Euler Class Topological Insulators

Theory and Numerical Results for Multi-Band Models

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September 28, 2025

Outline

Foundations Theoretical Framework

Results I: k.p Analysis of 3-Band Models

Results II: Bulk-Boundary Correspondence in a 4-Band Model

Outlook

Foundations Theoretical Framework

Historical Perspective

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- 2005–2007: The field exploded with the discovery of $\mathbb{Z}_2$ Topological Insulators, protected by time-reversal symmetry.
- 2017–Present: The frontier moved to classifying more subtle phases, leading to the concepts of higher-order and fragile topology.

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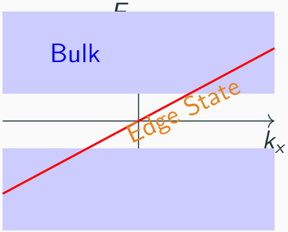
- The gap must close at the edge to support these states.



The Role of Symmetry in Band Theory

Symmetry constrains the form of the Hamiltonian and the wavefunctions.

- A tight-binding model can be Fourier transformed into the Bloch picture:

$$\hat{H} = \sum_{\mathbf{k} \in BZ} \sum_{\alpha,\beta} |\phi_{\alpha,\mathbf{k}}\rangle H_{\alpha\beta}(\mathbf{k}) \langle \phi_{\beta,\mathbf{k}}|$$

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- **Key Consequence:** Within a plane left invariant by the symmetry (the $C_2 T$ plane), the Hamiltonian is forced to be real and symmetric:

$$\mathbf{1} \cdot \tilde{H}^*(\mathbf{k}) \cdot \mathbf{1} = \tilde{H}(\mathbf{k})$$

  This reality condition makes the Chern number vanish and the Euler class the relevant invariant.

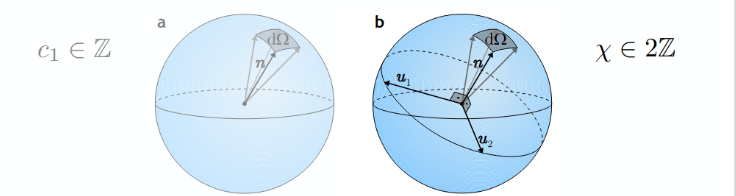
The Euler Class: Topology of Real Bundles

Chern Class (Complex Case)

- Describes complex vector bundles.
- Hamiltonian: $H(\mathbf{k}) = \mathbf{h}(\mathbf{k}) \cdot \boldsymbol{\sigma}$
- Curvature: $F_{ij} = \frac{1}{2}\mathbf{n} \cdot (\partial_i \mathbf{n} \times \partial_j \mathbf{n})$
- Invariant: $c_1 \in \mathbb{Z}$

Euler Class (Real Case, $C_2T^2 = +1$)

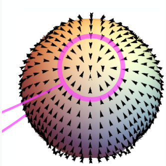
- Describes real oriented vector bundles.
- Hamiltonian: $H(\mathbf{k}) = 2\mathbf{n}\mathbf{n}^T - 1$
- Euler Form: $Eu_{ij} = \mathbf{n} \cdot (\partial_i \mathbf{n} \times \partial_j \mathbf{n})$
- Invariant: $\chi \in 2\mathbb{Z}$



Euler Class from Nodal Points Patches

Nodal Points Vortices:

- Per the Poincaré-Hopf theorem, the Euler class is the sum of the vorticities (winding numbers) of the vector field $\mathbf{n}(\mathbf{k})$ around its zeros.
- These zeros are the nodal points between bands.
- The number of stable nodal points is directly related to the Euler class: $\#NP = 2|\chi|$.

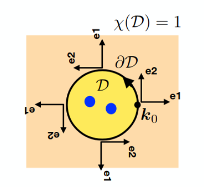


Patch Euler Number:

- The total Euler class can be computed by summing the local contributions from patches $\mathcal{D}_n$ around each nodal point.

$$\chi(\mathcal{D}) = \frac{1}{\pi}\sum_n \left[\int_{\mathcal{D}_n} Eu - \oint_{\partial\mathcal{D}_n} a\right]$$

$$= \sum_n W_n \in \mathbb{Z}$$



Results I: k.p Analysis of 3-Band Models

k.p Expansion of the Square Lattice Model

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- Start with a 3-band tight-binding Hamiltonian on the square lattice.
- Perform a k·p expansion around the Γ point to get a low-energy effective Hamiltonian.

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Model Hamiltonian:

$$H = \begin{pmatrix} -\frac{9}{4} + s_1^2 & s_1 s_2 & (1 - c_1 - c_2)s_1 \\ s_1 s_2 & \frac{7}{4} + s_2^2 & (1 - c_1 - c_2)s_2 \\ (1 - c_1 - c_2)s_1 & (1 - c_1 - c_2)s_2 & -2 + (c_1 + c_2 - 1)^2 \end{pmatrix}$$

where $c_i = \cos(\pi k_i)$, $s_i = \sin(\pi k_i)$

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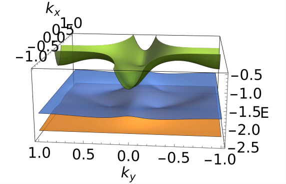
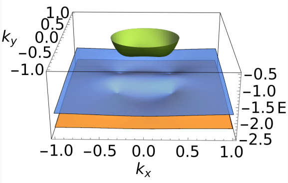
k.p Approximated Hamiltonian:

$$H_{kp} = \begin{pmatrix} -\frac{9}{4} + k_1^2 \pi^2 & k_1 k_2 \pi^2 & \frac{k_1 \pi}{2}(-2 + \pi^2(k_1^2 + k_2^2)) \\ k_1 k_2 \pi^2 & \frac{7}{4} + k_2^2 \pi^2 & \frac{k_2 \pi}{2}(-2 + \pi^2(k_1^2 + k_2^2)) \\ \frac{k_1 \pi}{2}(\dots) & \frac{k_2 \pi}{2}(\dots) & -2 + \frac{1}{4}(-2 + \pi^2(k_1^2 + k_2^2))^2 \end{pmatrix}$$

Original vs. Perturbed Spectrum:





Euler Form of the Square Lattice Model

The Euler Form:

- The Euler form is the integrand of the Euler class ($\chi = \frac{1}{2\pi} \int \mathbf{e}\, d^2k$).
- It can be viewed as the density of "topological charge" in the Brillouin zone.
- Its peaks indicate the momentum-space regions that contribute most to the non-trivial topology.

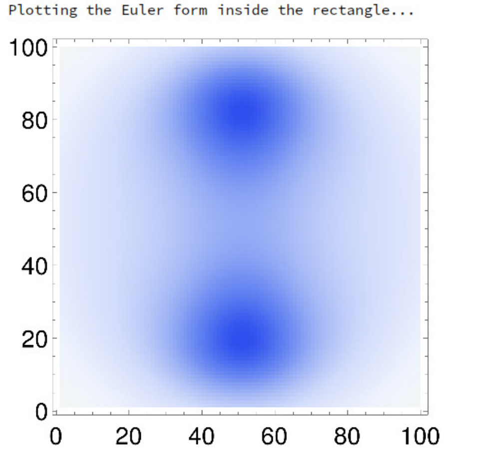


Figure: Euler form for the gapped square lattice model, showing strong peaks that integrate to $\chi = 2$.

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Model Hamiltonian:

$$H = \begin{pmatrix} \frac{4}{5} c_1 & c_{1-2} + c_{1+2} & c_2 + c_{21+2} \\ c_{1-2} + c_{1+2} & \frac{4}{5} c_2 & c_1 + c_{1+22} \\ c_2 + c_{21+2} & c_1 + c_{1+22} & \frac{4}{5} c_{1+2} \end{pmatrix}$$

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k.p Approximated Hamiltonian:

$$H_{kp} = \begin{pmatrix} \frac{4}{5}(1 - \frac{\pi^2 k_1^2}{2}) & 2 - \frac{\pi^2 (k_1^2 + k_2^2)}{4} & \ldots \\ 2 - \frac{\pi^2 (k_1^2 + k_2^2)}{4} & \frac{4}{5}(1 - \frac{\pi^2 k_2^2}{2}) & \ldots \\ \ldots & \ldots & \frac{4}{5}(1 - \frac{\pi^2 (k_1 + k_2)^2}{2}) \end{pmatrix}$$

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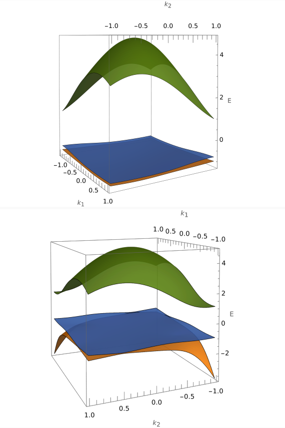

Euler Form of the Kagome Lattice Model

Topological Charge:

- The Euler form for the Kagome model shows a different distribution of topological charge.
- The integral confirms the Euler class is $\chi = 1$.
- This confirms k·p theory is a valid tool for analyzing fragile topology in different geometries.

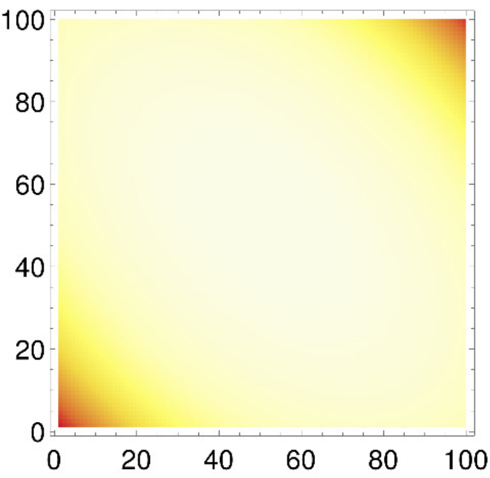

Figure: Euler form for the gapped Kagome model. The charge is concentrated near the BZ corners.

Results II: Bulk-Boundary Correspondence in a 4-Band Model

The 4-Band Model: Bulk Properties

Model Hamiltonian $H(k_1, k_2)$:

$$\begin{pmatrix} s_2 & s_1 & -1/4 & 1 - c_1 - c_2 \\ s_1 & -s_2 & -1 + c_1 + c_2 & 1/4 \\ -1/4 & -1 + c_1 + c_2 & s_2 & s_1 \\ 1 - c_1 - c_2 & 1/4 & s_1 & -s_2 \end{pmatrix}$$

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Bulk Analysis:

- A Wilson loop calculation confirms the model has a non-trivial bulk invariant.
- The 3D bulk band structure shows the system is fully gapped.

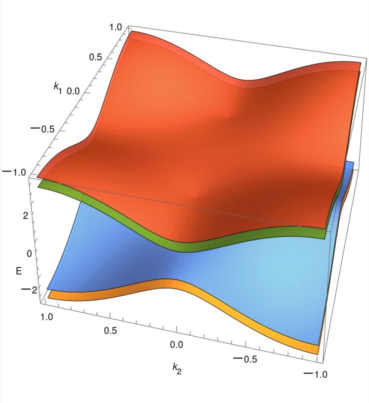

Figure: 3D Bulk Spectrum.

Ribbon Spectrum 1D Edge States

Method:

- Simulation on a ribbon geometry (open boundaries in one direction).

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Interpretation:

- These are the 1D edge states predicted by the non-trivial bulk. They are the first signature of the bulk-boundary correspondence.

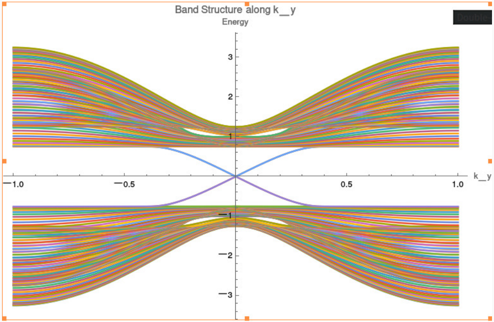

Figure: Ribbon spectrum showing protected 1D edge states crossing the bulk gap.

Finite Flake Simulation Corner States

Method:

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Result:

- The low-energy states show a remarkable concentration of their probability density $|\psi|^2$ at the edges.

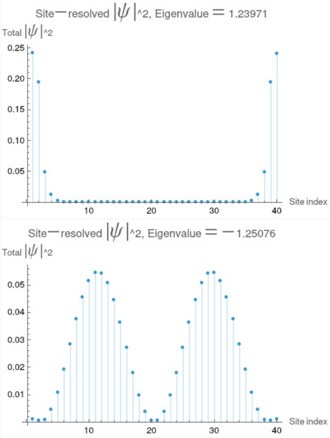

Figure: The resolved site $|\psi|^2$ shows a clear localization at the edges of the system(with 40 sites)

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- **4-Band Model:** Provides a complete demonstration of the higher-order bulk-boundary correspondence, from the gapped bulk to 1D edge states and finally to localized 0D corner modes.
- **Outlook:** This work validates the theoretical framework of fragile topology and provides tools to identify and engineer these phases in real materials and metamaterials.

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