Analysis of Algorithms

Skip Lists

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Outline

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Dictionaries

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A dictionary is a collection of elements; each of which has a unique search key.

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Example: Course records

Dictionary with member records

key ID	Student Name	HW1	
123	Stan Smith	49	$\cdots$
125	Sue Margolin	45	$\cdots$
128	Billie King	24	$\cdots$
	$\vdots$		
	$\vdots$		
190	Roy Miller	36	$\cdots$

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The dictionary ADT operations

Some operations on dictionaries

- size(): Returns the size of the dictionary.
- empty(): Returns TRUE if the dictionary is empty.
- findItem(key): Locates the item with the specified key.
- findAllItems(key): Locates all items with the specified key.
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Example of unordered dictionary

Example

Consider an empty unordered dictionary, we have then...

Operation	Dictionary	Output
InsertItem$(5, A)$	$\{(5, A)\}$	
InsertItem$(7, B)$	$\{(5, A), (7, B)\}$	
findItem(7)	$\{(5, A), (7, B)\}$	B
findItem(4)	$\{(5, A), (7, B)\}$	No Such Key
size$()$	$\{(5, A), (7, B)\}$	2
removeItem(5)	$\{(7, B)\}$	A
findItem(4)	$\{(7, B)\}$	No Such Key

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How to implement a dictionary?

There are many ways of implementing a dictionary

- Sequences / Arrays
  - Ordered
  - Unordered
- Binary search trees
- Skip lists
- Hash tables

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Unordered array

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Binary searches

Features

- Narrow down the search range in stages
- "High-low" game.

Binary searches

Example find Element(22)

2	4	5	7	8	9	12	**14**	17	19	22	25	27	28	33
$\uparrow$ LOW							$\uparrow$ MID							$\uparrow$ $HIGH$

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Recall binary search trees

Implement a dictionary with a BST

A binary search tree is a binary tree T such that:

- Each internal node stores an item (k, e) of a dictionary.
- Keys stored at nodes in the left subtree of v are less than or equal to k.
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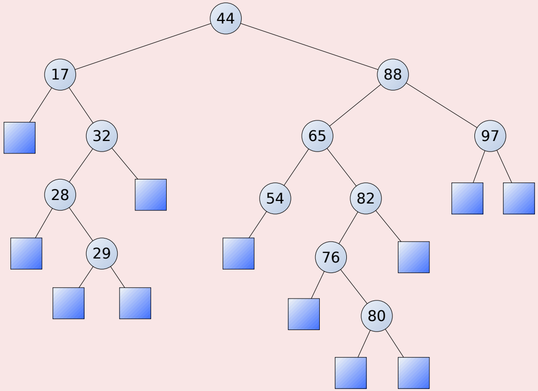
Binary searches Trees

Problem!!! Keeping a Well Balanced Binary Search Tree can be difficult!!!



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We want to have a

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Thus, we have the following possibilities

Unordered array complexities

Insertion: $O(1)$

Search: $O(n)$

Ordered array complexities

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We want something better!!!

For this

We will present a probabilistic data structure known as Skip List!!!

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Starting from Scratch

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- Imagine that you only require to have searches.
- A first approximation to it is the use of a link list for it ($\Theta(n)$ search complexity).
- Then, using this How do we speed up searches?

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- The Top is the high way system,

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1. The Bottom is the normal road system, L_2.
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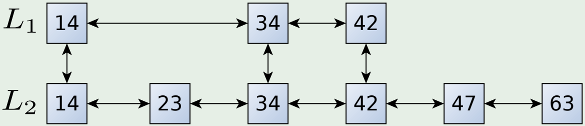
Example

High-Bottom Way System



Thus, we have...

The following rule

To Search first search in the top one (L_1) as far as possible, then go down and search in the bottom one (L_2).

We can use a little bit of optimization

We have the following worst cost

$$\text{Search Cost High-Bottom Way System} = \text{Cost Searching Top} + \ldots$$

$$\text{Cost Search Bottom}$$

Or

$$\text{Search Cost} = length\left(L_1\right) + \text{Cost Search Bottom}$$

The interesting part is "Cost Search Bottom"

This can be calculated by the following quotient:

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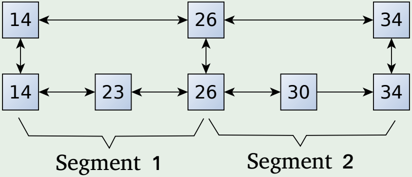
Why?

If we think we are jumping



Then cost of searching each of the bottom segments = 2

Thus the ratio is a "decent" approximation to the worst case search

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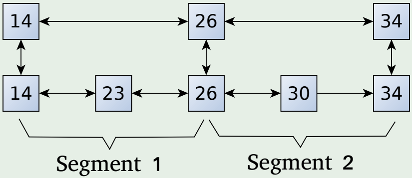

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Then, the cost for a search (when $length\left(L_2\right) = n$)

$$\text{Search Cost } = length\left(L_1\right) + \frac{length\left(L_2\right)}{length\left(L_1\right)} = length\left(L_1\right) + \frac{n}{length\left(L_1\right)} \tag{1}$$

Taking the derivative with respect to $length\left(L_1\right)$ and making the result equal 0

$$\frac{d\text{Search Cost}}{dlength\left(L_1\right)} = 1 - \frac{n}{length^2\left(L_1\right)} = 0$$

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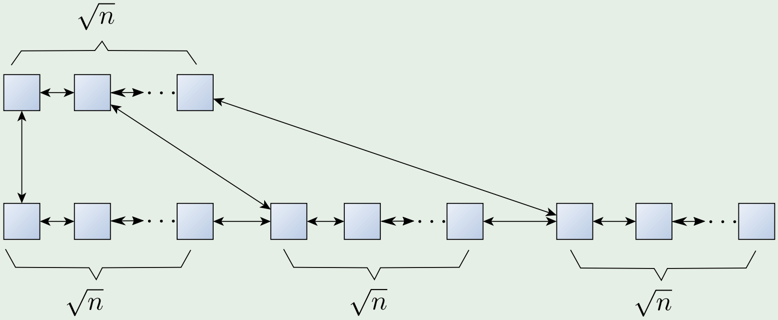
Data structure with a Square Root Relation

Something like this



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For a three layer link list data structure

We get a search cost of $3 \times \sqrt[3]{n}$

In general for k layers, we have

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Thus, if we make $k = \log_2 n$, we get

$$\begin{aligned}\text{Search Cost} &= \log_2 n \times \sqrt[\log_2 n]{n} \\ &= \log_2 n \times (n)^{1/\log_2 n} \\ &= \log_2 n \times (n)^{\log_n 2} \\ &= \log_2 n \times 2 \\ &= \Theta(\log_2 n)\end{aligned}$$

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Something Notable

We get the advantages of the binary search trees with a simpler architecture!!!

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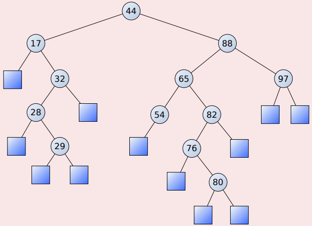
Binary Search Trees



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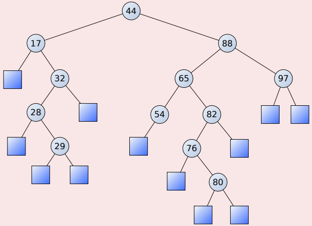

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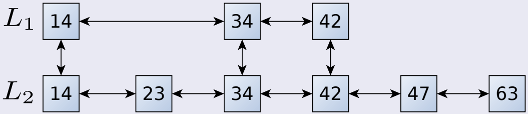

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Definition for Skip List

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Skip List Definition

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A skip list for a set S of distinct (key,element) items is a series of lists $S_0, S_1, ..., S_h$ such that:

- Each list S_i contains the special keys $+\infty$ and $-\infty$
- List S_0 contains the keys of S in nondecreasing order
- Each list is a subsequence of the previous one
  - $S_0 \supseteq S_1 \supseteq S_2 \supseteq ... \supseteq S_h$
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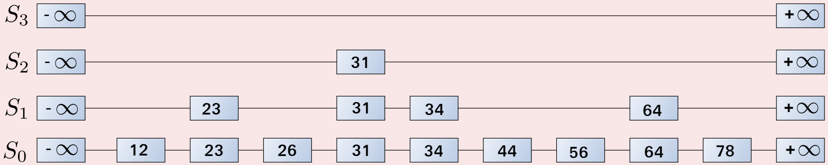
Example



Skip list search

We search for a key x in a skip list as follows

- We start at the first position of the top list.
- At the current position p, we compare x with $y == p.next.key$
  - $x == y$: we return $p.next.element$
  - $x > y$: we scan forward
  - $x < y$: we "drop down"
- If we try to drop down past the bottom list, we return $null$.

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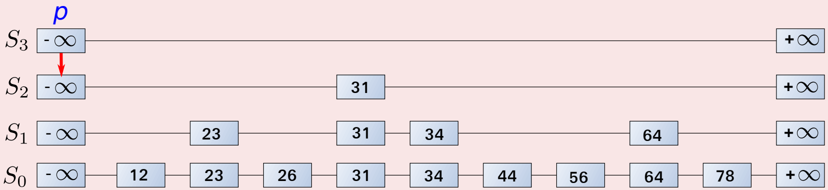
Example search for 78

$x < p.next.key$: "drop down"



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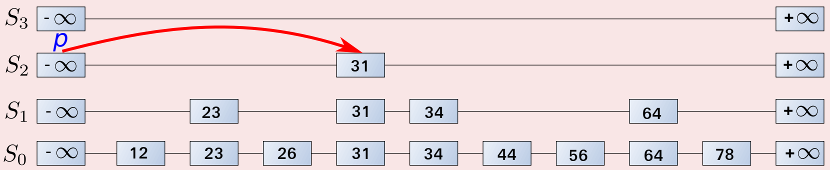

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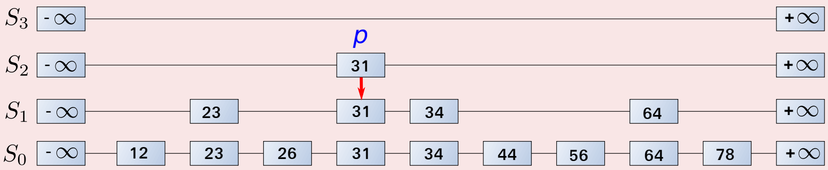

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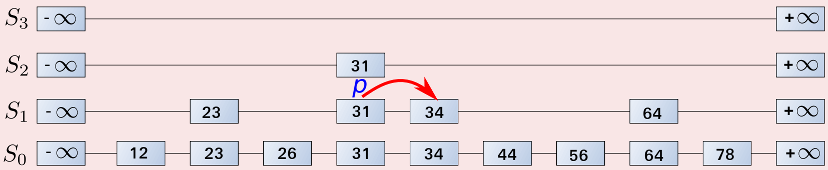

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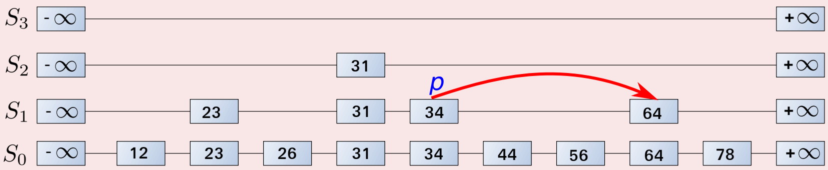

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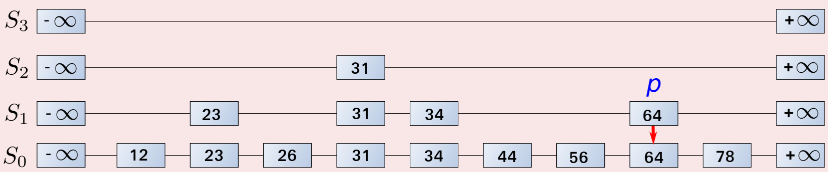

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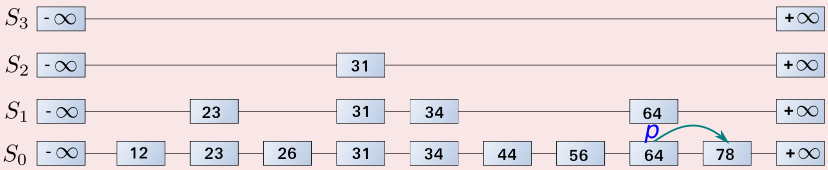
$x == y$: we return $p.next.element$



Outline

How do we implement this data structure?

We can implement a skip list with quad-nodes

A quad-node stores:

1. Entry Value
2. Link to the previous node
3. Link to the next node
4. Link to the above node
5. Link to the below node

Also we define special keys PLUS_INF and MINUS_INF, and we modify the key comparator to handle them.

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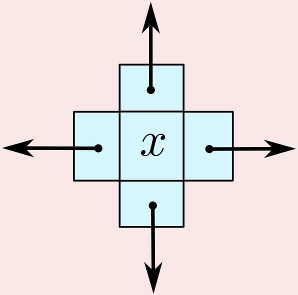
Quad-Node Example



Skip lists uses Randomization

Use of randomization

We use a randomized algorithm to insert items into a skip list.

Running time

We analyze the expected running time of a randomized algorithm under the following assumptions:

- The coins are unbiased.
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Outline

Insertion

To insert

To insert an entry $(key, object)$ into a skip list, we use a randomized algorithm:

1. We repeatedly toss a coin until we get tails:
   - We denote with i the number of times the coin came up heads.

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We have two cases

If $i \geq h$, we add to the skip list new lists $S_{h+1}, ..., S_{i+1}$

- Each containing only the two special keys.
- We search for x in the skip list and find the positions $p_0, p_1, ..., p_i$ of the items with largest key less than x in each lists $S_0, S_1, ..., S_i$.
- For $j \leftarrow 0, ..., i$, we insert item $(key, object)$ into list S_j after position p_j.

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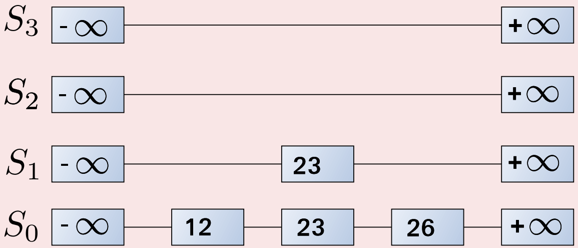
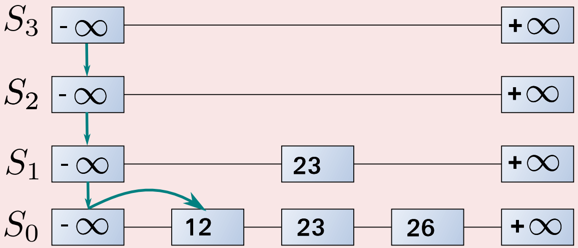
Example: Insertion of 15 in the skip list

First, we use $i = 2$ to insert S_3 into the skip list



Example: Insertion of 15 in the skip list

Clearly, you first search for the predecessor key!!!



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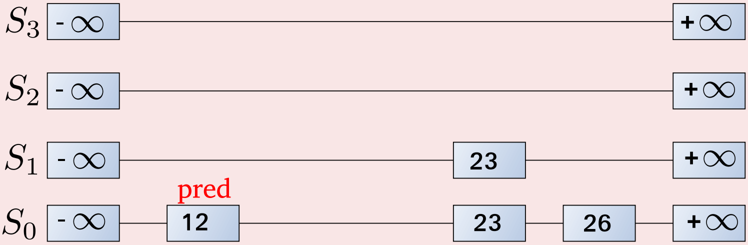
Insert the necessary Quad-Nodes and necessary information



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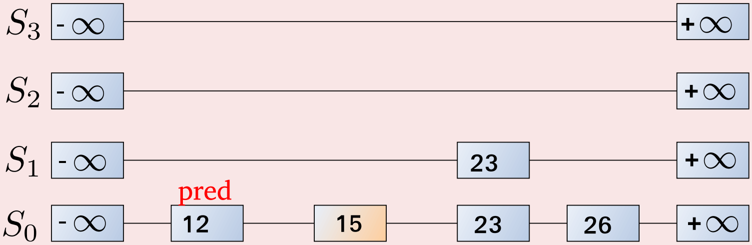

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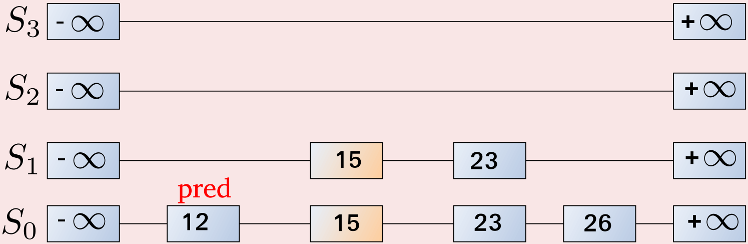
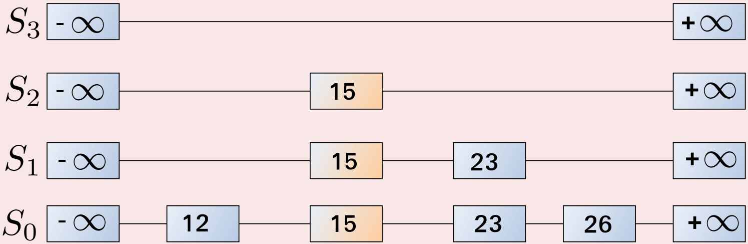


Example: Insertion of 15 in the skip list

Finally!!!



Outline

Deletion

To remove an entry with key x from a skip list, we proceed as follows

- We search for x in the skip list and find the positions $p_0, p_1, ..., p_i$ of the items with key x, where position p_j is in list S_j.
- We remove positions $p_0, p_1, ..., p_i$ from the lists $S_0, S_1, ..., S_i$.
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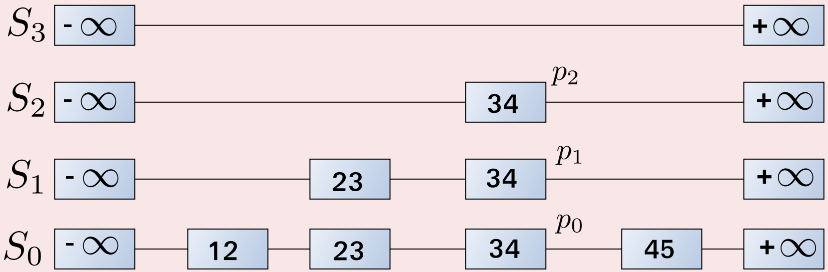
Example: Delete of 34 in the skip list

We search for 34 in the skip list and find the positions $p_0, p_1, ..., p_2$ of the items with key 34



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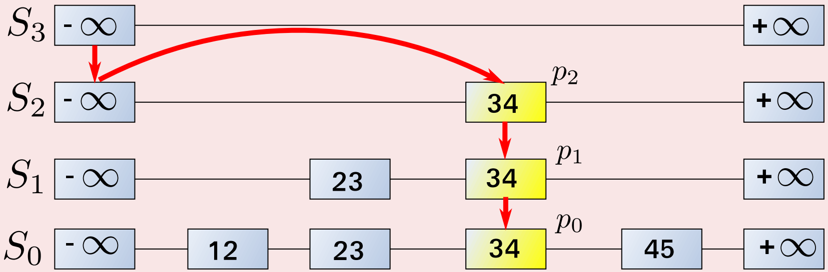
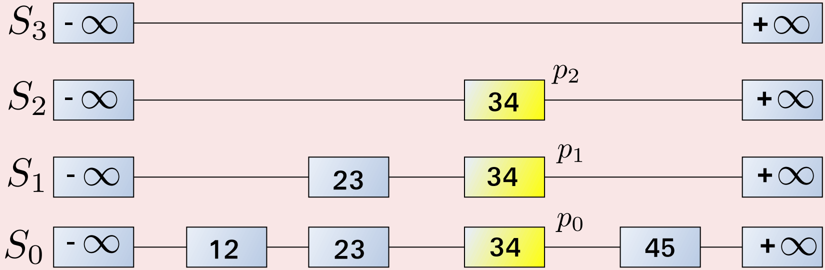


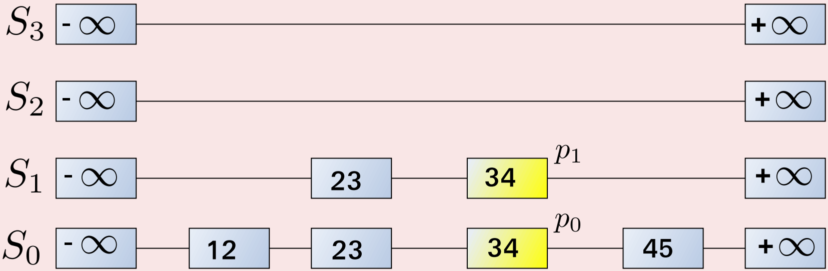
Example: Delete of 34 in the skip list

We start doing the deletion!!!



Example: Delete of 34 in the skip list

One Quad-Node after another



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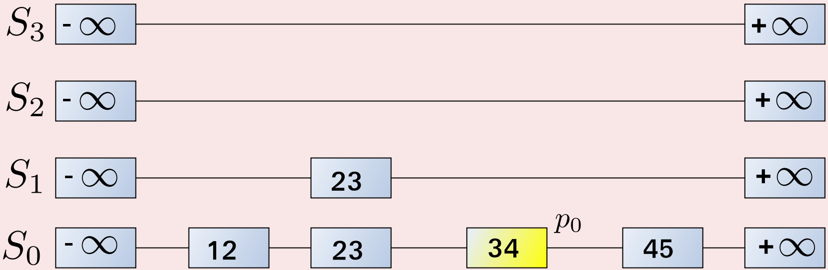

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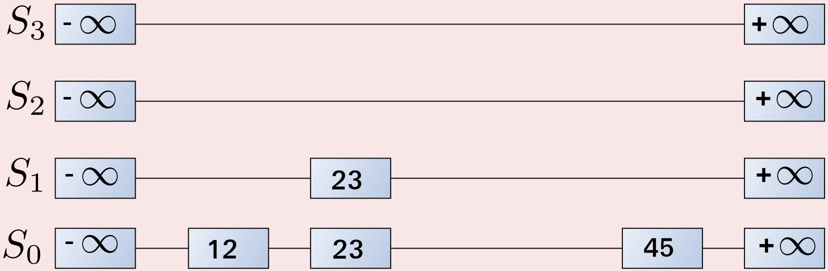

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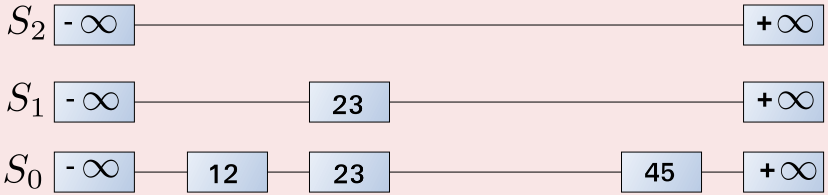
Remove One Level



Outline

Space usage

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The space used by a skip list depends on the random bits used by each invocation of the insertion algorithm.

Space : $O(n)$

Theorem

The expected space usage of a skip list with n items is $O(n)$.

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Proof

We use the following two basic probabilistic facts:

1. *Fact 1:* The probability of getting i consecutive heads when flipping a coin is $\frac{1}{2^i}$.
2. *Fact 2:* If each of n entries is present in a set with probability p, the expected size of the set is np.
    1. How? Remember $X = X_1 + X_2 + \ldots + X_n$ where X_i is an indicator function for event A_i = the i element is present in the set. Thus:

$$E[X] = \underbrace{\sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} Pr\{A_i\}}_{\text{Equivalence } E[X_A] \text{ and } Pr\{A\}} = \sum_{i=1}^{n} p = np$$

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   1. How? Remember $X = X_1 + X_2 + ... + X_n$ where X_i is an indicator function for event A_i = the i element is present in the set. Thus:

$$E[X] = \underbrace{\sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} Pr\{A_i\}}_{\text{Equivalence } E[X_A] \text{ and } Pr\{A\}} = \sum_{i=1}^{n} p = np$$

Space : $O(n)$

Theorem

The expected space usage of a skip list with n items is $O(n)$.

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We use the following two basic probabilistic facts:

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Proof

Now consider a skip list with n entries

Using Fact 1, an element is inserted in list S_i with a probability of

$$P\left[x \in S_i\right] = \frac{1}{2^i}$$

Now by Fact 2

The expected size of list S_i is

$$E\left[\left|S_i\right|\right] = \frac{n}{2^i}$$

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Proof

The expected number of nodes used by the skip list with height h

$$E\left[\text{Size Skip List}\right] = \sum_{i=0}^{h} \frac{n}{2^i} = n \sum_{i=0}^{h} \frac{1}{2^i}$$

Here, we have a problem!!! What is the value of h?

Height h

First

The running time of the search and insertion algorithms is affected by the height h of the skip list.

Second

We show that with high probability, a skip list with n items has height $O(\log n)$.

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We use the following Fact 3

We can view the level $l\left(x_i\right) = \max\left\{j | \text{where } x_i \in S_j\right\}$ of the elements in the skip list as the following random variable

$$X_i = l\left(x_i\right)$$

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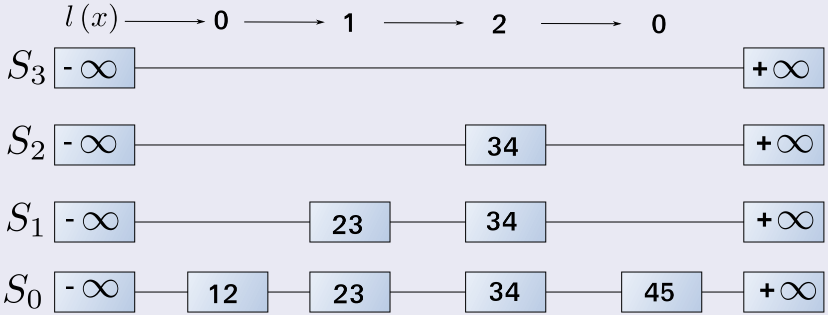
Example for $l(x_i)$

We have



BTW What is the geometric distribution?

k failures where

$$k = \{1, 2, 3, ...\}$$

Probability mass function

$$Pr\left(X = k\right) = \left(1 - p\right)^{k-1} p$$

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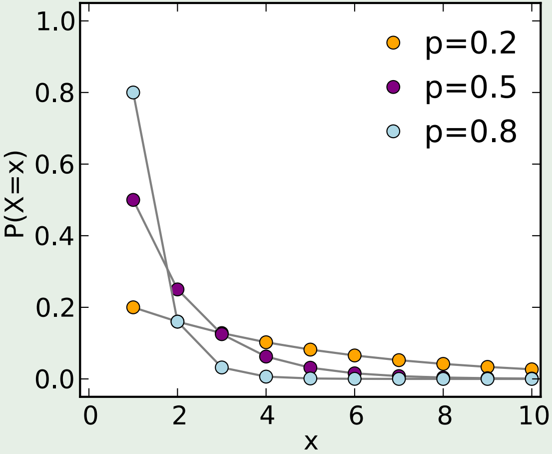
Probability Mass Function

For Different Probabilities



Then

We have the following inequality for the geometric variables

$$Pr\left[X_i > t\right] \leq (1-p)^t \;\; \forall i = 1, 2, ..., n$$

- If we assume we have a fair coin $p = \frac{1}{2}$

This is because

$$F(t) = P\left[X_i \leq t\right] = \sum_{i=1}^{t} (1-p)^{i-1} p$$

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$$\sum_{i=t+1}^{\infty} (1-p)^{i-1} p = p \sum_{i=t+1}^{\infty} (1-p)^{i-1}$$

$$= p \sum_{k=1, k=i-t}^{\infty} (1-p)^{k+t-1}$$

$$= p(1-p)^t \sum_{k=1, k=i-t}^{\infty} (1-p)^{k-1}$$

$$= (1-p)^t \frac{p}{1-p} \leq (1-t)^t \text{ Given the fair coin}$$

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Then, we have

Using our original formula

$$Pr\left[X_i > t\right] \leq \left(1 - t\right)^t$$

In this way, we have

Then, we have

$$Pr\left\{\max_i X_i > t\right\} \leq n(1-p)^t$$

How?

We have that

$$Pr\left\{\max_i X_i > t\right\} = Pr\left\{\max\left\{X_1, X_2, ..., X_n\right\} > t\right\}$$
$$= \sum_{i=1}^{n} Pr\left\{X_i > t \text{ and } X_i = \max\left\{X_1, X_2, ..., X_n\right\}\right\}$$

How?

- That one of the elements becomes the maximum in height and a height greater than t

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How?

- That one of the elements becomes the maximum in height and a height greater than t

Why?

Because the height of an element depends on independent event

- Each toss coin until tails is independent of the others!!!

Example

When having two lists

$$\{\max(X_1, X_2) > t\} = \{X_1 > t \text{ and } X_1 > X_2 \text{ or exclusive } X_2 > t \text{ and } X_2 > X_1\}$$

- Yes, you need to remember that the max is a single element not both...

Therefore

Then

$$
\begin{aligned}
Pr\left\{X_1 > t \text{ and } X_1 > X_2 \text{ or exclusive } X_2 > t\right\} = & Pr\left\{X_1 > t \text{ and } X_1 > X_2\right\} + \ldots \\
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Assuming exclusivity between phenomena $X_i > X_j$ and $X_i > t$

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Pr\left\{X_1 > t \text{ and } X_1 > X_2 \text{ or exclusive } X_2 > t\right\} = & P\left(X_1 > t\right) P\left(X_1 > X_2\right) + \ldots \\
& P\left(X_2 > t\right) P\left(X_2 > X_1\right) \\
\leq & P\left(X_1 > t\right) + P\left(X_2 > t\right)
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Therefore

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Pr\left\{X_1 > t \text{ and } X_1 > X_2 \text{ or exclusive } X_2 > t\right\} =& Pr\left\{X_1 > t \text{ and } X_1 > X_2\right\} + ...\\
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Finally, using this fact

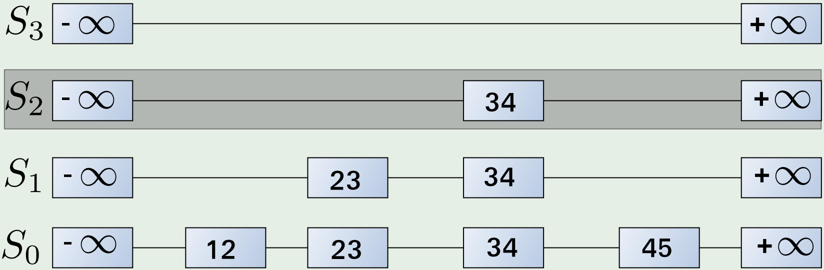
We have when summing over all events X_i

$$\sum_{i=1}^{n} Pr\left\{X_i > t\right\} \leq \sum_{i=1}^{n} \left(1-p\right)^t = n\left(1-p\right)^t$$

An Observation

The $\max_i X_i$

It represents the list with the one entry apart from the special keys.



Another One

Also REMEMBER!!!

We are talking about a fair coin, thus $p = \frac{1}{2}$.

Outline

Height: $3\log_2 n$ with probability at least $1 - \frac{1}{n^2}$

Theorem

A skip list with n entries has height at most $3\log_2 n$ with probability at least $1 - \frac{1}{n^2}$

Proof

Consider a skip list with n entires

By Fact 3, the probability that list S_t has at least one item (The $\max_i X_i > t$) is at most $\frac{n}{2^t}$.

$$P\left(|S_t| \geq 1\right) = P\left(\max_i X_i > t\right) \leq \frac{n}{2^t}.$$

By picking $t = 3\log n$

We have that the probability that $S_{3\log_2 n}$ has at least one entry is at most:

$$\frac{n}{2^{3\log_2 n}} = \frac{n}{n^3} = \frac{1}{n^2}.$$

Proof

Consider a skip list with n entires

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Finally

The expected number of nodes used by the skip list with height h

- Given that $h = 3\log_2 n$

$$\sum_{i=0}^{3\log_2 n} \frac{n}{2^i} = n \sum_{i=0}^{3\log_2 n} \frac{1}{2^i}$$

Given the geometric sum

$$S_m = \sum_{k=0}^{m} r^k = \frac{1 - r^{m+1}}{1 - r}$$

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The Upper Bound on the number of nodes

$$n\sum_{i=0}^{3\log_2 n}\frac{1}{2^i} = n\left(\frac{1-\left(\frac{1}{2}\right)^{3\log_2 n+1}}{1-1/2}\right)$$

$$= n\left(\frac{1-\frac{1}{2^{3\log_2 n+1}}}{1/2}\right)$$

$$= n\left(\frac{1-\frac{1}{\left(2^{\log_2 n}\right)^3 2}}{1/2}\right)$$

$$= n\left(\frac{1-\frac{1}{2n^3}}{1/2}\right) = n\left(\frac{2\left[2n^3-1\right]}{2n^3}\right)$$

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Finally

We have

$$\left(\frac{2n^3 - 1}{n^2}\right) = 2n - \frac{1}{n^2} \leq 2n$$

The Upper Bound with probability $1 - \frac{1}{n^2}$

$$2n - \frac{1}{n^2} \leq 2n = O(n)$$

Finally

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$$\left(\frac{2n^3-1}{n^2}\right)=2n-\frac{1}{n^2}\leq 2n$$

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$$2n-\frac{1}{n^2}\leq 2n=O\left(n\right)$$

Outline

Search and Insertion Times

Fact 4

The expected number of coin tosses required in order to get tails is 2:

$$\text{Given that } x \sim G\left(\frac{1}{2}\right) \Longrightarrow E\left[x\right] = \frac{1}{p} = 2 \text{ (Fair Coin Assumption)}$$

We use this

To prove that a search in a skip list takes $O(\log n)$ expected time.

- After all insertions require searches!!!

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Search and Insertions times

Search time

The search time in skip list is proportional to

the number of drop-down steps + the number of scan-forward steps

Drop-down steps

The drop-down steps are bounded by the height of the skip list and thus are $O(\log_2 n)$ with high probability.

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First

- When we scan forward in a list, the destination key does not belong to a higher list.

A scan-forward step is associated with a former coin toss that gave tails

- By Fact 4, in each list the expected number of scan-forward steps is 2.

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A scan-forward step is associated with a former coin toss that gave tails

- By Fact 4, in each list the expected number of scan-forward steps is 2.

Why?

Given the list S_i

- Then, the scan-forward intervals (Jumps between x_i and x_{i+1}) to the right of S_i are

$$I_1 = [x_1, x_2], I_2 = [x_2, x_3] \ldots I_k = [x_k, +\infty]$$

Then

These interval exist at level i if and only if all $x_1, x_2, \ldots, x_k$ belong to S_i.

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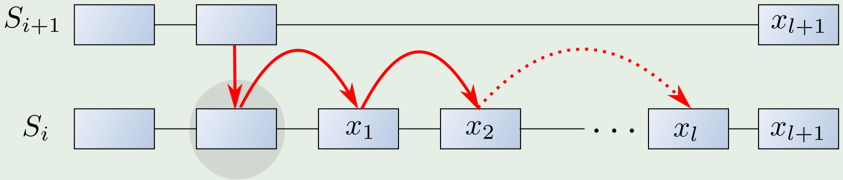
These interval exist at level i if and only if all $x_1, x_2, ..., x_k$ belong to S_i.

We introduce the following concept based on these intervals

Scan-forward siblings

These are element that you find in the search path before finding an element in the upper list.



Now

Given that a search is being done, S_i contains l forward siblings

It must be the case that given $x_1, ..., x_l$ scan-forward siblings, we have that

$$x_1, ..., x_l \notin S_{i+1}$$

and $x_{l+1} \in S_{i+1}$

Thus

We have

Since each element of S_i is independently chosen to be in S_{i+1} with probability $p = \frac{1}{2}$.

We have

The number of scan-forward siblings is bounded by a geometric random variable X_i with parameter $p = \frac{1}{2}$.

- Imagine the fact that you have multiple fails... then $x_1, ..., x_l \notin S_{i+1}$ is modeled by X_i

Thus, we have that

The expected number of scan-forward siblings is bounded by 2!!!

$$\text{Expected \# Scan-Forward Siblings at } i \leq \underbrace{E[X_i] = \frac{1}{1/2}}_{\text{Mean}} = 2$$

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- Imagine the fact that you have multiple fails... then $x_1, ..., x_l \notin S_{i+1}$ is modeled by X_i

Thus, we have that

The expected number of scan-forward siblings is bounded by 2!!!

$$\text{Expected } \# \text{ Scan-Fordward Siblings at } i \leq \underbrace{E\left[X_i\right] = \frac{1}{1/2}}_{\text{Mean}} = 2$$

Then

In the worst case scenario

A search is bounded by $O(\log_2 n) + 2\log_2 n = O(\log_2 n)$

An given that a insertion is a (search) + (deletion bounded by the height)

Thus, an insertion is bounded by $2\log_2 n + 3\log_2 n = O(\log_2 n)$

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Applications

We have

- Cyrus IMAP servers offer a "skiplist" backend Data Base implementation.
- Lucene uses skip lists to search delta-encoded posting lists in logarithmic time.
- Redis, an ANSI-C open-source persistent key/value store for Posix systems, uses skip lists in its implementation of ordered sets.
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  - The expected search, insertion and deletion time is $O(\log n)$
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Thanks

Questions?