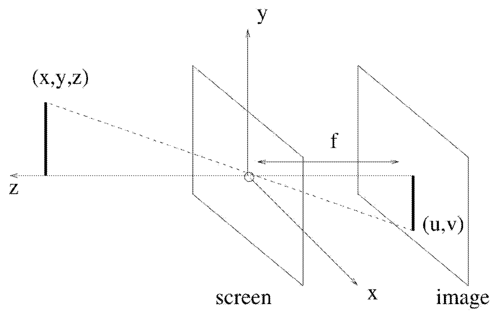
Pinhole camera model



$$\frac{x}{z} = -\frac{u}{f}$$
$$\frac{y}{z} = -\frac{v}{f}$$

Pinhole camera model

In matrix form:

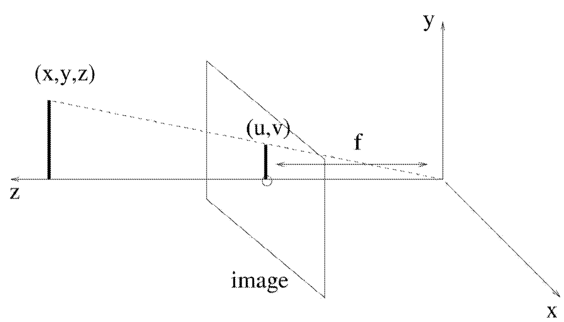
$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{z} \begin{bmatrix} -f & 0 \\ 0 & -f \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Without inversion:
Based on triangle similarity:

$$\frac{x}{z} = \frac{u}{f}$$
$$\frac{y}{z} = \frac{v}{f}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{z} \begin{bmatrix} +f & 0 \\ 0 & +f \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Alternative pinhole camera model



In matrix form:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{z} \begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Homogeneous coordinates

Moving from Cartesian to homogeneous coordinates:

2D → 2D H

$$(x, y) \leftrightarrow (x, y, 1)$$

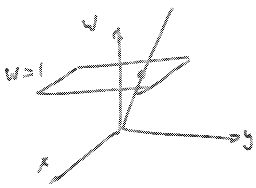
Equivalence class:

$$(x, y, 1) \leftrightarrow (kx, ky, k)$$

Moving from homogeneous to Cartesian coordinates:

2D H → 2D

$$(x, y, w) \leftrightarrow (x/w, y/w, 1)$$



$(x,y,0)$ = point at infinity
represents direction

Projection in homogenous coordinates

Perspective projection in homogeneous coordinates:

$$\alpha \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{aligned} u &= \frac{\alpha u}{\alpha} = \frac{fx}{z} \\ v &= \frac{\alpha v}{\alpha} = \frac{fy}{z} \end{aligned}$$

Balanced pinhole model:

$$\alpha \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{3 \times 4} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Projection in homogenous coordinates

* Matrix form:

$$p = QP$$

$$Q = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = [K \mid 0] = K[I \mid 0]$$

$$K = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$p = K[I \mid 0] \underline{P}$$

Transformations in homogenous coordinates

Affine transformations in homogeneous coordinates:

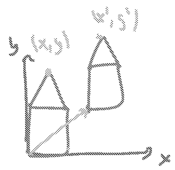
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \tag{17}$$

The matrix H is a combination of rotation, translation, and scale transformations combined by multiplication.

$$p' = Hp \tag{18}$$

3D Translation:

$$T(t_x, t_y, t_z) = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I & T \\ 0^T & 1 \end{bmatrix} \tag{19}$$



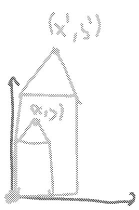
$$\begin{cases} x' = x + t_x \\ y' = y + t_y \end{cases}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

Transformations in homogenous coordinates

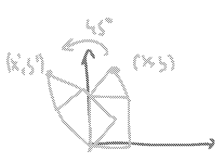
3D Scaling about the origin:

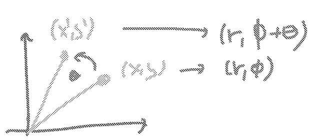
$$S(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} S & 0 \\ 0^T & 1 \end{bmatrix}$$



$$\begin{cases} x' = s_x \cdot x \\ y' = s_y \cdot y \end{cases}$$

Transformations in homogenous coordinates





$$\begin{cases} x = r \cos\phi \\ y = r \sin\phi \end{cases} \qquad \begin{cases} x' = r\cos(\phi+\theta) \\ y' = r\sin(\phi+\theta) \end{cases}$$

$$x' = \overbrace{r\cos\phi}^{x}\cos\theta - \overbrace{r\sin\phi}^{y}\sin\theta = x\cos\theta - y\sin\theta$$

$$y' = \underbrace{r\cos\phi}_{x}\sin\theta + \underbrace{r\sin\phi}_{y}\cos\theta = x\sin\theta + y\cos\theta$$

Transformations in homogenous coordinates

$$\overset{2D}{\begin{bmatrix} x' \\ y' \end{bmatrix}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \overset{2D}{\begin{bmatrix} x \\ y \end{bmatrix}}$$

$$\overset{2DH}{\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{2DH}{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}$$

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Transformations in homogenous coordinates

3D Rotation about the z axis:

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_z & 0 \\ 0^T & 1 \end{bmatrix}$$

3D Rotation about the x axis:

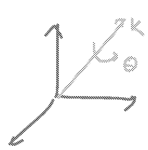
$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_x & 0 \\ 0^T & 1 \end{bmatrix}$$

3D Rotation about the y axis:

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_y & 0 \\ 0^T & 1 \end{bmatrix}$$

Transformations in homogenous coordinates

$$R_k(\theta) = R_x R_y R_z$$



Rodrigues formula:

$$\boxed{\mathbf{R} = \mathbf{I} + (\sin\theta)\mathbf{K} + (1-\cos\theta)\mathbf{K}^2}$$

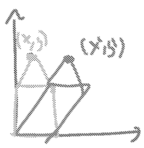
3D

$$\mathbf{K} = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}$$

$$R_k(\theta) = \left[\begin{array}{c|c} R & 0 \\ \hline 0^T & 1 \end{array}\right]$$

3DH

Transformations in homogenous coordinates



$$\begin{cases} y' = y \\ x' = x + Sh_x \, y \end{cases}$$

2D Shear along x relative to the origin:

$$SH_x(s_x) = \begin{bmatrix} 1 & s_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} SH & 0 \\ 0 & 1 \end{bmatrix}$$

Transformations in homogenous coordinates

Transformation properties:

1. Combine by matrix multiplication. $\longrightarrow p' = RTp$
2. Preserve the homogeneous coordinate.
3. Inverted by matrix inversion.

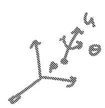
first

second

Examples:

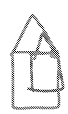
- rotation about arbitrary point:

$$R_{p,u}(\theta) = T(p)\, R_u(\theta)\, T(-p)$$



- scale about arbitrary point:

$$S_p(s_x, s_y, s_z) = T(p)\, S(s_x, s_y, s_z)\, T(-p)$$



Transformations in homogeneous coordinates

$$(TR)^{-1} = R^{-1} T^{-1}$$

$$T^{-1}(t_x, t_y, t_z) = T(-t_x, -t_y, -t_z)$$

$$S^{-1}(s_x, s_y, s_z) = S(1/s_x, 1/s_y, 1/s_z)$$

$$R_u^{-1}(\theta) = R_u(-\theta) = R_u^T(\theta)$$

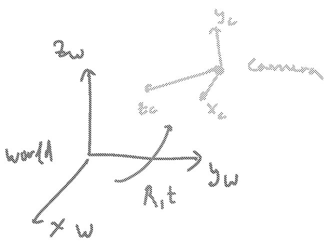
Transformations between coordinate systems

Let (x_w, y_w, z_w) be a world coordinate system. Let (x_c, y_c, z_c) be a camera coordinate system. Assume that the camera coordinate system is translated by t and rotated by R with respect to the world coordinate system.

A point $p^{(w)}$ in world coordinates is given by $p^{(c)}$ in camera coordinates. $p^{(c)}$ is related to $p^{(w)}$ by:

$$p^{(c)} = M_{c\leftarrow w} p^{(w)} \tag{26}$$

$$\begin{aligned} M_{c\leftarrow w} &= (T(t)R)^{-1} && (27)\\ &= R^{-1}T^{-1}(t) && (28)\\ &= R^T T(-t) && (29) \end{aligned}$$



$M_{c\leftarrow w}$: align camera with world (cancel translation then cancel rotation)

Transformations between coordinate systems

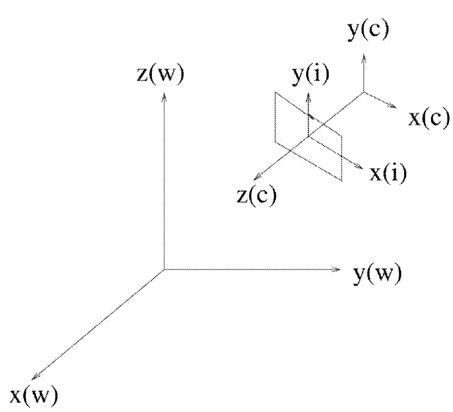
The rotation R is given by:

$$R = \begin{bmatrix} \hat{x}_c & \hat{y}_c & \hat{z}_c \end{bmatrix}$$

The inverse transformation is given by:

$$\begin{aligned} M_{w\leftarrow c} &= (M_{c\leftarrow w})^{-1}\\ &= (R^T T(-t))^{-1}\\ &= T(t)R \end{aligned}$$

General camera model



General camera model

Perspective projection in camera coordinates:

$$p^{(c)} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} P^{(c)} = K[I|0]P^{(c)} \quad (34)$$

The perspective projection we want:

$$p^{(i)} = QP^{(w)} \quad (35)$$

$$\begin{aligned} p^{(i)} &= M_{i \leftarrow c} p^{(c)} && (36) \\ &= M_{i \leftarrow c} K[I|0] P^{(c)} && (37) \\ &= M_{i \leftarrow c} K[I|0] M_{c \leftarrow w} P^{(w)} && (38) \end{aligned}$$

Assuming that the camera is translated by t and rotated by R with respect to the world ($R = [\ \hat{x}_c \quad \hat{y}_c \quad \hat{z}_c\ ]$):

General camera model

$$\begin{aligned} M_{c \leftarrow w} &= R^T T(-t) \\ &= \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}^T \begin{bmatrix} I & -t \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} R^* & T^* \\ 0 & 1 \end{bmatrix} \end{aligned}$$

The rotation/translation of the world with respect to the camera are extrinsic camera parameters:

$$\begin{aligned} R^* &= R^T \\ T^* &= -R^T t \end{aligned}$$

General camera model

Intrinsic camera parameters:

notation	meaning	units
k_u	scale in x relating pixels to mm	pixels/mm
k_v	scale in y relating pixels to mm	pixels/mm
u_0	translation of the principal point in x	pixels
v_0	translation of the principal point in y	pixels
f	focal length	mm
du, dv	focal length	pixels

The transformation $M_{i \leftarrow c}$ is composed of scale and translation:

$$M_{i \leftarrow c} = \begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

General camera model

Combining the transformation matrices:

$$\begin{aligned} p^{(i)} &= M_{i\leftarrow c} K [I|0] M_{c\leftarrow w} P^{(w)} \\ &= K^* [I|0] \begin{bmatrix} R^* & t^* \\ 0 & 1 \end{bmatrix} P^{(w)} \\ &= K^* [R^* | t^*] P^{(w)} \end{aligned}$$

The matrix K^* contains the intrinsic camera parameters:

$$\begin{aligned} K^* &= M_{i\leftarrow c} K \\ &= \begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} fk_u & 0 & u_0 \\ 0 & fk_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \alpha_u & 0 & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

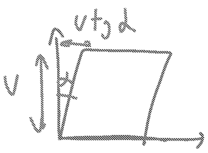
The parameters α_u, α_v specify scale in pixels.

General camera model

When allowing for shear:

$$\begin{aligned} K^* &= \begin{bmatrix} \alpha_u & \alpha_u \tan\alpha & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \alpha_u & s & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

The parameter s is skew (in pixels).



Radial lens distortion

$$p^{(i)} = \begin{bmatrix} 1/\lambda & & \\ & 1/\lambda & \\ & & 1 \end{bmatrix} K^* [R^* | T^*] P^{(w)}$$

$$\lambda = 1 + k_1 d + k_2 d^2$$

linear distortion coefficient

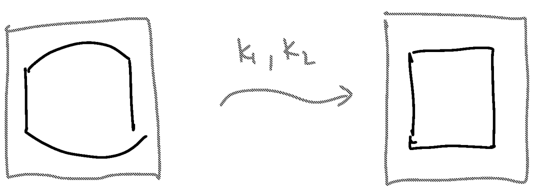
quadratic distortion coefficient

d = distance from center

larger shrink away from the center

Radial lens distortion

Warp the image to correct distortion
(warp using estimated distortion parameters)



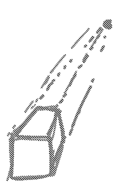
Weak perspective camera

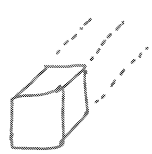
Perspective

$$\overset{3\times 4}{M}$$

weak perspective

$$M_\infty = \begin{array}{c} 3\times 4 \\ \begin{bmatrix} \text{———} & & & \\ \text{———} & & & \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$





Weak perspective camera

Weak perspective is correct when depth variation in the scene is small compared with distance from camera

depth variation

$$e = |M_\infty P - MP| = \frac{\Delta}{d_0}(MP - P_0)$$

distance from center

distance from camera

Affine camera

$$M_{affine} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

computational model