Analysis of Algorithms

Skip Lists

Andres Mendez-Vazquez

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Outline

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1. Dictionaries
   - Definitions
   - Dictionary operations
   - Dictionary implementation

2. Skip Lists
   - Why Skip Lists?
   - The Idea Behind All of It!!!
   - Skip List Definition
   - Skip list implementation
   - Insertion for Skip Lists
   - Deletion in Skip Lists
   - Properties
   - Search and Insertion Times
   - Applications
   - Summary

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- Uniqueness criteria may be relaxed (multi-set).
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Example: Course records

Dictionary with member records

key ID	Student Name	HW1	
123	Stan Smith	49	$\cdots$
125	Sue Margolin	45	$\cdots$
128	Billie King	24	$\cdots$
$\vdots$			
$\vdots$			
190	Roy Miller	36	$\cdots$

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The dictionary ADT operations

Some operations on dictionaries

- size(): Returns the size of the dictionary.
- empty(): Returns TRUE if the dictionary is empty.
- findItem(key): Locates the item with the specified key.
- findAllItems(key): Locates all items with the specified key.
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Example of unordered dictionary

Example

Consider an empty unordered dictionary, we have then...

Operation	Dictionary	Output
InsertItem$(5, A)$	$\{(5, A)\}$	
InsertItem$(7, B)$	$\{(5, A), (7, B)\}$	
findItem(7)	$\{(5, A), (7, B)\}$	B
findItem(4)	$\{(5, A), (7, B)\}$	No Such Key
size()	$\{(5, A), (7, B)\}$	2
removeItem(5)	$\{(7, B)\}$	A
findItem(4)	$\{(7, B)\}$	No Such Key

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How to implement a dictionary?

There are many ways of implementing a dictionary

- Sequences / Arrays
  - Ordered
  - Unordered
- Binary search trees
- Skip lists
- Hash tables

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Unordered array

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Binary searches

Features

- Narrow down the search range in stages
- "High-low" game.

Binary searches

Example find Element(22)

2	4	5	7	8	9	12	**14**	17	19	22	25	27	28	33
$\uparrow$ LOW							$\uparrow$ MID							$\uparrow$ $HIGH$

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Recall binary search trees

Implement a dictionary with a BST

A binary search tree is a binary tree T such that:

- Each internal node stores an item (k, e) of a dictionary.
- Keys stored at nodes in the left subtree of v are less than or equal to k.
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cinvestav
Unidad Guadalajara

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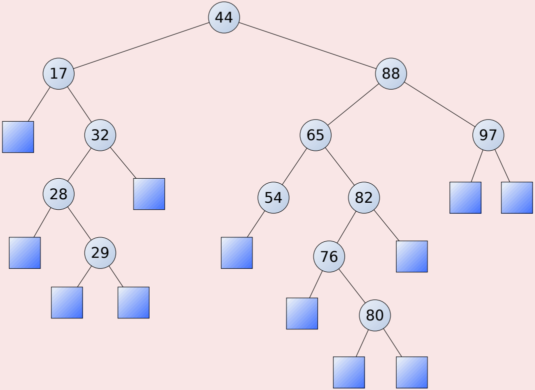
Binary searches Trees

Problem!!! Keeping a Well Balanced Binary Search Tree can be difficult!!!



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We want to have a

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Thus, we have the following possibilities

Unordered array complexities

Insertion: $O(1)$

Search: $O(n)$

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We want something better!!!

For this

We will present a probabilistic data structure known as Skip List!!!

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Starting from Scratch

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- Imagine that you only require to have searches.
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- Then, using this How do we speed up searches?

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- The Top is the high way system,

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1. The Bottom is the normal road system, L_2.
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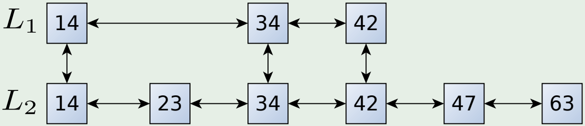
Example

High-Bottom Way System



Thus, we have...

The following rule

To Search first search in the top one (L_1) as far as possible, then go down and search in the bottom one (L_2).

We can use a little bit of optimization

We have the following worst cost

Search Cost High-Bottom Way System $=$ Cost Searching Top $+ \ldots$

Cost Search Bottom

Or

Search Cost $= length\left(L_1\right) +$ Cost Search Bottom

The interesting part is "Cost Search Bottom"

This can be calculated by the following quotient:

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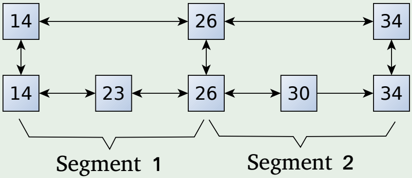
If we think we are jumping



Then cost of searching each of the bottom segments = 2

Thus the ratio is a "decent" approximation to the worst case search

$$\frac{length\left(L_2\right)}{length\left(L_1\right)} = \frac{5}{3} = 1.66$$

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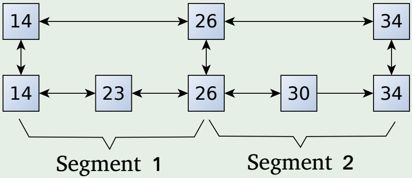

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Then, the cost for a search (when $length\left(L_2\right) = n$)

$$\text{Search Cost } = length\left(L_1\right) + \frac{length\left(L_2\right)}{length\left(L_1\right)} = length\left(L_1\right) + \frac{n}{length\left(L_1\right)} \quad (1)$$

Taking the derivative with respect to $length\left(L_1\right)$ and making the result equal 0

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Plugging back in (Eq. 1)

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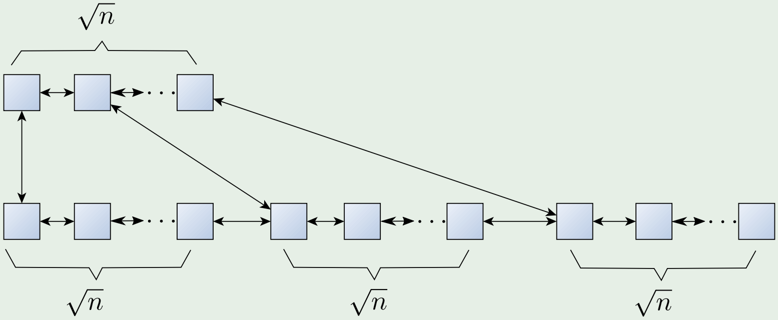
Data structure with a Square Root Relation

Something like this



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For a three layer link list data structure

We get a search cost of $3 \times \sqrt[3]{n}$

In general for k layers, we have

$$k \times \sqrt[k]{n}$$

Thus, if we make $k = \log_2 n$, we get

$$\begin{aligned}\text{Search Cost} &= \log_2 n \times \sqrt[\log_2 n]{n}\\ &= \log_2 n \times (n)^{1/\log_2 n}\\ &= \log_2 n \times (n)^{\log_n 2}\\ &= \log_2 n \times 2\\ &= \Theta(\log_2 n)\end{aligned}$$

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Something Notable

We get the advantages of the binary search trees with a simpler architecture!!!

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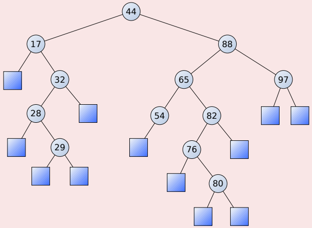
Binary Search Trees



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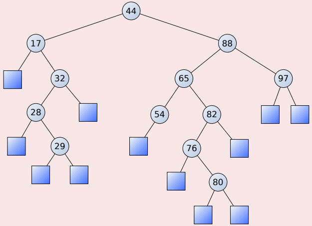

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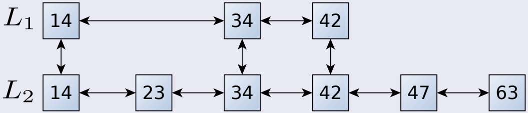

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We are ready to give a

Definition for Skip List

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- He is is an American computer scientist who invented the skip list and the Omega test for deciding Presburger arithmetic.
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Skip List Definition

Definition

A skip list for a set S of distinct (key,element) items is a series of lists $S_0, S_1, ..., S_h$ such that:

- Each list S_i contains the special keys $+\infty$ and $-\infty$
- List S_0 contains the keys of S in nondecreasing order
- Each list is a subsequence of the previous one
  - $S_0 \supseteq S_1 \supseteq S_2 \supseteq ... \supseteq S_h$
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Skip List Definition

Definition

A skip list for a set S of distinct (key,element) items is a series of lists $S_0, S_1, ..., S_h$ such that:

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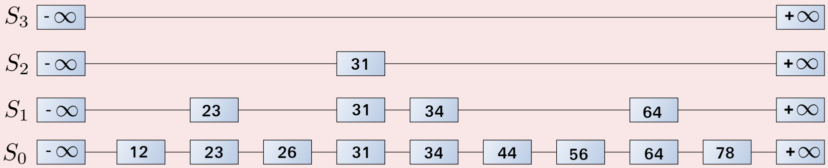
Example



Skip list search

We search for a key x in a skip list as follows

- We start at the first position of the top list.
- At the current position p, we compare x with $y == p.next.key$
  - $x == y$: we return $p.next.element$
  - $x > y$: we scan forward
  - $x < y$: we "drop down"
- If we try to drop down past the bottom list, we return $null$.

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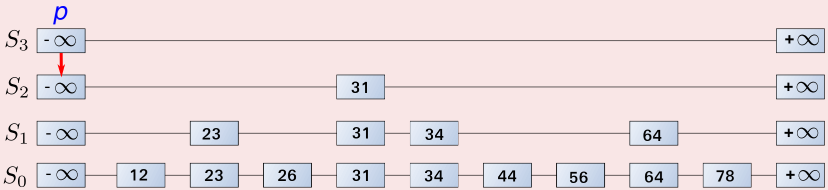
Example search for 78

$x < p.next.key$: "drop down"



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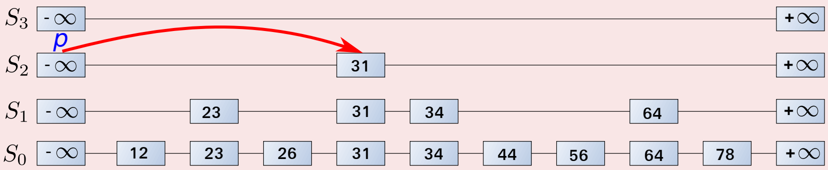

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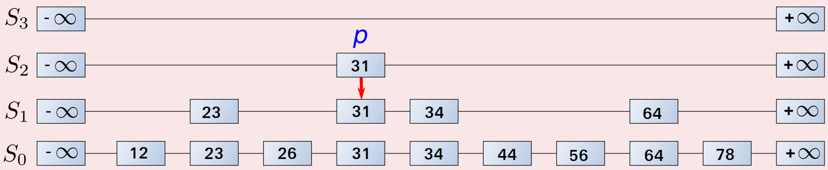

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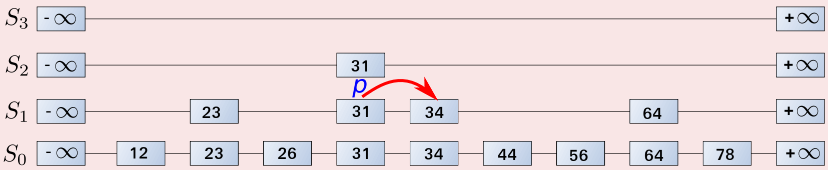

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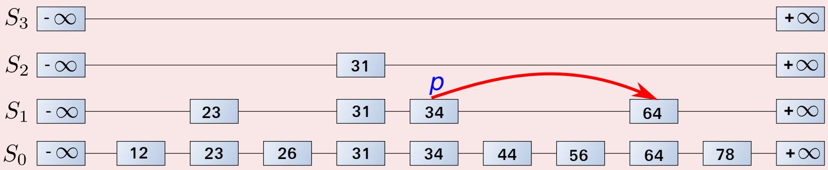

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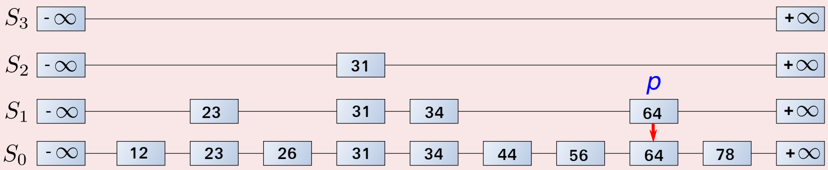

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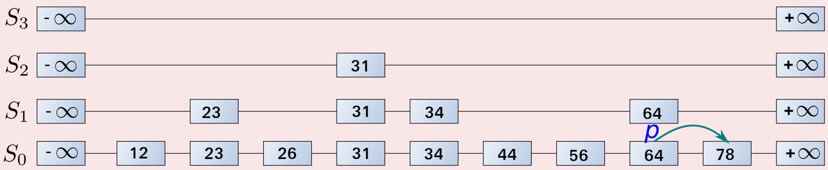
$x == y$: we return $p.next.element$



Outline

How do we implement this data structure?

We can implement a skip list with quad-nodes

A quad-node stores:

- Entry Value
- Link to the previous node
- Link to the next node
- Link to the above node
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Also we define special keys PLUS_INF and MINUS_INF, and we modify the key comparator to handle them.

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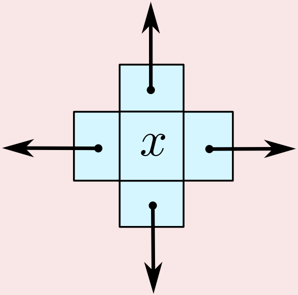
Quad-Node Example



Skip lists uses Randomization

Use of randomization

We use a randomized algorithm to insert items into a skip list.

Running time

We analyze the expected running time of a randomized algorithm under the following assumptions:

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Outline

Insertion

To insert

To insert an entry $(key, object)$ into a skip list, we use a randomized algorithm:

1. We repeatedly toss a coin until we get tails:
   - We denote with i the number of times the coin came up heads.
2. If $i \geq h$, we add to the skip list new lists $S_{h+1}, ..., S_{i+1}$:
   - Each containing only the two special keys.
3. We search for x in the skip list and find the positions $p_0, p_1, ..., p_i$ of the items with largest key less than x in each lists $S_0, S_1, ..., S_i$.
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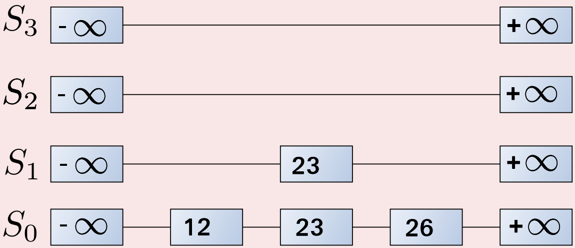
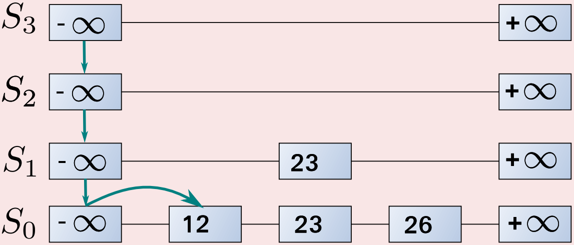
Example: Insertion of 15 in the skip list

First, we use $i = 2$ to insert S_3 into the skip list



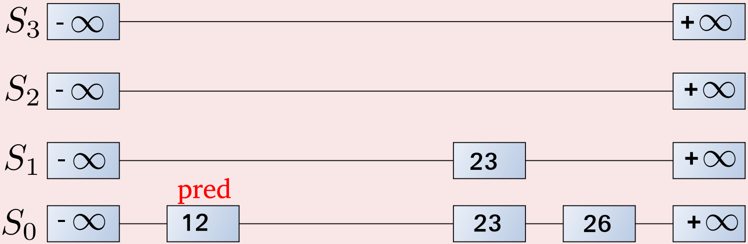
Example: Insertion of 15 in the skip list

Clearly, you first search for the predecessor key!!!



Example: Insertion of 15 in the skip list

Insert the necessary Quad-Nodes and necessary information



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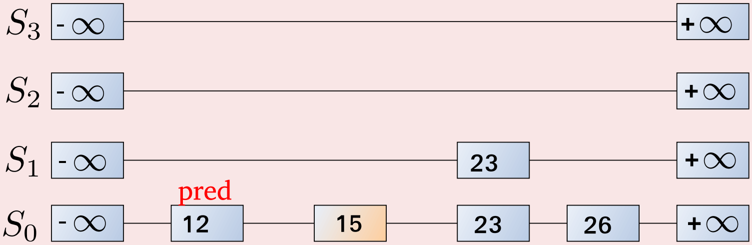

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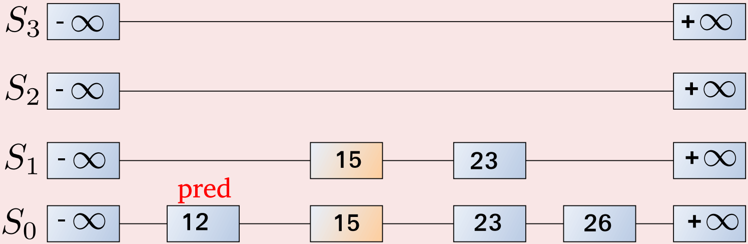
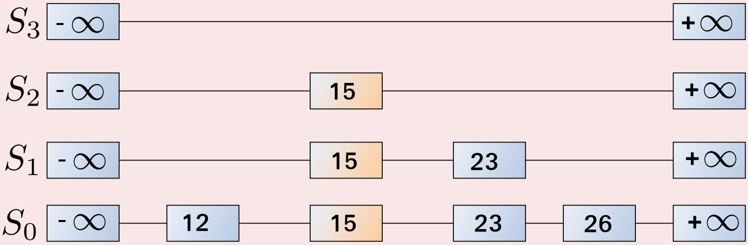


Example: Insertion of 15 in the skip list

Finally!!!



Outline

Deletion

To remove an entry with key x from a skip list, we proceed as follows

- We search for x in the skip list and find the positions $p_0, p_1, ..., p_i$ of the items with key x, where position p_j is in list S_j.
- We remove positions $p_0, p_1, ..., p_i$ from the lists $S_0, S_1, ..., S_i$.
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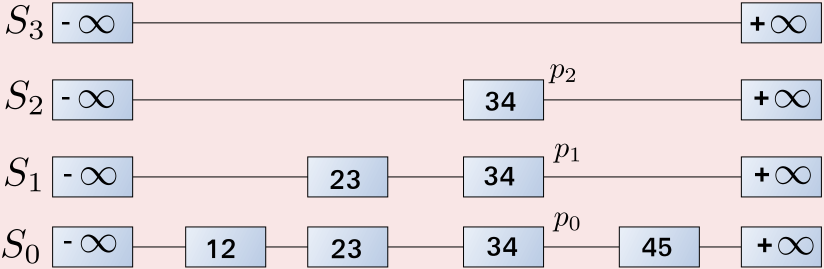
Example: Delete of 34 in the skip list

We search for 34 in the skip list and find the positions $p_0, p_1, ..., p_2$ of the items with key 34



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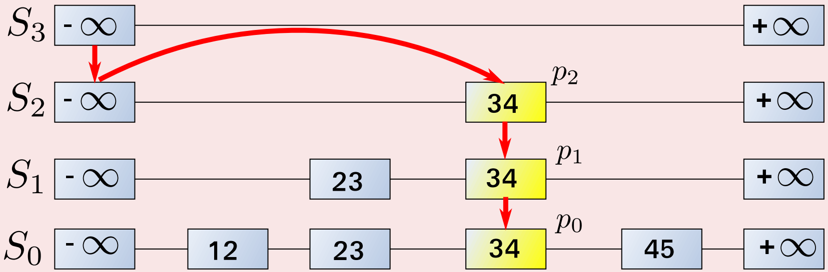
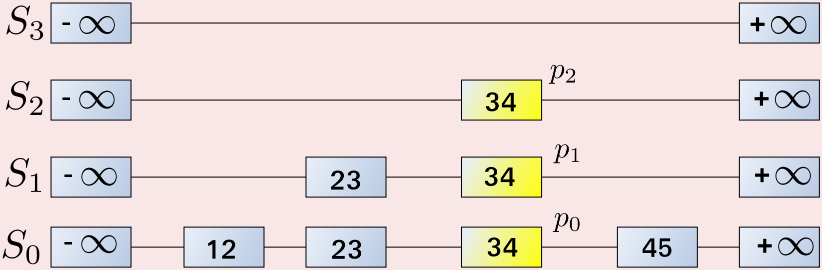


Example: Delete of 34 in the skip list

We start doing the deletion!!!



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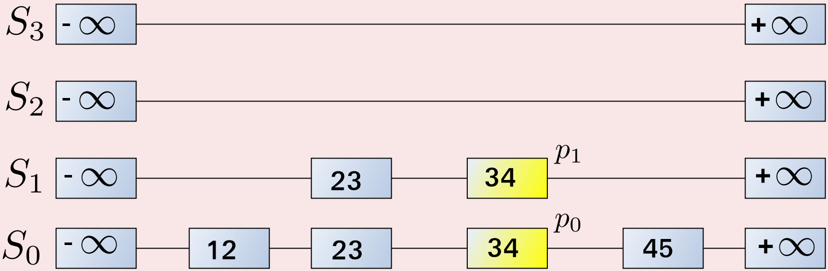
One Quad-Node after another



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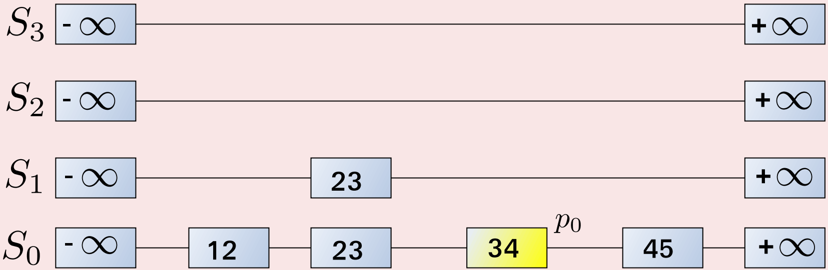

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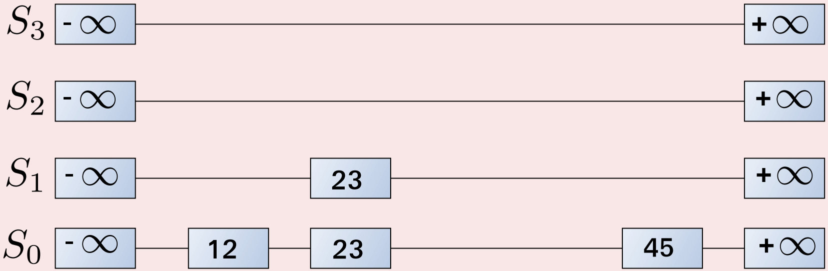

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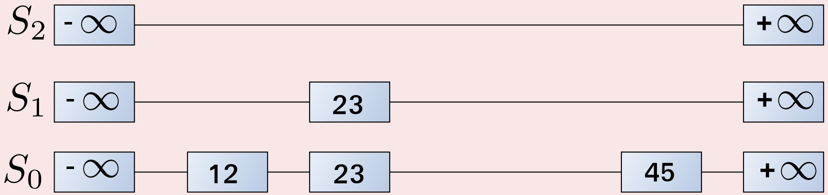
Remove One Level



Outline

Space usage

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The space used by a skip list depends on the random bits used by each invocation of the insertion algorithm.

Space : $O(n)$

Theorem

The expected space usage of a skip list with n items is $O(n)$.

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We use the following two basic probabilistic facts:

1. *Fact 1:* The probability of getting i consecutive heads when flipping a coin is $\frac{1}{2^i}$.
2. *Fact 2:* If each of n entries is present in a set with probability p, the expected size of the set is np.
   1. How? Remember $X = X_1 + X_2 + ... + X_n$ where X_i is an indicator function for event $A_i =$ the i element is present in the set. Thus:

$$E[X] = \underbrace{\sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} Pr\{A_i\}}_{\text{Equivalence } E[X_A] \text{ and } Pr\{A\}} = \sum_{i=1}^{n} p = np$$

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Space : $O(n)$

Theorem

The expected space usage of a skip list with n items is $O(n)$.

Proof

We use the following two basic probabilistic facts:

1. *Fact 1:* The probability of getting i consecutive heads when flipping a coin is $\frac{1}{2^i}$.
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Proof

Now consider a skip list with n entries

Using Fact 1, an element is inserted in list S_i with a probability of

$$\frac{1}{2^i}$$

Now by Fact 2

The expected size of list S_i is

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Proof

The expected number of nodes used by the skip list with height h

$$\sum_{i=0}^{h} \frac{n}{2^i} = n \sum_{i=0}^{h} \frac{1}{2^i}$$

Here, we have a problem!!! What is the value of h?

Height h

First

The running time of the search and insertion algorithms is affected by the height h of the skip list.

Second

We show that with high probability, a skip list with n items has height $O(\log n)$.

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We can view the level $l(x_i) = \max\{j | \text{where } x_i \in S_j\}$ of the elements in the skip list as the following random variable

$$X_i = l(x_i)$$

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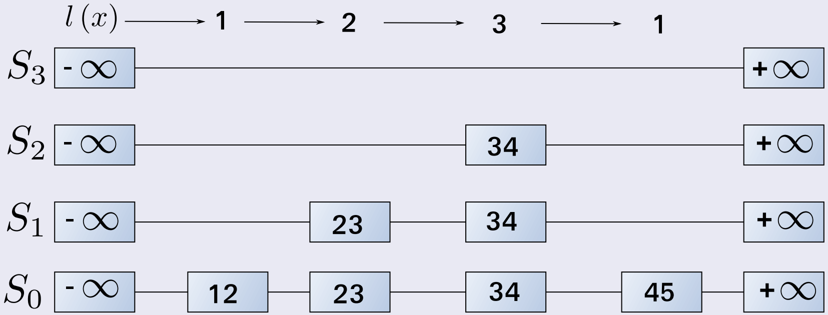
Example for $l\left(x_i\right)$

We have



BTW What is the geometric distribution?

k failures where

$$k = \{1, 2, 3, ...\}$$

Probability mass function

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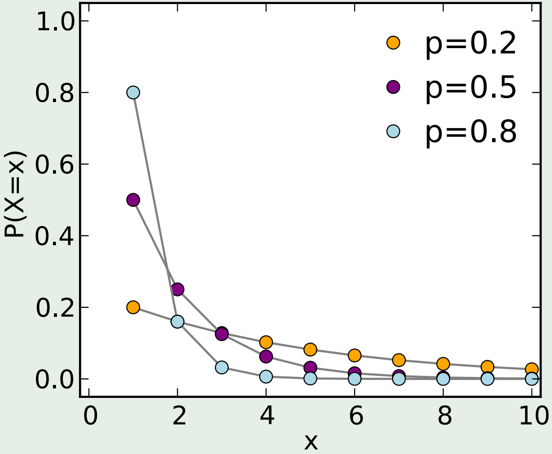
Probability Mass Function

For Different Probabilities



Then

We have the following inequality for the geometric variables

$$Pr\left[X_i > t\right] \leq (1-p)^t \;\; \forall i = 1, 2, ..., n$$

Because if the cdf $F(t) = P(X \leq t) = 1 - (1-p)^{t+1}$

Then, we have

$$Pr\left\{\max_i X_i > t\right\} \leq n(1-p)^t$$

This comes from $F_{\max_i X_i}(t) = (F(t))^n$

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How?

We have that

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\begin{aligned}
Pr\left\{\max_i X_i > t\right\} &= Pr\left\{\max\left\{X_1, X_2, ..., X_n\right\} > t\right\} \\
&= \sum_{i=1}^{n} Pr\left\{X_i > t \text{ and } X_i = \max\left\{X_1, X_2, ..., X_n\right\}\right\}
\end{aligned}
$$

Now, if we assume only two variables

$$
\left\{\max\left(X_1, X_2\right) > t\right\} = \left\{X_1 > t \text{ and } X_1 > X_2 \text{ or exclusive } X_2 > t \text{ and } X_2 > X_1\right\}
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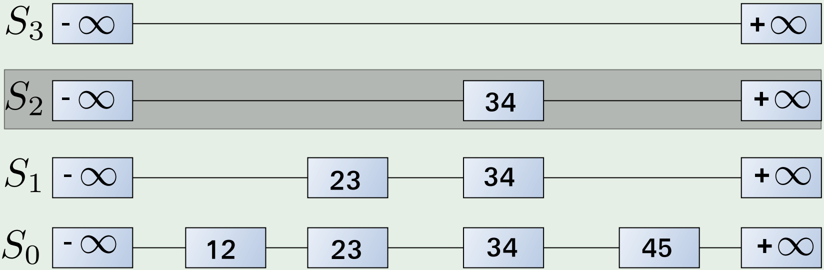
Observations

The $\max_i X_i$

It represents the list with the one entry apart from the special keys.



Observations

REMEMBER!!!

We are talking about a fair coin, thus $p = \frac{1}{2}$.

Height: $3\log_2 n$ with probability at least $1 - \frac{1}{n^2}$

Theorem

A skip list with n entries has height at most $3\log_2 n$ with probability at least $1 - \frac{1}{n^2}$

Proof

Consider a skip list with n entires

By Fact 3, the probability that list S_t has at least one item (The $\max_i X_i > t$) is at most $\frac{n}{2^t}$.

$$P\left(|S_t| \geq 1\right) = P\left(\max_i X_i > t\right) \leq \frac{n}{2^t}.$$

By picking $t = 3 \log n$

We have that the probability that $S_{3 \log_2 n}$ has at least one entry is at most:

$$\frac{n}{2^{3 \log_2 n}} = \frac{n}{n^3} = \frac{1}{n^2}.$$

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Finally

The expected number of nodes used by the skip list with height h

- Given that $h = 3\log_2 n$

$$\sum_{i=0}^{3\log_2 n} \frac{n}{2^i} = n \sum_{i=0}^{3\log_2 n} \frac{1}{2^i}$$

Given the geometric sum

$$S_m = \sum_{k=0}^{m} r^k = \frac{1 - r^{m+1}}{1 - r}$$

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The Upper Bound on the number of nodes

$$n\sum_{i=0}^{3\log_2 n}\frac{1}{2^i} = n\left(\frac{1-\left(\frac{1}{2}\right)^{3\log_2 n+1}}{1-1/2}\right)$$

$$= n\left(\frac{1-\frac{1}{2^{3\log_2 n+1}}}{1/2}\right)$$

$$= n\left(\frac{1-\frac{1}{\left(2^{\log_2 n}\right)^3 2}}{1/2}\right)$$

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Finally

We have

$$\left(\frac{2n^3-1}{n^2}\right)=2n-\frac{1}{n^2}\leq 2n$$

The Upper Bound with probability $1-\frac{1}{n^2}$

$$2n-\frac{1}{n^2}\leq 2n=O(n)$$

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Search and Insertion Times

Fact 4

The expected number of coin tosses required in order to get tails is 2:

$$\text{Given } x \sim G\left(\frac{1}{2}\right) \Longrightarrow E\left[x\right] = \frac{1}{2}$$

We use this

To prove that a search in a skip list takes $O(\log n)$ expected time.

- After all insertions require searches!!!

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Search time

The search time in skip list is proportional to

the number of drop-down steps + the number of scan-forward steps

Drop-down steps

The drop-down steps are bounded by the height of the skip list and thus are $O(\log_2 n)$ with high probability.

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Proof

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When we scan forward in a list, the destination key does not belong to a higher list.

A scan-forward step is associated with a former coin toss that gave tails

By Fact 4, in each list the expected number of scan-forward steps is 2.

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Why?

Given the list S_i

Then, the scan-forward intervals (Jumps between x_i and x_{i+1}) to the right of S_i are

$$I_1 = [x_1, x_2], I_2 = [x_2, x_3] \ldots I_k = [x_k, +\infty]$$

Then

These interval exist at level i if and only if all $x_1, x_2, \ldots, x_k$ belong to S_i.

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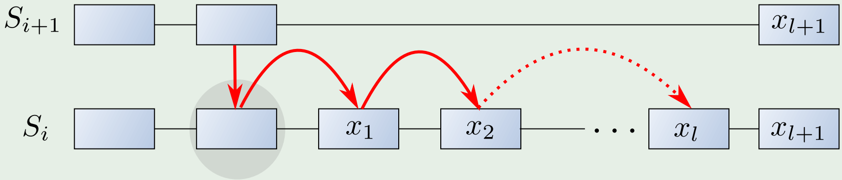
We introduce the following concept based on these intervals

Scan-forward siblings

These are element that you find in the search path before finding an element in the upper list.



Now

Given that a search is being done, S_i contains l forward siblings

It must be the case that given $x_1, ..., x_l$ scan-forward siblings, we have that

$$x_1, ..., x_l \notin S_{i+1}$$

and $x_{l+1} \in S_{i+1}$

Thus

We have

Since each element of S_i is independently chosen to be in S_{i+1} with probability $p = \frac{1}{2}$.

We have

The number of scan-forward siblings is bounded by a geometric random variable X_i with parameter $p = \frac{1}{2}$.

- Imagine the fact that you have multiple fails... then $x_1, ..., x_l \notin S_{i+1}$ is modeled by X_i

Thus, we have that

The expected number of scan-forward siblings is bounded by 2!!!

$$\text{Expected \# Scan-Forward Siblings at } i \leq \underbrace{E[X_i] = \frac{1}{1/2}}_{\text{Mean}} = 2$$

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In the worst case scenario

A search is bounded by $O(\log_2 n) + 2\log_2 n = O(\log_2 n)$

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Thus, an insertion is bounded by $2\log_2 n + 3\log_2 n = O(\log_2 n)$

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Applications

We have

- Cyrus IMAP servers offer a "skiplist" backend Data Base implementation.
- Lucene uses skip lists to search delta-encoded posting lists in logarithmic time.
- Redis, an ANSI-C open-source persistent key/value store for Posix systems, uses skip lists in its implementation of ordered sets.
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- In a skip list with n entries:
  - The expected space used is $O(n)$
  - The expected search, insertion and deletion time is $O(\log n)$
- Skip list are fast and simple to implement in practice.

Thanks

Questions?