Introduction to Deep Learning

Generative Adversarial Networks

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Outline

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Generative Models

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Such, we can sample from the model

$$data \sim p_{model}\left(x|\Theta\right)$$

Why the generative model?

Realistic samples generation and handling of missing data.

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Address the density estimation problem in unsupervised learning

- Yes unsupervised learning depends a lot on knn properties

It solves the problem of generating new data

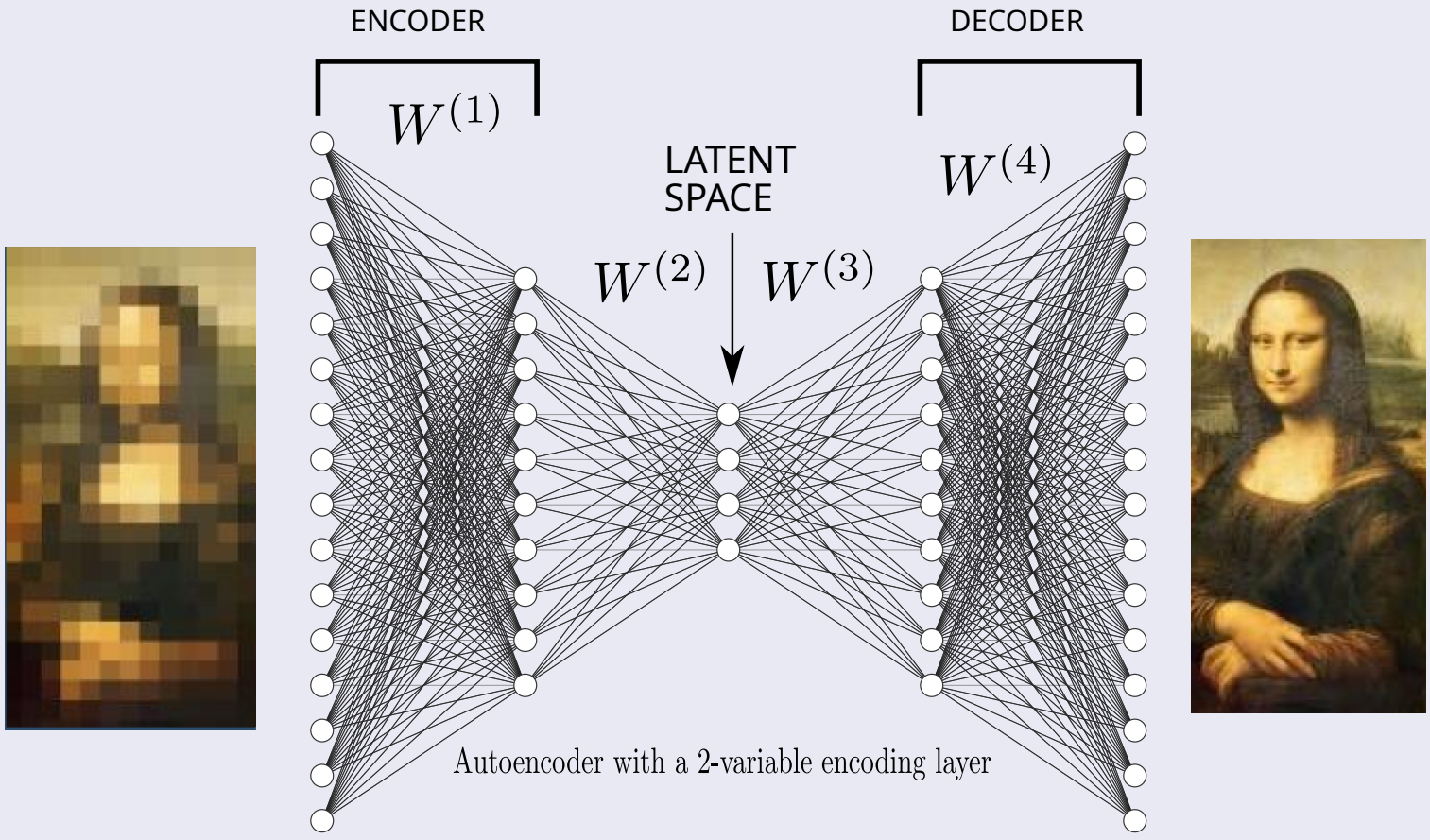
- Generate new data without human intervention

Outline

Remember Autoencoders

Do you remember?

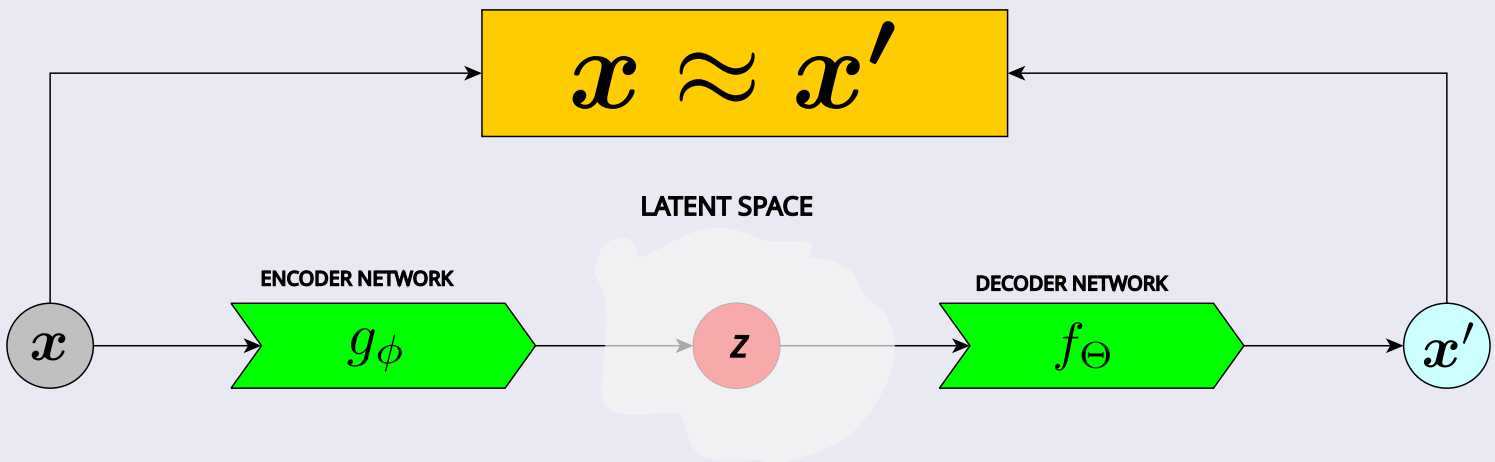
- "An autoencoder is a specific type of a neural network, which is mainly designed to encode the input into a compressed and meaningful representation, and then decode it back such that the reconstructed input is similar as possible to the original one."



Here, we have

With the following loss function

$$\mathcal{L}_{AE} = \frac{1}{n}\sum_{i=1}^{n} \|x - f_\Theta\left(g_\phi\left(x\right)\right)\|^2$$



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Samples, although reduced, are coming from a complex distribution

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Thus, we need as Markov Chain Monte Carlo Methods [1]

- In the case of the Cumulative Distribution Probability of $P_{complex}\left(z|\Theta\right)$ is not invertible or too complex
  - Which is almost always the case!!!

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Thus, Generative Adversarial Networks

- A escape way beyond Markov Chains [1]

Outline

This comes from the following paper

An notable work

- I. Goodfellow, J. Pouget-Abadie, M. Mirza, B. Xu, D. Warde-Farley, S. Ozair, A. Courville, and Y. Bengio, "Generative adversarial nets," in Advances in neural information processing systems, pp. 2672–2680, 2014.

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A expansion beyond Monte Carlo

- Given than GAN's do not depend on them

In addition

We are grateful to the notes by

- Scott Rome
  - "An Annotated Proof of Generative Adversarial Networks with Implementation Notes"
  - https://srome.github.io/An-Annotated-Proof-of-Generative-Adversarial-Networks-with-Implementation-Notes/

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The other is called the discriminator network (D)

- Also a MLP given that its inpiration was Variational Autoencoders

The Game

Something Notable

- The first player tries to generate a sample able to fool the discriminator player

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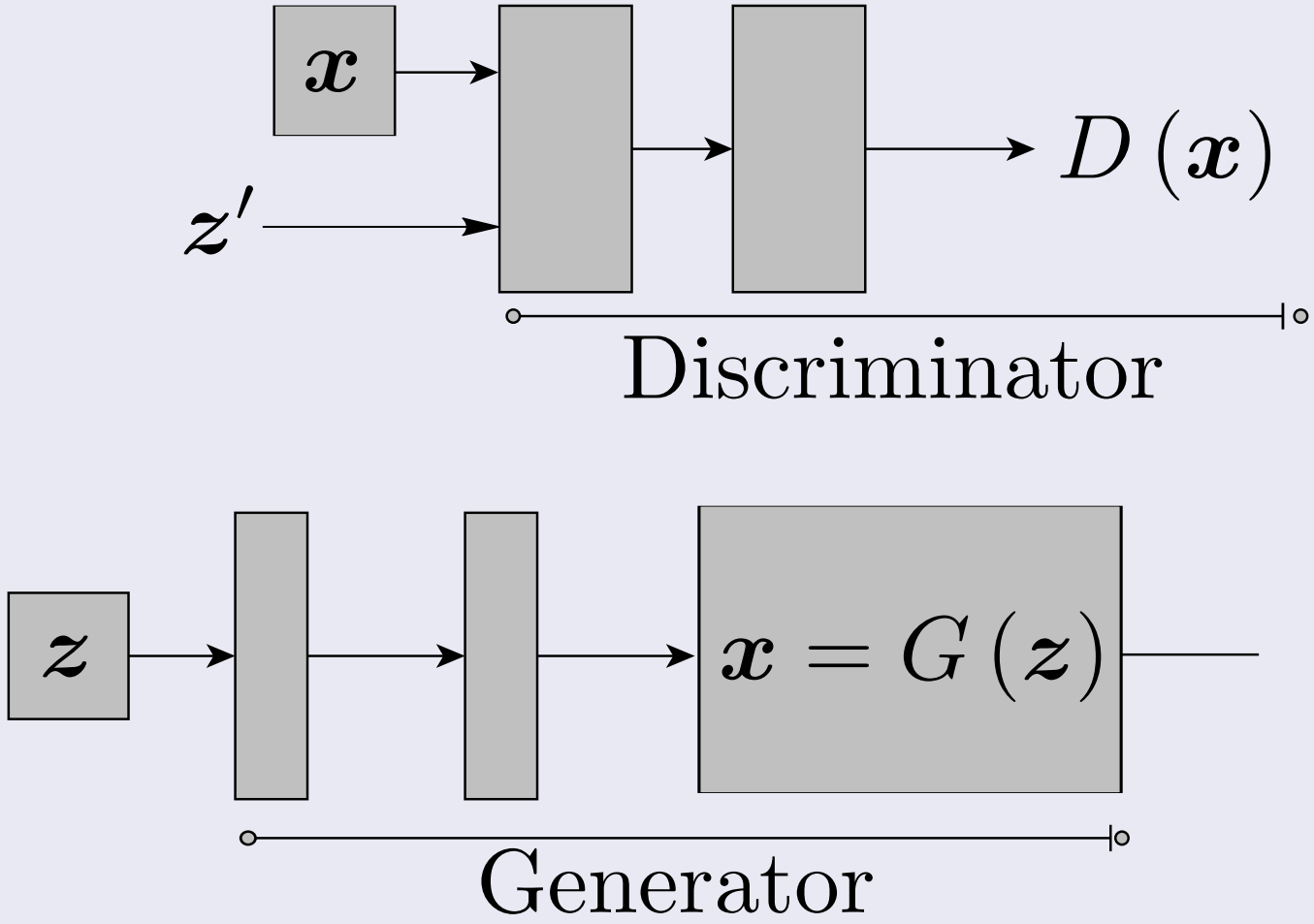
- The Generator works by trying to generate better fake results

The Discriminator tries to be better at finding

- if you have a real sample or not

Graphically

We have the following Basic Model



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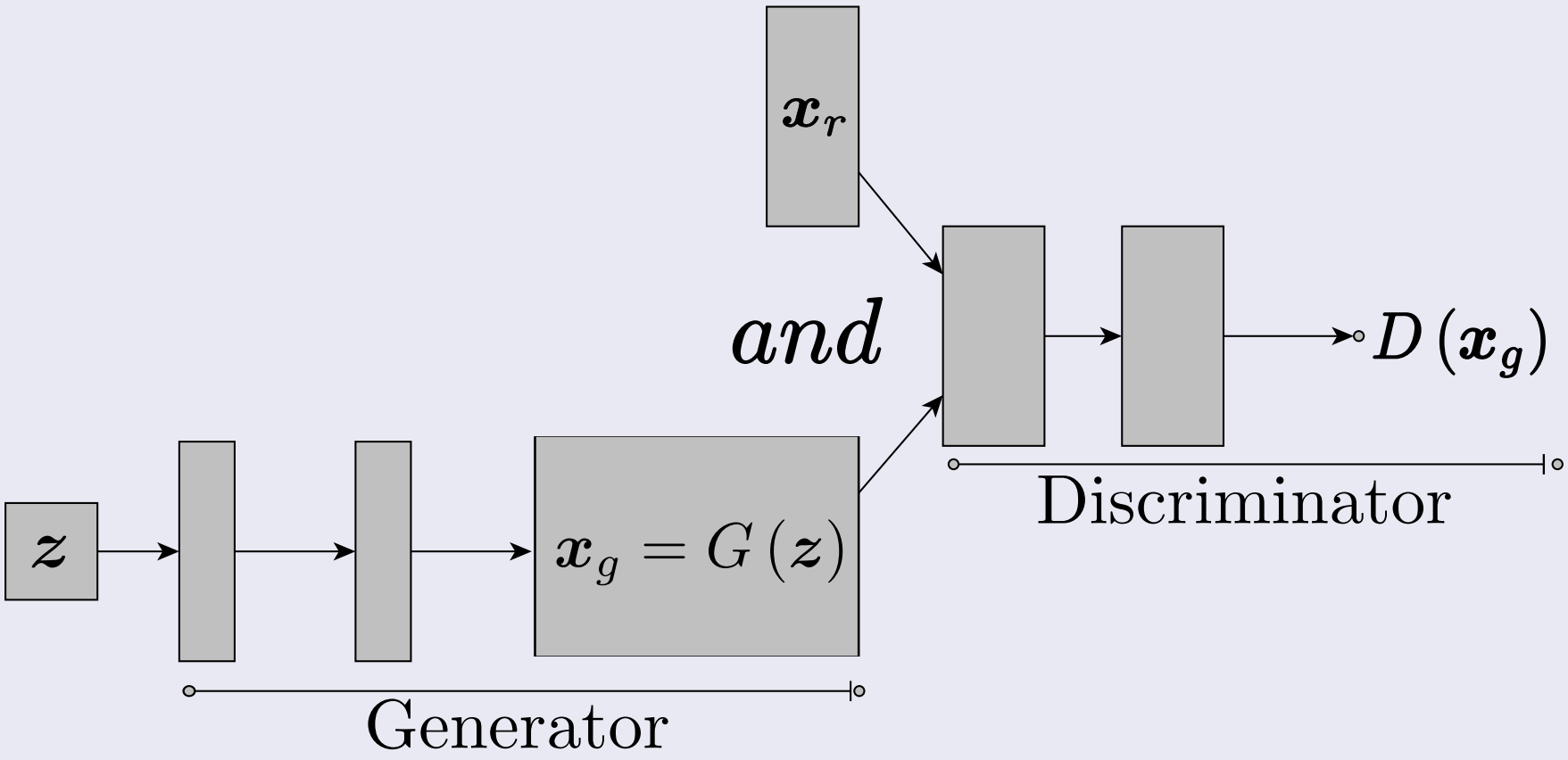
For this, we can use the **logistic cross-entropy loss** (We will explain more about this later)

$$\mathcal{L}_{LCE}(z,t) = L_{CE}(\sigma(z),t) = t\log(\sigma(z)) + (1-t)\log(1-\sigma(z))$$

Therefore, we have

The following architecture use this idea



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Basically, we have [2]

$$E_{x \sim p_{data}(x)} \left[\log \left(D \left(x \right) \right) \right]$$

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Properties

- Maximizing this term corresponds to D being able to precisely predict $D(x) = 1$ when $x \sim p_{data} \left(x \right)$

Now for the generator

The next term has to do with the Generator G tricking the discriminator.

- In particular, the term comes from the "negative class" of the log-loss function

$$E_{z \sim p_z(z)} \left[\log \left(1 - D \left(G \left(z \right) \right) \right) \right]$$

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$$E_{z \sim p_z(z)} \left[\log \left(1 - D \left(G \left(z \right) \right) \right) \right]$$

If this value is maximized because $\log(x) < 0$ when $x < 1$

- Meaning that $D(G(z)) \approx 0$ and G is NOT tricking D.

The Discriminator Goal

To combine these two concepts, the Discriminator's goal is to maximize

$$\max_D \left\{ E_{x \sim p_{data}(x)} \left[\log \left(D \left(x \right) \right) \right] + E_{z \sim p_z(z)} \left[\log \left(1 - D \left(G \left(z \right) \right) \right) \right] \right\}$$

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Given G

- which means that D properly identifies real and fake data points

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- which means that D properly identifies real and fake data points

Thus

- The optimal Discriminator in the sense of the above equation for a given G will be denoted D_G^*.

Meaning of this equation

Discriminant Training

- We have two parts the fake images and the real images, thus we can use $\log\left(D\left(x\right)\right)$ for the real images and the $\log\left(1-D\left(G\left(z\right)\right)\right)$ for the fake images

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Generative Training

- We need to generate images that need to pass fake images for real by using $\log\left(D\left(x\right)\right)$

For this we can use the following cost function BCE

$$-x\log\left(\sigma\left(z\right)\right)-\left(1-x\right)\log\left(1-\sigma\left(z\right)\right)$$

Thus the final cost function $V(G, D)$

We have that

$$V(G, D) = E_{x \sim p_{data}(x)} [\log (D(x))] + E_{z \sim p_z(z)} [\log (1 - D(G(z)))]$$

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Then, we have that

$$D_G^* = \arg \max_D V(G, D)$$

Now, G's aim is the reverse

The optimal G minimizes the previous equation when

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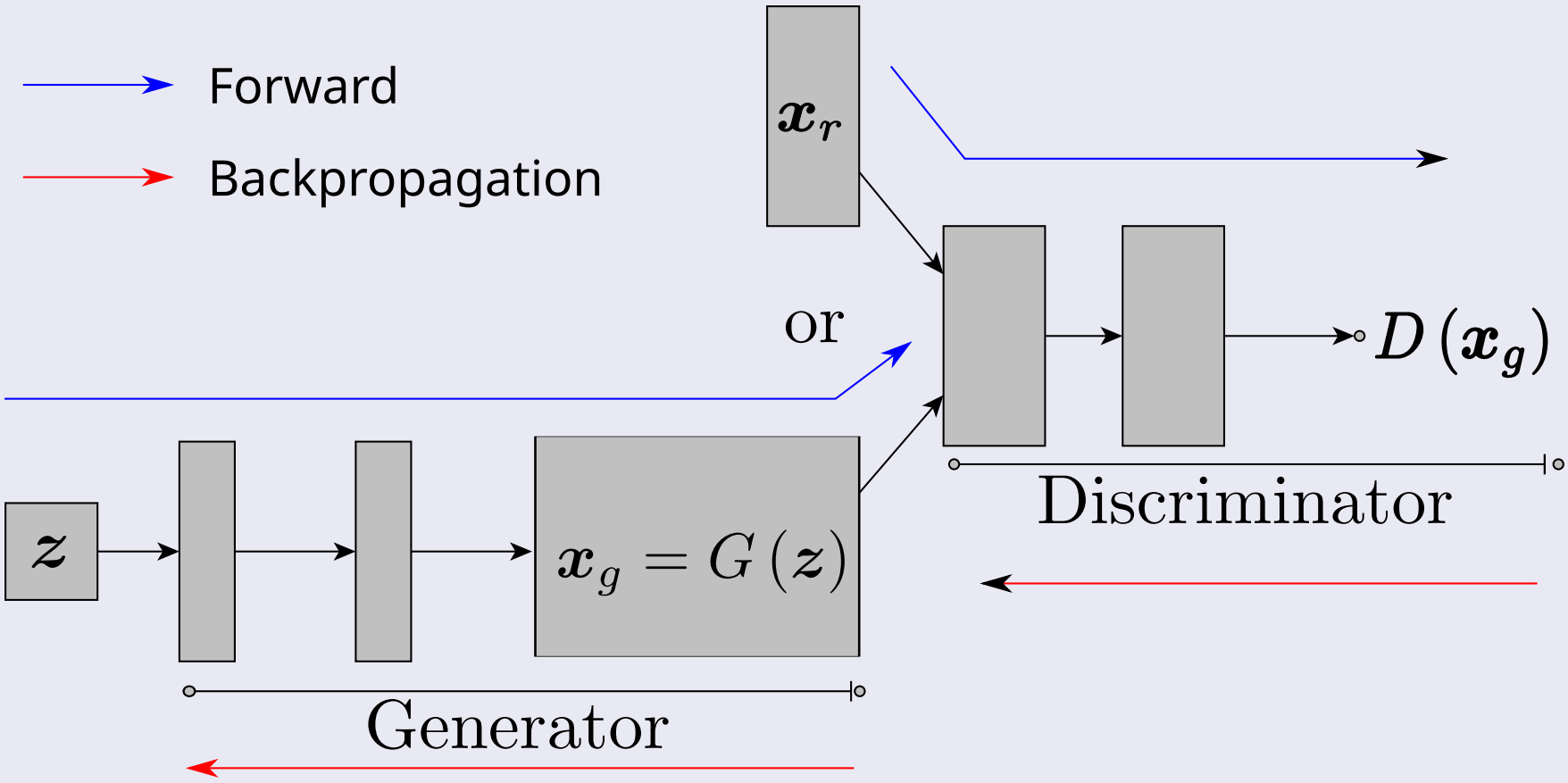
Basically

$$G^* = \arg\min_G V\left(G, D_G^*\right)$$

Therefore, we have two updates

First update the Discriminator



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One big idea from the GAN paper

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A really problematic proof

- We could try to replicate the proof incorrectly by using the classic change of variables of statistics
  - Needing an invertible G!!! Not a good idea

Yes, the following theorem does not work for GAN's

Theorem

- Suppose X is continuous with probability density function $f_x(x)$. Let $y = h(x)$ with h a strictly increasing continuously differentiable function with **inverse** $\boldsymbol{x = g(y)}$. Then $Y = h(X)$ defined by

$$P(c \leq Y \leq d) = P(a \leq Y \leq b),$$

where $a = g(c)$ and $b = g(d)$, is continuous with probability density function $f_Y(y)$ given by $f_Y(y) = f_X(g(y))\, g'(y)$.

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Clearly, we cannot use the previous theorem given its need of an inverse

- Not only that but the fact that h a strictly increasing continuously differentiable
  - This is too limiting!!!

Actually, you need the Law of the Unconscious Statistician

AKA LOTUS

- Given a random variable x with continuous distribution f_x then the expected value of a function $g : \mathbb{R} \to \mathbb{R}$ evaluated in x is

$$E_h\left[g\left(x\right)\right] = \int_{-\infty}^{\infty} g\left(x\right) f_x\left(x\right) dx$$

- h is the distribution of $g\left(x\right)$ which has support in the Radon–Nikodym Theorem [3].

Actually, we can apply it

Under continuous distribution and assuming $G(z) = x$ or $x \sim G$

$$E_{z \sim p_z(z)} \left[\log \left(1 - D \left(G \left(z \right) \right) \right) \right] = \int_x p_G \left(x \right) \log \left(1 - D \left(x \right) \right) dx$$

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Thus, we can do the following

$$\begin{aligned} V \left(G, D \right) &= \int_x p_{data} \left(x \right) \log D \left(x \right) dx + \int_z p \left(z \right) \log \left(1 - D \left(x \right) \right) dx \\ &= \int_x p_{data} \left(x \right) \log D \left(x \right) dx + \int_x p_G \left(x \right) \log \left(1 - D \left(x \right) \right) dx \\ &= \int_x \left[p_{data} \left(x \right) \log D \left(x \right) + p_G \left(x \right) \log \left(1 - D \left(x \right) \right) \right] dx \end{aligned}$$

Comments

If anybody wants to look at a great proof of Radon-Nikodym Theorem

- https://www.youtube.com/watch?v=sKQ6XLPymCg&t=2585s&ab_channel=InstitutodeMatem%C3%A1ticaPuraeAplicada

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A surprising property

- This makes the GAN not dependent on Markov Chains
  - MCMC's depend on having irreducible and aperiodic Markov Chains as targets [4]

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GAN's do not talk at all about such chains

- It could be included if necessary [5, 6]

But as the Bard comments

As always Horatio our philosophy falls shorts of our dreams

- "There are more things in heaven and Earth, Horatio, than are dreamt of in your philosophy."

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- As they say, we are the beach of the beginning of knowing the Universe

Another Beautiful Pebble has been lifted

- And it will take us to another new wonderful place

Now, which discriminator?

Given that the function can be rewritten as

$$f(y) = a \log y + b \log (1 - y)$$

Now, which discriminator?

Given that the function can be rewritten as

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How?

- First Derivative
- And Second Derivative Test

In this way, we have

Something Notable

$$V(G,D) \leq \int \max_{y} \left\{ p_{data}(x) \log y + p_G(x) \log(1-y) \right\} dx$$

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Under such bound, we have that

- If you let $D(x) = \frac{p_{data}(x)}{p_{data}(x)+p_G(x)}$ you achieve the maximum.

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Properties

- D is unique as well, as no other choice of D will achieve the maximum.

Outline

First than anything

The generator obtains the best when $p_G = p_{data}$

$$D_G^* = \frac{p_{data}}{p_{data} + p_G} = \frac{1}{2}$$

- This means that the Discriminator is completely confused.

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The authors then using this idea

- That such G is the solution of the minimax game

We have that

Theorem

- The global minimum of the virtual training criterion $C(G) = \max_D V(G, D)$ is achieved if and only if $p_G = p_{data}$.

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Proof - Backward Direction

- Assuming $p_G = p_{data}$

$$V(G, D_G^*) = \int_x p_{data}(x) \log\left(\frac{1}{2}\right) dx + p_G(x) \log\left(1 - \frac{1}{2}\right) dx$$

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Then, we have

$$\begin{aligned} V(G, D_G^*) &= -\log 2 \int_x p_{data}(x)\, dx - \log 2 \int_x p_G(x)\, dx \\ &= -2 \log 2 = -\log 4 \end{aligned}$$

A candidate for the minimum

We need to prove that is always the minimum

- For this, we drop the assumption that $p_G = p_{data}$.

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Observe that for any G

- We can plug D_G^* into $C(G) = \max_D V(G, D)$

$$C(G) = \int_x \left[p_{data}(x) \log \left(\frac{p_{data}(x)}{p_{data}(x) + p_G(x)} \right) + \ldots \right.$$
$$\left. p_G(x) \log \left(\frac{p_G(x)}{p_{data}(x) + p_G(x)} \right) \right] dx$$

Because

We have that

$$1 - D_G^* (x) = 1 - \frac{p_{data}(x)}{p_{data}(x) + p_G(x)}$$
$$= \frac{p_G(x)}{p_{data}(x) + p_G(x)}$$

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Now knowing that the minimum is $-\log 4$

$$C(G) = \int_x \left[\underbrace{(\log 2 - \log 2) p_{data}(x)}_{0} + p_{data}(x) \log \left(\frac{p_{data}(x)}{p_{data}(x) + p_G(x)} \right) \right.$$
$$\left. + \underbrace{(\log 2 - \log 2) p_G(x)}_{0} + p_G(x) \log \left(\frac{p_G(x)}{p_{data}(x) + p_G(x)} \right) \right] dx$$

We have the following

We ca rewrite $C(G)$

$$C(G) = -\log 2 \int_x \left[p_G(x) + p_{data}(x)\right] dx + \int \left[p_{data}(x) \left(\log 2 + \log \left[\frac{p_{data}(x)}{p_G(x) + p_{data}(x)}\right]\right) + p_G(x) \left(\log 2 + \log \left[\frac{p_G(x)}{p_G(x) + p_{data}(x)}\right]\right)\right] dx$$

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The first part of the equation

$$-\log 2 \left[\int_x \left[p_G(x) + p_{data}(x)\right] dx\right] = -2 \log 2 = -\log 4$$

Now, we have that

The following equality

$$\log 2 + \log\left[\frac{p_{data}(x)}{p_G(x) + p_{data}(x)}\right] = \log\left[2\frac{p_{data}(x)}{p_G(x) + p_{data}(x)}\right]$$
$$= \log\left[\frac{p_{data}(x)}{\frac{p_G(x)+p_{data}(x)}{2}}\right]$$

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$$= \log\left[\frac{p_{data}(x)}{\frac{p_G(x)+p_{data}(x)}{2}}\right]$$

Therefore, we have

$$C(G) = -\log 4 + \int_x p_{data}(x) \log\left[\frac{p_{data}(x)}{\frac{p_G(x)+p_{data}(x)}{2}}\right] dx + \int_x p_G(x) \log\left[\frac{p_G(x)}{\frac{p_G(x)+p_{data}(x)}{2}}\right] dx$$

A Series of Kullback-Leiber Divergences

We have that

$$C(G) = -\log 4 + KL\left[p_{data} || \frac{p_G(x) + p_{data}(x)}{2}\right] + KL\left[p_G || \frac{p_G(x) + p_{data}(x)}{2}\right]$$

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The Kullback-Leibler divergence is always non-negative

- we immediately see that $-\log 4$ is the global minimum of $C(G)$

Final Steps

If we prove only one G acheives this

- Because $p_G = p_{data}$ would be the unique point where $C(G) = -\log 4$.

Final Steps

If we prove only one G acheives this

- Because $p_G = p_{data}$ would be the unique point where $C(G) = -\log 4$.

we have to notice a second identity

- The Jenson-Shannon divergence

Jensen-Shannon Divergence

The Jensen–Shannon Divergence (JSD)

- $JSD : \mathcal{M}_+^1(A) \times \mathcal{M}_+^1(A) \to [0, \infty)$ is defined as

$$JSD(P||Q) = \frac{1}{2} KL(P||M) + \frac{1}{2} KL(Q||M)$$

- With $M = \frac{1}{2}(Q + P)$

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- With $M = \frac{1}{2}(Q + P)$

Property

$$0 \leq JSD(P||Q) \leq 1$$

$JSD(P||Q) = 0$

- When $P = Q$

Therefore, we have that

Thus the minimum $-\log 4$ happens

- When $p_G = p_{data}$ as

$$KL\left[p_{data}||p_{data}\right] + KL\left[p_G||p_G\right] = 0$$

Now, the Last Almost Theorem

Theorem

- If G and D have enough capacity, and at each step of the minimax game, the discriminator is allowed to reach its optimum D^* given G, and p_g is updated so as to improve the criterion

$$V\left(G, D^*\right) = E_{x \sim p_{data}(x)}\left[\log\left(D_G^*\left(x\right)\right)\right] + E_{x \sim p_g}\left[\log\left(1 - D_G^*\left(x\right)\right)\right]$$

then p_g converges to p_{data}.

Proof

Consider $V(G, D) = U(p_g, D)$ as a function of p_g as done in the above criterion.

- Note that $U(p_g, D)$ is convex in p_g

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How this happens? Given two different probability generators p_{g_1} and p_{g_2}

- Given $\lambda \in (0,1)$, we have that $\lambda p_{g_1} + (1-\lambda) p_{g_2}$

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How this happens? Given two different probability generators p_{g_1} and p_{g_2}

- Given $\lambda \in (0,1)$, we have that $\lambda p_{g_1} + (1-\lambda) p_{g_2}$

Then

$$U(\lambda p_{g_1} + (1-\lambda) p_{g_2}, D) = \int (\lambda p_{g_1} + (1-\lambda) p_{g_2}) \log(1 - D_G(x)) + const$$

Here

We can see that for U, $E_{x \sim p_{data}(x)} \left[\log \left(D_G \left(x \right) \right) \right]$ is a constant or convex by definition

$$const = E_{x \sim p_{data}(x)} \left[\log \left(D_G \left(x \right) \right) \right]$$

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$$const = E_{x\sim p_{data}(x)}\left[\log\left(D_G\left(x\right)\right)\right]$$

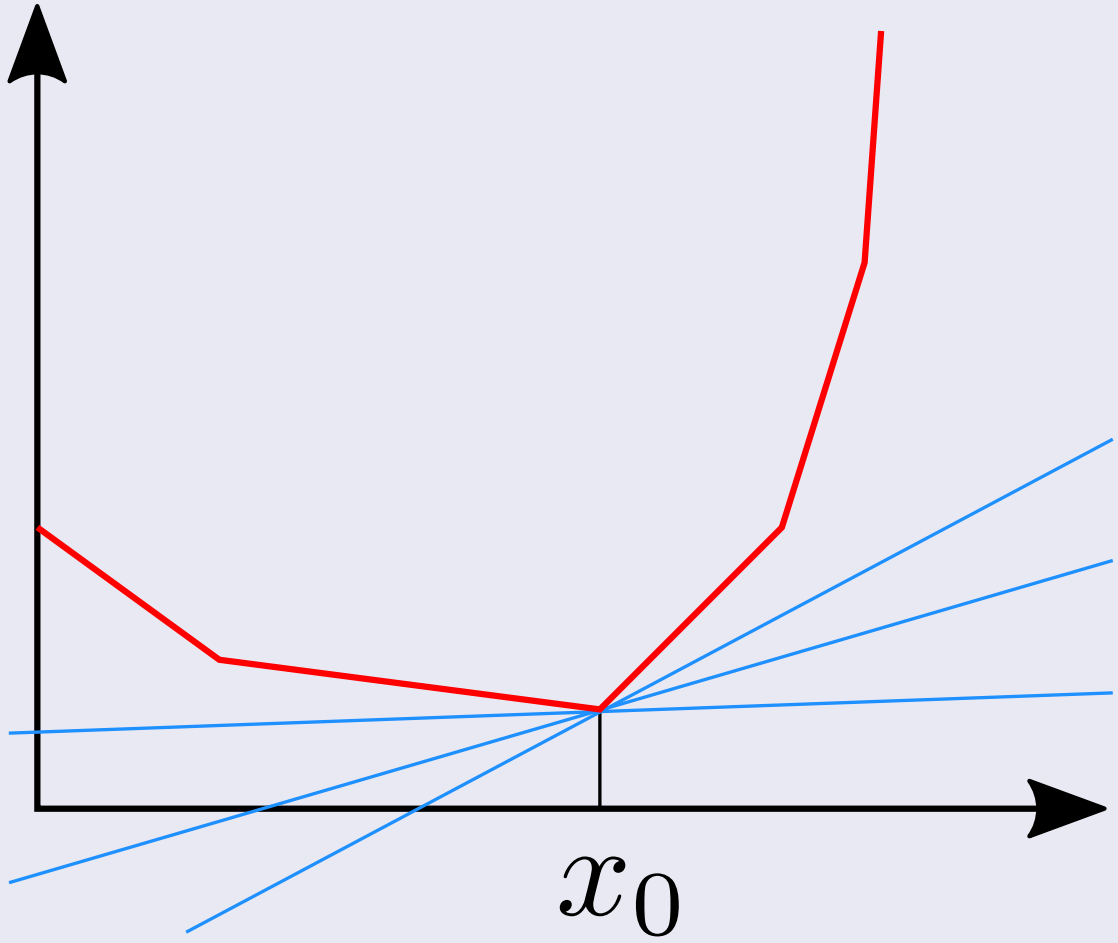
Then, we have that

$$\int \left(\lambda p_{g_1} + \left(1-\lambda\right)p_{g_2}\right)\log\left(1-D_G\left(x\right)\right) = \lambda E_{p_{g_1}}\left[1-D_G\left(x\right)\right] + ...$$
$$\left(1-\lambda\right)E_{p_{g_2}}\left[1-D_G\left(x\right)\right]$$

- Or a convex or concave function

Here, we introduce the concept of subderivatives

Basically used when convex functions are not derivable



Or a more mathematical version

Definition

- A subderivative of a convex function $f : I \to \mathbb{R}$ at a point x_0 in the open interval I is a real number c such that

$$f(x) - f(x_0) \geq c(x - x_0)$$

for all $x \in I$

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Therefore if we want to improve p_g we improve the generator

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Important!!!

- Here, $U(p_g, D^*)$ is convex and also concave at the same time

Therefore, for this special case

When looking at infimum, we use the supremum, remember $\min_G \max_D V(G, D)$

- And we know that

$$\sup \max_D V(G, D)$$

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When looking at infimum, we use the supremum, remember $\min_G \max_D V(G, D)$

- And we know that

$$\sup \max_D V(G, D)$$

Therefore, we define the following function

$$f(x) = \sup_{p_g} U(p_g(x), D^*)$$

- It is a convex function

Then

we can look at the argument of the sup for a fixed x

- Then, when getting the optimal argument
  $p_g^* (x) = \arg\sup_{p_g} U (p_g (x), D^*)$

Then

we can look at the argument of the sup for a fixed x

- Then, when getting the optimal argument $p_g^*(x) = \arg\sup_{p_g} U(p_g(x), D^*)$

Properties of the subdifferential

- A point $p_g^*(x)$ is a global minimum of a convex function f if and only if $\partial U\left(p_g^*(x), D^*\right) = 0$ is contained in the subdifferential set $\partial f(x)$

$$\partial U\left(p_g^*(x), D^*\right) = 0 \in \partial f(x)$$

Which is true because

$$\partial U\left(p_g^*(x), D^*\right) = 0$$

- Which happens given that at the optimal $U\left(p_g^*(x), D^*\right) = -\log 4$

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- Which happens given that at the optimal $U\left(p_g^*(x), D^*\right) = -\log 4$

Now a big problem we cannot equate this to a convergence by gradient descent

- Actually, we can only use the sub-gradient method

Sub-gradient method

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function with domain $\mathbb{R}^n$

- The subgradient method uses the iteration

$$x_{k+1} = x_k - \alpha_k g_k$$

  - Where g_k are sub-gradient of $f(x)$

Theorem

We have that

- For constant step size and constant step length, the subgradient method is guaranteed to converge to within some range of the optimal value

$$\lim f_{best}^{k} - U\left(p_{g}^{*}\left(x\right), D^{*}\right) < \epsilon$$

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For the diminishing step size rule

- The algorithm is guaranteed to o converge to the optimal value i.e.

$$\lim_{k->\infty} f\left(x_k\right) = U\left(p_g^*\left(x\right), D^*\right)$$

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The meaning of the reason of saying

- "with sufficiently small updates of p_g , p_g converges to p_x"

Here, we have a small situation

We have that the last theorem still needs more work

- Because different types of $U(p_g, D)$ will imply different results

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As they say no free lunch

- More study is needed before saying more about this

Outline

Implementation

Something Notable

- Most implementations do not explicitly perform the minimax optimization

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First, we define two models

- The first is the Discriminator $D(x)$
- The second is the Adversarial net $A(z) = D(G(z))$

Implementation

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- Most implementations do not explicitly perform the minimax optimization

First, we define two models

- The first is the Discriminator $D(x)$
- The second is the Adversarial net $A(z) = D(G(z))$

Note the following

- Notice, the adversarial net is also a classifier!!!

Thus

You can define

- 1 to be a **real image**
- 0 to be a **fake image**

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Train is done in two steps

- Train the Discriminator D to discriminate between real images and fake generated images via a standard 0-1 classification loss function.

Thus

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- 1 to be a **real image**
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Train is done in two steps

- Train the Discriminator D to discriminate between real images and fake generated images via a standard 0-1 classification loss function.

Then,

- Freeze the weights of D and train the adversarial network A with generated images with their labels forced to be 1.

GAN original algorithm

Minibatch stochastic gradient descent training

- **for** training iterations do
- **for** k steps **do**
- Sample minibatch of m noise samples $z_i \sim p_g(z)$
- Sample minibatch of m examples $x_i \sim p_{data}(x)$
- update by stochastic ascent
- $\nabla_{\theta_D} \frac{1}{m} \sum_{i=1}^{m} \left[\log\left(D\left(x_i\right)\right) + \log\left(1 - D\left(G\left(z_i\right)\right)\right) \right]$
- Sample minibatch of m noise samples $z_i \sim p_g(z)$
- Update generator
- $$\nabla_{\theta_G} \frac{1}{m} \sum_{i=1}^{m} \log\left(1 - D\left(G\left(z_i\right)\right)\right)$$

Clearly, there are other methods

Please take a look at

- https://github.com/soumith/ganhacks
  - How to Train a GAN? Tips and tricks to make GANs work
- Jabbar, Abdul, Xi Li, and Bourahla Omar. "A survey on generative adversarial networks: Variants, applications, and training." ACM Computing Surveys (CSUR) 54.8 (2021): 1-49.

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Last Word

- But not worry they are difficult to train!!!

Outline

Conditional GAN (CGAN) [7]

The objective function of a two-player minimax game

$$\min_G \max_D V(D,G) = E_{x\sim p_{data}(x)}[\log D(x|y)] + E_{z\sim p_z(z)}[1-\log D(G(z|y))]$$

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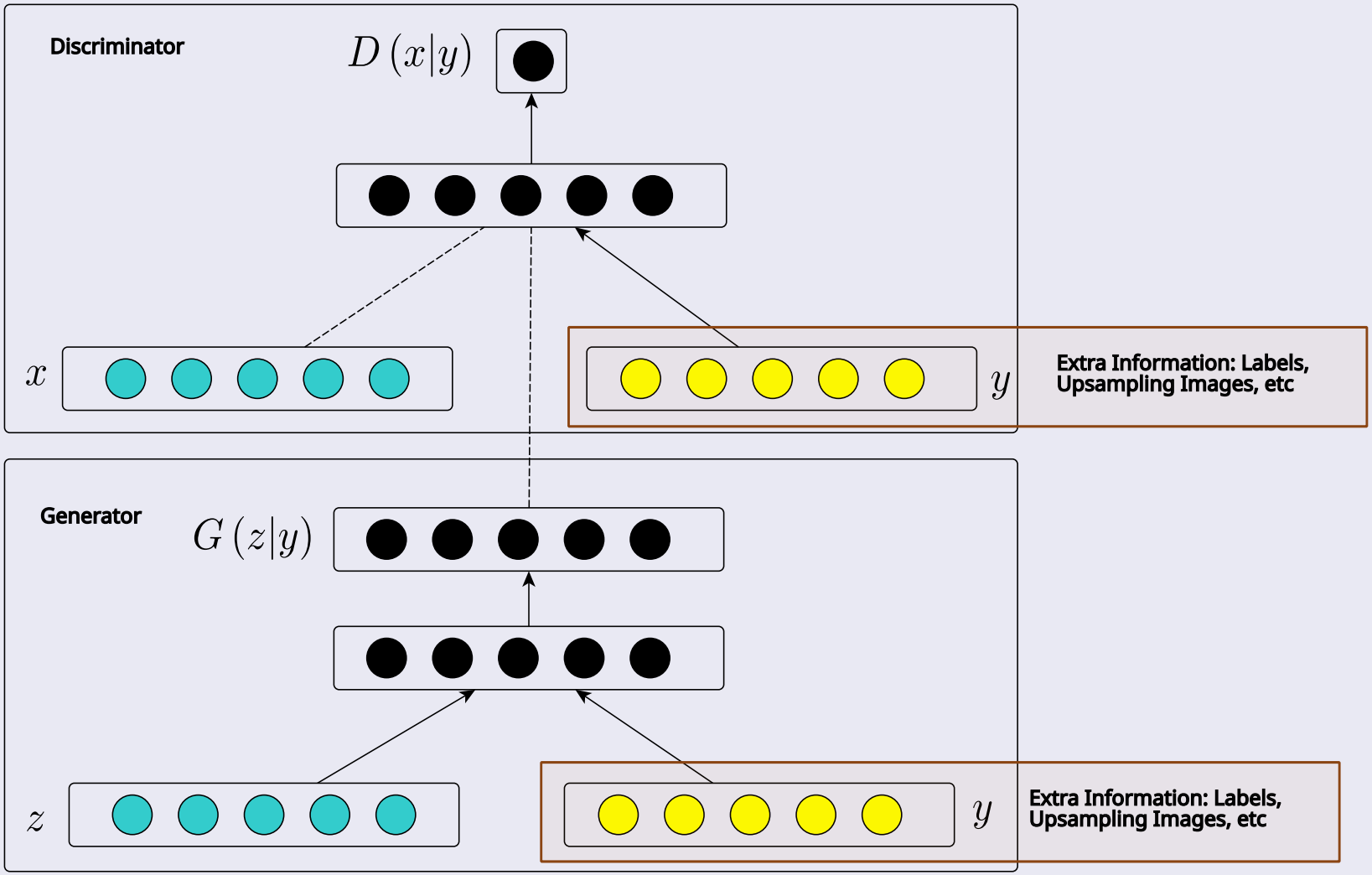
This is extra information

- y could be any kind of auxiliary information, such as
  - class labels or
  - data from other modalities.

Graphical Representation

We have the following representation



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Deep Convolutional GAN (DCGAN) [8]

DCGAN was the first structure that practiced de-convolutional neural networks (de-CNN)

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- For stabilization purposes... remember the transpose Convolution

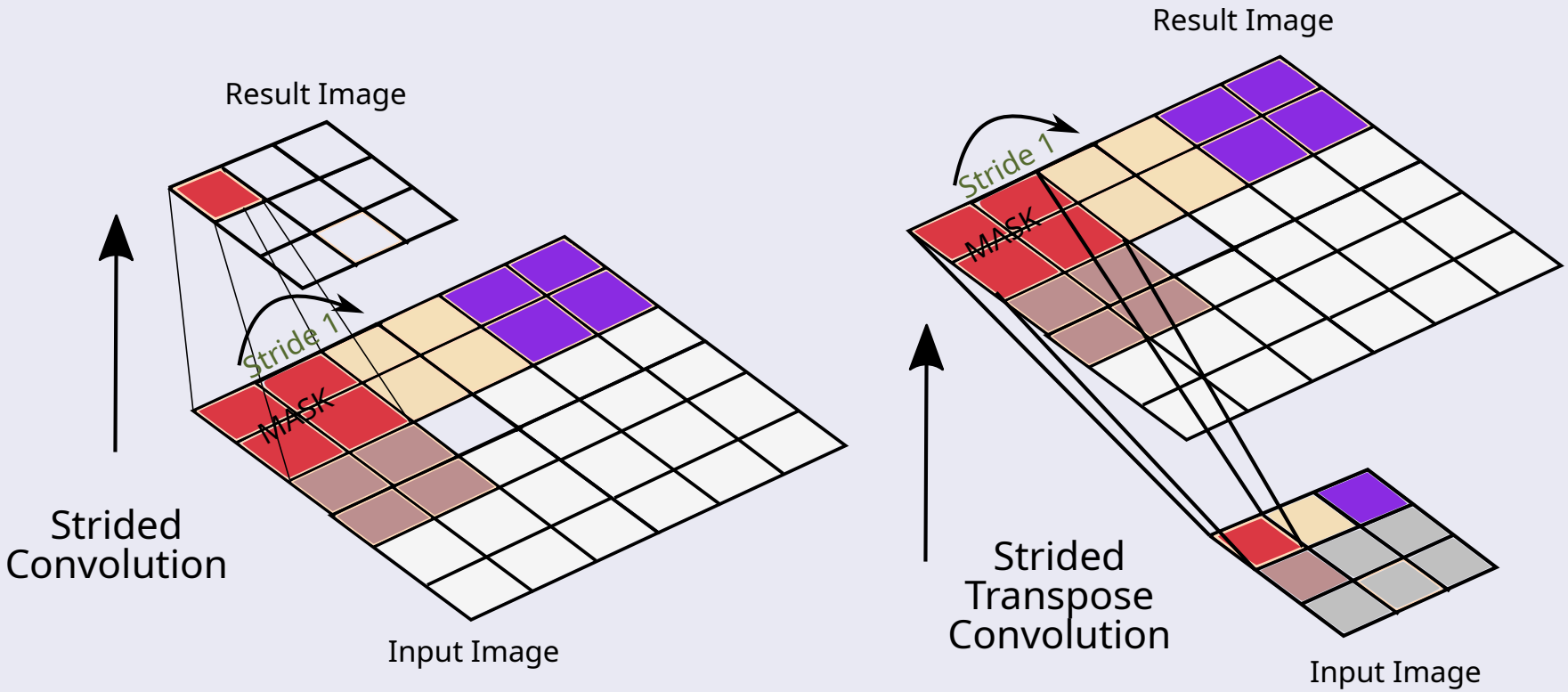
These frameworks consist of two networks

- One network works as a CNN called the generator
- The other network works as a de-CNN called discriminator

Thus

We have the following Architecture

- Architecture guidelines for stable Deep Convolutional GANs
  - Replace any pooling layers with strided convolutions (discriminator) and fractional-strided convolutions (generator).



Furthermore

Batch Normalization

- Use batch normalization in both the generator and the discriminator.

Furthermore

Batch Normalization

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Finally

- Remove fully connected hidden layers for deeper architectures.
- Use ReLU activation in generator for all layers except for the output, which uses Tanh.
- Use LeakyReLU activation in the discriminator for all layers.

Outline

Laplacian GAN [9]

A Combination of

- The CGAN
- Laplacian Pyramid

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Laplacian Pyramid

- Somewhat convoluted process
  - First a Gaussian Pyramid is built

$$G(I) = [I_0, I_1, ..., I_K]$$

Here, we have that

$I_0 = I$ the original image

- Here, we use a down sampling operation d which blurs and decimates a $\frac{j}{2} \times \frac{j}{2}$image I,
  - We could use a Convolution

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Where, we have that the next image I_i is a $\frac{j}{2} \times \frac{j}{2}$

- We also need an up sampling operator u which smooths and expand an image from $\frac{j}{2} \times \frac{j}{2}$ to $\frac{j}{2} \times \frac{j}{2}$
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Using the down sampling, it is easy to build

$$G(I) = [I_0, I_1, ..., I_K]$$

Now, Laplacian Pyramid $\mathscr{L}(\mathscr{I})$

They are constructed by taking the difference between adjacent levels in the Gaussian pyramid

$$h_k = \mathscr{L}_k(I) = \mathcal{G}_k(I) - u(\mathcal{G}_{k+1}(I)) = I_k - u(I_{k+1})$$

- Intuitively, each level captures image structure present at a particular scale.

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The final level of the Laplacian pyramid h_K is not a difference image

- but a Low-Frequency residual equal to the final Gaussian pyramid, $h_K = I_K$

And here comes Deep Learning

Basically, we generate two sequences

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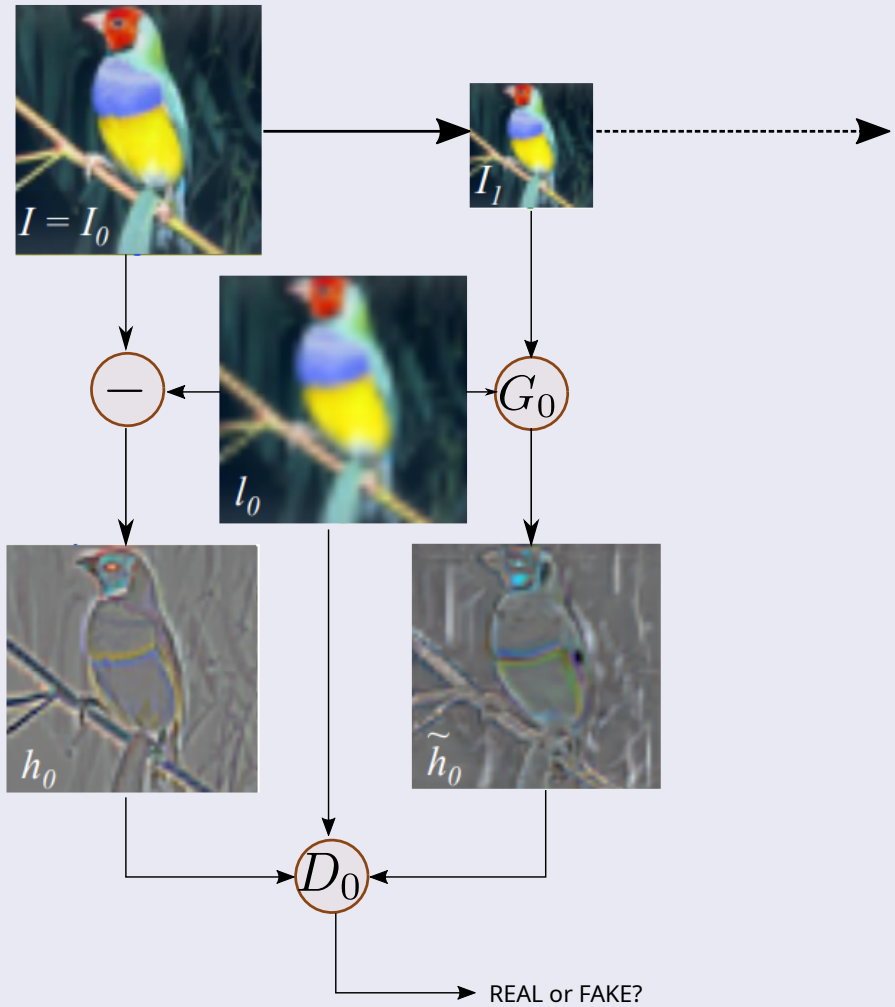
When reconstructing the image or sampling from the generatives

$$\begin{aligned}\widetilde{I}_k =& u\left(\widetilde{I}_{k+1}\right) + \widetilde{h}_k \\ =& u\left(\widetilde{I}_{k+1}\right) + G_k\left(z_k, u\left(\widetilde{I}_{k+1}\right)\right)\end{aligned}$$

- Where $G_k\left(z_k, u\left(\widetilde{I}_{k+1}\right)\right) = G\left(z_k | u\left(\widetilde{I}_{k+1}\right)\right)$

Training of Generative Model

Starting at $k = 0$



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The heuristic computes the error for a given validation image at level k

$$L_k\left(I_k\right) = \min_{\{z_j\}} \left\| G_k\left(z_k, u\left(I_{k+1}\right)\right) - h_k \right\|_2$$

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In other words, the heuristic is asking

- **Are any of the generated residual images close to the ground truth?**

However

The cost functions used at each D_k and G_k

- It is the Binary Cross Entropy - Fake or Real

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However

- We want to train all that in an automatic fashion

The Possible Loss Function

The Final Loss Function

$$
\begin{aligned}
L = & E_{x \sim p_{data}(x)} \left[\log D \left(\widetilde{I}_k | u \left(I_{k+1} \right) \right) \right] + ... \\
& E_{z \sim p_z(z)} \left[1 - \log D \left(G \left(z | u \left(I_{k+1} \right) \right) \right) \right] + ... \\
& \sum \min_{\{z_j\}} \left\| G_k \left(z_k | u \left(\widetilde{I}_{k+1} \right) \right) - h_k \right\|_2
\end{aligned}
$$

Outline

Information Maximizing GAN [10]

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- We do not know what the optimal representations are for solving a specific problem.
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- We can use unsupervised learning for representation learning.

Basically

- We are learning reusable feature representations

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Tries to decompose the noise z in two parts

- z, which is treated as source of incompressible noise
- c, which we will call the latent code

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In this way

- c will target the salient structured semantic features of the data distribution.

Therefore

We denote the set of structured latent variables by

- $c_1, c_2, ..., c_L$ with $P(c_1, c_2, ..., c_L) = \prod_{i=1}^{L} P(c_i)$ or you can get each of them independently

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Thus, the proposal

- To have a generator with two variables $G(z, c)$

The proposal is to find codes with high information between c and $G(z, c)$

- For this, we can look at mutual information

Here

We have that

$$I(X;Y) = H(X) - H(X|Y)$$

- Where $\mathrm{H}(X) = -\sum_{i=1}^{n} \mathrm{P}(x_i) \log(x_i)$ and

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Which has the following intuition

- If X and Y are independent, then $I(X;Y) = 0$,
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Which has the following intuition

- If X and Y are independent, then $I(X;Y) = 0$,
  - because knowing one variable reveals nothing about the other

If X and Y are related by a deterministic, invertible function

- Then maximal mutual information is attained.

Given that $I(X;Y) \geq 0$

Consequently

- $H(X) \geq H(X|Y)$

Given that $I(X;Y) \geq 0$

Consequently

- $H(X) \geq H(X|Y)$

Or if you want

- $-\sum_{i=1}^{n} \mathrm{P}(x_i) \log(x_i) \geq -\sum_{i=1}^{n} \mathrm{P}(x_i|y_i) \log(x_i|y_i)$

Using the previous interpretation

This interpretation makes it easy to formulate a cost

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In other words

- The information in the latent code c should not be lost in the generation process.

This allows to develop a new cost function

The new cost function

$$\min_{G} \max_{D} V_1(D, G) = V(D, G) - \lambda I(c, G(z, c))$$

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- The mutual information term $I(c, G(z,c))$ is hard to maximize as you need $P(c|x)$

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However

- The mutual information term $I(c, G(z,c))$ is hard to maximize as you need $P(c|x)$

Thus, we use a variational approach

- For this, we define an auxiliary distribution $Q(c|x)$ to approximate $P(c|x)$

We have this

The following equation

$$
\begin{aligned}
I\left(c; G\left(z, c\right)\right) =& H\left(c\right) - H\left(c|G\left(z, c\right)\right) \\
=& \mathbb{E}_{x \sim G(z,c)} \left[\mathbb{E}_{c' \sim P(c|x)} \log P\left(c'|x\right)\right] + H\left(c\right) \\
=& \mathbb{E}_{x \sim G(z,c)} \left[\underbrace{D_{KL}\left(P\left(\cdot|x\right)\right) \| Q\left(\cdot|x\right)}_{\geq 0} + \mathbb{E}_{c' \sim P(c|x)} \log Q\left(c'|x\right)\right] + H\left(c\right) \\
\geq& \mathbb{E}_{x \sim G(z,c)} \left[\mathbb{E}_{c' \sim P(c|x)} \log Q\left(c'|x\right)\right] + H\left(c\right)
\end{aligned}
$$

They proved the following lemma

Lemma

- For random variables X, Y and function $f(x, y)$ under suitable regularity conditions:

$$\mathbb{E}_{x\sim X, y\sim Y|x}\left[f\left(x, y\right)\right] = \mathbb{E}_{x\sim X, y\sim Y|x, x'\sim X|y}\left[f\left(x', y\right)\right]$$

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Thus, we can prove that

$$\begin{aligned} L_I\left(G, Q\right) =& \mathbb{E}_{c\sim P(x), x\sim G(z,c)}\left[\log Q\left(c|x\right)\right] + H\left(c\right) \\ =& \mathbb{E}_{x\sim G(z,c)}\left[\mathbb{E}_{c'\sim P(c|x)}\left[\log Q\left(c'|x\right)\right]\right] + H\left(c\right) \\ \leq& I\left(c; G\left(z, c\right)\right) \end{aligned}$$

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This is easy to approximate using Monte Carlo simulation

- In particular, $L_I\left(G, Q\right)$ can be maximized w.r.t. Q directly and w.r.t. G via the re-parametrization trick.

In this way

We have that

$$\min_{G,Q} \max_{D} V_{InfoGAN}(D,G,Q) = V(D,G) - \lambda L_I(G,Q)$$

Outline

They asked [11]

What is the meaning of learning a distribution?

- We always go for a parametric family of densities $(P_\theta)_{\theta \in \mathbb{R}^d}$ when we have a data set $\{x_i\}_{i=1}^N$

$$\max_{\theta \in \mathbb{R}^d} \frac{1}{N} \sum_{i=1}^{N} \log P_\theta (x_i)$$

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Thus, given the $\mathbb{P}_{data}$ (Over all possible data) and $\mathbb{P}_\theta$ final min result

- We want to

$$\min_{\mathbb{P}_\theta} KL \left[\mathbb{P}_{data} || \mathbb{P}_\theta \right]$$

They asked the following

How to improve

- on the various ways to define a distance or divergence $\rho\left(\mathbb{P}_{data}, \mathbb{P}_{\theta}\right)$

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Given a sequence $\{\theta_t\}$ with limit θ

- Continuity means that when a sequence of parameters θ_t converges to θ, the distributions $\mathbb{P}_{\theta_t}$ also converge to $\mathbb{P}_{\theta}$.

Thus, they asked for

A weaker distance

- The weaker this distance, the easier it is to define a continuous mapping from θ-space to $\mathbb{P}_\theta$-space

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After all given a weaker distance ρ

- We would love to have the following loss function

$$\mathscr{L} : \mathbb{R}^d \to \mathbb{R}, \mathscr{L}(\theta) = \rho(\mathbb{P}_\theta, \mathbb{P}_{data})$$

Then, they studied elementary distances and divergences

The Total Variation (TV) distance

$$\delta\left(\mathbb{P}_\theta, \mathbb{P}_{data}\right) = \sup_{A \in \Sigma} \left|\mathbb{P}_\theta\left(A\right) - \mathbb{P}_{data}\left(A\right)\right|$$

- Actually over a measurable space (X, Σ)

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The Kullback-Leibler (KL) divergence

$$KL\left(\mathbb{P}_{data} || \mathbb{P}_\theta\right) = \int \log\left(\frac{\mathbb{P}_{data}\left(x\right)}{\mathbb{P}_\theta\left(x\right)}\right) \mathbb{P}_{data}\left(x\right) d\mu\left(x\right)$$

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The Jensen-Shannon (JS) divergence

$$JS\left(\mathbb{P}_{data}, \mathbb{P}_\theta\right) = KL\left(\mathbb{P}_{data} || \mathbb{P}_m\right) + KL\left(\mathbb{P}_\theta || \mathbb{P}_m\right)$$

- where $\mathbb{P}_m = \frac{\mathbb{P}_\theta + \mathbb{P}_{data}}{2}$

Finally

The Earth-Mover (EM) distance or Wasserstein-1

$$W\left(\mathbb{P}_{data}, \mathbb{P}_{\theta}\right) = \inf_{\gamma \in \prod\left(\mathbb{P}_{data}, \mathbb{P}_{\theta}\right)} \mathbb{E}_{(x,y)\sim\gamma}\left[\|x - y\|\right]$$

- where $\prod\left(\mathbb{P}_{data}, \mathbb{P}_{\theta}\right)$ is the set of all joint distributions $\gamma\left(x, y\right)$ whose marginals are respectively $\mathbb{P}_{data}, \mathbb{P}_{\theta}$.

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Something Notable

- Intuitively, $\gamma\left(x, y\right)$ indicates how much "mass" must be transported from x to y in order to transform the distributions $\mathbb{P}_{data}$ into the distribution $\mathbb{P}_{\theta}$.

Here they did a proof pointing to the Wasserstein-1

Theorem

- Let $\mathbb{P}$ be a distribution on a compact space $\mathcal{X}$ and $(\mathbb{P}_n)_{n\in\mathbb{N}}$ be a sequence of distributions on $\mathcal{X}$. Then, considering all limits as $n \to \infty$

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Theorem

- Let $\mathbb{P}$ be a distribution on a compact space $\mathcal{X}$ and $(\mathbb{P}_n)_{n\in\mathbb{N}}$ be a sequence of distributions on $\mathcal{X}$. Then, considering all limits as $n \to \infty$

The following statements are equivalent

1. $W(\mathbb{P}_n, \mathbb{P}_0) \to 0$
2. $\mathbb{P}_n \xrightarrow{\mathcal{D}} \mathbb{P}_0$ where $\xrightarrow{\mathcal{D}}$ represents convergence in distribution for random variables.

In this way, we have

The Kantorovich-Rubinstein duality [12]

$$
\begin{aligned}
W(P,Q) &= \inf_{\pi \in \prod(P,Q)} \mathbb{E}\left[\|x-y\|_2\right] \\
&= \sup_{\|h\|_L \leq 1} \left[\mathbb{E}_{x\sim P}\left[h(x)\right] - \mathbb{E}_{x\sim Q}\left[h(x)\right]\right]
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Applied to the GAN problem

$$\sup_{\|f\|_L \leq 1} \left[\mathbb{E}_{x\sim \mathbb{P}_r}\left[f(x)\right] - \mathbb{E}_{x\sim \mathbb{P}_\theta}\left[f(x)\right]\right]$$

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$$\sup_{\|f\|_L \leq 1} \left[\mathbb{E}_{x\sim \mathbb{P}_r}\left[f(x)\right] - \mathbb{E}_{x\sim \mathbb{P}_\theta}\left[f(x)\right]\right]$$

Where, we have that

- $f : \mathcal{X} \to \mathbb{R}$ a 1-Lipschitz function or Lipschitz continous with derative bounded by 1

Therefore

Therefore, if we have a parameterized family of functions $\{f_w\}_{w \in \mathcal{W}}$

- that are all K-Lipschitz for some K

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Therefore, if we have a parameterized family of functions $\{f_w\}_{w \in \mathcal{W}}$

- that are all K-Lipschitz for some K

we could consider solving the problem

$$\max_{w \in \mathcal{W}} \left[\mathbb{E}_{x \sim \mathbb{P}_{data}} \left[f_w \left(x \right) \right] - \mathbb{E}_{z \sim p(z)} \left[f_w \left(G_\theta \left(z \right) \right) \right] \right]$$

Therefore, we have that

Theorem

- Let $\mathbb{P}_{data}$ be any distribution. Let $\mathbb{P}_\theta$ be the distribution of $G_\theta(Z)$ with Z a random variable with density p and G_θ a function satisfying If it is continuous in θ, thus it is $W(\mathbb{P}_{data}, \mathbb{P}_\theta)$. Then, there is a solution $f : \mathcal{X} \to \mathbb{R}$ to the problem

$$\max_{w \in \mathcal{W}} \left[\mathbb{E}_{x \sim \mathbb{P}_{data}} [f_w(x)] - \mathbb{E}_{z \sim p(z)} [f_w(G_\theta(z))] \right]$$

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$$\max_{w \in \mathcal{W}} \left[\mathbb{E}_{x \sim \mathbb{P}_{data}} [f_w(x)] - \mathbb{E}_{z \sim p(z)} [f_w(G_\theta(z))] \right]$$

And we have that

$$\nabla_\theta W(\mathbb{P}_{data}, \mathbb{P}_\theta) = -\mathbb{E}_{z \sim p(z)} [\nabla_\theta f(G_\theta(z))]$$

Therefore

We have an algorithm

- That allows to obtain an approximation to the $\mathbb{P}_{data}$ using the Wasserstein loss W

For more in the subject

Take a look at

- Jabbar, Abdul, Xi Li, and Bourahla Omar. "A survey on generative adversarial networks: Variants, applications, and training." ACM Computing Surveys (CSUR) 54.8 (2021): 1-49.

Outline

Anime Character Generation

At Computer Games

- The creation of animated characters in computer games is expensive

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At Computer Games

- The creation of animated characters in computer games is expensive

There have great results using

- Progressive Structure-Conditional GAN (PS-CGAN) [13] generates high resolution (e.g.1024 X 1024), and full-body anime characters with particular sequences of poses.

Outline

Image Super-Resolution

The purpose of image super-resolution (SR) is to up-scale the images of low-resolution (LR)

- because images with high-resolution (HR) have better visual qualities.

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The purpose of image super-resolution (SR) is to up-scale the images of low-resolution (LR)

- because images with high-resolution (HR) have better visual qualities.

Examples of this techniques are

- Deep Enhanced SRGAN (ESRGAN) [14]
- Super Resolution Dual-GAN (SRDGAN) [15]
- etc

Example

We have that from [14]

Figure 7: (a) Ground truth, (b) SRGAN result, and (c) ESRGAN result are shown in the figure, which shows that ESRGAN delivers better visual performance than SRGAN. Images from [100].

Outline

Physics-Informed Deep Learning (PIDL)

Recetly, we have that [16]

- Given a labeled data set $\{(x_u, y_u)\}_{i=1}^{N_u}$

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Recetly, we have that [16]

- Given a labeled data set $\{(x_u, y_u)\}_{i=1}^{N_u}$

We are interested in learning a neural network model, $\widehat{y} = f_\theta(x)$

- We also want to ensure that $\widehat{y}$ satisfies K physics-based equations, $\left\{R^{(k)}(x, y) = 0\right\}_{k=1}^{K}$
  - on a larger set of unlabeled instances $\left\{x_{u_j}\right\}_{j=1}^{N_f}$ with $N_f \gg N_u$

Using a Physics-Based Loss

We define the following

$$\mathcal{L}(\theta) = \frac{1}{N_u} \sum_{i=1}^{N_u} \| y_{u_i} - \widehat{y}_{u_i} \| + \frac{\lambda}{N_f} \sum_{j=1}^{N_f} \sum_{k=1}^{K} R^{(k)} \left(x_{f_j}, \widehat{y}_{f_j} \right)^2$$

Furthermore

Using the previous equation, we have

$$\mathcal{L}_G(\theta) = \frac{1}{N_u}\sum_{i=1}^{N_u} D\left(x_{u_i}, \widehat{y}_{u_i}\right) + \frac{\lambda}{N_f}\sum_{j=1}^{N_f}\sum_{k=1}^{K} R^{(k)}\left(x_{f_j}, \widehat{y}_{f_j}\right)^2$$

$$\mathcal{L}_D(\theta) = \frac{1}{N_u}\sum_{i=1}^{N_u} \log D\left(x_{u_i}, \widehat{y}_{u_i}\right) + \frac{1}{N_u}\sum_{i=1}^{N_u} \log\left[1 - D\left(x_{u_i}, y_{u_i}\right)\right]$$

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Where

- $\widehat{y}_{u_i} = G\left(x_{u_i}, z_{u_i}\right)$ and $\widehat{y}_{f_j} = G\left(x_{f_j}, z_{f_j}\right)$

Example

From https://physicsbaseddeeplearning.org/intro.html

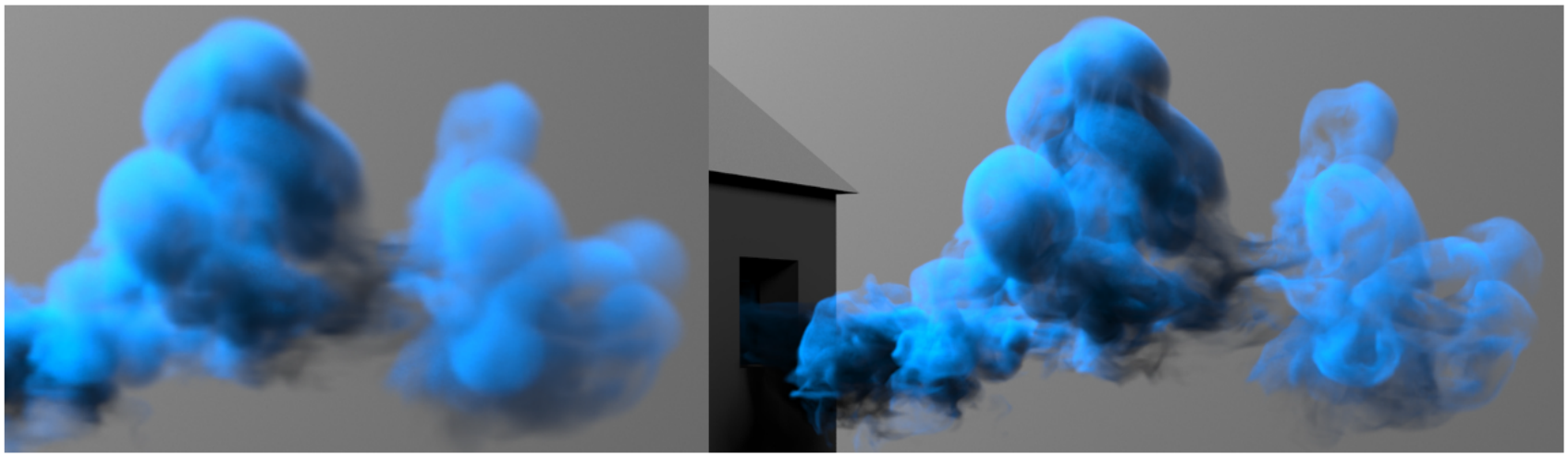

Fig. 31 GANs were shown to work well for tasks such as the inference of super-resolution solutions where the range of possible results can be highly ambiguous.

Conclusion

The explosion of GAN

- It has been incredible

Conclusion

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And Recently with the definition of

- Out Distribution and Adversarial Attacks

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The explosion of GAN

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And Recently with the definition of

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We are going on the direction of

- Synthetic Data!!!


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