Probability Review

July 21, 2014

1 Introduction

Probability is one of those fields with many applications. In Analysis of algorithms, we use probability as a way of counting. After all it is easier to deal with certain events in an algorithm as random events that are happening because of an underlying probability. Therefore, anybody who want to compute the complexities of any algorithm, once the traditional methods of counting have failed, will require to use probabilities.

2 Posterior Probability

You can see this as a reduction in the sample space that requires a renormalization. After all

$$\sum_{\forall A \subseteq S} P(A|B) = 1 \tag{1}$$

For example in the following image, we have that idea expressed as intersection of sets:

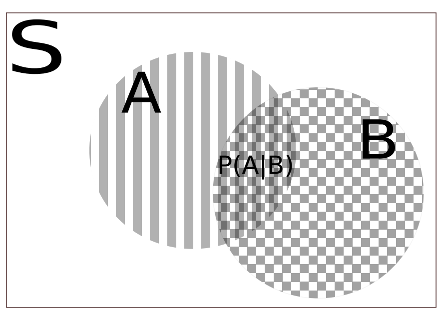

Figure 1: The idea of Posterior Probability

3 Law of Total Probability Example

This is a quite simple, but enlightening example of the use of total probabilities

Example

- My mood can take one of two values : happy or sad.
- The weather then takes one of three values Rainy, Sunny, Cloudy.
- Then it is possible to calculate the probability of being happy through the values in the weather. How? Using total probability:
  - $P(Happy) = P(Happy|Rainy) + P(Happy|Sunny) + P(Happy|Cloudy)$
  - $P(Sad) = P(Sad|Rainy) + P(Sad|Sunny) + P(Sad|Cloudy)$

3.1 Application: Bayes Rule

From the rule:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \cong P(B|A)P(A)$$

we have that each term has a special intuitive idea. For example, $P(A|B)$ is the probability of event A after the event B, the posterior. This is quite interesting because $P(B|A)$, the likelihood. can be used to calculate the probability of the event B once we see data from event A. This is know as the likelihood of the event B based of evidence A. Using this basic ideas, we can start playing the game of estimating B:

$$\begin{array}{c} B \text{ depends on } P(b|A) \\ A \text{ depends on } P(A) \end{array} \tag{2}$$

where the last quantity, $P(A)$ the prior, represents our past/prior understanding of the event A. This is the classical game of the egg and the chicken!!!

3.2 Example of the Bayes Rule

A simple example of the usage of Bayes is the following one:

- We know that Meningitis (M) causes a stiff neck 50% of the time.
- A patient comes with a Stiff Neck (S). Then, what is the probability that she/he has meningitis?
- An we know the following:
  - The prior probability $P(M) = \frac{1}{50{,}000}$.
  - The prior probability of $P(S) = \frac{1}{20}$.
- Then, What?

3.3 Example of Independece

Throwing a dice after another is a classic example of independence. Other examples can be seen in the casrd games.

4 Random Variables

A random variable is a function that samples events in S to the $\mathbb{R}$, $X : \{A|A \subseteq S\} \to \mathbb{R}$. This can be seen in figure 2.

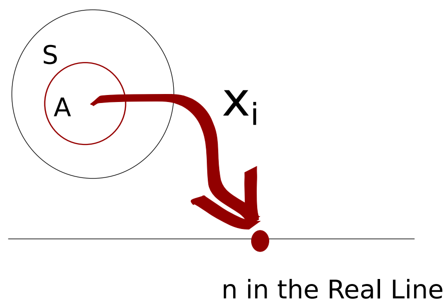

Figure 2: The mapping of the random variable X_i.

Now, How is the probability function of the random variable is being defined from the probability function of the original sample space? Suppose the following

4.1 Example

Given the following formulation:

- Suppose the sample space is $S = \{s_1, s_2, ..., s_n\}$
- Suppose the range of the random variable $X =< x_1, x_2, ..., x_m >$
- Then, we observe $X = x_i$ if and only if the outcome of the random experiment is an $s_j \in S$ s.t $X(s_j) = x_j$ or

$$P(X = x_j) = P(s_j \in S|X(s_j) = x_j)$$

What is P(X=2)? You can easily thing on this in the following way:

1. Given a one in how many positions you can put it if you think that you have 50 place holders? 50
2. What about the second one? 49
3. Then, we have that

$$P(X = 2) = \frac{50 * 49}{2^{50}}$$

4.2 Random Variables

4.2.1 Probability Mass Functions (PMF) / Discrete Random Variable

An example of a pmf is the function:

$$P_X(k|p,n) = \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k}$$

$$k = 0, 1, 2, ..., n$$

with plot

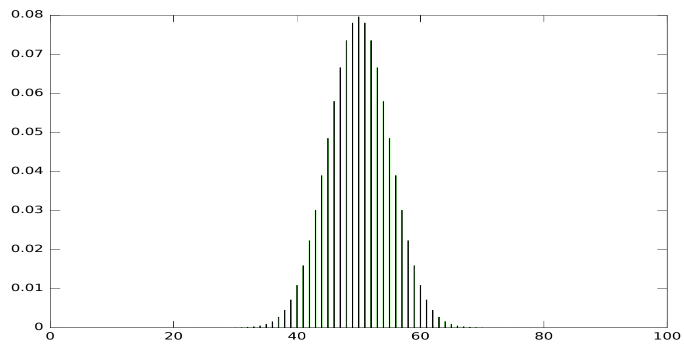

Figure 3: Binomial distribution plot

Example

Consider the experiment of throwing a pair of dices. Then X=*"sum of the numbers after throwing the dices."* Now, for *X=5,* we have that $A_5 = \{(1,4), (4,1), (2,3)\,(3,1)\}$. Therefore, we have that

$$P(X=5) = P(A_5) = \sum_{s:X(s)=5} P(s) = P((1,4)) + ... + P(3,1) = \frac{4}{36} = \frac{1}{9}$$

4.2.2 Probability Density Function (PDF)/ Continuous Random Variable

An example of a pdf is the function:

$$P_X(x|\lambda) = \lambda \exp^{-\lambda x}$$

$$\lambda > 0, x \in [0, \infty)$$

with plot

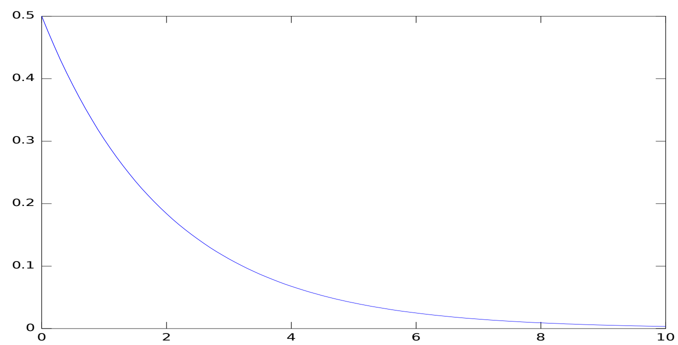

Figure 4: Exponential distribution plot

4.2.3 Example of a CDF

- Given a discrete probability, we have that

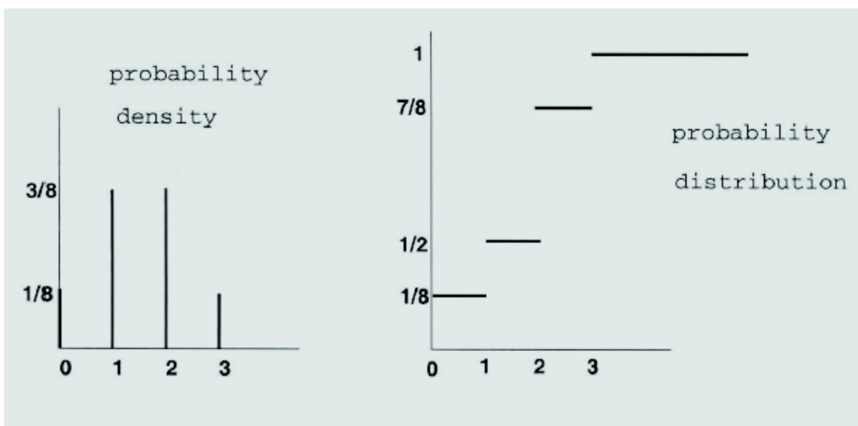

Figure 5: Example of the discrete case

- Given a Gaussian pdf, we have the following cdf:

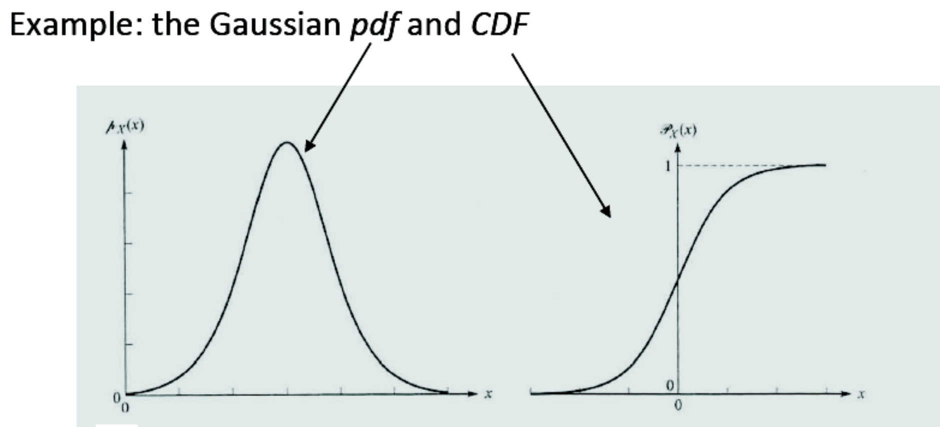

Figure 6: From pdf to cdf

4.3 Expected Value

Example Let X denote the outcome of a fair die roll

$E(X) = 1 \times \frac{1}{6} + 2\frac{1}{6} + ... + 6\frac{1}{6} = 3.5$

In addition, it is possible to talk of a sample mean $\overline{x}$ for a random variable X given by

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

where $x_i$denotes the $i - th$ measurement of X.

4.4 Variance

In a similar way, you can talk of a sample variance by using the formula:

$$\overline{Var}(X) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x}).$$