Lab 3
September 19, 2016
System Properties

INSTRUCTIONS:
All lab submissions include a written report and source code in the form of an m-file. The report contains all plots, images, and figures specified within the lab. All figures should be labeled appropriately. Answers to questions given in the lab document should be answered in the written report. ***The written report must be in PDF format***. Submissions are done electronically through **my.ECE**.

1 System Response

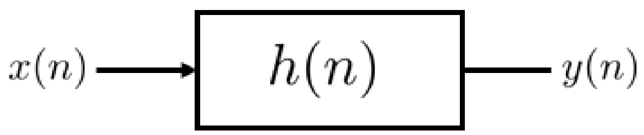


Figure 1: A linear, shift-invarient system.

A linear, shift-invariant (LSI) system can be characterized by its impulse response $h(n)$. From the impulse response, the output $y(n)$ to any input $x(n)$ can be expressed as a convolution. Given that $x(n)$ is length L and $h(n)$ is length M, the length of $y(n)$ is $L + M - 1$. Discrete convolution can be performed in the frequency domain using DFT:

$$y(n) = \mathrm{IDFT}\{\{\mathrm{DFT}\{x(n)\} \cdot \mathrm{DFT}\{h(n)\}\} \tag{1}$$

However, using the DFT implies that the frequency domain is discrete (in addition to periodic). By duality, $x(n)$ is also periodic where $x(n)$ supplied to the DFT is assumed to be a single period of the signal. Convolution of two periodic signals is known as *circular convolution*.

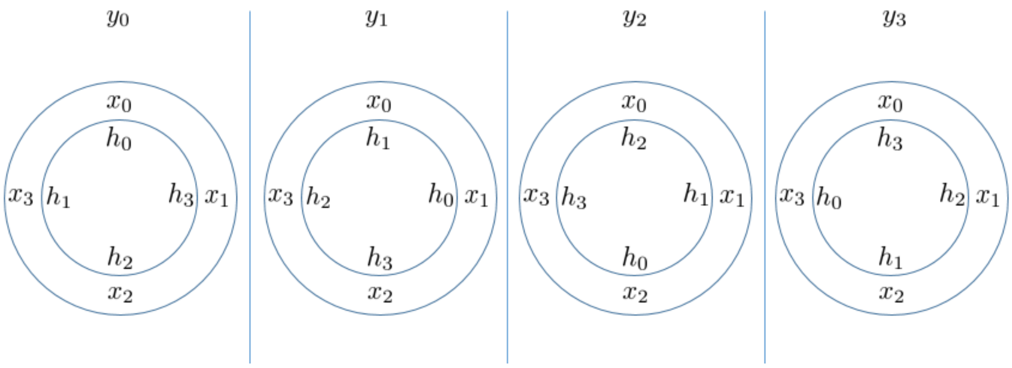


Figure 2: A graphical interpretation of circular convolution. Each output is a sum of the inputs that line up. For example, $y_0 = h_0x_0 + h_3x_1 + h_2x_2 + h_1x_3$. When performing circular convolution, both inputs must be the same length. If this is not the case, the shorter sequence is zero-padded to the length of the longer sequence.

To obtain regular convolution from circular convolution, both $x(n)$ and $h(n)$ must be zero-padded to length $L + M - 1$ prior to taking the DFT. This ensures that the length of $y(n)$ is also $L + M - 1$.

Report Item: Write a function called **myDFTConv** that uses the **fft** to perform a convolution. Use this function to find the convolution of $x(n)$ and $h(n)$ and plot the results with **stem**. Verify your answer with **conv**. Using the fact that **fft** is $\mathcal{O}(N \log N)$, what is the approximate order of complexity of **myDFTConv**?

$$x(n) = \{-1, 2, 1, 5, 4\}$$
$$h(n) = \{1, 2, 3, 2\}$$

The impulse response to a system can be determined by setting the input to an impulse. For a discrete system, a discrete impulse (also known as Kroenecker delta function) can be expressed as

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & \text{else} \end{cases} \tag{2}$$

Report Item: Implement the following difference equation as a function called **sys1** that takes arguments a and $x(n)$ and returns $y(n)$.

$$y(n) = ay(n-1) + 0.3x(n) - 2x(n-1)$$

Let $a = 2$ and plot the impulse response $h(n)$ using **stem** for $N = 64$. Is this system stable? Causal?

2 System Properties

A system can be described by several useful properties: Linearity, Shift-Invariance, Causality, and BIBO stability. A system is *linear* if scaling the input produces an output with the same scaling.

$$ax(n) \longrightarrow ay(n) \tag{3}$$

A system is *shift-invariant* if a shift in the input produces an identical shift in the output

$$x(n-n_0) \longrightarrow y(n-n_0) \tag{4}$$

A system is *causal* if $h(n) = 0$ for $n \leq 0$. That is, it does not produce an output until given an input. And the output must come after the input. A system if BIBO stable if a bounded input produces a bounded output. This implies that $\sum_{n=-\infty}^{\infty} |h(n)| \leq \infty$.

The properties of linearity and shift-invariance provide the basis for the convolution property of LSI systems. An *eigensequence* is defined as an input that produces an output which is a scaled version of the input (analogous to eigenvectors in linear algebra). The eigensequence of an LSI system is a complex exponential function. This can be shown by using the convolution property of LSI systems. Let $x(n) = e^{i\omega_0 n}$. Using convolution, the output can be expressed as

$$\begin{aligned} y(n) &= \sum_{m=-\infty}^{\infty} h(m)x(n-m) && (5)\\ &= \sum_{m=-\infty}^{\infty} h(m)e^{i\omega_0(n-m)} && (6)\\ &= e^{i\omega_0 n} \sum_{m=-\infty}^{\infty} h(m)e^{-i\omega_0 m} && (7)\\ &= x(n)H(\omega_0) && (8) \end{aligned}$$

Hence, complex exponentials can be used to excite individual frequencies of a transfer function. This may be obvious from the fact that fourier transforms are sinusoidal decompositions of a signal.

Report Item: Implement the following difference equation as a function called **sys2** that takes arguments a and $x(n)$ and returns $y(n)$.

$$y(n) = ay(n-1) + x(n)^2 \tag{9}$$

Set $a = 2$ and plot the impulse response $h(n)$ using **stem** for $N = 64$. Repeat for $a = 0.5$. For each case, is the system causal? Stable? Linear? Can you find the output to either system by convolving $x(n)$ with $h(n)$?

3 Z-Transform

For LSI systems, causality and stability as well as other useful properties can be determined from the z-transform of the system. The z-transform can be expressed as

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n} \tag{10}$$

where $z = \sigma + i\omega$ is complex. By letting $z = e^{i\omega}$, the z-transform becomes

$$X(e^{i\omega}) = \sum_{n=0}^{N-1} x(n) e^{-i\omega n} \tag{11}$$

which is the same as the DTFT. Any LSI system can be expressed as a difference equation of the form

$$y(n) + \sum_{k=1}^{M-1} a_k y(n-k) = \sum_{l=0}^{N-1} b_l x(n-l) \tag{12}$$

The z-transform of (12) is

$$Y(z) + \sum_{k=1}^{M-1} a_k Y(z) z^{-k} = \sum_{l=0}^{N-1} b_l X(z) z^{-l} \tag{13}$$

from which the transfer function $H(z) = Y(z)/X(z)$ can be written as

$$H(z) = \frac{\sum_{l=0}^{N-1} b_l z^{-l}}{1 + \sum_{k=1}^{M-1} a_k z^{-k}} \tag{14}$$

The poles and zeros of $H(z)$ are the solutions to $\sum_{l=0}^{N-1} b_l z^{-l} = 0$ and $1 + \sum_{k=1}^{M-1} a_k z^{-k} = 0$, respectively. Since both the numerator and denominator are polynomials, they can be written be factorized into

$$H(z) = \frac{\prod_{l=0}^{N-1} (1 - z_l z^{-1})}{\prod_{k=1}^{M-1} (1 - p_k z^{-1})} \tag{15}$$

where p_k denotes a pole and z_l, a zero. Using (15), the magnitude can be expressed as

$$|H(z)| = \frac{\prod_{l=0}^{N-1} |1 - z_l z^{-1}|}{\prod_{k=1}^{M-1} |1 - p_k z^{-1}|} \tag{16}$$

where $|1 - z_l z^{-1}| \equiv |z - z_l|$ is the distance between z and z_l. A useful representation of the z-transform is the pole-zero plot. For $h(n) = \{1, 3, 4, 5, 3, 1\}$, the pole-zero plot is

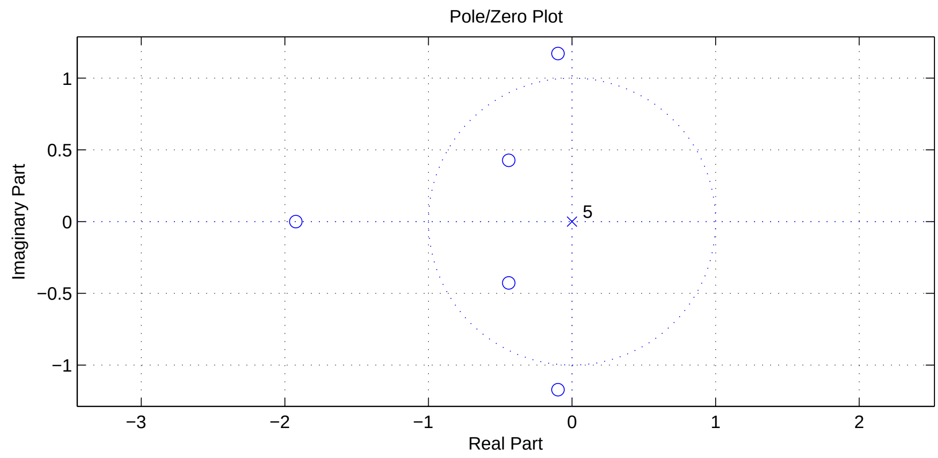


Figure 3: A pole-zero plot. The circles indicate zeros and the crosses indicate poles. For any system, there is an equal number of poles and zeros.

The unit-circle corresponds to the DTFT. By tracing around the unit circle and measuring the distance between a point $z = e^{i\omega_0}$ for $\omega = \omega_0$, one can estimate the magnitude response of the filter using simple geometry. For example, at $\omega = \pi/4$, the distance between the observation point (marked by the red circle) and the poles and zeros are shown in Figure 4.

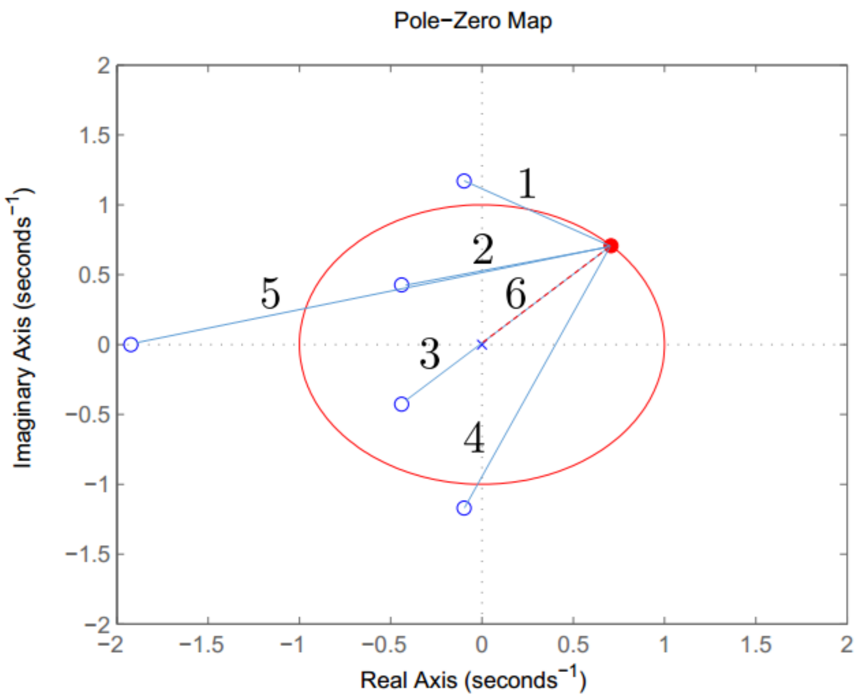
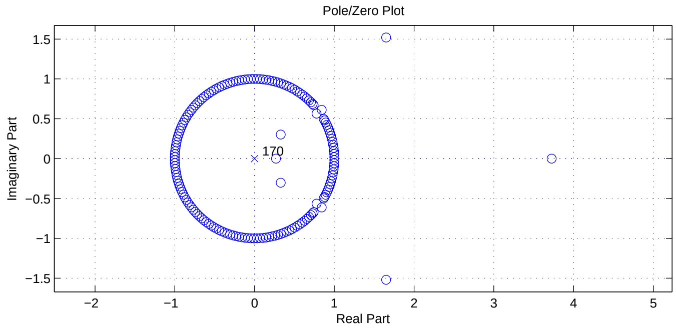


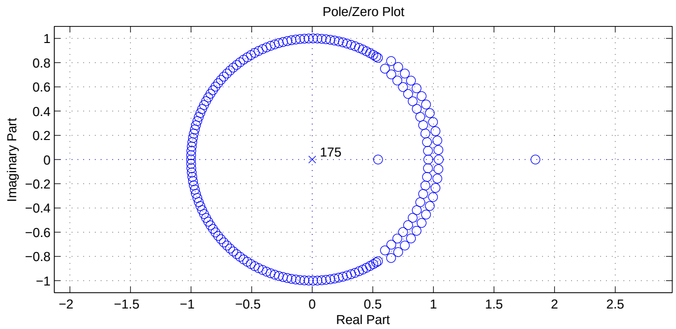
Figure 4: Geometric interpretation of magnitude response from a pole-zero plot.

The magnitude response is the product of the lengths of blue lines marked 1, 2, 3, 4, and 5 divided by the product of the red dashed lines marked by 6. In this case, all 5 poles lie on the origin, so the length of the dashed red lines are 1. Hence, for this system, the poles do not affect the magnitude response of the system. In general, the magnitude response of a system becomes larger when the observation point is closer to a pole or far from a zero. The magnitude response becomes smaller when an observation point is farther from a pole or closer to a zero. This enables one to guess the behavior of system from its pole-zero plot.

Report Item: Given the following pole-zero plots, draw the ***approximate*** magnitude response (**Hint**: Use the fact that the DTFT is a function of $e^{i\omega}$ and start at $\omega = 0$ and work your way to $\omega = \pi$. Remember that the magnitude response must be zero at a zero.):



(a)



(b)

Transfer functions can be defined in MATLAB using **tf**(b, a) where $b = [b_0, b_1, \ldots, b_{N-1}]$ are the numerator coefficients and $a = [1, a_1, \ldots, a_{M-1}]$ are the denominator coefficients corresponding to

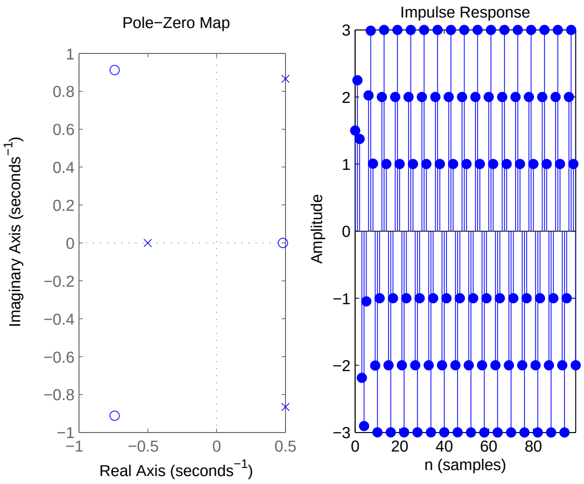
$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_{N-1} z^{-(N-1)}}{1 + a_1 z^{-1} + \cdots + a_{M-1} z^{-(M-1)}} \tag{17}$$

The object returned from **tf** can be used in **pzplot** to obtain the pole-zero plot of the transfer function. The impulse response of $h(z)$ can be found using **impz**(b, a).

```
b = [3,3,2,-2];
a = [2,-1,1,1];
S = tf(b,a);
N = 100;

figure;
subplot(121);
pzplot(S);
subplot(122);
impz(b,a,N);
```

(a) Example code using **pzplot** and **impz**.



(b) Pole-zero and impulse response plots.

The stability of a system can be ascertained from its pole-zero plot. For a causal system, if ***all*** poles are contained within unit circle, then the system is stable. If even one pole lies on or outside the unit circle, then the system is unstable.

Report Item: For each transfer function below, plot the pole-zero plot and impulse response of length $N = 20$ using **pzplot** and **impz**. Also note whether each system is stable or unstable.

$$H_1(z) = 2 + 5z^{-2} + 4z^{-3} - 3z^{-6} \tag{18}$$

$$H_2(z) = 3 + 2z^{-1} - 2z^{-3} \tag{19}$$

$$H_3(z) = \frac{z^{-3} + z^{-6} - 2z^{-7}}{12 + z^{-1} + 4z^{-3}} \tag{20}$$

If a pole lies on the unit circle, the system is generally unstable. But if the input does not excite this frequency, then it is possible to obtain an output that does not tend towards infinity from an unstable system.

Report Item: Consider the system

$$H(z) = \frac{z}{(z + e^{\frac{-i8\pi}{10}})(z + e^{\frac{i8\pi}{10}})} \tag{21}$$

Is this system BIBO stable? For what inputs is this system unstable? For what inputs is this system stable? Plot the pole-zero plot and impulse response of this system for $N = 35$ using **pzplot** and **impz**.