the vectors $\{y_1, y_2, ..., y_{n-m}\}$ so that the extended set of n vectors $\{x_1, x_2, ..., x_m, y_1, y_2, ..., y_{n-m}\}$ is a basis.
- In a finitely generated vector space, the cardinality of every basis is the same and is equal to the dimension of the vector space.

3.1.8 Subspace of a vector space

- A subset W of a vector space V over a field F is called a subspace of V if W is a vector space over F with the same operations of addition and multiplication as defined for V.
- **Theorem :** Let V be a vector space and W be a subset of V. Then W is a subspace of V if and only if the following three conditions hold for operations defined in V.
    1. $\mathbf{0} \in W$
    2. $x + y \in W$ whenever $x \in W$ and $y \in W$
    3. $\alpha x \in W$ whenever $x \in W$ and $\alpha \in F$