

Lecture 1

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1.1 Definitions

1.1.1 Subfield

- A subfield of a field K is a subset L of K that is a field with respect to the field operations inherited from K .
- Example : \mathbb{R} is a subfield of \mathbb{C}

1.1.2 Characteristic of a field

- If F is a field, it may be possible to add the unit element(**1**) to itself a finite number of times to obtain **0**. That is,

$$\mathbf{1} + \mathbf{1} + \mathbf{1} + \dots + \mathbf{1} = \mathbf{0}$$

- This is not possible in the field of complex numbers.
- In such cases where it is not possible to obtain **0** by adding **1** a finite number of times then the field F is a field of *characteristic zero*.
- Otherwise, the least n such that adding **1** n times results in **0** is called the *characteristic* of the field.

1.1.3 Ring

- A ring is a set R with two binary operations $+$, \cdot such that :
 1. Both operations are closed.
 2. R is *abelian* under addition.
 3. Multiplication is distributed over addition on both left and right.
 4. Multiplication is associative
- Note that **1** might not be an element of the ring.
- In case $\mathbf{1} \in R$, then R is called a ring with unity.
- A commutative division ring is called a field.

1.1.4 Finite Fields

- If the number of elements in a field is finite, then the field is called a finite field.
- Example : If $\mathbb{Z}_n = \{x; 0 \leq x < n\}$, then \mathbb{Z}_p is a finite field if p is a prime number.
- Also note that \mathbb{Z}_4 is not a field.

1.2 Vector Spaces

1.2.1 Definition

- Elements of a field are called scalars.
- $(V, +, \cdot)$ is called a vector space over a field K if :
 1. $(V, +)$ is an abelian group.
 2. $\alpha \in F$ and $x, y \in V$, then : $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$
 3. $\alpha, \beta \in F$ and $x \in V$, then : $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$
 4. $\alpha, \beta \in F$ and $x \in V$, then : $(\alpha\beta) \cdot x = \alpha(\beta \cdot x)$
 5. $\forall x \in V, 1 \cdot x = x$
- Example : n-tuple space

1.2.2 Example

- Take n-tuple space as an example.
- Let V be the set of all ordered n-tuples of elements of any field F for a fixed integer n . That is,

$$V = \{(a_1, a_2, \dots, a_n) : a_i \in F\}$$

- Then V is a vector space over F , with the following \cdot and $+$:
 1. Let $x = (a_1, a_2, \dots, a_n)$ and $y = (b_1, b_2, \dots, b_n)$
 2. $x + y = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$ (Addition)
 3. $\alpha x = (\alpha a_1, \alpha a_2, \dots, \alpha a_n)$ (Scalar Multiplication)
 4. $x = y$ iff $\forall i \in \{1, 2, \dots, n\}, a_i = b_i$

1.2.3 Properties

1. A vector space over a field K can be regarded as a vector space over any of its subfield(S) of F
2. $F(F)$ is a vector space over any field F .
 - \mathbb{R} is not a vector space over \mathbb{C} as it is not closed under scalar multiplication.
3. Set $f(x)$ of polynomials over a field F is a vector space. (With conventional addition and multiplication)

4. The set of all convergent sequences is a vector space over the field of real numbers.
5. The set of all finite matrices with real elements is a vector space over real numbers
6. Let K be an arbitrary field. Let X be any non-empty set. Consider the set V of all functions from X to K . The sum of any two functions $f, g \in V$ is the function $f + g \in V$ defined by :

$$(f + g)(x) = f(x) + g(x)$$

Where the scalar product with $\alpha \in K$, $f \in V$, $\alpha f \in V$ is defined by :

$$(\alpha f)(x) = \alpha f(x)$$

is a vector space over the field K .