Analysis of Algorithms

Complexity and Sorting

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Outline

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1. Algorithmic Complexity Analysis
   - Introduction
   - Asymptotic Bounds
   - Correctness of Algorithms
   - Excercises

Algorithm efficiency

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- Memory usage.
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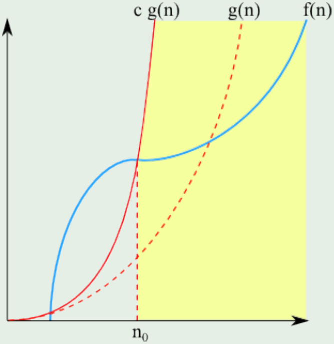
Big O (Upper bound)

Intuition

- Let $f(n)$ and $g(n)$ be two real valued functions, lets build intuition on the meaning of $f(n) \in O(g(n))$.



Definition

- $f(n) \in O(g(n))$ if there exists $c, n_0 > 0$ such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.

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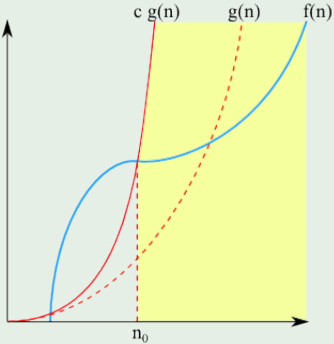

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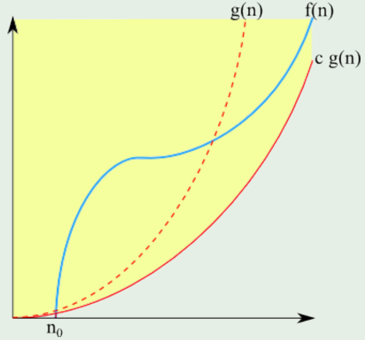

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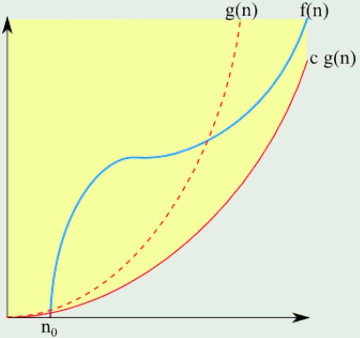

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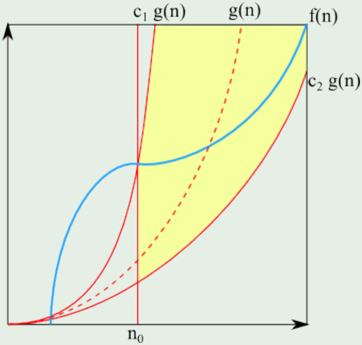

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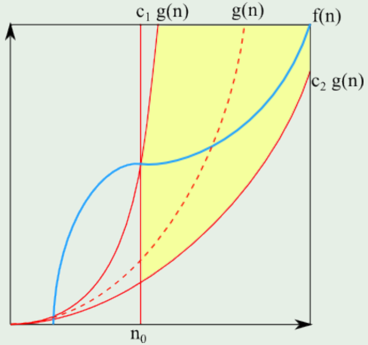

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Correctness of an algorithm

Loop invariant and loop conditional

- A loop invariant is a condition that is necessarily true immediately before and immediately after each iteration of a loop.
- A loop conditional is a statements that controls the termination of the loop.
- Both loop invariant and loop conditional must be different conditions.

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Well, now you know the basics. Time to work!

From Dasgupta's Algorithms book, exercise 0.1

Using the definition in each of the following situations indicate wether $f = O(g)$, or $f = \Omega(g)$, or both (in which case $f = \Theta(g)$).

1. $f(n) = n - 100$, $g(n) = n - 200$
2. $f(n) = n2^n$, $g(n) = 3^n$

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Exercise

Let's try this one!

Show that $\sum_{k=1}^{n} \frac{1}{k^2}$ is bounded by a constant. (help me here!).

Exercise

From Cormen's book exercise 2.3-4

We can express insertion sort as a recursive procedure as follows. In order to sort $A[1...n]$, we recursively sort $A[1...n-1]$ and then insert $A[n]$ into the sorted array. Write a recurrence for the running time of this recursive version of insertion sort.

Exercise

From Cormen's book exercise 3.1-7

Prove that $o(g(n)) \cap \omega(g(n))$ is the empty set.

Exercise

From Cormen's book exercise 3.2-8

Show that $k \ln k = \Theta(n)$ implies that $k = \Theta\left(\frac{n}{\ln n}\right)$.

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From Cormen's book exercise 4.3-1, 4.3-6

1. Show that the solution of $T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) + 1$ is $O(\lg n)$.
2. Show that the solution of $T(n) = 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1$ is $\Omega(n \lg n)$.

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Exercise

From Cormen's book exercise 2.1-3

Consider the searching problem:

Input: A sequence of n numbers $A = \langle a_1, a_2, ..., a_n \rangle$ and a value v.

Output: An index i such that $v = A[i]$ or the special value NIL if v does not appear in A.

Write pseudocode for linear search, which scans through the sequence, looking for v. Using a loop invariant, prove that your algorithm is correct. Make sure that your loop invariant fulfills the three necessary properties.