Introduction to Artificial Intelligence

Reinforcement Learning

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Outline

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Reinforcement Learning is learning what to do

What does Reinforcement Learning want?

- Maximize a numerical reward signal

The learner is not told which actions to take

- A discovery must be performed to obtain the actions that yield the most rewards

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- As the optimal control of incompletely-known **Markov Decision Processes** (MDP).

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- Interact with the environment to learn by using punishments and rewards.

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A mobile robot

decides whether it should enter a new room

- in search of more trash to collect
- or going back to charge its battery

It needs to take a decision based on the

- current charge level of its battery
- how quickly it can arrive to its base

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A K-Armed Bandit Problem

Definition

- You are faced repeatedly with a choice among k different options, or actions.

After each choice you receive a numerical reward chosen

- From an stationary probability distribution depending on the actions.

Definition Stationary Probability

- A stationary distribution of a Markov chain is a probability distribution that remains unchanged in the Markov chain as time progresses.

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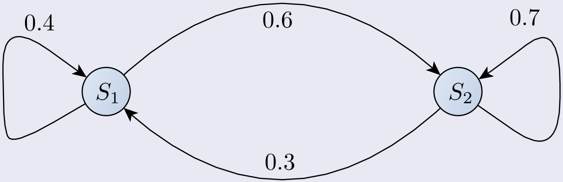
Example

A Two State System



Therefore

Each of the k actions has an expected value

$$q_*\left(a\right) = E\left[R_t | A_t = a\right]$$

Something Notable

- If you knew the value of each action,
  - It would be trivial to solve the k-armed bandit problem.

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What do we want?

To find the estimated value of action a at time t, $Q_t(a)$

$$|Q_t(a) - q_*(a)| < \epsilon \text{ with } \epsilon \text{ as small as possible}$$

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Greedy Actions

Intuitions

- To choose the greatest $E\left[R_t|A_t = a\right]$

This is know as Exploitation

- You are exploiting your current knowledge of the values of the actions.

Contrasting to Exploration

- When you select the non-gready actions to obtain a better knowledge of the non-gready actions.

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An Interesting Conundrum

It has been observed that

- Exploitation maximizes the expected reward on the one step,
- Exploration may produce the greater total reward in the long run.

Thus, the idea

- The value of a state s is the total amount of reward an agent can expect to accumulate over the future, starting from s.

Thus, Values

- They indicate the long-term profit of states after taking into account
  - The states that follow and their rewards.

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Therefore, the parts of a Reinforcement Learning

A Policy

- It defines the learning agent's way of behaving at a given time.
  - The policy is the core of a reinforcement learning system
  - In general, policies may be stochastic

  - The system's sole objective is to maximize the total reward over the long run
  - This encourage Exploitation
  - It is immediate ... you might want something with more long term

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A Model of the Environment

- This simulates the behavior of the environment
  - Allowing to make inferences on how the environment can behave

Something Notable

- Models are used for planning
  - Meaning we made decisions before they are executed
  - Thus, Reinforcement Learning Methods using models are called model-based

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Limitations

Given that

- Reinforcement learning relies heavily on the concept of state

Therefore, it has the same problem as in Markov Decision Processes

- Defining the state to obtain a better representation of the search space.

This is where the creative part of the problems come to be

- As we have seen, solutions in AI really a lot in the creativity of the solution...

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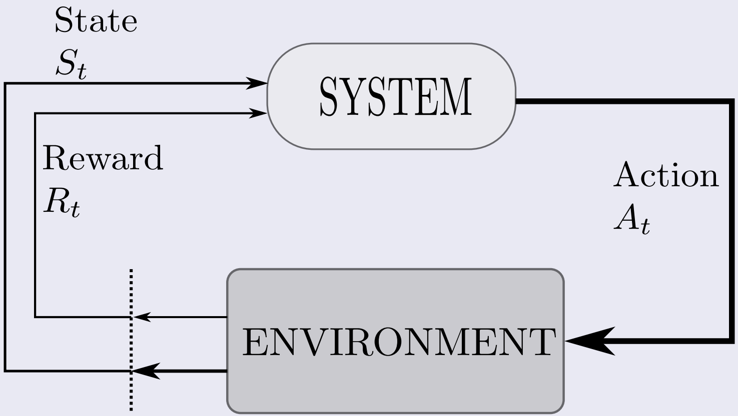
The System–Environment Interaction

In a Markov Decision Process



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Markov Process and Markov Decision Process

Definition of Markov Process

- A sequence of states is Markov if and only if the probability of moving to the next state s_{t+1} depends only on the present state s_t

$$P\left[s_{t+1}|s_t\right] = P\left[s_{t+1}|s_1, ..., s_t\right]$$

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- We always talk about time-homogeneous Markov chain in Reinforcement Learning:
  - the probability of the transition is independent of t

$$P\left[s_{t+1} = s'|s_t = s\right] = P\left[s_t = s'|s_{t-1} = s\right]$$

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Formally

Definition (Markov Process)

- a Markov Process (or Markov Chain) is a tuple $(\mathcal{S}, \mathcal{P})$, where
  1. $\mathcal{S}$ is a finite set of states
  2. $\mathcal{P}$ is a state transition probability matrix. $\mathcal{P}_{ss'} = P\left[s_{t+1} = s' | s_t = s\right]$

Thus, the dynamic of the Markov process is as follow

$$s_0 \xrightarrow{\mathcal{P}_{s_0 s_1}} s_1 \xrightarrow{\mathcal{P}_{s_1 s_2}} s_2 \xrightarrow{\mathcal{P}_{s_2 s_3}} s_3 \longrightarrow \cdots$$

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The Goal of Reinforcement Learning

We want to maximize the expected value of the return

- i.e. To obtain the optimal policy!!!

Thus, as in our previous study of MDP's

$$G_t = R_{t+1} + \alpha R_{t+2} + \cdots = \sum_{k=0}^{\infty} \alpha^k R_{t+k+1}$$

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Remarks

Interpretation

- The discounted future rewards can be interpreted as the **current value of future rewards**.

The reward α

- α close to 0 leads to "short-sighted" evaluation... you are only looking for immediate rewards!!!
- α close to 1 leads to "long-sighted" evaluation... you are willing to wait for a better reward!!!

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Why to use this discount factor?

Basically

1. Discount factor avoids infinite returns in cyclic Markov process given the property of the geometry series.
2. There is certain uncertainty about the future reward.
3. If the reward is financial, immediate rewards may earn more interest than delayed rewards.
4. Animal/human behavior shows preference for immediate reward.

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A Policy

Using our MDP ideas...

- A policy π is a distribution over actions given states

$$\pi\left(a|s\right) = P\left[A_t = a|S_t = s\right]$$

  - A policy guides the choice of action at a given state.
  - And it is time independent

Then, we introduce two value functions

- State-value function.
- Action-value function.

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&= E_\pi[R_{t+1} + \alpha G_{t+1} | S_t = s] \\
&= \underbrace{E_\pi[R_{t+1} | S_t = s]}_{\text{Immediate Reward}} + \underbrace{E_\pi[\alpha v_\pi(S_{t+1}) | S_t = s]}_{\text{Discount value of successor state}}
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The Action-Value Function $q_{\pi}(s,a)$

It is the expected return starting from state s

- Starting from state s, taking action a, and then following policy π

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We can also obtain the following decomposition

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To simplify notations, we define

$$\mathcal{R}_s^a = E\left[R_{t+1} | S_t = s, A_t = a\right]$$

Then, we have

$$v_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s)\, q_\pi(s,a)$$

$$q_\pi(s,a) = \mathcal{R}_s^a + \alpha \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s')$$

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Then

Expressing $q_\pi(s,a)$ in terms of $v_\pi(s)$ in the expression of $v_\pi(s)$ - The Bellman Equation

$$v_\pi(s) = \sum_{a\in\mathcal{A}} \pi(a|s)\left[\mathcal{R}_s^a + \alpha \sum_{s'\in\mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s')\right]$$

The Bellman equation relates the state-value function

- of one state with that of other states.

Then

Expressing $q_\pi(s,a)$ in terms of $v_\pi(s)$ in the expression of $v_\pi(s)$ - The Bellman Equation

$$v_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left[\mathcal{R}_s^a + \alpha \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s') \right]$$

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Bellman equation for $q_\pi(s,a)$

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How do we solve this problem?

- We have the following solutions

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Outline

Action-Value Methods

One natural way to estimate this is by averaging the rewards actually received

$$Q_t(a) = \frac{\sum_{i=1}^{t-1} R_i I_{A_i=a}}{\sum_{i=1}^{t-1} I_{A_i=a}}$$

Nevertheless

- If the denominator is zero, we define $Q_t(a)$ at some default value

Now, as the denominator goes to infinity, by the law of large numbers,

$$Q_t(a) \longrightarrow q^*(a)$$

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A Simple Alternative is to act greedily most of the time

- every once in a while, say with small probability ϵ, select randomly from among all the actions with equal probability.

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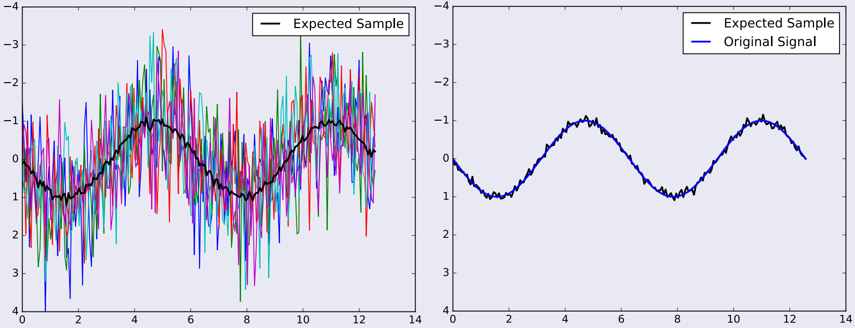
It is like the Mean Filter

We have a nice result



Example

In the following example

- The $q^*(a)$ were selected according to a Gaussian Distribution with mean 0 and variance 1.
- Then, when an action is selected, the reward R_t was selected from a normal distribution with mean $q_*(A_t)$ and variance 1.

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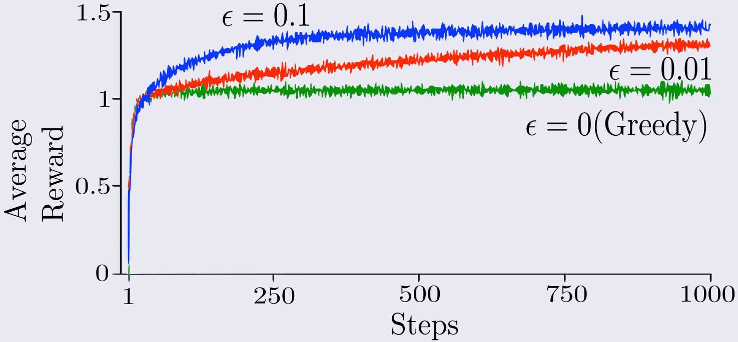
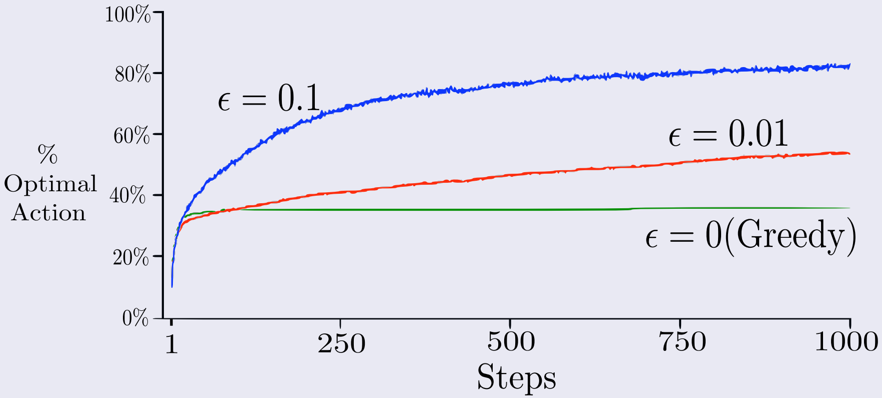


Now, as the sampling goes up

We get



Outline

Imagine the following

Let R_i denotes the reward after the i^{th} of this action

$$Q_n = \frac{R_1 + R_2 + \cdots + R_{n-1}}{n-1}$$

The problem of this technique

- The Amount of Memory and Computational Power to keep updates the estimates as more reward happens

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We need something more efficient

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Q_{n+1} &= \frac{1}{n}\sum_{i=1}^{n} R_n \\
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Outline

General Formula in Reinforcement Learning

The Pattern Appears almost all the time in RL

$$\text{New Estimate} = \text{Old Estimate} + \text{Step Size}\left[\text{Target} - \text{Old Estimate}\right]$$

Looks a lot as a Gradient Descent if we decide Assume a Taylor Series

$$f(x) = f(a) + f'(a)(x-a) \Longrightarrow f'(a) = \frac{f(x) - f(a)}{x-a}$$

Then

$$x_{n+1} = x_n + \gamma f'(a) = x_n + \gamma\left[\frac{f(x) - f(a)}{x-a}\right] = x_n + \gamma'\left[f(x) - f(a)\right]$$

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A Simple Bandit Algorithm

Initialize for $a = 1$ to k

1. $Q(a) \leftarrow 0$
2. $N(a) \leftarrow 0$

Loop until certain threshold γ

1. $A \leftarrow \begin{cases} \arg\max_a Q(a) & \text{with probability } 1-\epsilon \\ \text{A random action} & \text{with probability } \epsilon \end{cases}$
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3. $N(A) \leftarrow N(A) + 1$
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Remarks

Remember

- This is works well for a stationary problem

For more in non-stationary problems

- "Reinforcement Learning: An Introduction" by Sutton et al. at chapter 2.

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Dynamic Programming

As in many things in Computer Science

- We love the Divide Et Impera...

Then

- In Reinforcement Learning, Bellman equation gives recursive decompositions, and value function stores and reuses solutions.

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Policy Evaluation

For this, we assume as always

- The problem can be solved with the Bellman Equation

We start from an initial guess v_1

- Then, we apply Bellman Iteratively

$$v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_\pi$$

For this, we use synchronous updates

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At each iteration $k+1$, for all states $s \in \mathcal{S}$

- We update v_{k+1} from $v_k(s')$ according to Bellman Equations, where s' is a successor state of s:

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- The iterative process stops when the maximum difference between value function at the current step and that at the previous step is smaller than some small positive constant.

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Our ultimate goal is to find the optimal policy

- For this, we can use policy iteration process

Algorithm

1. Initialize π randomly
2. Repeat until the previous policy and the current policy are the same.
   1. Evaluate v_π by policy evaluation.
   2. Using synchronous updates, for each state s, let

$$\pi(s) = \arg\max_{a \in \mathcal{A}} q(s, a) = \arg\max_{a \in \mathcal{A}} \left[\mathcal{R}_s^a + \alpha \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s') \right]$$

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- This policy iteration algorithm will improve the policy for each step.

Assume that

- We have a deterministic policy π and after one step we get π'.

We know that the value improves in one step because

$$q_{\pi}\left(s, \pi'\left(s\right)\right) = \max_{a \in \mathcal{A}} q_{\pi}\left(s, a\right) \geq q_{\pi}\left(s, \pi\left(s\right)\right) = v_{\pi}\left(s\right)$$

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We can see that

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Then

If improvements stop

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This means that the Bellman equation has been satisfied

$$v_\pi\left(s\right) = \max_{a \in \mathcal{A}} q_\pi\left(s, a\right)$$

- Therefore, $v_\pi\left(s\right) = v_*\left(s\right)$ for all $s \in \mathcal{S}$ and π is an optimal policy.

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Outline

Once the policy has been improved

We can iteratively a sequence of monotonically improving policies and values

$$\pi_0 \xrightarrow{E} v_{\pi_0} \xrightarrow{I} \pi_1 \xrightarrow{E} v_{\pi_1} \xrightarrow{I} \pi_2 \xrightarrow{E} \cdots \xrightarrow{I} \pi_* \xrightarrow{E} v_*$$

- with $E = evaluation$ and $I = improvement$.

Policy Iteration (using iterative policy evaluation)

Step 1. Initialization

- $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

Step 2. Policy Evaluation

1. Loop:
2. $\quad \Delta \leftarrow 0$
3. $\quad$ Loop for each $s \in \mathcal{S}$
4. $\quad\quad v \leftarrow V(s)$
5. $\quad\quad V(s) \leftarrow \sum_{s',r} p(s', r|s, \pi(s)) [r + \alpha V(s')]$
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7. until $\Delta < \theta$ (A small threshold for accuracy)

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Further

Step 3. Policy Improvement

1. Policy-stable $\leftarrow$ True
2. For each $s \in S$:
3. old_action $\leftarrow \pi(s)$
4. $\pi(s) \leftarrow \sum_{s',r} p(s'.r|s,a)\left[r + \alpha V(s')\right]$
5. if old_action$\neq \pi(s)$ then policy-stable $\leftarrow$ False
6. If policy-stable then stop and return $V \approx v_*$ and $\pi \approx \pi_*$ else go to 2.

Outline

Jack's Car Rental

Setup

- We have a Car's Rental with two places (s_1, s_2).
- And each time Jack is available he rents it out and is credited \$10 by the company.
- If Jack's is out of cars at that location, then the business is lost.
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We choose a probability for modeling requests and returns

Additionally, renting at each location has a Poisson distribution

$$P\left(n|\lambda\right)=\frac{\lambda^{n}}{n!}e^{-\lambda}$$

We simplify the problem by having an upper-bound $n = 20$

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We have the following $\lambda' s$ for request and returns

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Actually, it defines how many cars are requested or returned in a time interval (One day)

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Actually, it defines how many cars are requested or returned in a time interval (One day)

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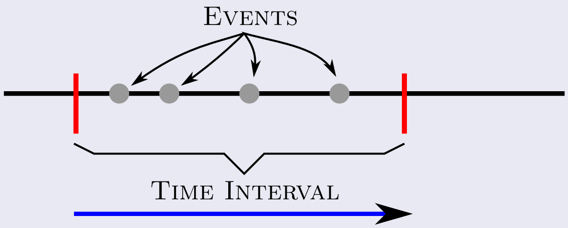
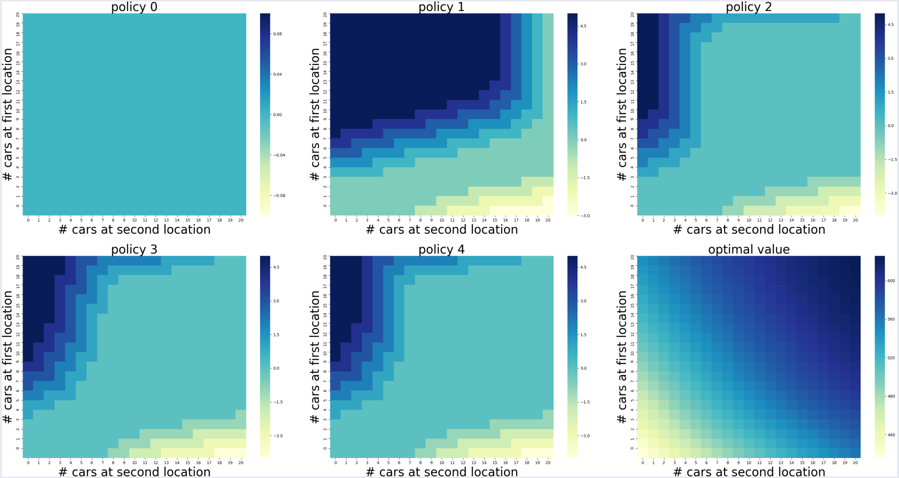


Thus, we have the following running

With a $\alpha = 0.9$



Outline

History

There is an original earlier work

- By Arthur Samuel and it was used by Sutton to invent Temporal Difference-Lambda

It was famously applied by Gerald Tesauro

- To build TD-Gammon
  - A program that learned to play backgammon at the level of expert human players

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Temporal-Difference (TD) Learning

Something Notable

- Possibly the central and novel idea of reinforcement learning,

It has similarity with Monte Carlo Methods

- They use experience to solve the prediction problem (For More look at Sutton's Book)

Monte Carlo methods wait until G_t is known

$$V(S_t) \leftarrow V(S_t) + \gamma \left[G_t - V(S_t) \right] \text{ and } G_t = \sum_{k=0}^{\infty} \alpha^k R_{t+k+1}$$

- Let us call this method constant-α Monte Carlo.

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Fundamental Theorem of Simulation

Cumulative Distribution Function

- With every random variable, we associate a function called Cumulative Distribution Function (CDF) which is defined as follows:

$$F_X(x) = P(f(X) \leq x)$$

- With properties:
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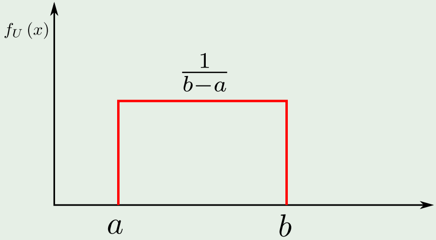
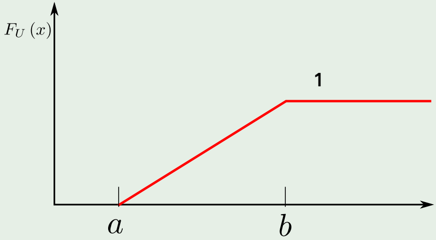
Example Uniform Distribution





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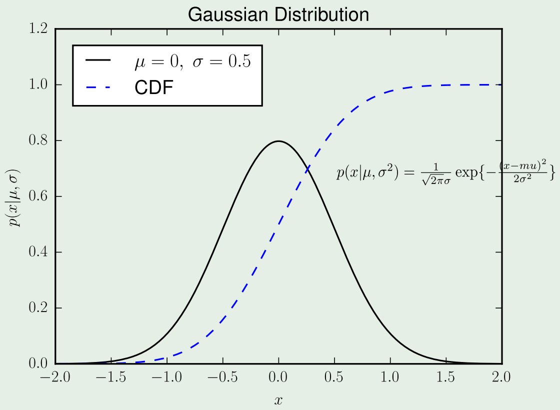

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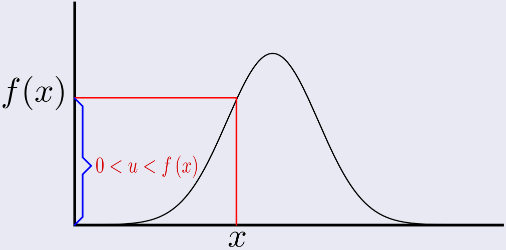

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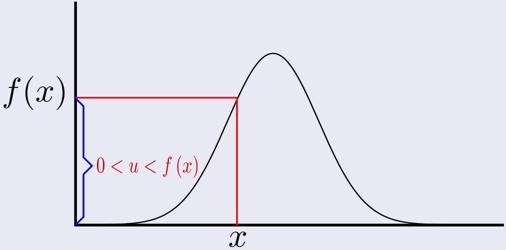

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One thing that is made clear by this theorem is that we can generate X in three ways

1. First generate $X \sim f$, and then $U|X = x \sim U\left\{u|u \leq f(x)\right\}$, but this is useless because we already have X and do not need X.
2. First generate U, and then $X|U = u \sim U\left\{x|u \leq f(x)\right\}$.
3. Generate (X, U) jointly, a smart idea.
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Markov Chains

The random process $X_t \in S$ for $t = 1, 2, ..., T$ has a Markov property iif

$$p\left(X_T | X_{T-1}, X_{T-2}, ..., X_1\right) = p\left(X_T | X_{T-1}\right)$$

Finite-state Discrete Time Markov Chains $|S| < \infty$

- It can be completly specified by the transition matrix.
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The Idea

Draw an i.i.d. set of samples $\left\{ \boldsymbol{x}^{(i)} \right\}_{i=1}^{N}$

From a target density $p\left(\boldsymbol{x}\right)$ on a high-dimensional $\mathcal{X}$.

- The set of possible configurations of a system.

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Thus, we can use those samples to approximate the target density

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with

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Where Variance of $f(\boldsymbol{x})$

$$\sigma_f^2 = \mathbb{E}_{p(\boldsymbol{x})}\left[f^2(\boldsymbol{x})\right] - I^2(f) < \infty$$

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By Robert & Casella 1999, Section 3.2

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What if we do not wait for G_t

TD methods need to wait only until the next time step

- At time $t+1$, it uses the observed reward R_{t+1} and the estimate $V(S_{t+1})$

The simplest TD method

$$V(S_t) \leftarrow V(S_t) + \gamma\left[R_{t+1} + \alpha V(S_{t+1}) - V(S_t)\right]$$

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- There are TD with n-step TD methods

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TD as a Bootstrapping Method

Quite similar to Dynamic

- Additionally, we know that

$$\begin{aligned} v_\pi(s) &= E_\pi\left[G_t|S_t\right] \\ &= E_\pi\left[R_{t+1} + \alpha G_{t+1}|S_t\right] \\ &= E_\pi\left[R_{t+1} + \alpha v_\pi(S_{t+1})\,|S_t\right] \end{aligned}$$

Monte Carlo methods use an estimate of $v_\pi(s) = E_\pi\left[G_t|S_t\right]$

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And it is used to improve the policy

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Finally

If the array V does not change during the episode,

- The Monte Carlo Error can be written as a sum of TD errors

$$\begin{aligned}
\epsilon = G_t - V(S_t) &= R_{t+1} + \alpha G_{t+1} - V(S_t) + \alpha V(S_{t+1}) - \alpha V(S_{t+1}) \\
&= \delta_t + \alpha \left(G_{t+1} - V(S_{t+1})\right) \\
&= \delta_t + \alpha \delta_{t+!} + \alpha^2 \left(G_{t+2} - V(S_{t+2})\right) \\
&= \sum_{k=t}^{T-1} \alpha^{k-t} \delta_k
\end{aligned}$$

Optimality of TD(0)

Under batch updating

- TD(0) converges deterministically to a single answer independent
  - as long as γ is chosen to be sufficiently small.

$$V(S_t) \leftarrow V(S_t) + \gamma \left[G_t - V(S_t) \right] \text{ and}$$

Outline

Q-learning: Off-Policy TD Control

One of the early breakthroughs in reinforcement learning

- The development of an off-policy TD control algorithm, Q-learning (Watkind, 1989)

It is defined as

$$Q\left(S_t, A_t\right) \leftarrow Q\left(S_t, A_t\right) + \gamma\left[R_{t+1} + \alpha \max_a Q\left(S_{t+1}, a\right) - Q\left(S_t, A_t\right)\right]$$

Here

- The learned action-value function, Q, directly approximates q_* the optimal action-value function
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The policy still has an effect

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- All that is required for correct convergence is that all pairs continue to be updated

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Q-learning (Off-Policy TD control) for estimating $\pi \approx \pi_*$

Algorithm parameters

- $\alpha \in (0, 1]$ and small $\epsilon > 0$

Initialize

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Then

Loop for each episode

1. Initialize S
2. Loop for each step of episode:
3. Choose A from S using policy derived from Q (for example ϵ-greedy)
4. Take action A, observe R, S'
$$Q(S, A) \leftarrow Q(S, A) + \gamma \left[R + \alpha \max_a Q(S', a) - Q(S, A) \right]$$
5. $S \leftarrow S'$
6. Until S is terminal

Outline

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Backgammon is a major game

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- An estimated value $\hat{v}(s, \boldsymbol{w})$ is used to estimate the probability of winning from state s.

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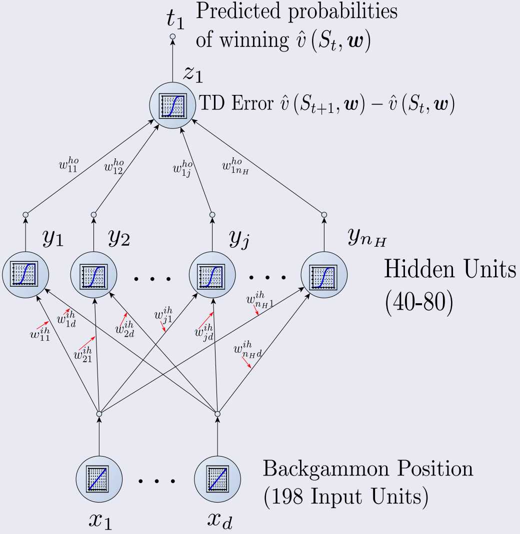
Implementation of the value function in TD-Gammon

They took the decision to use a standard multilayer ANN



The interesting part

Tesauro's representation

- For each point on the backgammon board
  - Four units indicate the number of white pieces on the point

Basically

- If no white pieces, all four units took zero values,
- if there was one white piece, the first unit takes a one value
- And so on...

Therefore, given four white pieces and another four black pieces at each of the 24 points

- We have 4*24+4*24=192 units

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Then

We have that

- Two additional units encoded the number of white and black pieces on the bar (Each took the value $n/2$ with n the number of the bar)
- Two other represent the number of black and white pieces already (these took the value $n/15$)
- two units indicated in a binary fashion the white or black turn.

Outline

TD(λ) by Back-Propagation

TD-Gammon uses a semi-gradient of the $TD(\lambda)$

- $TD(\lambda)$ has the following general update rule:

$$\boldsymbol{w}_{t+1} = \boldsymbol{w}_t + \gamma \left[R_{t+1} + \alpha \hat{v}\left(S_{t+1}, \boldsymbol{w}_t\right) - \hat{v}\left(S_t, \boldsymbol{w}_t\right) \right] \boldsymbol{z}_t$$

  - with $\boldsymbol{w}_t$ is the vector of all modifiable parameters

$\boldsymbol{z}_t$ is a vector of eligibility traces

- Eligibility traces unify and generalize TD and Monte Carlo methods.
- It works as Long-Short Term Memory

The rough idea is that when a component of $\boldsymbol{w}_t$ produces an estimate

- $\boldsymbol{z}_t$ is bumped out and then begins to fade away.

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How this is done?

Update Rule

$$\boldsymbol{z}_t = \alpha\lambda \boldsymbol{z}_{t-1} + \nabla \hat{v}\left(S_t, \boldsymbol{w}_t\right) \text{ with } \boldsymbol{z}_0 = 0$$

Something Notable

- The gradient in this equation can be computed efficiently by the back-propapagation procedure

For this, we set the error in the ANN

$$\hat{v}\left(S_{t+1}, \boldsymbol{w}_t\right) - \hat{v}\left(S_t, \boldsymbol{w}_t\right)$$

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