Analysis of Algorithms

Dealing with NP Problems: Intelligent Exponential Search and Approximation Algorithms

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Outline

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The Dilemma

We face the following

- NP problems need solutions in real-life!!!
- We only know exponential algorithms for them!!!
- What do we do?

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Ways of dealing with NP problems

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- Fourth, it may still be possible to find near-optimal solutions in polynomial time (either in the worst case or on average).

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Three main branches attacking the NP Problems

Intelligent Exponential Search

Procedures that are exponential time in the worst-case, but with the right design can be very efficient on typical instances.

- They are a trade-off between optimality, completeness, accuracy and precision and running time

Three main branches attacking the NP Problems

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Algorithms guaranteed to find a "near optimal" solution, under a certain bound, in polynomial time.

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We have something like this

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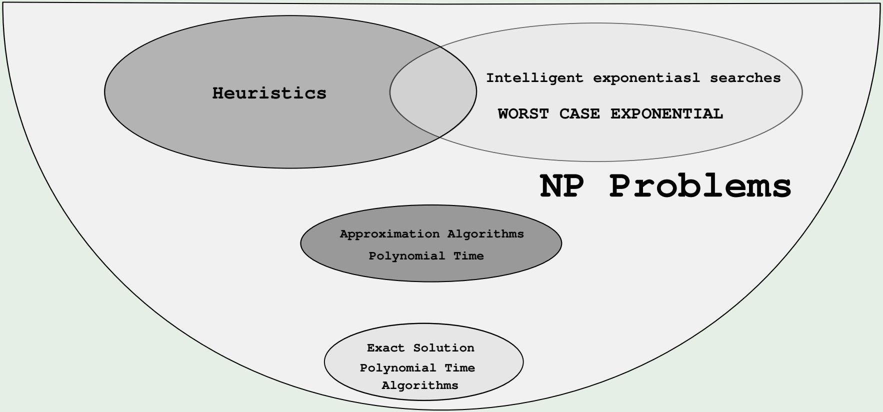

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Backtracking

Something Notable

Backtracking is based on that it is often possible to reject a solution by looking at just a small portion of it.

Example

If an instance of SAT contains the clause $C_i = (x_1 \vee x_2)$, then all assignments with $x_1 = x_2 = 0$ can be instantly eliminated.

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Example

Pruning Example

Given the possible values that you can give to two literals:

x_1	x_2
1	1
1	0
0	1
0	**0**

It is possible to prune a quarter of the entire search space... Can this be systematically exploited?

An example of exploiting this idea in SAT solvers

Consider the following Boolean formula $\phi(w, x, y, z)$

$$(w \vee x \vee y \vee z) \wedge (w \vee \neg x) \wedge (x \vee \neg y) \wedge (y \vee \neg z) \wedge (z \vee \neg w) \wedge (\neg w \vee \neg z)$$

We start branching in one variable, we can choose w

Note: This selection does not violate any of the clauses of $\phi(w, x, y, z)$

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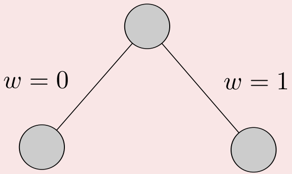

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Now

The partial assignment $w = 0, x = 1$ violates the clause $(w \lor \neg x)$

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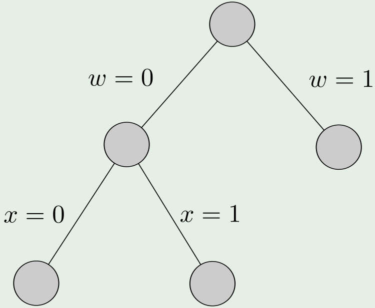

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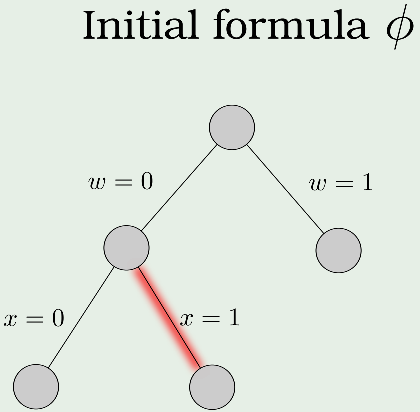

In addition

What if $w = 0, x = 0$

Then, the following clauses are satisfied

1. $\neg w = 1$
2. $\neg x = 1$

Thus, we have the following left

1. Before
   1. $(w \lor x \lor y \lor z) \land (w \lor \neg x) \land (x \lor \neg y) \land (y \lor \neg z) \land (z \lor \neg w) \land (\neg w \lor \neg z)$
2. After
   1. $(0 \lor 0 \lor y \lor z) \land (0 \lor 1) \land (0 \lor \neg y) \land (y \lor \neg z) \land (z \lor 1) \land (1 \lor \neg z)$

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Finally

We have the following reduced number of equations

$(y \vee z), (1), (\neg y), (y \vee \neg z), (1), (1) \Leftrightarrow \boldsymbol{(y \vee z), (\neg y), (y \vee \neg z)}$

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Clearly

We prune that part of the search tree.

Note: we use "()≡(0)" to point out to a "empty clause" ruling out satisfiability.

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The decisions we need to make in backtracking

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Which subproblem to expand next.

Second

Which branching variable to use.

Remark

The benefit of backtracking lies in its ability to eliminate portions of the search space.

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The test needs to look at the subproblem to declare quickly if

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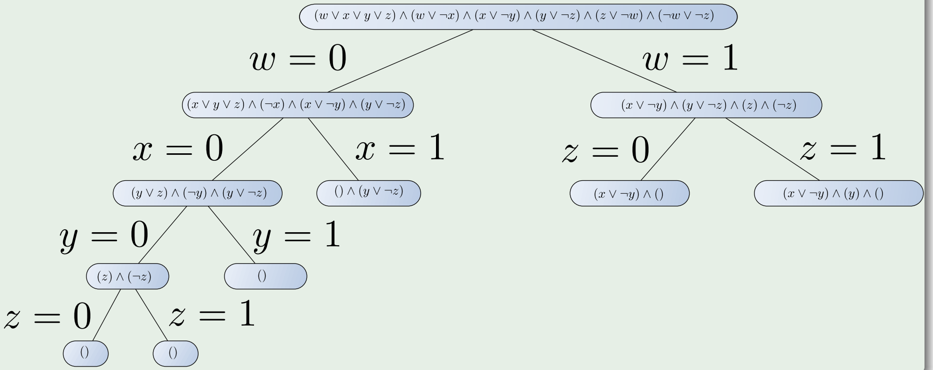

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Pseudo-code for Backtracking

We have

BACKTRACKING(P_0)

1. Start with some problem P_0
2. Let $\mathcal{S} = \{P_0\}$, the set if active subproblems
3. While $\mathcal{S} \neq \emptyset$
4. **choose** a subproblem $P \in \mathcal{S}$ and remove it from $\mathcal{S}$
5. **expand** it into smaller subproblems $P_1, P_2, ..., P_k$
6. For each P_i
7. if test(P_i) succeeds: halt and return the branch solution
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With the right test, expand, and choose routines

- Backtracking can be remarkably effective in practice

Further

- The backtracking algorithm we showed for SAT is the basis of many successful satisfiability programs

For Example, 2SAT problems

- It is a conjunction (a Boolean and operation) of clauses,
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- If presented with a 2 SAT instance,
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These problems are known as

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First, imagine a minimization problem

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Lower Bound

We use the info in the problem

To design an efficient way to approximate the solution.

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BRANCH-AND-BOUND(P_0)

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4. *While* $\mathcal{S} \neq \emptyset$
5. **choose** *a subproblem (Partial Solution)* $P \in \mathcal{S}$ *and remove it from* $\mathcal{S}$
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Example: The Traveling Salesman Problem

A partial solution to the TSP

It is a simple path $a \rightsquigarrow b$ passing through some vertices's $S \subseteq V$ with $a, b \in S$.

- Denote this as $[a, S, b]$

The subproblem

It is necessary to find the best completion of the tour i.e. the cheapest complementary path $b \rightsquigarrow a$ with intermediate vertices in $V - S$.

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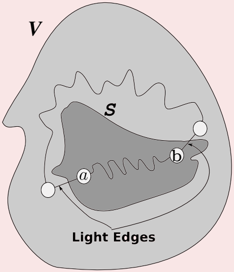

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TSP on a graph $G = (V, E)$ with edge lengths $w_e > 0$

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- S includes the endpoints a and b.

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$$b \rightsquigarrow y \rightsquigarrow a, \text{ such that } y \in V - S \text{ (Intermediate Vertex)}$$

Initial problem is the form

$$[a, \{a\}, a] \textbf{ For any } a \in V$$

At each step of the branch-and-bound

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- We extend a particular solution $[a, S, b]$ by a single edge (b, x) where $x \in V - S$

Then

The Subproblem

- The corresponding subproblem is to find the best completion of the tour, that is, the cheapest complementary path

$$b \rightsquigarrow y \rightsquigarrow a, \text{ such that } y \in V - S \text{ (Intermediate Vertex)}$$

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Thus, Given $a \rightsquigarrow b$

Leading to $|V - S|$ subproblems

- Of the form $[a, S \cup \{x\}, x]$

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The remainder of the tour consists

- A path through $V - S +$ edges from a and b to $V - S$

1. The lightest edge from a to $V - S$.
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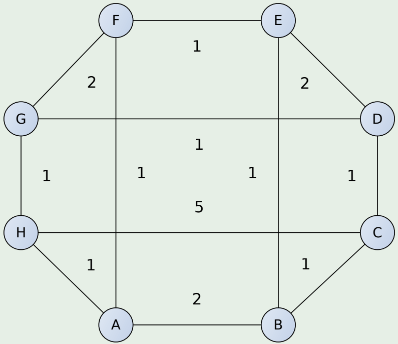
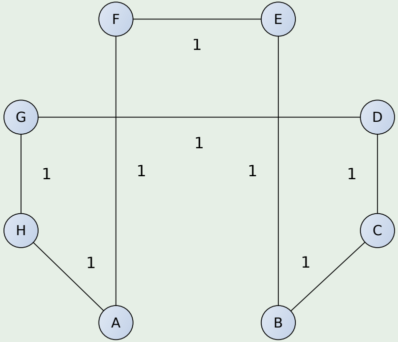
Example

A graph where a brute force approach will take 7!=5,040



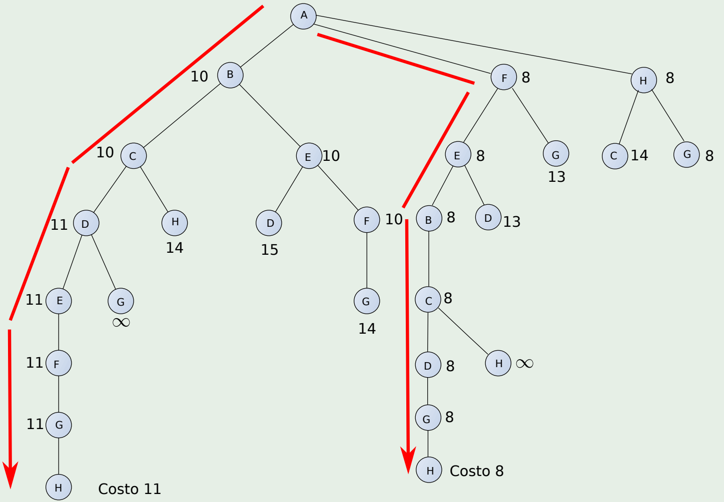
Example

TSP Solution



Example

Branch and Bound Solution



This is a quite simple lower bound

Clearly

- The lightest edges and the minimum spanning tree is a good estimation

Something Notable

- Notice how just 28 partial solutions are considered

Instead of

- The $7! = 5{,}040$ that would arise in a brute-force search.

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Outline

Approximation Algorithms

Remarks

- In practice, near-optimality is often good.
- An algorithm that returns near-optimal solutions is called an approximation algorithm.

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Definition of approximation algorithms

Framework

Suppose that we are working on an optimization problem in which each potential solution has a positive cost.

- We wish to find a near-optimal solution.

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We say that an algorithm for a problem has an approximation ratio of $\rho(n)$ if:

- For any input of size n the cost C of the solution produced by the algorithm is within a factor of $\rho(n)$ of the cost C^* of an optimal solution:

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Further

The definitions of the approximation ratio and of a $\rho(n)$-**approximation algorithm** apply to both minimization and maximization problems.

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The two possibilities

For a maximization problem

We have that $0 < C \leq C^*$, then the ratio C^*/C gives the factor by which the cost of an optimal solution is larger than the cost of the approximate solution.

For a minimization problem

We have that $0 < C^* \leq C$, then the ratio C/C^* gives the factor by which the approximate solution is larger than the cost of an optimal solution.

Properties

The approximation ratio of an approximation algorithm is never less than 1.

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Properties

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Approximation algorithms have the following characteristics

- They have polynomial times.
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We will look at...

Examples

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- The Traveling Salesman Problem.

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All the theory of approximation algorithms.

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Outline

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Vertex Cover Problem

Definition of a vertex cover

A vertex cover of an undirected graph $G = (V, E)$ is a subset $V' \subseteq V$ such that if (u, v) is an edge of G. then either $u \in V'$ or $v \in V'$ (Or both).

Thus

The size of a vertex cover is the number of vertices in it.

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Thus

The size of a vertex cover is the number of vertices in it.

Vertex Cover Problem

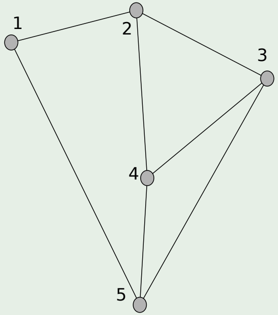
Definition

The vertex-cover problem is to find a vertex cover of minimum size in a given undirected graph.

Example of Vertex Cover Problem

Example

Determine the smallest subset of vertex that "Cover" the graph on the right.



Vertex cover

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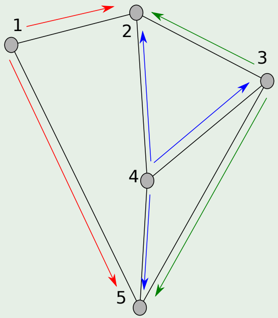


ANSWER: $\{1, 3, 4\}$

Approximation vertex cover algorithm

Now, we have

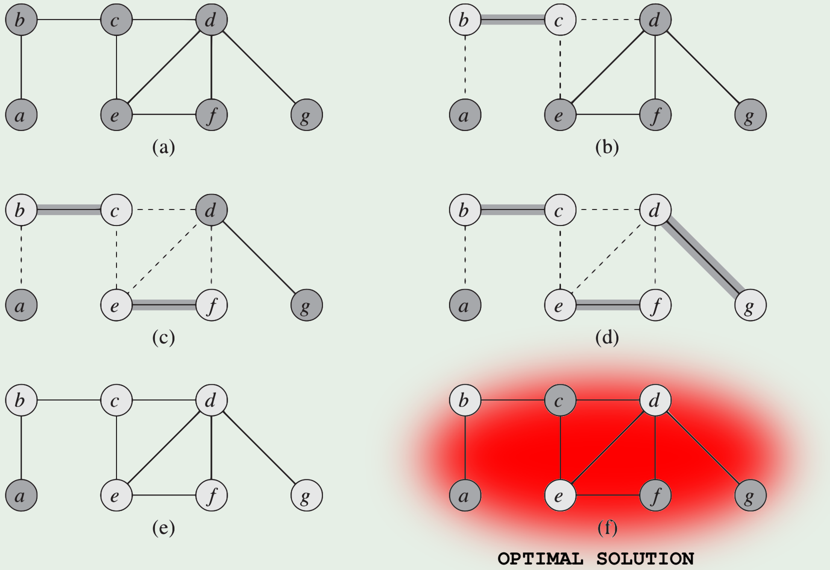
APPROX-VERTEX-COVER(G)

1. $C = \emptyset$
2. $E' = G.E$
3. while $E' \neq \emptyset$
4. let (u, v) be an arbitrary edge of E'
5. $C = C \cup \{u, v\}$
6. remove from E' every edge incident on either u or v
7. return C

Complexity $O(V + E)$

Example

We have



Theorem

Theorem 35.1

APPROX-VERTEX-COVER is a polynomial-time 2-approximation algorithm.

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- We know that the algorithms return a vertex cover until it finishes, and it runs in poly-time.
- Now, we only need to prove that APPROX-VERTEX-COVER returns a vertex cover of at most twice the size of the optimal cover.

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Proof

First

- We call C as the set of vertices that is returned by the approximation algorithm.
- Now, given the set of edges A produced by the algorithm.
- Now, given an optimal cover C^*:
  - C^* must include at least one endpoint of each edge in A.
  - No two edges in A share an endpoint, thus no two edges are covered by the same vertex from C^*:

$$|C^*| \geq |A|$$

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Proof

Second

Now, each execution in line 4 picks an edge for which neither of its endpoints is already in C, thus we have

$$
\begin{aligned}
|C| &= 2|A| \\
&\leq 2|C^*|
\end{aligned}
$$

Proof

Finally, we have

$$\frac{|C|}{|C^*|} \leq 2$$

Remarks

Did you notice the trick?

We do not know the size of C^*, but we obtain a lower bound for it

Actually

The set A is a **maximal matching** in the graph G.

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What is a Maximal Matching?

Definition

Given a graph $G = (V, E)$, a matching M in G is a set of pairwise non-adjacent edges; that is, no two edges share a common vertex.

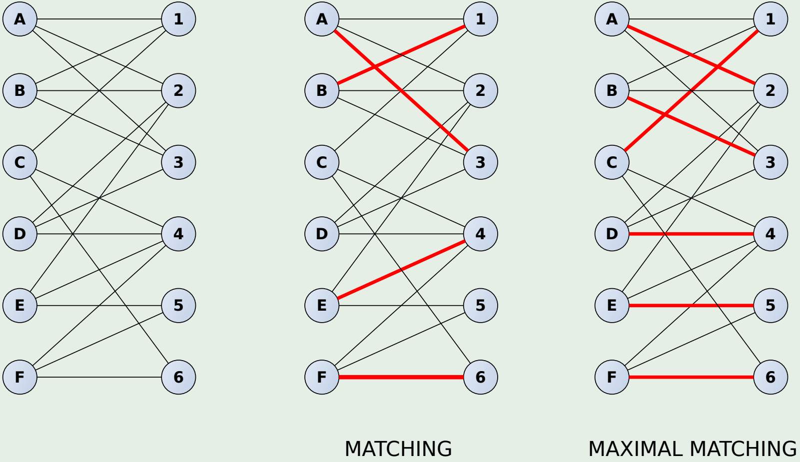
Maximal Matching

Definition

A Maximal Matching is a matching M of a graph G with the property that if any edge not in M is added to M, it is no longer a matching.

Example

Here, we can see an example



Outline

The Traveling-Salesman Problem

Given a complete undirected graph $G = (V, E)$

- With a no-negative integer cost function $c(u, v)$ associated to each edge $(u, v) \in E$
- We need to find a Hamiltonian cycle of G with minimum cost.

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Extra notation

As an extension of our notation

Let $c(A)$ denote the total cost of the edges in a subset $A \subseteq E$

$$c(A) = \sum_{(u,v) \in A} c(u, v)$$

The extra assumption: The triangle inequality

Assume

:

We will assume that the cost function satisfies the triangle inequality for all its vertices $u, v, w \in V$ then:

$$c(u, w) \leq c(u, v) + c(v, w)$$

The 2-approximation algorithm

Pseudocode

APPROX-TSP-TOUR(G, c, r)

1. select a vertex $r \in G.V$ to be a "root" vertex.
2. compute a minimum spanning tree T for G from root r using the MST-PRIM(G, c, r).
3. Let H be a list of vertices, ordered according to when they are first visited in a preorder tree walk of T.
4. return the Hamiltonian cycle H

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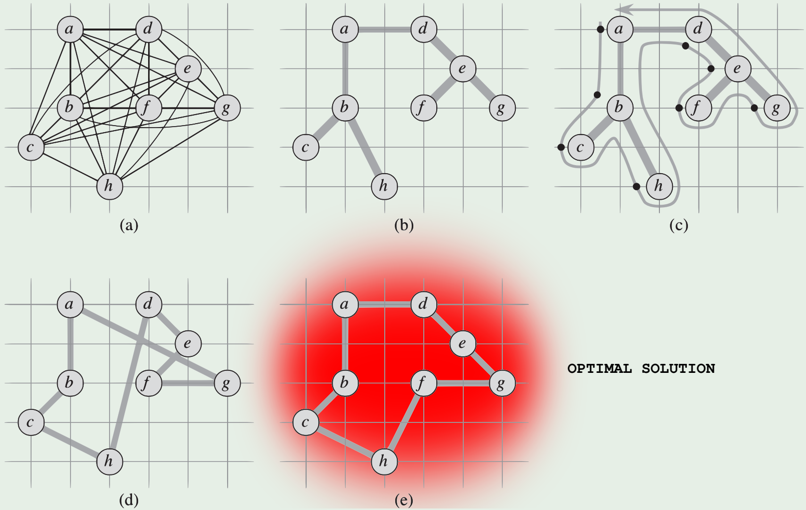
Example

The root is a



(a) (b) (c) (d) (e)

Theorem that supports the claim

Theorem 35.2

APROX-TSP-TOUR is a polynomial-time 2-approximation algorithm for the traveling salesman problem with the triangle inequality.

Proof

First

- Let H^* denote an optimal tour for a set of vertices,
  - H^* is a cycle.
  - If from H^* we erase an edge, H^* becomes a tree.
- Let T the minimum spanning tree computed at line 2

Then:

$$c(T) \leq c(H^*)$$

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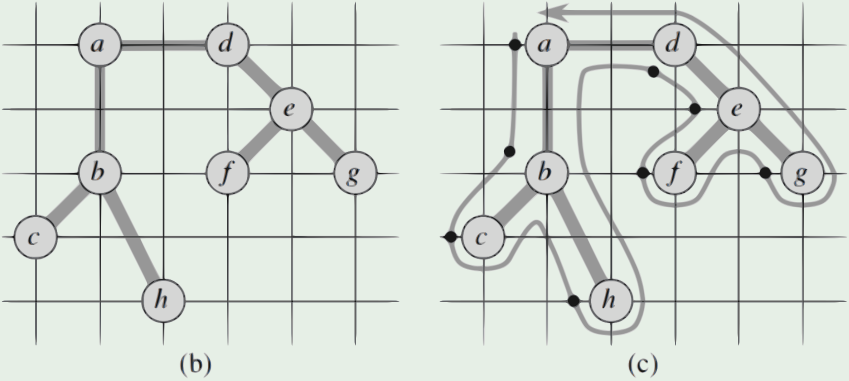
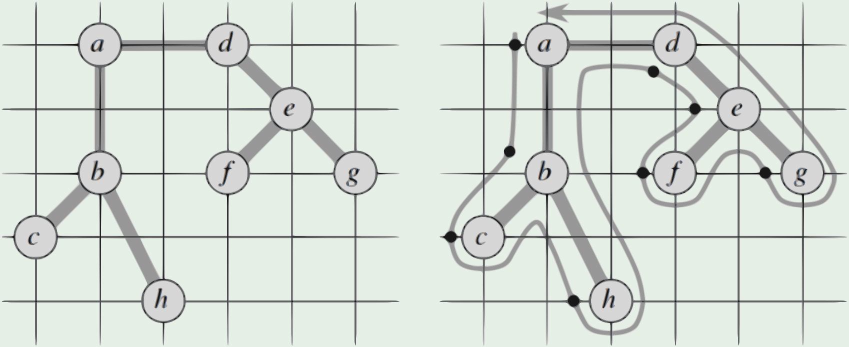
Full walk

A full walk W of T lists how many times a vertex has been visited



Full walk

A full walk $W = a, b, c, b, h, b, a, d, e, f, e, g, e, d, a$



(b) (c)

Then, we have

Since the full walk traverses every edge of T exactly twice

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In addition

However, W is not a tour... What can we do?

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Something Notable

By the triangle inequality, however, we can delete a visit to any vertex from W.

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Using the triangle inequality the cost does not increase.

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Deletion of vertices

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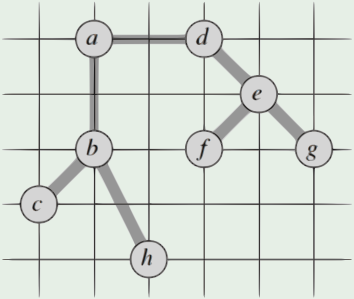
Thus

We get

a, b, c, h, d, e, f, g

Minimum Spanning Tree

From our example



(b)

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Thus

$$c(H) \leq c(W) \leq 2c(T) \leq 2c(H^{*})$$

Finally

We have that

$$\frac{c\left(H\right)}{c\left(H^{*}\right)} \leq 2$$

However

Given the General Traveling-Salesman Problem

If we drop the assumption that the cost function c satisfies the triangle inequality.

Theorem 35.3

If $P \neq NP$, then for any constant $\rho \geq 1$, there is no polynomial-time approximation algorithm with approximation ratio ρ for the general traveling-salesman problem.

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If $P \neq NP$, then for any constant $\rho \geq 1$, there is no polynomial-time approximation algorithm with approximation ratio ρ for the general traveling-salesman problem.

Excercises

From Cormen's book solve

- 35.1-1
- 35.1-2
- 35.1-4
- 35.2-1
- 35.2-2
- 35.2-3
- 35.2-5