Analysis of Algorithms

Divide and Conquer

Andres Mendez-Vazquez

September 8, 2020

Outline

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Divide an Conquer

Divide et impera

A classic technique based on the multi-based recursion.

- Until you reach a base case!!

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That Divide and Conquer works by recursively breaking down the problem into subproblems and solving those subproblems recursively.

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$$Recursion \equiv Iteration \tag{1}$$

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Gauss and the Beginning

Carl Friedrich Gauss (1777–1855)

He devised a way to multiply two imaginary numbers as

$$(a+bi)(c+di) = ac + (ad+bc)\,i - bd \tag{2}$$

By realizing that

$$bc + ad = (a+b)(c+d) - ac - bd \tag{3}$$

Thus minimizing the number of multiplications from four to three.

Actually

We can represent binary numbers like 1001 as $1000 + 01 = 2^2 \times 10 + 01$

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Thus

We can represent numbers x, y as

- $x = x_L \circ x_R = 2^{n/2} x_L + x_R$
- $y = y_L \circ y_R = 2^{n/2} y_L + y_R$

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We can represent numbers x, y as

- $x = x_L \circ x_R = 2^{n/2} x_L + x_R$
- $y = y_L \circ y_R = 2^{n/2} y_L + y_R$

Thus, the multiplication can be found by using

$$xy = \left(2^{n/2} x_L + x_R\right)\left(2^{n/2} y_L + y_R\right) = 2^n x_L y_L + 2^{n/2}\left(x_L y_R + x_R y_L\right) + x_R y_R \quad (4)$$

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However

if we use the Gauss's trick, we only need $x_L y_L$, $x_R y_R$, $(x_L + x_R)(y_L + y_R)$ to calculate the multiplication:

- $x_L y_R + x_R y_L = (x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R$

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Now, You have this...

We have that

xy can be calculated by using the two parts, Left and Right.

Then

Thus, each $x_L x_L$, $x_L y_R$, $x_R y_L$ and $x_R y_R$ can be calculated in a similar way

Recursion

This is know as a Recursive Procedure!!!

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Complexities

Old Multiplication

$$T(n) = 4T\left(\frac{n}{2}\right) + \text{ Some Work} \quad (5)$$

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- For new style multiplications $O\left(n^{\log_2 3}\right)$

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In a really unclever way!!!

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Thus, improving speedup!!!

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Recursion is the base of Divide and Conquer

This is the natural way we do many things

We always attack smaller versions first of the large one!!!

Stephen Cole Kleene

- He defined the basics about the use of recursion.

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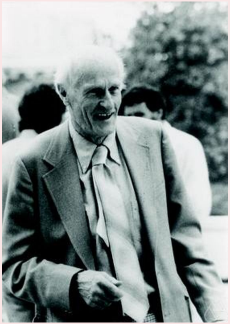

Kleene and Company

Some facts about him

- Stephen Cole Kleene (January 5, 1909 – January 25, 1994) was an American mathematician.
- One of the students of Alonzo Church!!!
  - Church is best known for the lambda calculus, Church–Turing thesis and proving the undecidability of the use of an algorithm to say Yes(Valid) or No(No Valid) to a first order logic statement on a FOL System (Proposed by David Hilbert).

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- This theory subsequently helped to provide the foundations of theoretical computer science.

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- Sometimes it is difficult to define an object explicitly.
- It may be easy to define this object in smaller version of itself.
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We can use recursion to define sequences, functions, and sets.

Example

- $a_n = 2^n$ for $n = 0, 1, 2, \ldots \Longrightarrow 1, 2, 4, 8, 16, 32, \ldots$
- Thus, the sequence can be defined in a recursive way:

$$a_{n+1} = 2 \times a_n \tag{7}$$

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Recursively Defined Functions

First

Assume T is a function with the set of nonnegative integers as its domain.

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We use two steps to define T:

Basis step:
Specify the value of $T(0)$.

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Give a rule for $T(x)$ using $T(y)$ where $0 \leq y < x$.

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Example

Can you give me the following?

Give an inductive definition of the factorial function $T(n) = n!$.

Base case

Which is the base case?

Recursive case

What is the recursive case?

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Example

Consider

Consider $S \subseteq \mathbb{Z}$ defined by...

Basis Step

$3 \in S$

Recursive Step

If $x \in S$ and $y \in S$, then $x + y \in S$.

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Example

Elements

- $3 \in S$
- $3 + 3 = 6 \in S$
- $6 + 3 = 9 \in S$
- $6 + 6 = 12 \in S$
- $\cdots$

Outline

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Split problem into a number of subproblems.

Conquer

Solve each subproblem recursively.

Combine

The solution of the problems into the solution of the original problem.

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Time Complexities

Definition

- Given an input as a string where the problem is being encoded using an alphabet Σ,
  - The **time complexity** quantifies the amount of time taken by an algorithm to run as a function on the length of such string.

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The Divide and Conquer of Merge Sort

Merge-Sort(A, p, r)

1. if $p < r$ then
2. $\quad q \leftarrow \left\lfloor \frac{p+r}{2} \right\rfloor$
3. $\quad$ Merge-Sort(A, p, q)
4. $\quad$ Merge-Sort$(A, q+1, r)$
5. $\quad$ MERGE(A, p, q, r)

Explanation

Divide part into the conquer!!!

The Divide and Conquer of Merge Sort

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Explanation

The combine part!!!

Merge Sort

- Merge(A, p, q, r)
  1. $n_1 \leftarrow q - p + 1$, $n_2 \leftarrow r - p$
  2. let $L[1, 2, ..., n_1 + 1]$ and $R[1, 2, ..., n_2 + 1]$ be new arrays.
  3. for $i \leftarrow 1$ to n_1
  4. $\quad L[i] \leftarrow A[p + i - 1]$
  5. for$j \leftarrow 1$ to n_2
  6. $\quad R[i] \leftarrow A[q + j]$
  7. $L[n_1 + 1] \leftarrow \infty$
  8. $R[n_2 + 1] \leftarrow \infty$
  9. $i \leftarrow 1$, $j \leftarrow 1$
  10. for $k \leftarrow p$ to r
  11. $\quad$ if $L[i] \leq R[j]$ then
  12. $\quad\quad A[k] \leftarrow L[i]$
  13. $\quad\quad i \leftarrow i + 1$
  14. $\quad$ else
  15. $\quad\quad A[k] \leftarrow R[j]$
  16. $\quad\quad j \leftarrow j + 1$

Explanation

- Copy all to be merged lists into two containers.

Merge Sort

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  1. $n_1 \leftarrow q - p + 1$, $n_2 \leftarrow r - p$
  2. let $L[1, 2, ..., n_1 + 1]$ and $R[1, 2, ..., n_2 + 1]$ be new arrays.
  3. for $i \leftarrow 1$ to n_1
  4. $L[i] \leftarrow A[p + i - 1]$
  5. for$j \leftarrow 1$ to n_2
  6. $R[i] \leftarrow A[q + j]$
  7. $L[n_1 + 1] \leftarrow \infty$
  8. $R[n_2 + 1] \leftarrow \infty$
  9. $i \leftarrow 1$, $j \leftarrow 1$
  10. for $k \leftarrow p$ to r
  11.   if $L[i] \leq R[j]$ then
  12. $A[k] \leftarrow L[i]$
  13. $i \leftarrow i + 1$
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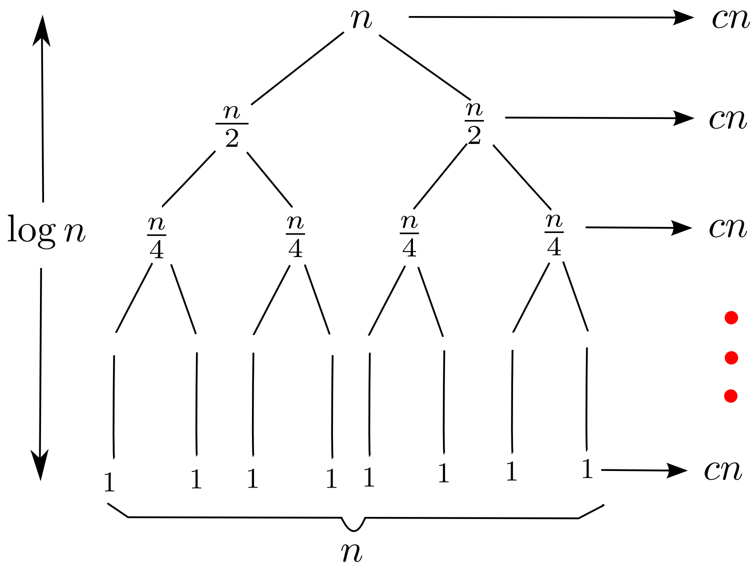
Explanation

- Merging part.

The Merge Sort Recursion Cost Function



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Using Church-Turing Thesis

Every computable function from natural numbers to natural numbers is recursive and computable.

YES!!!

We can use recursive functions to represent the TOTAL number of steps carried when computing an ALGORITHM

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We can use recursive functions to represent the TOTAL number of steps carried when computing an ALGORITHM

Thus, we have

Each Step for ONE Merging takes...

A certain constant time c!!!

Thus, if we merge n elements

Total time at level 1 of recursion:

$$cn \tag{8}$$

In addition...

We have that the recursion split each work by

$$\frac{1}{2^i}, \text{ for } i = 1, ..., \log n \tag{9}$$

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Thus, we have the following Recursion

Base Case $n = 1$

$$T(n) = c \tag{10}$$

Where c stands for a constant in the number of time units or assembly instructions per line!!!

Recursive Step $n > 1$

$$2T\left(\frac{n}{2}\right) + cn \tag{11}$$

Finally

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ 2T\left(\frac{n}{2}\right) + cn & \text{if } n > 1 \end{cases} \tag{12}$$

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Recursion and Induction

Something Notable

When a sequence is defined recursively, mathematical induction can be used to prove results about the sequence.

For Example

We want

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Something Notable

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Again Insertion Sort - Proving the Sorting Property

Data: Unsorted Sequence A

Result: Sort Sequence A

`Insertion Sort(A)`

for $j \leftarrow 2$ **to** *lenght(A)* **do**

$\quad key \leftarrow A[j]$;

$\quad$ `// Insert` $A[j]$ `Insert` $A[j]$ `into the sorted sequence` $A[1, ..., j-1]$

$\quad i \leftarrow j - 1$;

$\quad$ **while** $i > 0$ *and* $A[i] > key$ **do**

$\quad\quad A[i+1] \leftarrow A[i]$;

$\quad\quad i \leftarrow i - 1$;

$\quad$ **end**

$\quad A[i+1] \leftarrow key$

end

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You have an initial input n

- Input of n elements.
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- That the condition is true for one element!!!
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Termination

We need

- To prove that the property is TRUE for n elements.
  - At the end of the algorithm $A[1, ..., n]$ is a sorted

Outline

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For example, Insertion Sort (Thanks to Luis Rodriguez Oracle Master 2012)

First, we define the following sets with sorted elements

- $Less = \langle x_1, ..., x_k | x_i < key, i = 1, .., k \rangle$
- $Greater = \langle x_1, ..., x_m | x_j > key, j = 1, ..., m \rangle$
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Before we enter to the inner while loop, we have

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Proof From Lambda Calculus

- Assume languages IT (with Iterative constructs only) and REC (with Recursive constructs only).
- Simulate a universal Turing machine using IT, then simulate a universal Turing machine using REC.
- The existence of the simulator programs guarantees that both IT and REC can calculate all the computable functions.

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Let's go back to first principles

- We can look at our problem of complexities as bounding functions for approximation.

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Big O

Definition (Big O - Upper Bound)

For a given function $g(n)$:

$$O(g(n)) = \{ f(n) \mid \text{There exists } c > 0 \text{ and } n_0 > 0 \text{ s.t. } 0 \leq f(n) \leq c g(n) \ \forall n \geq n_0 \}$$

Example

Big O

Definition (Big O - Upper Bound)

For a given function $g(n)$:

$$O(g(n)) = \{f(n) | \text{ There exists } c > 0 \text{ and } n_0 > 0 \text{ s.t. } 0 \leq f(n) \leq cg(n) \ \forall n \geq n_0\}$$

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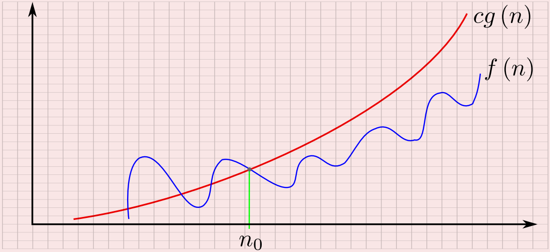

Big Ω

Definition (Big Ω - Lower Bound)

For a given function $g(n)$:

$$\Omega(g(n)) = \{f(n) |\ \text{There exists } c > 0 \text{ and } n_0 > 0$$
$$\text{s.t. } 0 \leq cg(n) \leq f(n)\ \forall n \geq n_0\}$$

Example

Big Ω

Definition (Big Ω - Lower Bound)

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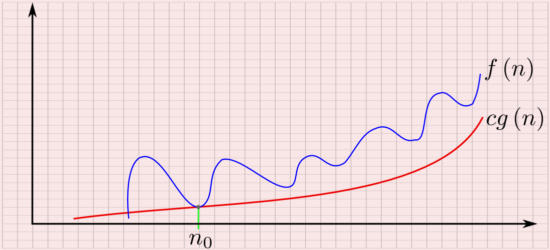

Big Θ

Definition (Big Θ - Tight Bound)

For a given function $g(n)$:

$$\Theta(g(n)) = \{f(n) | \text{ There exists } c_1 > 0, c_2 > 0 \text{ and } n_0 > 0 \text{ s.t. } 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \ \forall n \geq n_0\}$$

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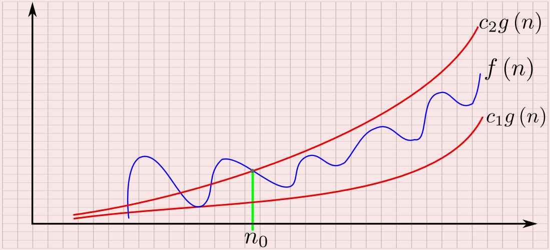

Outline

Can we relate this with practical examples?

You could say

This is too theoretical!

However, this is not the case!!

Look at this java code...

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Look at this java code...

Example: Step count of Insertion Sort in Java

Counting when $A.length = n$

```
// Sort A assume is full
public int[] InsertionSort(int[] A){            Step
 // Initial Variables                            0
 int B[] = new int[A.length];                    1
 int size = 1;                                   1
 int i, j, t;                                    1
 // Initialize the Array B                       0
 B[0]=A[0];                                      1
 for(i = 1; i < A.length; i++){                  n
   t = A[i];                                     n-1
   for(j=size-1;
       j>=0&&t<B[j];j--)                         i+1
    {
     //shift to the right                        0
        B[j+1]=B[j];}                            i
    B[j+1]=t;                                    n-1
    size++;                                      n-1
  }
 return B;                                       1
}
```

The Result

Step count for body of for loop is

$$6 + 3(n-1) + n + \sum_{i=1}^{n-1}(i+1) + \sum_{j=1}^{n-1}(i) \tag{13}$$

The summation

They have the quadratic terms n^2.

Complexity

Insertion sort complexity is $O(n^2)$

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What does this means for insertion sort?

We have

$$6+3\,(n-1)+n+\sum_{i=1}^{n-1}(i+1)+\sum_{j=1}^{n-1}(i)=...$$

$$3+4n+\frac{n\,(n-1)}{2}+n-1+\frac{n\,(n-1)}{2}=...$$

$$2+5n+n(n-1)=...$$

$$n^2+4n+2\leq n^2+4n^2+2n^2$$

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$$n^2 + 4n + 2 \leq n^2 + 4n^2 + 2n^2$$

Thus

$$n^2 + 4n + 2 \leq 7n^2 \quad (14)$$

With $T_{insertion}(n) = n^2 + 4n + 2$ describing the number of steps for insertion when we have n numbers.

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Actually

For $n_0 = 2$

$$2^2 + 4 \times 2 + 2 = 14 < 7 \times 2^2 = 28 \tag{15}$$

Graphically

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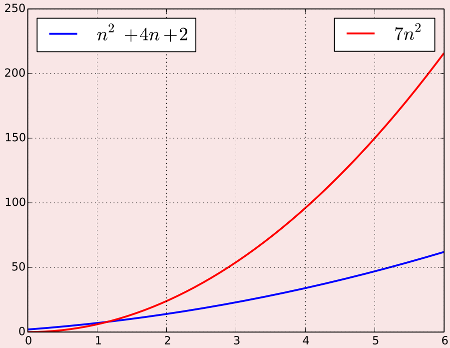

Meaning

First

Time or number of operations does not exceed cn^2 for a constant c on any input of size n (n suitably large).

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- Is $O(n^2)$ too much time?
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Outline

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Then

We have the following

n	n	$n \log n$	n^2	n^3	n^4
1000	1 micros	10 micros	1 milis	1 second	17 minutes
10,000	10 micros	130 micros	100 milis	17 minutes	116 days
10^6	1 milis	20 milis	17 minutes	32 years	3×10^7years

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n	n^{10}	2^n
1000	3.2×10^{13} years	3.2×10^{283} years
10,000	???	???
10^6	?????	?????

The Reign of the Non Polynomial Algorithms

Outline

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Little o Bound

Definition

For a given function $g(n)$:

$$o(g(n)) = \{f(n) \mid \text{For any } c > 0 \text{ there exists } n_0 > 0 \text{ s.t. } 0 \leq f(n) < cg(n)\ \forall n \geq n_0\}$$

Observations

It is not tight.

- For example, We have that $2n = o(n^2)$, but $2n^2 \neq o(n^2)$.

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Little o Bound

Not only that

Under the definition, we have for any $f(n) \in o(g(n))$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

Little ω Bound

Definition

For a given function $g(n)$:

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Under the definition, we have for any $f(n) \in \omega(g(n))$

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Outline

Interpretation

How do you interpret $f(n) = O(n^2)$?

It means that $f(n)$ belongs to $O(n^2)$

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$\exists f(n) \in \Theta(n)$ such that:

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Outline

Properties

Equivalence

For any two functions $f(n)$ and $g(n)$, we have that $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

Transitivity

$f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$ then $f(n) = \Theta(h(n))$

Reflexivity

$f(n) = \Theta(f(n))$

Symmetry

$f(n) = \Theta(g(n)) \Longleftrightarrow g(n) = \Theta(f(n))$

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- In the case of the first part, it is easy to see that for any given c exist a n_0 such that $\frac{1}{\frac{n_o}{2}} < c$.
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Outline

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Ok, we have the basics...

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2. Recursion-Tree Method
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Example

Solve the following recurrence

$$T(n) = 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n \tag{16}$$

I decide to do the following GUESS

Guess that $T(n) = O(n \log n)$!!!

For this

We assume that the bound holds for $\left\lfloor \frac{n}{2} \right\rfloor < n$ (Remember Inductive Hypothesis!!!).

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Thus

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$$\begin{aligned} T(n) &\leq 2c\left\lfloor\frac{n}{2}\right\rfloor\log_2\left(\left\lfloor\frac{n}{2}\right\rfloor\right)+n\\ &\leq 2c\times\frac{n}{2}\times\log_2\left(\frac{n}{2}\right)+n\\ &= cn\log_2\left(\frac{n}{2}\right)+n \end{aligned}$$

Remember the following

$$\begin{aligned} \log_2\left(\frac{n}{2}\right) &= \log_2 n-\log_2 2\\ &= \log_2 n-1 \end{aligned}$$

Finally, we have

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$$T(n) \leq cn\log_2 n - cn + n$$

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Now, we need to have that

$$- cn + n \leq 0$$

$$n \leq cn$$

$$1 \leq n$$

$T(n)$

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$$
\begin{aligned}
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$$

Then, as long $c \geq 1$, we have that

$$
\begin{aligned}
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Subtleties

What about ?

$$T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + T\left(\left\lceil \frac{n}{2} \right\rceil\right) + 1$$

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Here

We can guess that $T(n) = O(n)$

$$
\begin{aligned}
T(n) \leq & c\left\lfloor \frac{n}{2} \right\rfloor + c\left\lceil \frac{n}{2} \right\rceil + 1 \\
= & cn + 1 \\
= & O(n)
\end{aligned}
$$

Incorrect!!!

- After all $cn + 1$ is not cn.

We can overcome this problem by assuming a $d \geq 0$ and then "guessing" $T(n) \leq cn - d$

$$
\begin{aligned}
T(n) \leq & \left(c\left\lfloor \frac{n}{2} \right\rfloor - d\right) + \left(c\left\lceil \frac{n}{2} \right\rceil - d\right) + 1 \\
= & cn - 2d + 1
\end{aligned}
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Then

- if we select $d \geq 1 \Rightarrow 0 \geq 1 - d$.

This means that $cn - 2d + 1 \leq cn - d$

- Therefore, $T(n) \leq cn - d = O(n)$.

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Outline

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The Recursion-Tree Method

Surprise

- Sometimes is hard to do a good guess.
- For example $T(n) = 3T\left(\frac{n}{4}\right) + cn^2$

The Recursion-Tree Method

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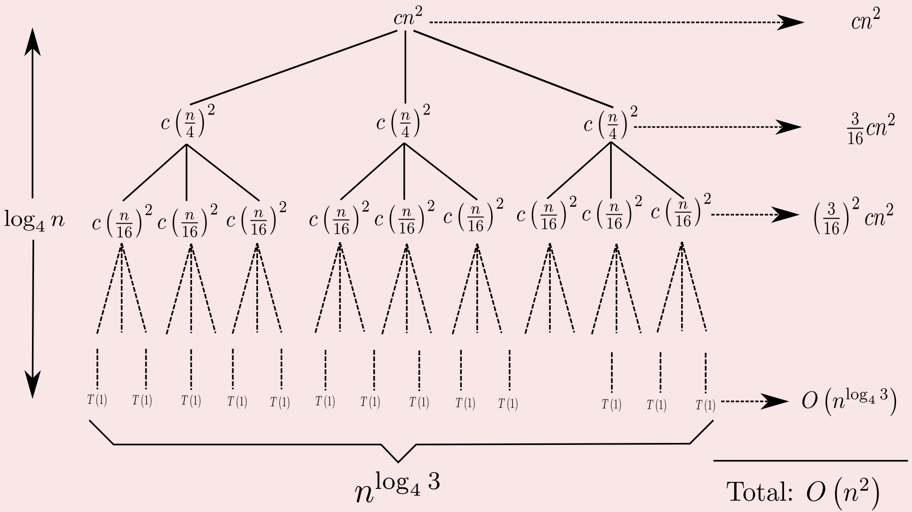
The Recursion-Tree Method

Therefore, we draw the recursion tree



Using the previous expansion, we count!!!

Counting Again!!!

- A subproblem for a node at depth i is $n/4^i$, then once

$$n/4^i = 1 \Rightarrow i = \log_4 n \tag{18}$$

- At each level $i = 0, 1, 2, ..., \log_4 n - 1$ the cost of each node is

$$c\left(\frac{n}{4^i}\right)^2 \tag{19}$$

- At each level $i = 0, 1, 2, ..., \log_4 n - 1$ the total cost of the work is

$$3^i c\left(\frac{n}{4^i}\right)^2 = \left(\frac{3}{16}\right)^i cn^2 \tag{20}$$

- At depth $\log_4 n$, we have this many nodes

$$3^{\log_4 n} = n^{\log_4 3} \tag{21}$$

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Now, we add all this counts!!!

Then, we have that

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\begin{aligned}
T(n) &= \sum_{i=0}^{log_4 n-1} \left(\frac{3}{16}\right)^i cn^2 + n^{\log_4 3} \\
&< \sum_{i=0}^{\infty} \left(\frac{3}{16}\right)^i cn^2 + n^{\log_4 3} \\
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Outline

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The Master Theorem

Theorem - Cookbook for solving $T(n) = aT\left(\frac{n}{b}\right) + f(n)$

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the non-negative integers by the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) \tag{22}$$

where we interpret $\frac{n}{b}$ as $\left\lfloor \frac{n}{b} \right\rfloor$ or $\left\lceil \frac{n}{b} \right\rceil$. Then $T(n)$ can be bounded asymptotically as follows:

1. If $f(n) = O\left(n^{\log_b a - \epsilon}\right)$ for some constant $\epsilon > 0$. Then $T(n) = \Theta\left(n^{\log_b a}\right)$.
2. If $f(n) = \Theta\left(n^{\log_b a}\right)$, then $T(n) = \Theta\left(n^{\log_b a} \lg n\right)$.
3. If $f(n) = \Omega\left(n^{\log_b a + \epsilon}\right)$ for some constant $\epsilon > 0$ and if $af\left(\frac{n}{b}\right) \leq cf(n)$ for some $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.

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We will prove a simplified version

Simplified Master Method

If $T(n) = aT\left(\left\lceil \frac{n}{b} \right\rceil\right) + O(n^d)$ for some constants $a > 0$, $b > 1$, and $d \geq 0$ then

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$

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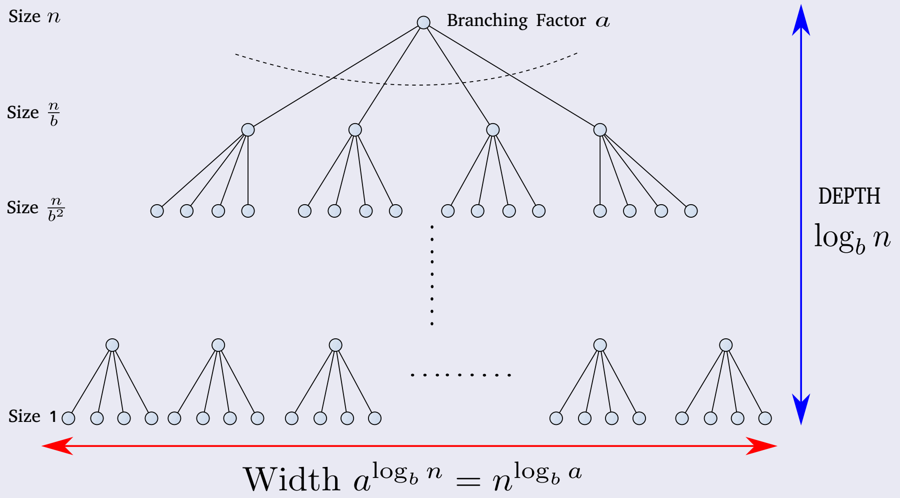
The Branching

Recursive Expansion



Proof

First, for convenience assume $n = b^p$

- Now we can notice that the size of the subproblems are decreasing by a factor of b at each recursive step.

Something Notable

- This means that the size of each subproblems is $\frac{n}{b^i}$ at level i.

Thus, in order to reach the bottom you need to have subptoblems of size 1.

$$\frac{n}{b^i} = 1 \Rightarrow i = \log_b n$$

- where i = height of the recursion three.

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Therefore

Now, given that the branching factor is a

- We have at the k^{th}level a^ksubproblems, each of size $\frac{n}{b^k}$.

Then, the work at level k is

$$T(n) = O(n^d) \times \left(\frac{a}{b^d}\right)^0 + O(n^d) \times \left(\frac{a}{b^d}\right)^2 + ... + O(n^d) \times \left(\frac{a}{b^d}\right)^{log_b n}$$

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Then, we have that

For a $g(m) = 1 + c + c^2 + ... + c^m$

1. if $c < 1$ then $g(m) = \Theta(1)$
2. if $c = 1$ then $g(m) = \Theta(m)$
3. if $c > 1$ then $g(m) = \Theta(c^m)$

If $c < 1$ then $g(m) = \Theta(1)$

If $\frac{a}{b^d} < 1$,

- Then, we have that $a < b^d$ or $\log_b a < d$ (Case one of the theorem). Then, $T(n) = O(n^d)$.

If $c = 1$ then $g(m) = \Theta(m)$

If $\frac{a}{b^d} = 1$

- Then we have that $a = b^d$ or $\log_b a = d$ (Case two of the theorem).

Then

- We have that $g(n) = \left(\frac{a}{b^d}\right)^0 + \left(\frac{a}{b^d}\right)^2 + ... + \left(\frac{a}{b^d}\right)^{log_b n}$ is $\Theta(\log_b n)$.

Now

- $T(n) = O(n^{\log_b a} \log_b n) = O\left(n^{\log_n a} \log_2 n\right)$ because b can only be greater or equal to two.

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If $c > 1$ then $g(m) = \Theta(c^m)$

If $\frac{a}{b^d} > 1$

- Then we have that $a > b^d$ or $\log_b a > d$ (Case three of the theorem).

Then

- We have
$$n^d \times \left(\frac{a}{b^d}\right)^{\log_b n} = n^d \times \left(\frac{a^{\log_b n}}{\left(b^{\log_b n}\right)^d}\right) = a^{\log_b n} = a^{(\log_a n)(\log_b a)} = n^{\log_b a}$$

Thus

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Using the Master Theorem

Consider the following recursion

$$T(n) = 9T\left(\frac{n}{3}\right) + n$$

We have that

$a = 9$, $b = 3$ and $f(n) = n$

Thus

$n^{\log_3 9} = \Theta(n^2)$ and $f(n) = O(n^{\log_3 9 - \epsilon})$ with $\epsilon = 1$

Then, we use then the case 1 of the Master Theorem

$$T(n) = O\left(n^2\right) \tag{23}$$

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$a = 9$, $b = 3$ and $f(n) = n$

Thus

$n^{\log_3 9} = \Theta(n^2)$ and $f(n) = O(n^{\log_3 9 - \epsilon})$ with $\epsilon = 1$

Then, we use then the case 1 of the Master Theorem

$$T(n) = O\left(n^2\right) \tag{23}$$