# Analysis of Algorithms

## Divide and Conquer

Andres Mendez-Vazquez

January 10, 2018

# Outline



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## Divide et impera

A classic technique based on the multi-based recursion.

- Until you reach a base case!!!

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Given the fact of the following equivalence:

$$Recursion \equiv Iteration \quad (1)$$

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# Gauss and the Beginning

## Carl Friedrich Gauss (1777–1855)

He devised a way to multiply two imaginary numbers as

$$(a + bi)(c + di) = ac + (ad + bc)i - bd \quad (2)$$

## By realizing that

$$bc + ad = (a + b)(c + d) - ac - bd \quad (3)$$

Thus minimizing the number of multiplications from four to three.

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## We can represent numbers $x, y$ as

- $x = x_L \circ x_R = 2^{n/2} x_L + x_R$
- $y = y_L \circ y_R = 2^{n/2} y_L + y_R$

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**Thus, the multiplication can be found by using**

$$xy = \left(2^{n/2} x_L + x_R\right)\left(2^{n/2} y_L + y_R\right) = 2^n x_L y_L + 2^{n/2}\left(x_L y_R + x_R y_L\right) + x_R y_R \quad (4)$$

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**However**

if we use the Gauss's trick, we only need $x_L y_L$, $x_R y_R$, $(x_L + x_R)(y_L + y_R)$ to calculate the multiplication:

- $x_L y_R + x_R y_L = (x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R$

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# Now, You have this...

**We have that**

$xy$ can be calculated by using the two parts, Left and Right.

**Then**

Thus, each $x_L x_L$, $x_L y_R$, $x_R y_L$ and $x_R y_R$ can be calculated in a similar way

**Recursion**

This is know as a Recursive Procedure!!!

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# Recursion is the base of Divide and Conquer

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We always attack smaller versions first of the large one!!!

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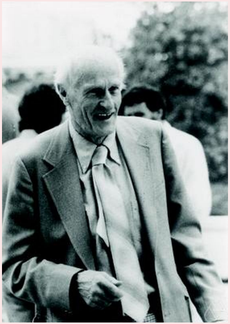

# Kleene and Company

## Some facts about him

- Stephen Cole Kleene (January 5, 1909 – January 25, 1994) was an American mathematician.
- One of the students of Alonzo Church!!!
  - Church is best known for the lambda calculus, Church–Turing thesis and proving the undecidability of the use of an algorithm to say Yes(Valid) or No(No Valid) to a first order logic statement on a FOL System (Proposed by David Hilbert).

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We can use recursion to define sequences, functions, and sets.

## Example

- $a_n = 2^n$ for $n = 0, 1, 2, \ldots \Longrightarrow 1, 2, 4, 8, 16, 32, \ldots$
- Thus, the sequence can be defined in a recursive way:

$$a_{n+1} = 2 \times a_n \quad (7)$$

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# Example

**Can you give me the following?**

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**Base case**

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## Recursively Defined Sets and Structures

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# Example

**Consider**

Consider $S \subseteq \mathbb{Z}$ defined by...

**Basis Step**

$3 \in S$

**Recursive Step**

If $x \in S$ and $y \in S$, then $x + y \in S$.

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# Example

## Elements

- $3 \in S$
- $3+3=6 \in S$
- $6+3=9 \in S$
- $6+6=12 \in S$
- $\cdots$

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# Divide and Conquer

**Divide**

Split problem into a number of subproblems.

**Conquer**

Solve each subproblem recursively.

**Combine**

The solution of the problems into the solution of the original problem.

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# Time Complexities

## Definition

- Given an input as a string where the problem is being encoded using an alphabet $\Sigma$,
  - The **time complexity** quantifies the amount of time taken by an algorithm to run as a function on the length of such string.

# The Divide and Conquer of Merge Sort

**Merge-Sort$(A, p, r)$**

1. if $p < r$ then
2. $\quad q \leftarrow \left\lfloor \frac{p+r}{2} \right\rfloor$
3. $\quad$ Merge-Sort$(A, p, q)$
4. $\quad$ Merge-Sort$(A, q+1, r)$
5. $\quad$ MERGE$(A, p, q, r)$

**Explanation**

Divide part into the conquer!!!

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**Explanation**

The combine part!!!

# Merge Sort

- Merge(A, p, q, r )

$n_1 \leftarrow q - p + 1$, $n_2 \leftarrow r - p$
let $L[1, 2, ..., n_1 + 1]$ and
$R[1, 2, ..., n_2 + 1]$ be new arrays.
for $i \leftarrow 1$ to $n_1$
    $L[i] \leftarrow A[p + i - 1]$
for$j \leftarrow 1$ to $n_2$
    $R[i] \leftarrow A[q + j]$
$L[n_1 + 1] \leftarrow \infty$
$R[n_2 + 1] \leftarrow \infty$
$i \leftarrow 1$, $j \leftarrow 1$
for $k \leftarrow p$ to $r$
    if $L[i] \leq R[j]$ then
        $A[k] \leftarrow L[i]$
        $i \leftarrow i + 1$
    else
        $A[k] \leftarrow R[j]$
        $j \leftarrow j + 1$

**Explanation**

- Copy all to be merged lists into two containers.

# Merge Sort

- Merge(A, p, q, r )

```
n1 ← q − p + 1, n2 ← r − p
let L[1, 2, ..., n1 + 1] and
R[1, 2, ..., n2 + 1] be new arrays.
for i ← 1 to n1
      L[i] ← A[p + i − 1]
forj ← 1 to n2
      R[i] ← A[q + j]
L[n1 + 1] ← ∞
R[n2 + 1] ← ∞
i ← 1, j ← 1
for k ← p to r
       if L[i] ≤ R[j] then
             A[k] ← L[i]
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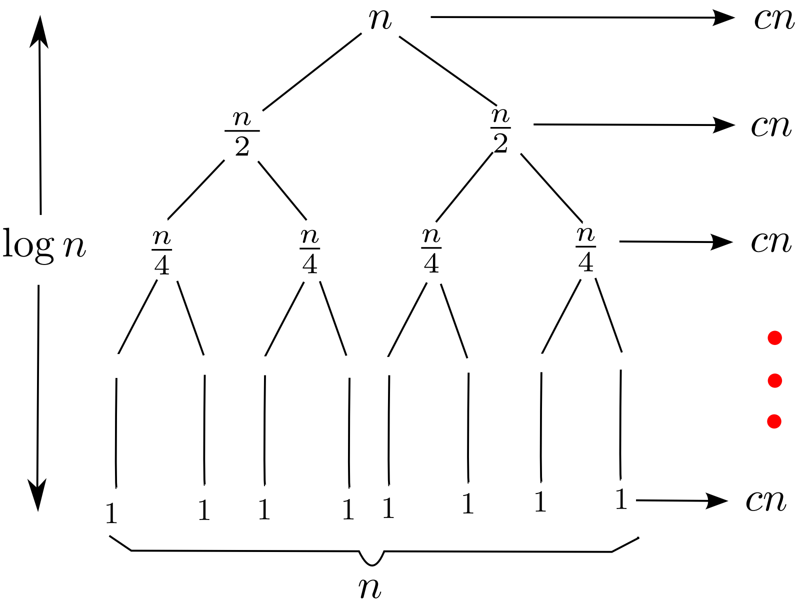
**Explanation**

- Merging part.

# The Merge Sort Recursion Cost Function



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# Recursive Functions

## Using Church-Turing Thesis

**Every computable function from natural numbers to natural numbers is recursive and computable.**

## YES!!!

We can use recursive functions to represent the TOTAL number of steps carried when computing an ALGORITHM

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# Thus, we have

## Each Step for ONE Merging takes...

A certain constant time $c$!!!

## Thus, if we merge $n$ elements

Total time at level 1 of recursion:

$$cn \tag{8}$$

## In addition...

We have that the recursion split each work by

$$\frac{1}{2^i}, \text{ for } i = 1, ..., \log n \tag{9}$$

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# Thus, we have the following Recursion

## Base Case $n = 1$

$$T(n) = c \tag{10}$$

Where $c$ stands for a constant in the number of time units or assembly instructions per line!!!

## Recursive Step $n > 1$

$$2T\left(\frac{n}{2}\right) + cn \tag{11}$$

## Finally

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ 2T\left(\frac{n}{2}\right) + cn & \text{if } n > 1 \end{cases} \tag{12}$$

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# Thus, we have the following Recursion

**Base Case $n = 1$**

$$T(n) = c \tag{10}$$

Where $c$ stands for a constant in the number of time units or assembly instructions per line!!!

**Recursive Step $n > 1$**

$$2T\left(\frac{n}{2}\right) + cn \tag{11}$$

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# Recursion and Induction

## Something Notable

When a sequence is defined recursively, mathematical induction can be used to prove results about the sequence.

# For Example

**We want**

To show that the set $S$ is the set $A$ of all positive integers that are multiples of 3.

**First $A \subseteq S$**

Show that if $\forall k \geq 1$ $P(k)$ is true, then $P(k+1)$ is true

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# Definition of Structural induction

## Basis Step

- Assume $j$ is an element specified in the base step of the definition.
- Show that $\forall j$, $P(j)$ is true.

- Assume $k_1, k_2, ..., k_m$ are elements used to construct an element $k$ in the recursive step of the definition.
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# Again Insertion Sort - Proving the Sorting Property

**Data:** Unsorted Sequence $A$

**Result:** Sort Sequence $A$

`Insertion Sort(A)`

**for** $j \leftarrow 2$ **to** *lenght(A)* **do**

  $key \leftarrow A[j]$;

  `// Insert` $A[j]$ `Insert` $A[j]$ `into the sorted sequence` $A[1, ..., j-1]$

  $i \leftarrow j - 1$;

  **while** $i > 0$ *and* $A[i] > key$ **do**

    $A[i+1] \leftarrow A[i]$;

    $i \leftarrow i - 1$;

  **end**

  $A[i+1] \leftarrow key$

**end**

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# The Structure of the Inductive Proof for a Loop

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- Input of $n$ elements.
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# Termination

## We need

- To prove that the property is TRUE for $n$ elements.
  - At the end of the algorithm $A[1, ..., n]$ is a sorted

# Outline



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# For example, Insertion Sort (Thanks to Luis Rodriguez Oracle Master 2012)

**First, we define the following sets with sorted elements**

- $Less = \langle x_1, ..., x_k | x_i < key, i = 1, .., k \rangle$
- $Greater = \langle x_1, ..., x_m | x_j > key, j = 1, .., m \rangle$
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- Once $j > length(A)$, we get out of the outer loop and $j = n + 1$.
- Then, using the maintenance procedure we have that the sub-array $A[1...n]$ is sorted as we wanted.

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## Properties

A Turing-complete system is called Turing equivalent if every function it can compute is also Turing Computable.

- It computes precisely the same class of functions as do Turing machines.

# Recursion $\equiv$ Iteration

**Then**

Since you can build a Turing complete language using strictly iterative structures and a Turning complete language using only recursive structures, then the two are therefore equivalent.

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## Proof From Lambda Calculus

- Assume languages IT (with Iterative constructs only) and REC (with Recursive constructs only).
- Simulate a universal Turing machine using IT, then simulate a universal Turing machine using REC.
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**Important**

- We use **RECURSIVE** procedures, when we begin to solve new problems so we can understand them.
- Then, we move everything to **ITERATIVE** procedures for speed!!!

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# Outline



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# Introduction

**Let's go back to first principles**

- We can look at our problem of complexities as bounding functions for approximation.

**Can we do better?**

Asymptotic Approximation... We will see a little bit more as the course goes...

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# Big $O$

**Definition (Big $O$ - Upper Bound)**

For a given function $g(n)$:

$$O(g(n)) = \{f(n) \mid \text{There exists } c > 0 \text{ and } n_0 > 0 \text{ s.t. } 0 \leq f(n) \leq cg(n) \ \forall n \geq n_0\}$$

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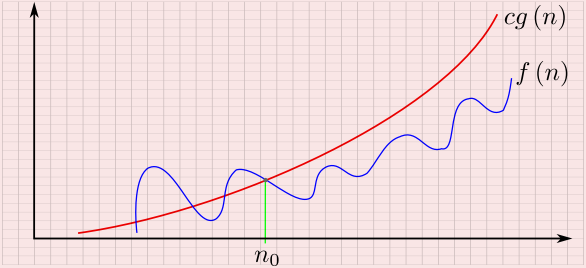

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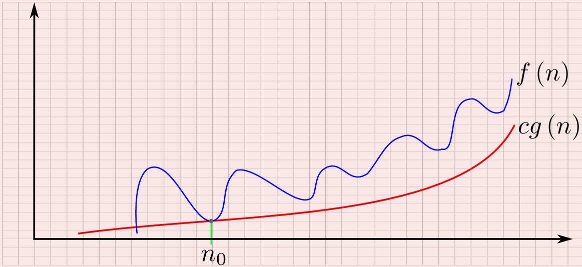

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**Definition (Big $\Theta$ - Tight Bound)**

For a given function $g(n)$:

$$\Theta(g(n)) = \{f(n) |\text{ There exists } c_1 > 0, c_2 > 0 \text{ and } n_0 > 0 \\ \text{s.t. } 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)\ \forall n \geq n_0\}$$

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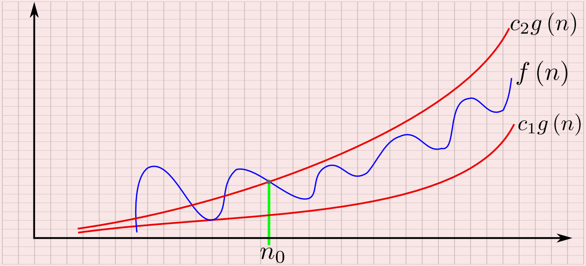

# Outline



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# Can we relate this with practical examples?

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This is too theoretical!

**However, this is not the case!!**

Look at this java code...

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# Example: Step count of Insertion Sort in Java

## Counting when $A.length = n$

```
// Sort A assume is full
public int[] InsertionSort(int[] A){          Step
 // Initial Variables                          0
 int B[] = new int[A.length];                  1
 int size = 1;                                 1
 int i, j, t;                                  1
 // Initialize the Array B                     0
 B[0]=A[0];                                    1
 for(i = 1; i < A.length; i++){                n
   t = A[i];                                   n-1
   for(j=size-1;
       j>=0&&t<B[j];j--)                       i+1
    {
     //shift to the right                      0
       B[j+1]=B[j];}                           i
    B[j+1]=t;                                  n-1
    size++;                                    n-1
  }
 return B;                                     1
}
```

# The Result

**Step count for body of for loop is**

$$6 + 3(n-1) + n + \sum_{i=1}^{n-1}(i+1) + \sum_{j=1}^{n-1}(i) \qquad (13)$$

**The summation**

They have the quadratic terms $n^2$.

**Complexity**

Insertion sort complexity is $O\left(n^2\right)$

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# What does this means for insertion sort?

## We have

$$6 + 3(n-1) + n + \sum_{i=1}^{n-1} (i+1) + \sum_{j=1}^{n-1} (i) = \ldots$$

$$3 + 4n + \frac{n(n-1)}{2} + n - 1 + \frac{n(n-1)}{2} = \ldots$$

$$2 + 5n + n(n-1) = \ldots$$

$$n^2 + 4n + 2 \leq n^2 + 4n^2 + 2n^2$$

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## Thus

$$n^2 + 4n + 2 \leq 7n^2 \tag{14}$$

With $T_{insertion}(n) = n^2 + 4n + 2$ describing the number of steps for insertion when we have $n$ numbers.

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# Actually

**For $n_0 = 2$**

$$2^2 + 4 \times 2 + 2 = 14 < 7 \times 2^2 = 28 \tag{15}$$

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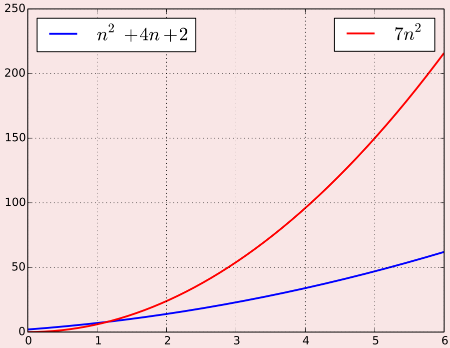

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## First

Time or number of operations does not exceed $cn^2$ for a constant $c$ on any input of size $n$ ($n$ suitably large).

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- Is $O(n^2)$ too much time?
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# Then

## We have the following

| $n$ | $n$ | $n \log n$ | $n^2$ | $n^3$ | $n^4$ |
|---|---|---|---|---|---|
| 1000 | 1 micros | 10 micros | 1 milis | 1 second | 17 minutes |
| 10,000 | 10 micros | 130 micros | 100 milis | 17 minutes | 116 days |
| $10^6$ | 1 milis | 20 milis | 17 minutes | 32 years | $3 \times 10^7$years |

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| $n$ | $n^{10}$ | $2^n$ |
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| 1000 | $3.2 \times 10^{13}$ years | $3.2 \times 10^{283}$ years |
| 10,000 | ??? | ??? |
| $10^6$ | ????? | ????? |

**The Reign of the Non Polynomial Algorithms**

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# Outline



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# Little $o$ Bound

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For a given function $g(n)$:

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## Observations

It is not tight.

- For example, We have that $2n = o\left(n^2\right)$, but $2n^2 \neq o(n^2)$.

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Under the definition, we have for any $f(n) \in o(g(n))$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

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# Properties

## Equivalence

For any two functions $f(n)$ and $g(n)$, we have that $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

## Transitivity

$f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$ then $f(n) = \Theta(h(n))$

## Reflexivity

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**Solve the following recurrence**

$$T(n) = 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n \tag{16}$$

**I decide to do the following GUESS**

Guess that $T(n) = O(n \log n)$!!!

**For this**

We assume that the bound holds for $\left\lfloor \frac{n}{2} \right\rfloor < n$ (Remember Inductive Hypothesis!!!).

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&\leq 2c \times \frac{n}{2} \times \log_2 \left( \frac{n}{2} \right) + n \\
&= cn \log_2 \left( \frac{n}{2} \right) + n
\end{aligned}
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## Remember the following

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\begin{aligned}
\log_2 \left( \frac{n}{2} \right) &= \log_2 n - \log_2 2 \\
&= \log_2 n - 1
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# Subtleties

**What about ?**

$$T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + T\left(\left\lceil \frac{n}{2} \right\rceil\right) + 1$$

**Here, we have a problem**

Look at the Board!!!

# Outline



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# The Recursion-Tree Method

## Surprise

- Sometimes is hard to do a good guess.
- For example $T(n) = 3T\left(\frac{n}{4}\right) + cn^2$

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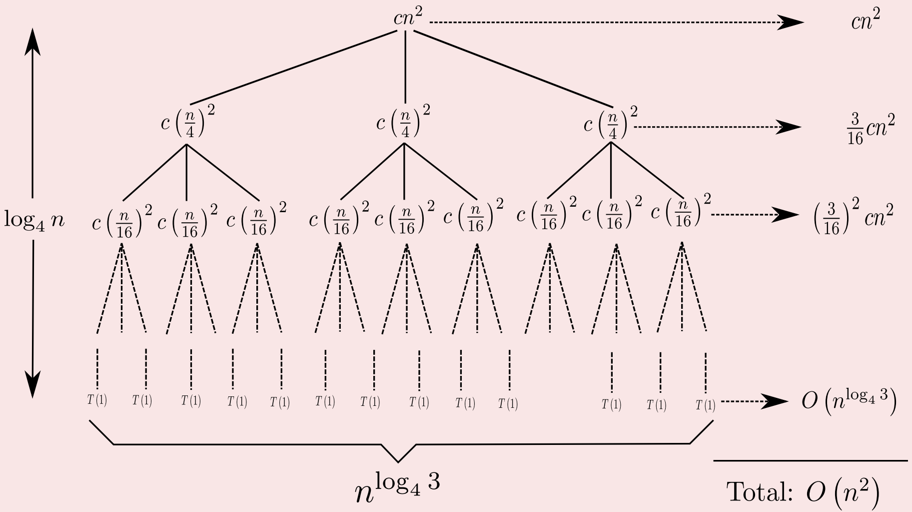
# The Recursion-Tree Method

## Therefore, we draw the recursion tree



# Using the previous expansion, we count!!!

## Counting Again!!!

- A subproblem for a node at depth $i$ is $n/4^i$, then once

$$n/4^i = 1 \Rightarrow i = \log_4 n \tag{18}$$

- At each level $i = 0, 1, 2, ..., \log_4 n - 1$ the cost of each node is

$$c\left(\frac{n}{4^i}\right)^2 \tag{19}$$

- At each level $i = 0, 1, 2, ..., \log_4 n - 1$ the total cost of the work is

$$3^i c\left(\frac{n}{4^i}\right)^2 = \left(\frac{3}{16}\right)^i cn^2 \tag{20}$$

- At depth $\log_4 n$, we have this many nodes

$$3^{\log_4 n} = n^{\log_4 3} \tag{21}$$

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# Outline

# The Master Theorem

## Theorem - Cookbook for solving $T(n) = aT\left(\frac{n}{b}\right) + f(n)$

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the non-negative integers by the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) \tag{22}$$

where we interpret $\frac{n}{b}$ as $\left\lfloor \frac{n}{b} \right\rfloor$ or $\left\lceil \frac{n}{b} \right\rceil$. Then $T(n)$ can be bounded asymptotically as follows:

1. If $f(n) = O\left(n^{\log_b a - \epsilon}\right)$ for some constant $\epsilon > 0$. Then $T(n) = \Theta\left(n^{\log_b a}\right)$.
2. If $f(n) = \Theta\left(n^{\log_b a}\right)$, then $T(n) = \Theta\left(n^{\log_b a} \lg n\right)$.
3. If $f(n) = \Omega\left(n^{\log_b a + \epsilon}\right)$ for some constant $\epsilon > 0$ and if $af\left(\frac{n}{b}\right) \leq cf(n)$ for some $c < 1$ and all sufficiently large $n$, then $T(n) = \Theta(f(n))$.

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# We will prove a simplified version

## Simplified Master Method

If $T(n) = aT\left(\left\lceil \frac{n}{b} \right\rceil\right) + O(n^d)$ for some constants $a > 0$, $b > 1$, and $d \geq 0$ then

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$

## Proof at the Board

Look at this

# Using the Master Theorem

## Consider the following recursion

$$T(n) = 9T\left(\frac{n}{3}\right) + n$$

## We have that

$a = 9$, $b = 3$ and $f(n) = n$

## Thus

$n^{\log_3 9} = \Theta(n^2)$ and $f(n) = O(n^{\log_3 9 - \epsilon})$ with $\epsilon = 1$

## Then, we use then the case 1 of the Master Theorem

$$T(n) = O\left(n^2\right) \tag{23}$$

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