THE POP-STACK OPERATOR ON ORNAMENTATION LATTICES (EXCERPT)

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Abstract.

1. Introduction

1.1. **Ornamentation Lattices.**

Laplante-Anfossi [15] recently introduced remarkable polytopes called *operahedra*. These polytopes are special examples of *graph associahedra* [5, 17] and *poset associahedra* [13], and they generalize both the associahedra and permutohedra. Laplante-Anfossi also considered certain posets obtained by orienting the 1-skeletons of operahedra, and he asked if these posets are lattices. Defant and Sack [11] answered Laplante-Anfossi's question in the affirmative, naming his posets *operahedron lattices*. A key idea that Defant and Sack employed was to embed an operahedron lattice into the product of an interval in the weak order on the symmetric group and another lattice. This other lattice, which is the primary focus of this article, is a new structure called an *ornamentation lattice*.

Let PT_n denote the set of rooted plane trees with n nodes. Let $\mathsf{T} \in \mathrm{PT}_n$. We view T both as a graph and as a poset in which every non-root element is covered by exactly 1 element (so the root is the unique maximal element). We denote the partial order on T by $\leq_{\mathsf{T}}$. For $v \in \mathsf{T}$, we write

$$\Delta_{\mathsf{T}}(v) = \{v' \in \mathsf{T} : v' \leq_{\mathsf{T}} v\} \quad \text{and} \quad \nabla_{\mathsf{T}}(v) = \{v' \in \mathsf{T} : v \leq_{\mathsf{T}} v'\}.$$

An *ornament* of T is a set of nodes that induces a connected subgraph of T. Let $\mathrm{Orn}(\mathsf{T})$ denote the set of ornaments of T. Two sets are said to be *nested* if one is a subset of the other. An *ornamentation* of T is a function $\delta\colon \mathsf{T} \to \mathrm{Orn}(\mathsf{T})$ such that

- for every $v \in \mathsf{T}$, the unique maximal element of $\delta(v)$ is v;
- for all $v, v' \in \mathsf{T}$, the ornaments $\delta(v)$ and $\delta(v')$ are either nested or disjoint.

Let $\mathcal{O}(\mathsf{T})$ denote the set of ornamentations of T. We view $\mathcal{O}(\mathsf{T})$ as a poset, where the partial order $\leq$ is defined so that $\delta \leq \delta'$ if and only if $\delta(v) \subseteq \delta'(v)$ for all $v \in \mathsf{T}$. Then $\mathcal{O}(\mathsf{T})$ is a lattice whose meet operation $\wedge$ is such that

$$(\delta \wedge \delta')(v) = \delta(v) \cap \delta'(v)$$

for all $v \in \mathsf{T}$. The minimum element $\delta_{\min}$ and maximum element $\delta_{\max}$ of $\mathcal{O}(\mathsf{T})$ satisfy

$$\delta_{\min}(v) = \{v\} \quad \text{and} \quad \delta_{\max}(v) = \Delta_{\mathsf{T}}(v)$$

for all $v \in \mathsf{T}$. See Figure 1.

Ornamentation lattices naturally generalize Tamari lattices. Indeed, the n-th Tamari lattice is isomorphic to the ornamentation lattice of an n-element chain. Tamari lattices have numerous interesting properties; we will generalize some of these properties to the much broader family of ornamentation lattices.

1.2. **Pop-Stack Operators.**

The *pop-stack operator* of a finite lattice L is the map $\mathsf{Pop}\colon L \to L$ defined by

$$\mathsf{Pop}(x) = \bigwedge(\{x\} \cup \{y \in L : y \lessdot x\}).$$

Initial works on this operator focused on the case where L is the weak order on the symmetric group [1, 2, 7, 8, 16, 18, 20], where it can be interpreted as a deterministic sorting procedure that makes use

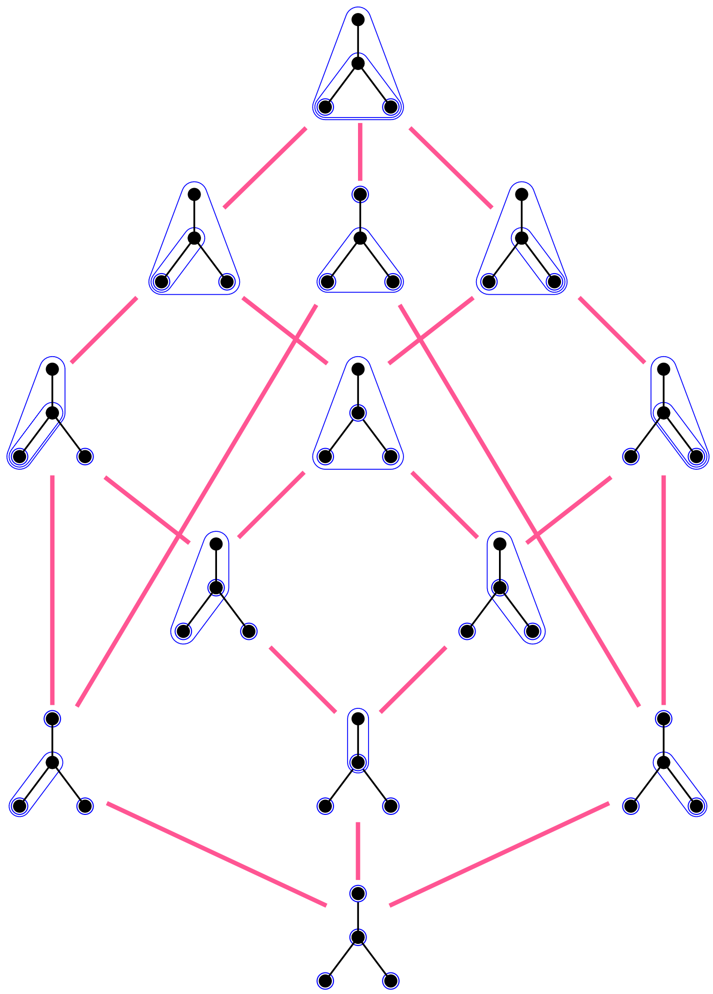

FIGURE 1. The ornamentation lattice of a rooted tree with 4 vertices.

of a data structure called a *pop-stack*. Several recent articles have studied pop-stack operators on other interesting lattices, especially from a dynamical point of view [4, 6, 9, 10, 12, 14]. For example, Defant [10] and Hong [14] found that the pop-stack operator on a Tamari lattice has particularly well behaved dynamics. Choi and Sun [6] and Barnard, Defant, and Hanson [4] later generalized these results to the much broader class of *Cambrian lattices*. In this article, we generalize the results about Tamari lattices in a different direction by considering pop-stack operators of ornamentation lattices.

Given a finite lattice L and an element $x \in L$, we let $\mathrm{Orb}_{\mathsf{Pop}}(x) = \{x, \mathsf{Pop}(x), \mathsf{Pop}^2(x), \dots\}$ denote the *forward orbit* of x under Pop. If t is a sufficiently large integer, then $\mathsf{Pop}^t(x)$ is the minimum element $\hat{0}$ of L. We are interested in $|\mathrm{Orb}_{\mathsf{Pop}}(x)|$. When considering the pop-stack operator, one of the most well studied quantities associated to L is

$$\max_{x \in L} |\mathrm{Orb}_{\mathsf{Pop}}(x)|$$

(see, e.g., [4, 9, 10, 20]). For example, Barnard, Defant, and Hanson found that if L is a Cambrian lattice of a finite Coxeter group W, then $\max_{x \in L} |\mathrm{Orb}_{\mathsf{Pop}}(x)|$ is equal to the Coxeter number of W. In particular, if L is the n-th Tamari lattice (which is a Cambrian lattice of the symmetric group S_n), then this number is n.

Another important aspect of the pop-stack operator is its image, which has been studied for a variety of lattices in [4, 6, 12, 14, 19]. Defant and Williams found numerous different interpretations of the size of the image of Pop when L is a semidistributive lattice (or, more generally, a *semidistrim* lattice). Notably, when L is semidistributive, the image of Pop is in bijection with the facets of a simplicial complex called the *canonical join complex* of L (see [3, 4]). Hong [14] characterized and enumerated Pop images on Tamari lattices. Choi and Sun [6] subsequently obtained analogous results for a variety of other lattices. Barnard, Defant, and Hanson then unified several of these results by providing a simple Coxeter-theoretic description of the image of Pop on an arbitrary Cambrian lattice of a finite Coxeter group. Once again, we aim to generalize Hong's results concerning Tamari lattices in a different direction by studying Pop images on ornamentation lattices.

1.3. **Main Results.** Let $\mathsf{T} \in \mathrm{PT}_n$. Unless otherwise stated, we will write Pop for the pop-stack operator on the ornamentation lattice $\mathcal{O}(\mathsf{T})$. The *depth* of an element $v \in \mathsf{T}$ is the quantity $\mathrm{depth}_{\mathsf{T}}(v) = |\nabla_{\mathsf{T}}(v)| - 1$. Let $\mathcal{M}_{\mathsf{T}}$ denote the set of maximal chains in T.

Theorem 1.1. *Let $\mathsf{T} \in \mathrm{PT}_n$, and let $\mathfrak{r}$ be the root of T. Then*

$$\max_{\delta \in \mathcal{O}(\mathsf{T})} |\mathrm{Orb}_{\mathsf{Pop}}(\delta)| = \max_{C \in \mathcal{M}_{\mathsf{T}}} \min_{v \in C \setminus \{\mathfrak{r}\}} (|\Delta_{\mathsf{T}}(v)| + 2\,\mathrm{depth}_{\mathsf{T}}(v) - 1).$$

If T is a chain with n nodes, then $\mathcal{O}(\mathsf{T})$ is the n-th Tamari lattice. In this special case, Theorem 1.1 tells us that the maximum size of a forward orbit of Pop on the n-th Tamari lattice is n, which agrees with one of the results in [10].

We next describe the image of Pop in ornamentation lattices. Let $\delta \in \mathcal{O}_{\mathsf{T}}$. For convenience, we add to T an "imaginary" node ω, and let $\delta(\omega) = \{\omega\} \cup \mathsf{T}$. We impose that the ornament hung at ω is left unaffected by Pop. Define the semilattice Ω_δ^* whose elements are $\mathsf{T} \cup \{\omega\}$, ordered by $v >_{\Omega_\delta^*} u \Leftrightarrow \delta(v) \supseteq \delta(u)$. We say that a node v *wraps* u in δ if $v \gtrdot_{\Omega_\delta^*} u$. We then say that v *hugs* u if v wraps u, and there exists a child $u' \in \mathrm{ch}_{\delta(u)}(u)$ such that $\Delta_{\delta(u)}(u') = \Delta_{\delta(v)}(u')$. In other words, v hugs u if v's ornament dominates u's ornament, and leaves no "slack" – extra nodes contained in $\delta(v)$ but not in $\delta(u)$ – to one of u's children in $\delta(u)$. Note that a node may be hugged by the imaginary node ω under our definition. The following theorem characterizes ornamentations lying in the image of Pop.

Theorem 1.2. *Let $\delta \in \mathcal{O}_{\mathsf{T}}$. Then $\delta \in \mathsf{Pop}(\mathcal{O}_{\mathsf{T}})$ if and only if, for each $u \in \mathsf{T}$, $|\delta(u)| = 1$ or u is not hugged by any $v \in \mathsf{T} \cup \{\omega\}$.*

Later in this paper, we also give a necessary condition for ornamentations lying in the image of Pop^k. The condition we give is also sufficient in the case of the n-node chain graph C_n, and as such it allows us to characterize the image of Pop^k in the n-Tamari lattice. We also find that the size of the image $|\mathsf{Pop}^k(\mathcal{O}_{\mathsf{C}_n})| = |\mathsf{Pop}^k(\mathrm{Tam}_n)|$ takes the form of the generalized Catalan numbers

$$C_{n,k} = \sum_{i=1}^{n-k} C_{i-1,k} C_{n-i,k}.$$

When $k = 1$, we see that the sizes of the images $|\mathsf{Pop}(\mathrm{Tam}_n)|$ give the Motzkin numbers, generalizing a result of Hong [14, Theorem 1.2].

We also prove that every ornamentation lattice $\mathcal{O}(\mathsf{T})$ is in fact semdistributive. This means that $\mathcal{O}(\mathsf{T})$ has a canonical join complex, the number of facets of which is $|\mathsf{Pop}(\mathcal{O}(\mathsf{T}))|$. See [12, Corollary 9.10] for other interpretations of the size of the image of Pop on a semidistributive lattice.

(End of excerpt)

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