Isotropic distribution in Spherical polar coordinates
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To study random gravitational wave sources in the sky, it is not correct to establish their nature (location and orientation) by drawing values from uniform distributions for (θ, ϕ) ($\theta \equiv$ co-declination: $\delta = \pi/2 - \theta$) and ϕ (the azimuth). The appropriate probability distribution calculations for these parameters are well explained in Ref.[1].

Assuming that the distribution of sources (the same works for random isotropic orientation) is isotropic, the probability density function must be invariant under rotation: there should be no preferred directions. If $p(\Omega)\mathrm{d}\Omega$ is the pdf for finding a source in the differential solid angle $\mathrm{d}\Omega$, then isotropy means that $p(\Omega)\mathrm{d}\Omega$ = constant, and correct normalisation gives $p(\Omega) = 1/(4\pi)$.

Changing the variables to co-declination θ and right ascension ϕ:

$$p(\theta, \phi)|d\delta d\phi| = p(\Omega)|d\Omega| \tag{1}$$

and recalling that $d\Omega = \sin\theta \; \mathrm{d}\theta \; \mathrm{d}\phi$, we obtain:

$$p(\theta, \phi) = \frac{\sin\theta}{4\pi} \tag{2}$$

we can marginalise over $\phi \in [0,2\pi)$:

$$p(\theta) = p(\theta|\phi) = \int_0^{2\pi} p(\theta, \phi) d\phi = \frac{\sin\theta}{2} \tag{3}$$

equally we can marginalise over $\theta \in [0,\pi)$:

$$p(\phi) = p(\phi|\theta) = \int_0^{\pi} p(\theta, \phi) d\theta = \frac{1}{2\pi} \tag{4}$$

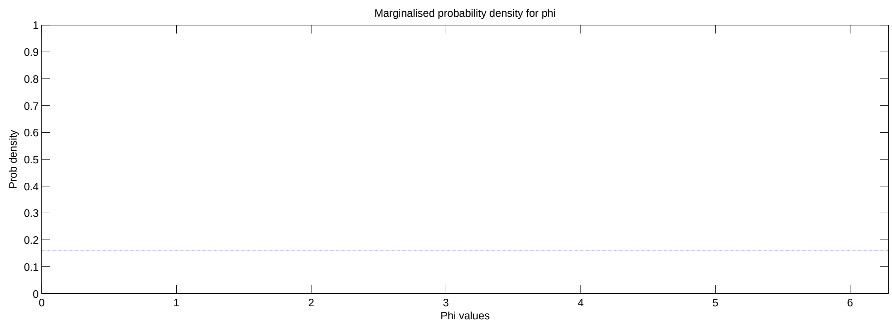
We notice that ϕ is distributed uniformly, but θ is not.

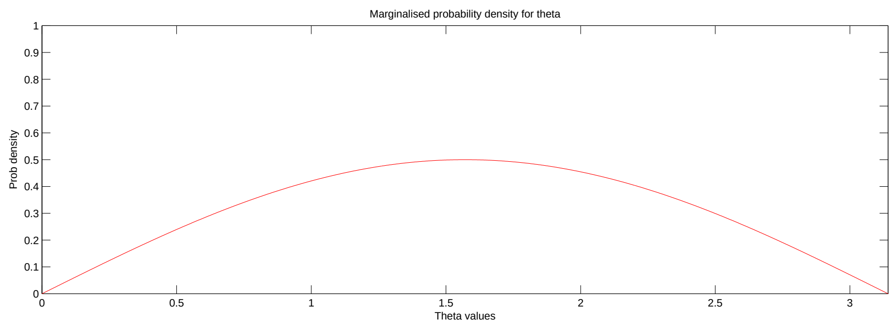
Isotropy means that selecting a value for θ we do not impose any constraint on ϕ and viceversa. So the joint probability distribution is the multiplication of the two:

$$p(\theta, \phi) = p(\theta)p(\phi) = \frac{\sin\theta}{2}\frac{1}{2\pi} = \frac{\sin\theta}{4\pi} \tag{5}$$

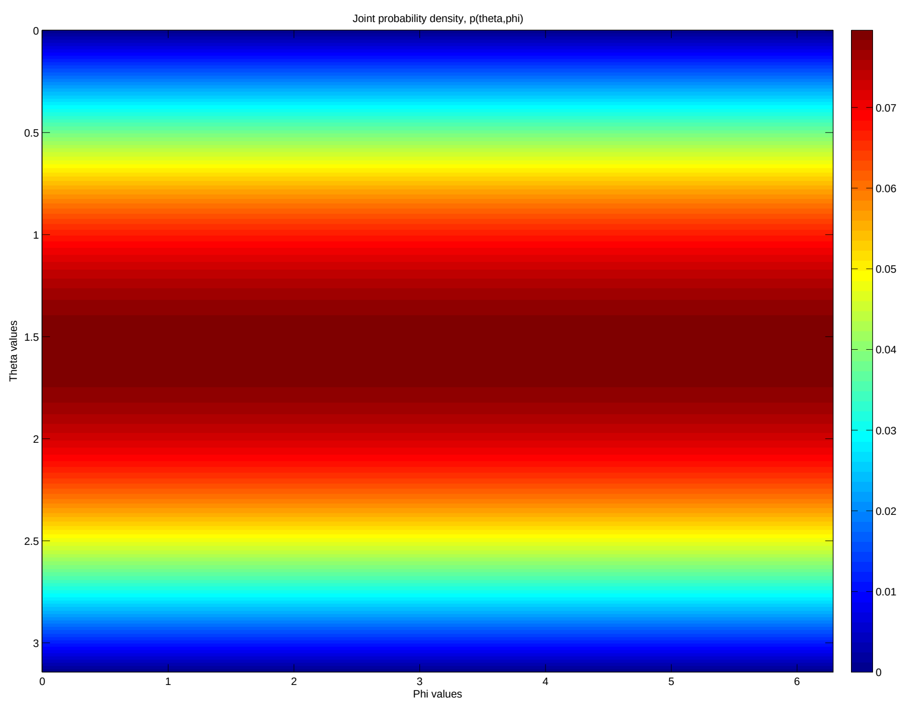
and we recover the original expression.

The figure below shows both marginalised distributions:





The joint distribution is not uniform in 2D as can be seen in the figure below:

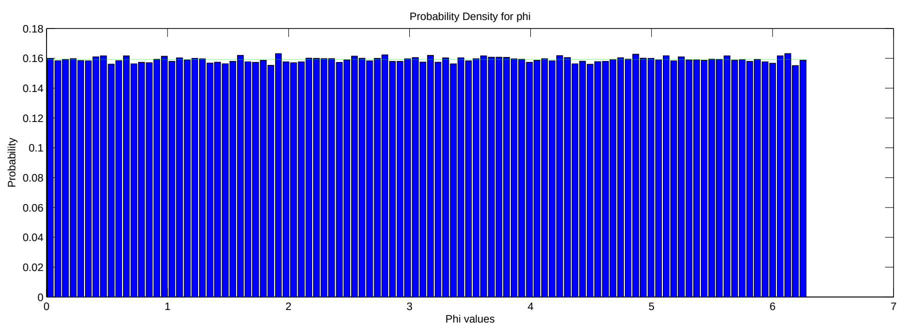


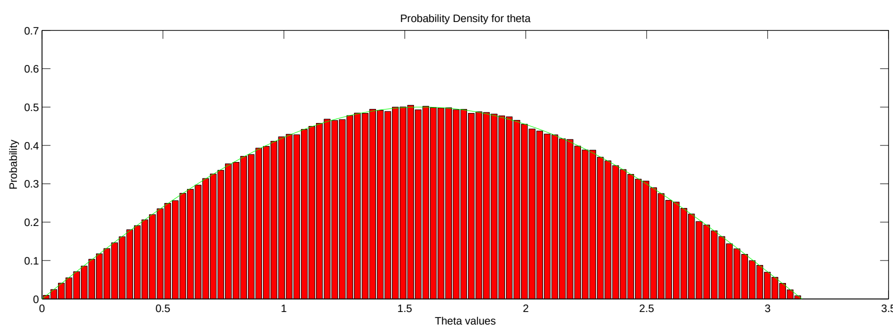
Another way of looking at this, it would be as follows (see ref.[2]):

In case of a gravitational wave source location/orientation the prior should represent an equal probability of the spin/symmetry axis pointing in any direction on the sphere surrounding it, i.e. a uniform prior on area. The element of area as expressed in spherical polar coordinates (θ, ϕ) is:

$$dA = \sin\theta d\theta d\phi = -d(\cos\theta) d\phi \tag{6}$$

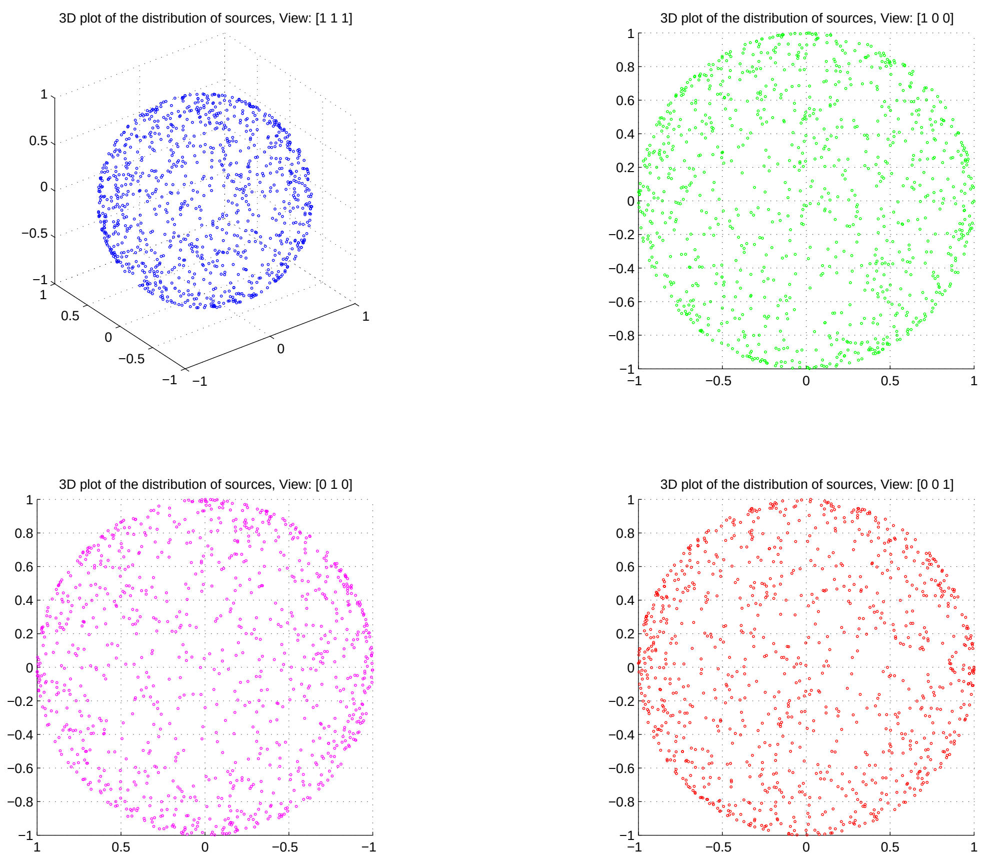
but the area is positive so we can write $\mathrm{d}A = \mathrm{d}(\cos\theta)\ \mathrm{d}\phi$. From here, we see that the prior should be uniform on the parameters $\cos\theta$ and ϕ. If we gave random values to these two "parameters" [calculate θ by doing $\theta = \mathrm{acos}(\mathrm{unifrnd}(1,1,1))$] and plotted an histogram to see the occurrence of each value we would get the next figure. Previously calculated analytical functions are overlaid with a green colour line. The resemblance is good.





COUNTING STATISTICS

To prove that this distribution is isometric and is valid to assign more or less the same number of sources to each solid angle, we proceed on counting the number of sources contained within each quarter (e.g. quarter $1 \equiv x > 0, y > 0,\ z > 0$). The next figure shows a 3D view of the distribution of sources. Not too many sources (1000) have been plotted here to get a clearer view.



We counted the number of sources in each quarter for the original distribution:
Quarter 1/Quarter 2/Quarter 3/Quarter 4/Quarter 5/Quarter 6/Quarter 7/Quarter 8
122/119/109/149/121/137/129/114.

There are differences in number of source from quarter to quarter but these even up when including more sources.
For Quarter 1/Quarter 2/Quarter 3/Quarter 4/Quarter 5/Quarter 6/Quarter 7/Quarter 8
6351/6203/6219/6252/6343/6063/6317/6252.

To make sure that the distribution is istropic under any rotation, we rotate all the sources an arbitrary angle in the range $[0, 2\pi)$ respect to a vector oriented randomly. Mathematically the rotation can be described as the overall combination of three different partial rotations α, β, γ around the unitary vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$.

$$rot_x(\alpha) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{vmatrix}$$

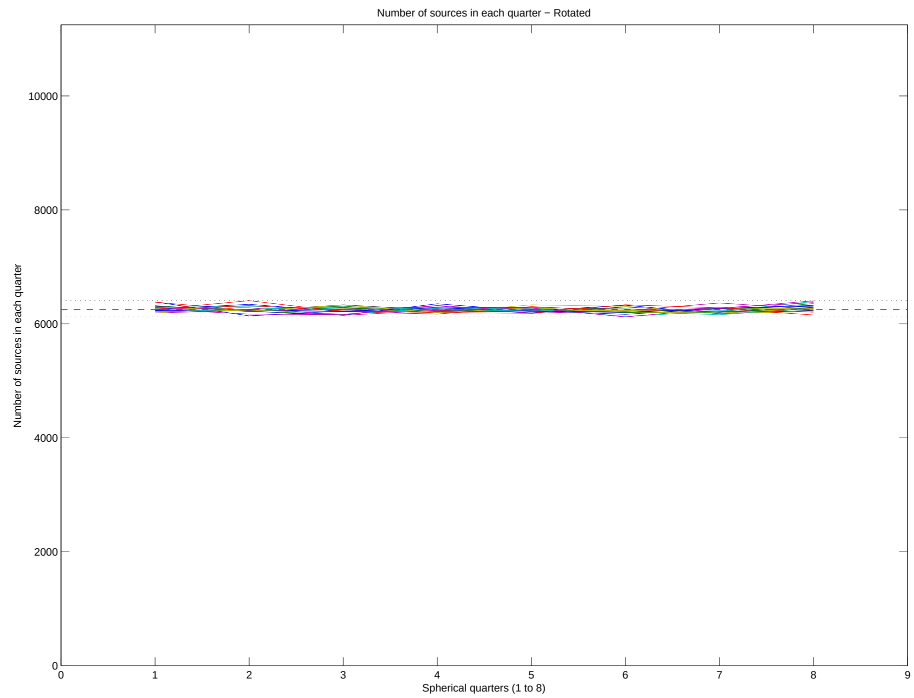
$$rot_y(\beta) = \begin{vmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{vmatrix}$$

$$rot_z(\gamma) = \begin{vmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

The combined rotation can be expressed as:

$$rot_{xyz}(\alpha,\beta,\gamma) = \begin{vmatrix} \cos\gamma\cos\beta & \cos\beta\sin\gamma & -\sin\beta \\ \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\cos\beta \\ \sin\beta\cos\alpha\cos\gamma + \sin\alpha\sin\gamma & \sin\beta\cos\alpha\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\cos\beta \end{vmatrix}$$

The next figure below resumes the results of 20 rotations. A few of the angles we consider on each rotation are listed here:
Alpha / Beta / Gamma (radians)
1) 0.8319 / 4.8116 / 2.3340
2) 0.1395 / 1.8280 / 4.7525
3) 3.9208 / 2.6501 / 2.7315
...
18) 2.5303 / 3.6739 / 4.7368
19) 2.5421 / 1.5358 / 4.3449
20) 4.6709 / 0.2511 / 1.5291



Summarising the results for all the rotations we have:
(numbers indicate sources per spherical quarter):
The exact value per quarter is (total/8) is: 6250
The mean value of the table is: 6250
The max value of the table is: 6407
The min value of the table is: 6122
Max deviation is : 285 (4.56% of the mean value)

References

1. 'Numerical Astronomy' course, Graham Woan and Martin Hendry, Questions and solutions II, http://radio.astro.gla.ac.uk/numerical/index.html
2. 'Applications of Markov Chain Monte Carlo Methods to Continuous Gravitational Wave Data Analysis', PhD. thesis, J.Veitch, 2007.