

Lecture 1

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1.1 Definitions

1.1.1 Cartesian Product

- Suppose G is a non-empty set, Then :

$$G \times G = \{(a, b); a \in G, b \in G\}$$

1.1.2 Binary Operation

- If $f : G \times G \rightarrow G$, then f is said to be a binary operation on the set G
- We often use the symbols $+$, \times , \cdot , \circ , etc to denote binary operations.
- For e.g., '+' is a binary operation in G only iff

$$\forall a, b \in G, a + b \in G \text{ and } a+b \text{ is unique}$$

1.1.3 Conventions

- \mathbb{N} - Set of Natural Numbers
- \mathbb{Z} - Set of Integers
- \mathbb{Q} - Set of Rational Numbers
- \mathbb{R} - Set of Real Numbers
- \mathbb{C} - Set of Complex Numbers

1.1.4 Algebraic Structure or Algebraic System

- A non-empty set G equipped with one or more binary operations is called an algebraic structure.
- Suppose $*$ is a binary operation on G , then $(G, *)$ is an algebraic structure.
- E.g.
 - $(\mathbb{N}, +)$
 - $(\mathbb{R}, +, \cdot)$

1.1.5 Group

- Suppose S is a non-empty set and let $*$ be a binary operation defined on S .
- i.e. $*$: $S \times S \rightarrow S$
- We say $(S, *)$ is a group if it satisfies the following properties :
 - $\forall a, b, c \in S, a * (b * c) = (a * b) * c$
 - $\exists z \in S$ such that, $\forall a \in S, a * z = z * a = a$ (Identity)
 - $\forall a \in S, \exists a^{-1} \in S$ such that, $a * a^{-1} = a^{-1} * a = z$ (Inverse)

1.1.6 Abelian/Commutative Group

- If $(S, *)$ is a group such that $\forall a, b \in S, a * b = b * a$ ($*$ is Commutative), then $(S, *)$ is called an Abelian group or Commutative Group.

1.1.7 Field

- Suppose F is a non-empty set equipped with two binary operations called addition and multiplication, denoted by '+' and '.', respectively.
- That is, $\forall a, b \in F$, we have : $a + b \in F$ and $a \cdot b \in F$.
- Then the algebraic structure $(F, +, \cdot)$ is called a field, if the following properties are satisfied :
 1. Addition is commutative. i.e. $\forall a, b \in F, a + b = b + a$
 2. Addition is associative. i.e. $\forall a, b, c \in F, a + (b + c) = (a + b) + c$
 3. $\exists \mathbf{0} \in F$ (called zero), such that $\forall a \in F, a + \mathbf{0} = \mathbf{0} + a = a$
 4. $\forall a \in F, \exists (-a) \in F$, such that : $a + (-a) = \mathbf{0}$
 5. Multiplication is commutative. i.e. $\forall a, b \in F, a \cdot b = b \cdot a$
 6. Multiplication is associative. i.e. $\forall a, b, c \in F, a \cdot (b \cdot c) = (a \cdot b) \cdot c$
 7. $\exists \mathbf{1} \in F$ (called one), such that $\forall a \in F, a \cdot \mathbf{1} = \mathbf{1} \cdot a = a$
 8. $\forall a \in F \setminus \{\mathbf{0}\}, \exists a^{-1} \in F \setminus \{\mathbf{0}\}$, such that : $a \cdot a^{-1} = \mathbf{1}$
 9. Multiplication Distributes over addition, i.e. $\forall a, b, c \in F, a \cdot (b + c) = a \cdot b + a \cdot c$ (left distribution) and $\forall a, b, c \in F, (a + b) \cdot c = a \cdot c + b \cdot c$ (right distribution)
- Notice that property 1-4 essentially states that $(F, +)$ is abelian. Similarly properties 5-8 states that (F, \cdot) is abelian.
- Note that $\mathbf{0}$ is called that Zero element of the field(F) and $\mathbf{1}$ is called the Unity element of the field(F).
- Equivalently, $(F, +, \cdot)$ is a field iff
 1. $(F, +)$ is an abelian group.
 2. (F, \cdot) is an abelian group.
 3. Addition and Multiplication are linked by distributive property for both left and right distribution.
- Equivalently, A commutative division ring is a field.

- Examples of fields :
 1. \mathbb{Q} : set of Rational numbers.
 2. \mathbb{R} : set of Real numbers.
 3. \mathbb{C} : set of Complex numbers.