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CS512 Spring 2022.

Assignment 3

Q1. Line detection

Ans a) The problem of using the slope & y-intercept as line parameters is that the parameter space size is not determined. For the line $y = ax + b$, the range of a will be 0 to ∞ (as $\frac{\Delta y}{\Delta x} = \infty$ as $\Delta x = 0$) & it will not be able to represent vertical lines. And so, the size of parameter space will be infinite.

Q1 b)

Ans. $\theta = 45^\circ$

$d = 10$

$ax + by + c = 0$.

i.e. $(\cos\theta)x + (\sin\theta)y - d = 0$.

replacing $\theta = 45^\circ$ & $d = 10$,

$\cos 45 x + \sin 45 y - 10 = 0$.

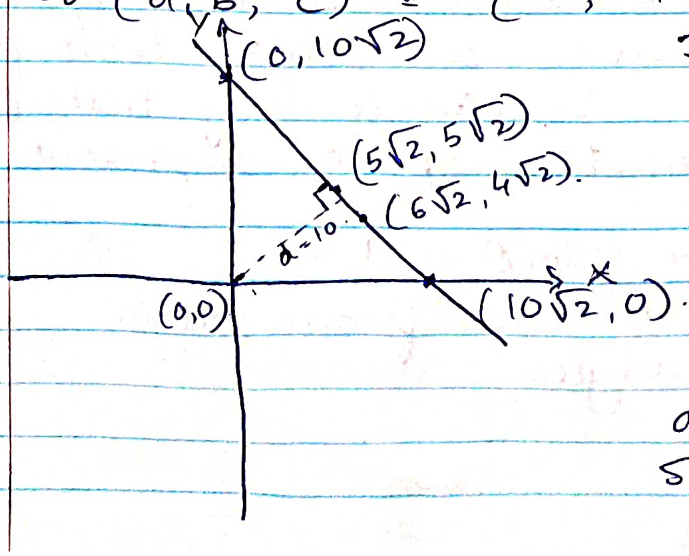
$\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y - 10 = 0$.

$x + y - 10\sqrt{2} = 0$.

~~$x =$~~ $y = -x + 10\sqrt{2}$

So $(a, b, c) \equiv (1, 1, -10\sqrt{2})$

If we consider
$a = 1$
$b = -1$
$c = 10\sqrt{2}$
then points

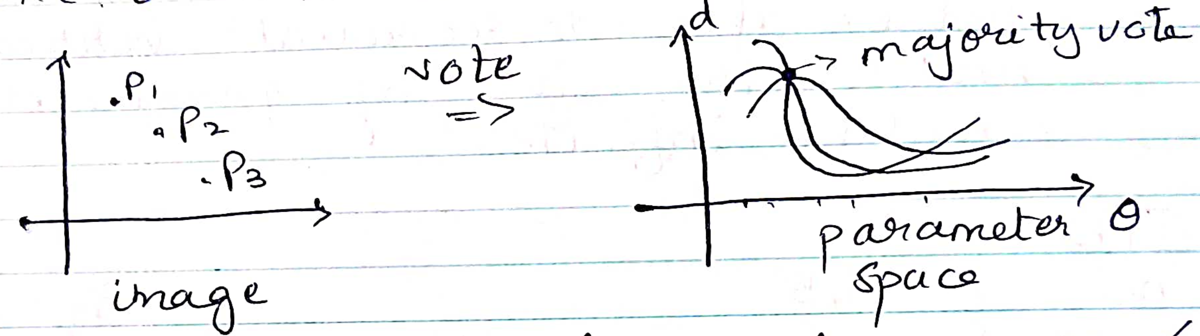


If $x = 0$, $y = 10\sqrt{2}$.
If $y = 0$, $x = 10\sqrt{2}$.
If $x = 5\sqrt{2}$, $y = 5\sqrt{2}$.
If $x = 6\sqrt{2}$, $y = 4\sqrt{2}$

So, the points $(0, 10\sqrt{2})$, $(10\sqrt{2}, 0)$, $(5\sqrt{2}, 5\sqrt{2})$, $(6\sqrt{2}, 4\sqrt{2})$ are the points which satisfy the explicit line equation.

(2)

Q1.c)
Ans. When we use polar representation of lines
$x\cos\theta + y\sin\theta = d$,
the vote for point (x, y) will be:
- for each θ_i, vote $d_i = x\cos\theta_i + y\sin\theta_i$
- For all possible values of θ, we compute d
i.e. $\theta = 0, \ldots 360$



The vote is in the form of ~~a~~ waves ~~(sine)~~ (sinusoidal)

Q1d)
Ans. Multiple points on a line in the image correspond to a single point in parameter space (the parameters of the line)
We detect lines by casting votes in the parameter space. Each point in the image defines a line in parameter space & when multiple lines representing these points ~~th~~ have an intersection point, the distance & angle of that point in the parameter space represent the line

$x\cos\theta + y\sin\theta - d = 0$.
where d = distance from origin
θ = angle.

(2a)

Q1.e) If the bin size is smaller, it is more accurate but slower. Larger bins are more efficient but provide less localization and are less accurate. Smaller bins may not give intersection.

Q1.f)
Ans. If the normal at each voting point is known (n), then we can know θ too. If we know θ & d accurately, each point will have single vote & will be much faster. For our advantage, we compute the gradient vector at voting point.

Point (x, y) with normal θ votes for
$d = x\cos(\theta + \alpha\Delta\theta) + y\sin(\theta + \alpha\Delta\theta)$
where $\alpha \in [-1, 1]$.

Q1.g)
Ans. When using Hough transform for circles, the votes will be circle & we will scan r & θ to compute a & b. And so, the number of parameters (a, b, r) will be three. So, the number of dimensions of parameter space will be 3.

2b

Q1.h)

Ans. $\theta \in [45°, 135°]$

Line equation :- $x\cos\theta + y\sin\theta = d$

Dividing the above equation by $\sin\theta$, we get

$$\frac{x\cos\theta}{\sin\theta} + \frac{y\sin\theta}{\sin\theta} = \frac{d}{\sin\theta}$$

Rearranging the equation, we get

$$\cancel{y = - x}$$

$$y = -\frac{\cos\theta}{\sin\theta}x + \frac{d}{\sin\theta}$$

As $x \in [0, n]$ & $\theta \in [45°, 135°]$

If $x = 0$ & $\theta = 45°$

$$y = -\frac{\cos 45}{\sin 45}(0) + \frac{d}{\sin 45}$$

$$y = \frac{d}{1/\sqrt{2}}$$

$$y = \sqrt{2}d.$$

If $x = 0$ & $\theta = 135°$

$$y = -\frac{\cos 135°}{\sin 135°}(0) + \frac{d}{\sin 135°}$$

$$y = \frac{d}{1/\sqrt{2}} = \sqrt{2}d.$$

Pixel point $= (0, \sqrt{2}d)$.

2c

Q1. i)

Ans. $x\cos\theta + y\sin\theta = d$

Dividing above equation by $\cos\theta$, we get

$$\frac{x\cos\theta}{\cos\theta} + \frac{y\sin\theta}{\cos\theta} = \frac{d}{\cos\theta}$$

Rearranging,

$$x = -\frac{\sin\theta}{\cos\theta}y + \frac{d}{\cos\theta}$$

As $y \in [0, m]$ & $\theta \in [-45°, 45°]$

If $y = 0$ & $\theta = -45°$

$$x = -\frac{\sin(-45)}{\cos(-45)}(0) + \frac{d}{\cancel{\cos 45}\ \cos(-45)}$$

$$= \frac{d}{1/\sqrt{2}}$$

$$x = \sqrt{2}d$$

If $y = 0$ & $\theta = 45°$

$$x = -\frac{\sin 45}{\cos 45}(0) + \frac{d}{\cos 45}$$

$$x = \frac{d}{1/\sqrt{2}}$$

$$x = \sqrt{2}d.$$

Pixel point is $(\sqrt{2}d, 0)$

③

Q2. a)

Ans. $y = ax+b$ .. - the disadvantage of this equation is that it is not good for vertical lines as 'a' tends from 0 to ∞. though it is good for nearly horizontal lines. ~~So, the~~ Also, another disadvantage is that the geometric distance between the predicted and real point does not get minimized and so, the fitting will not be accurate. Vertical lines can not be fitted accurately using this equation and also same is the case for lines with higher slope.

Q2.b)

Ans.

$L^T p = 0$.

$L = (a, b, c)^T \quad p = (x, y, 1)$

$\therefore L = (1, 2, -2)^T \quad p = (x, y, 1)$

$L^T p = 0$

$$\therefore \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \begin{bmatrix} x & y & 1 \end{bmatrix} = 0.$$

$x + 2y - 2 = 0 \quad \therefore$

$\therefore \quad L = [1, 2, -2]$

Q2.c) 1

Ans. First, we convert the points to homogeneous coordinates i.e. $\{x_i, y_i\}_{i=1}^{m}$ is converted to $P_i = (x_i, y_i, 1)$

$$E(L) = \sum_i (L^T P_i)^2$$

$$L^* = \underset{L}{\operatorname{argmin}}\ E(L) -$$

(5)

(4)

If all points lie on the same line l, then $E(l) = 0$ i.e. $(l^T p_i)^2$

$l^T x = 0$

$E(L) = \sum l^T p_i p_i^T l = l^T \sum p_i p_i^T L = l^T S L$

$\nabla E(l) = 0 \Rightarrow Sl = 0 \quad (\nabla E(L) = \cancel{2} S L = 0)$

Here, L ~~to~~ will be eigenvector of S belonging to zero/smallest eigenvalue.

Correlation matrix $S = \sum p_i p_i^T$

$$= \begin{bmatrix} \sum x_i^2 & \sum x_i y_i & \sum x_i \\ \sum x_i y_i & \sum y_i^2 & \sum y_i \\ \sum x_i & \sum y_i & n \end{bmatrix}$$

Q2.d) Given points: $\{(0,1), (1,3), (2,6)\}$

Corr matrix $S = \begin{bmatrix} \sum x_i^2 & \sum x_i y_i & \sum x_i \\ \sum x_i y_i & \sum y_i^2 & \sum y_i \\ \sum x_i & \sum y_i & n \end{bmatrix}$

$$= \begin{bmatrix} 0+1+4 & 3+12 & 1+2 \\ 3+12 & 1+9+36 & 1+3+6 \\ 0+1+2 & 1+3+6 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 15 & 3 \\ 15 & 46 & 10 \\ 3 & 10 & 3 \end{bmatrix}$$

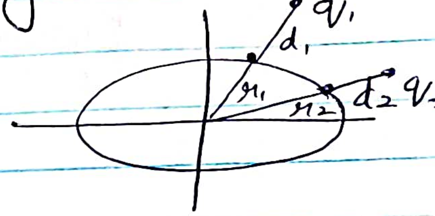
Q2.e) Implicit equation for conic curve:-

$ax^2 + bxy + cy^2 + dx + ey + f = 0$

The constraint that guarantees that the model will be an ellipse is $b^2 - 4ac < 0$

Q2.f) When P_i is on the ellipse : $L^T P_i = 0$.
Ans. Equation that needs to be solved for fitting an ellipse using algebraic distance is

$$E(L) = \sum_{i=1}^{m} (L^T P_i)^2.$$



algebraic distance $q_i \equiv L^T x_i \approx \dfrac{d_i}{d_i + r_i}$

d_1 & d_2 are geometric distance so $d_1 \approx d_2$

but $q_1 > q_2$ as $\dfrac{d_1}{d_1 + r_1} > \dfrac{d_2}{d_2 + r_2}$. (as $r_2 > r_1$)

This shows that points close to the short axis of the ellipse affect more.

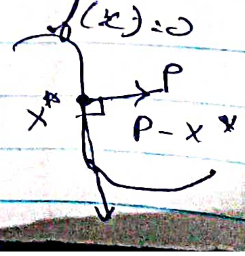
Q2.g) When fitting an ellipse, the objective
Ans. function which needs to be minimized using geometric distance is

$$\underset{\text{geometric distance}}{d(P, f)} = |P - x^*| = \frac{|f(P)|}{|\nabla f(x^*)|} \approx \underset{\text{algebraic distance}}{\frac{|f(P)|}{|\nabla f(P)|}}$$

i.e.

$$E(L) = \sum_i \frac{|f(P_i; L)|}{|\nabla f(P_i; L)|} \begin{array}{l} \longrightarrow \text{algebraic distance} \\ \rightarrow \text{gradient} \end{array}$$

The problem here is how to find the closest point (x^*) to point P
(on the curve).



⑥

Q2.h)
Ans.

$$\phi_s = \begin{bmatrix} x(s) \\ y(s) \end{bmatrix}$$

Error functional

$$E[\phi(s)] = \int_{\phi_s} \Big(\alpha(s) \underbrace{E_{cont} + \beta(s) E_{curv}}_{\text{internal energy}} + \underbrace{\gamma(s) E_{img}}_{\text{external energy}} \Big) ds$$

$\alpha(s)$, $\beta(s)$ & $\gamma(s)$ are coefficients

$E_{cont} \rightarrow$ E continuity

$E_{curv} \rightarrow$ E curvature

$E_{img} \rightarrow$ image energy.

Q2 i) $\phi(s) \rightarrow \{P_i\}_{i=1}^{n}$

Ans. Continuity of discrete curve :-

$$E_{cont} = \left|\frac{d\phi}{ds}\right|^2 = |P_{i+1} - P_i|^2$$

Curvature of discrete curve :-

$$E_{curv} = \left|\frac{d^2\phi}{ds^2}\right|^2 = |(P_{i+1} - P_i) - (P_i - P_{i-1})|^2$$

$$= |P_{i+1} - 2P_i + P_{i-1}|^2$$

Q2 j) The continuity of active contours will be $|P_{i+1} - P_i| - d$ to allow for sharp corners.

⑦

Q3. a) $(1,2)\ (3,4)$

Ans. Correlation ~~a~~ matrix $S = \sum P_i P_i^T$

$$= \begin{bmatrix} \sum x_i^2 & \sum x_i y_i & \sum x_i \\ \sum x_i y_i & \sum y_i^2 & \sum y_i \\ \sum x_i & \sum y_i & n \end{bmatrix}$$

$$= \begin{bmatrix} 1+9 & 2+12 & 1+3 \\ 2+12 & 4+16 & 2+4 \\ 1+3 & 2+4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 14 & 4 \\ 14 & 20 & 6 \\ 4 & 6 & 2 \end{bmatrix}$$

Given gradient vectors, their 2×2 corr matrix will be.

$$= \cancel{\begin{bmatrix} 10 & 4 \\ 4 & 2 \end{bmatrix}}$$

$$= \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}$$

Q3. b)

$(10,10)\quad(20,20)$

Converting to homogeneous coordinates

$P_1 \equiv (10,10,1)$ & $P_2 \equiv (20,20,1)$

$$E(L) = \sum_i (L^T P_i)^2$$

$$E(L) = L^T \underbrace{\sum P_i P_i^T}_{S} L = L^T S L$$

$\nabla E(L) = 0 \Rightarrow SL = 0.$

For the implicit line equation, we will first find out a, b.

$$x = A^{-1} b$$

where $x = \begin{bmatrix} a \\ b \end{bmatrix}$ $A = \begin{bmatrix} \sum x_i^2 & \sum x_i \\ \sum x_i & m \end{bmatrix}$ $b = \begin{bmatrix} \sum x_i y_i \\ \sum y_i \end{bmatrix}$

$$A = \begin{bmatrix} 500 & 30 \\ 30 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 500 \\ 30 \end{bmatrix} \quad |A| = 1000 - 900 = 100.$$

$$A^{-1} = \begin{bmatrix} 2/100 & -30/100 \\ -30/100 & 500/100 \end{bmatrix} = \begin{bmatrix} 1/50 & -3/10 \\ -3/10 & 5 \end{bmatrix}$$

$$\therefore x = \begin{bmatrix} 1/50 & -3/10 \\ -3/10 & 5 \end{bmatrix} \begin{bmatrix} 500 \\ 30 \end{bmatrix} = \begin{bmatrix} 500/50 - \frac{90}{10} \\ -1500/10 + 150 \end{bmatrix}$$

$$= \begin{bmatrix} 10 - 9 \\ -150 + 150 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \therefore a = 1, \; b = 0$$

So, line eqn $y = ax + b$ will become

$y = x$. $\therefore$ rearranging

$\underline{x - y = 0}$.

Q3. c) Line coefficients $(a, b, c) = (1, 2, 3)$

$\cancel{y =}\ ax + by + c = 0.$

$x + 2y + 3 = 0.$

if $x = 2$

then $2 + 2y + 3 = 0$

$\therefore 2y = -5$

$y = \frac{-5}{2} = -2.5$

$\therefore (x, y) = (2, -2.5)$

OR.

$ax + by + c = 0$

$x + 2y - 3 = 0$

if $x = 2$

then $2 + 2y \cancel{+} -3 = 0$

$2y = 1$

$y = \frac{1}{2} = 0.5.$

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Q3. d) (1,1)

Vote by point (x, y):

for each θ_i vote $d_i = x\cos\theta_i + y\sin\theta_i$

as $\theta = 0$ & point $= (1,1)$

$\therefore d = \cancel{x\cos\theta}\ x\cos 0 + y\sin 0$

$= (1)\cos 0 + (1)\sin 0$

$d = \cancel{(1)(0) + (1)(1)}\ (1)(1) + (1)(0)$

$d = 1$

Q3. e) (1,2) (3,4)

In homogeneous coordinates

$P_1 = (1, 2, 1)$ $P_2 = (3, 4, 1)$.

$E(L) = L^T \Sigma P_i P_i^T L = L^T S L$.

$S = \Sigma P_i P_i^T$

$$= \begin{bmatrix} \Sigma x_i^2 & \Sigma x_i y_i & \Sigma x_i \\ \Sigma x_i y_i & \Sigma y_i^2 & \Sigma y_i \\ \Sigma x_i & \Sigma y_i & n \end{bmatrix}$$ OR

$$= \underset{P_1}{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}} \underset{P_1^T}{\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}} + \underset{P_2}{\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}} \underset{P_2^T}{\begin{bmatrix} 3 & 4 & 1 \end{bmatrix}}$$

$$= \cancel{\begin{bmatrix} 1 & 4 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 9 & 12 & 3 \\ 12 & 16 & 4 \\ 3 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 14 & 4 \\ 14 & 20 & 6 \\ 4 & 6 & 2 \end{bmatrix}$$

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Q3.f)

Ans. Distance from a point p to implicit curve is given by $f(x)=0$.

$$d(p,f) = |p - x^*|$$ $\longrightarrow$ closest point

distance of p to f

Given: $f(p) = 1$ $\quad \nabla f(x^*) = 2$

$$d(p,f) = |p - x^*| = \frac{|f(p)|}{|\nabla f(x^*)|} = \frac{1}{2} = 0.5$$

$\therefore$ geometric distance of point p from implicit curve f is 0.5. & algebraic distance is 1.

Q3g)

Ans. Approximated geometric distance:

$$d(p,f) = |p - x^*| = \frac{|f(p)|}{|\nabla f(x^*)|} \approx \frac{|f(p)|}{|\nabla f(p)|}$$

Given: $f(p) = 1$ $\quad \nabla f(p) = 2$.

$$\therefore d(p,f) = \frac{1}{2} = 0.5$$

$\therefore$ approximated geometric distance is 0.5

Q3 h) $p_1 = (1,2)$ $\quad p_2 = (2,3)$ $\quad p_3 = (3,4)$

p_{i-1} $\quad p_i$ $\quad p_{i+1}$

$$E_{cont} = |p_{i+1} - p_i|^2 = \{(4-3)^2 + (3-2)\}^2$$

$$= 1 + 1 = 2$$

$$E_{curv} = |(p_{i+1} - p_i) - (p_i - p_{i-1})|^2$$

$$= |(p_{i+1} - 2p_i + p_{i-1})|^2$$

$$= (3 - 2\times2 + 1)^2 + (4 - 2\times3 + 2)^2$$

$= 0 + 0 = 0.$

$E_{cont} = 2$
& $E_{curv} = 0.$

Q3. i) Ans. For a point with high curvature, set $\beta = 0$ to guarantee tight fitting of an active contour

Q4.a) Ans Outliers are data points that differ significantly from other data points and can cause serious problems in statistical analysis. If ot outliers are considered while fitting a model, the fitted model will be incorrect. Outliers influence the model fitting value estimates and so, it is necessary to detect them in order to improve the model.

Q 4.b) Ans. The standard least squares or MSE function is given as:

$$E(\theta) = \Sigma d^2 (x_i; \theta)$$

The objective function for robust estimation is given as:

$$E(\theta) = \Sigma S_\sigma(d(x_i; \theta))$$

where $S_\sigma(x) = x^2$

More the outliers, bigger will be the loss function so according to Geman McClure estimator

$$S_\sigma(x) = \frac{x^2}{x^2 + \sigma^2}$$

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In standard least square objective function outliers will influence the model more and have higher value but in robust estimation, the influence of the outlier will be reduced ($S_\sigma(x) = \frac{x^2}{x^2+\sigma^2}$)

Q 4.c)
Ans. Geman McClure function for robust estimation is given as :-

$$S_\sigma(x) = \frac{x^2}{x^2+\sigma^2}$$

By using this function, the influence of the outliers on the model will be reduced.
If σ is chosen incorrectly, we will not get correct results.
If σ is too small, the number of points considered will be limited and will give a wrong fit
If σ is too large, all points will be considered (outliers too) and will give a wrong fit.
As we know that large $\sigma \rightarrow$ include more points
small $\sigma \rightarrow$ include fewer points,
We will start with large bandwidth parameter σ and then decrease as converging.

Q 4.d) If $x=1$ & $\sigma=1$ then

$$S_\sigma(x) = \frac{x^2}{x^2+\sigma^2} = \frac{1^2}{1^2+1^2} = \frac{1}{2} = \underline{\underline{0.5}}$$

Then, $E(\Theta) = \Sigma(0.5)(d(x_i;\Theta))$

(13)

Q4. e)

Ans. Principle of RANSAC algorithm

- To perform multiple experiments, choose best results (best estimate) and use small sets of points in hope that atleast one set will not have outliers.

RANSAC algorithm:

* Repeat k times:
  - Draw n points uniformly at random (with replacement)
  - Fit a model to points
  - Find inliers in entire set [distance < t]
  - Recompute model (if ~~all~~ atleast d inliers)
  - Update parameters (k, t)
* Choose best solution:
  - largest consensus set (or smallest error)

The number of points drawn at each attempt should be small because if too many points are drawn, there will be more outliers. Also, we need only minimum points to fit the model.

Q4. f) Parameters of RANSAC algo:

Ans.

n = # points drawn at each evaluation

d = minimum # points needed to estimate model

k = # trials (many experiments)

t = distance threshold to identify inliers.

p: probability that atleast one experiment does not have outliers

w: probability that a point is an inlier

(14)

No. of Trials :

$$\boxed{k = \frac{\log(1-p)}{\log(1-w^n)}}$$

where probability that all k experiments failed :

$$(1-p) = (1-w^n)^k$$

$$\log(1-p) = k\log(1-w^n)$$

$$w \leftarrow \frac{\#\text{ inliers}}{\#\text{ points}}$$

$$\therefore \quad k = \frac{\log(1-p)}{\log(1-w^n)}$$

Q 4. g) $p = 0.99$ (atleast one experiment not having outlier

$w = 0.9$

$$k = \frac{\log(1-p)}{\log(1-w^n)}$$

$$k = \frac{\log(1-0.99)}{\log(1-0.9^n)}$$

If $n=1$

$$k = \frac{\log(1-0.99)}{\log(1-0.9)} = \frac{\log(0.01)}{\log(0.1)} = \frac{-2}{-1} = 2.$$