Euler Class Topological Insulators

Theory and Numerical Results for Multi-Band Models

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Outline

Foundations and Theoretical Framework
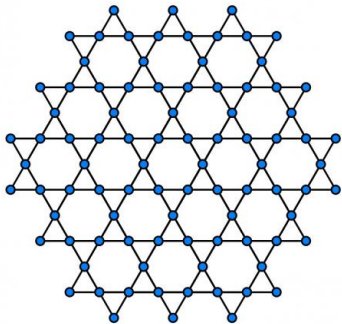
Results I: k.p Analysis of 3-Band Models

Results II: Bulk-Boundary Correspondence in a 4-Band Model

Outlook

Foundations and Theoretical Framework

Historical Perspective on 2D Topological Insulators

Kagome lattice

- **1980s – Quantum Hall Effect**
  - Integer quantization of Hall conductance
  - Topological origin: Chern number
  - Robust to disorder/impurities

- **2005–2007 – $\mathbb{Z}_2$ Topological Insulators**
  - Kane–Mele model (graphene + SOC)
  - $\mathbb{Z}_2$ invariant (time-reversal symmetry)
  - BHZ model $\rightarrow$ HgTe/CdTe QWs (1st experiment)

- **2010s – Classification Era**
  - Tenfold Way, K-theory classification
  - 2D $\rightarrow$ 3D TIs, crystalline phases
  - Many experimental confirmations

- **2017–Present – Beyond Stable Topology**
  - Higher-order topology (hinge/corner states)
  - Fragile topology (band representations)
  - Symmetry-based refinements

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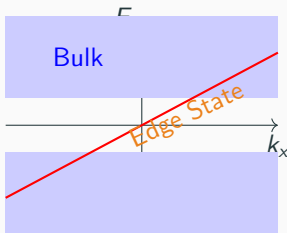# The Bulk-Boundary Correspondence (Chern Case)

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- This bulk property dictates that there must be C conducting 1D channels (edge states) at the boundary.

- The gap must close at the edge to support these states.

The Role of Symmetry in Band Theory

Symmetry constrains the form of the Hamiltonian and the wavefunctions.

- A tight-binding model can be Fourier transformed into the Bloch picture:

$$\hat{H} = \sum_{\mathbf{k} \in BZ} \sum_{\alpha,\beta} |\phi_{\alpha,\mathbf{k}}\rangle H_{\alpha\beta}(\mathbf{k}) \langle \phi_{\beta,\mathbf{k}}|$$

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- $C_2 T$ **symmetry** is an anti-unitary symmetry $\mathcal{A} = UK$ that is crucial for real topological phases.

- **Key Consequence:** Within a plane left invariant by the symmetry (the $C_2 T$ plane), the Hamiltonian is forced to be real and symmetric: This reality condition makes the Chern number vanish and the Euler class the relevant invariant.
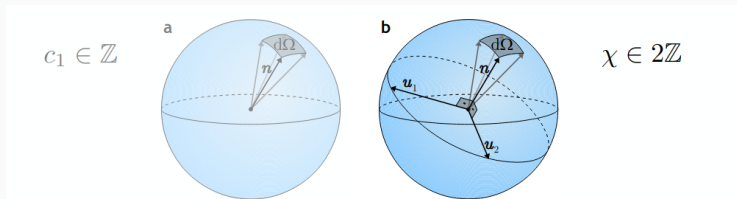
The Euler Class: Topology of Real Bundles

Chern Class (Complex Case)

- Describes complex vector bundles.
- Hamiltonian: $H(\mathbf{k}) = \mathbf{n}(\mathbf{k}) \cdot \boldsymbol{\sigma}$
- Curvature:
  $F_{ij} = \frac{1}{2}\mathbf{n} \cdot (\partial_i \mathbf{n} \times \partial_j \mathbf{n})$
- Invariant: $c_1 \in \mathbb{Z}$(integral of curvature over the whole Brillouin zone)

Euler Class (Real Case, $C_2 T^2 = +1$)

- Describes real oriented vector bundles.
- Hamiltonian: $H(\mathbf{k}) = 2\mathbf{n}\mathbf{n}^T - 1$
- Euler Form:
  $Eu_{ij} = \mathbf{n} \cdot (\partial_i \mathbf{n} \times \partial_j \mathbf{n})$
- Invariant: $\chi \in \mathbb{Z}$(integral of curvature over the whole Brillouin zone)

$c_1 \in \mathbb{Z}$

$\chi \in 2\mathbb{Z}$

Euler Class from Nodal Points

Nodal Points & Vortices:

- Per the Poincaré-Hopf theorem, the Euler class is the sum of the vorticities (winding numbers) of the tangent vector fields as $\mathbf{n}(\mathbf{k})$ covers the whole sphere.

- These zeros are the nodal points between bands.

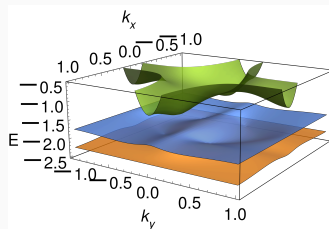- The number of stable nodal points is directly related to the Euler class:

$$\#NP = 2|\chi|$$

Figure 1: Nodal points as vortices of $\mathbf{n}(\mathbf{k})$

Patch Euler Number:

- The total Euler class can be computed by summing the local contributions from patches $\mathcal{D}_n$ around each nodal point.

$$\chi(\mathcal{D}) = \frac{1}{\pi} \sum_n \left[\int_{\mathcal{D}_n} Eu - \oint_{\partial\mathcal{D}_n} a \right]$$

- we use the patch Euler class as the criterium to obtain the minimal polynomial theory that preserves the topology of the original lattice model
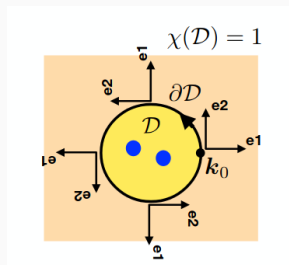

Figure 2: Patch contributions add up to total Euler class

Results I: k.p Analysis of 3-Band Models

k.p Expansion of the Kagome Lattice 3 band Model

Method:

- For kagome lattice , the euler class is always an odd number, lowest value is 1.

- Apply k·p expansion to a 3-band Kagome lattice model.

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Model Hamiltonian:

$$H = \begin{pmatrix} \frac{4}{5}c_1 & c_{1-2} + c_{1+2} & c_2 + c_{21+2} \\ c_{1-2} + c_{1+2} & \frac{4}{5}c_2 & c_1 + c_{1+22} \\ c_2 + c_{21+2} & c_1 + c_{1+22} & \frac{4}{5}c_{1+2} \end{pmatrix}$$

where $c_i = \cos(\pi k_i)$, $c_{1\pm2} = \cos(\frac{\pi}{2}(k_1 \pm k_2))$, etc.

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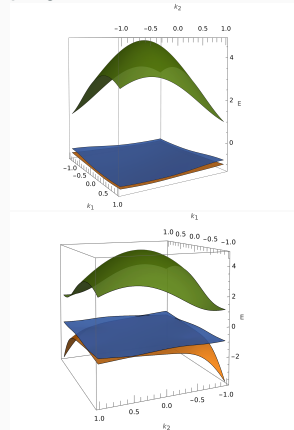- Apply k·p expansion to a 3-band Kagome lattice model.

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minimal k.p Approximated Hamiltonian:

$$H_{kp} = \begin{pmatrix} \frac{4}{5}(1 - \frac{\pi^2 k_1^2}{2}) & 2 - \frac{\pi^2(k_1^2+k_2^2)}{4} & \cdots \\ 2 - \frac{\pi^2(k_1^2+k_2^2)}{4} & \frac{4}{5}(1 - \frac{\pi^2 k_2^2}{2}) & \cdots \\ \cdots & \cdots & \frac{4}{5}(1 - \frac{\pi^2(k_1+k_2)^2}{2}) \end{pmatrix}$$

lattice model vs. polynomial model:

k.p Expansion of the Square Lattice Model

Method:

- On square lattice, for 3 band model the euler classs is always an even number, the lowest value can be 2.

- Perform a k·p expansion around the Γ point to get a low-energy effective Hamiltonian.

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Model Hamiltonian:

$$H = \begin{pmatrix} -\frac{9}{4} + s_1^2 & s_1 s_2 & (1 - c_1 - c_2)s_1 \\ s_1 s_2 & \frac{7}{4} + s_2^2 & (1 - c_1 - c_2)s_2 \\ (1 - c_1 - c_2)s_1 & (1 - c_1 - c_2)s_2 & -2 + (c_1 + c_2 - 1)^2 \end{pmatrix}$$

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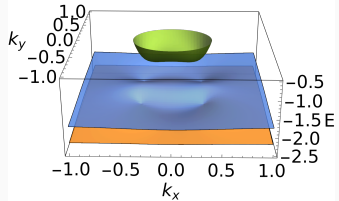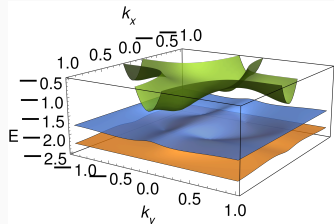
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$$H_{kp} = \begin{pmatrix} -\frac{9}{4} + k_1^2 \pi^2 & k_1 k_2 \pi^2 & \frac{k_1 \pi}{2}(-2 + \pi^2(k_1^2 + k_2^2)) \\ k_1 k_2 \pi^2 & \frac{7}{4} + k_2^2 \pi^2 & \frac{k_2 \pi}{2}(-2 + \pi^2(k_1^2 + k_2^2)) \\ \frac{k_1 \pi}{2}(\dots) & \frac{k_2 \pi}{2}(\dots) & -2 + \frac{1}{4}(-2 + \pi^2(k_1^2 + k_2^2))^2 \end{pmatrix}$$

lattice model vs. polynomial model:

Conclusion: k·p Theory

- Result for the kagome case with Euler class 1: the polynomial theory is at least second order in the momentum components
- Result for square lattice with Euler class 2: the polynomial theory is at least quartic in kx, ky
- This provides a starting point for an effective field theory of Euler class topological insulators.
- **Outlook:** We aim to investigate a gauge theory for lattice deformation using this minimal k·p theory.

Results II: Bulk-Boundary Correspondence in a 4-Band Model

The 4-Band Model: Bulk Properties

- square lattice again, but now with Euler class one! which was not allowed in the 3-band case.
- Euler class 1 in each two-band subspaces means 2 nodal points connecting the bands

Model Hamiltonian $H(k_1, k_2)$:

$$\begin{pmatrix} s_2 & s_1 & -1/4 & 1 - c_1 - c_2 \\ s_1 & -s_2 & -1 + c_1 + c_2 & 1/4 \\ -1/4 & -1 + c_1 + c_2 & s_2 & s_1 \\ 1 - c_1 - c_2 & 1/4 & s_1 & -s_2 \end{pmatrix}$$

where $c_i = \cos(k_i \pi)$, $s_i = \sin(k_i \pi)$

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Bulk Analysis:

- A Wilson loop calculation confirms the model has a non-trivial bulk invariant.
- The 3D bulk band structure shows the system is fully gapped.

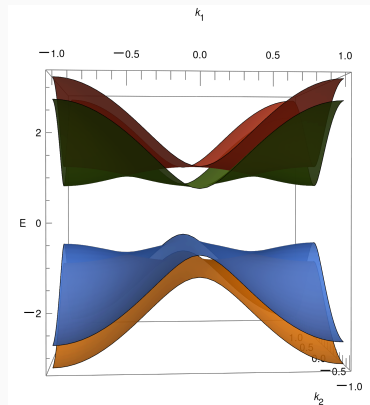

Figure: 3D Bulk Spectrum.

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Ribbon Geometry: Square Lattice

Concept:

- A ribbon geometry is a finite-width strip of the lattice, infinite (or periodic) along one direction.
- Allows study of **edge states** while keeping a bulk region.
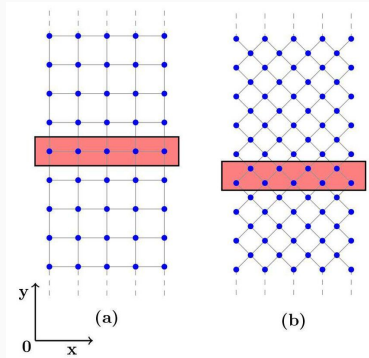- Edge states appear localized at the boundaries, while bulk states remain in the interior.

Figure 3: Ribbon geometry with edges highlighted. Bulk in gray, edges in red.

Method:

- Simulation on a ribbon geometry (Finite in one direction and infinite in another).

Ribbon Spectrum 1D Edge States

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Interpretation:

- These are the 1D edge states predicted by the non-trivial bulk. They are the first signature of the bulk-boundary correspondence.(Helical edge states)
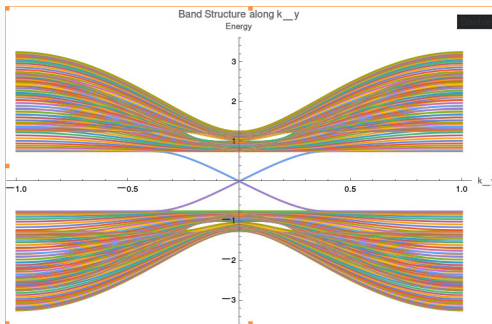

Figure: Ribbon spectrum showing protected 1D edge states crossing the bulk gap.

Ribbon Simulation edge States

Method:

- We simulate a 2D
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Ribbon Simulation edge States

Method:

- We simulate a 2D ribbon of the 4-band model, finite in x direction , infinite and periodic in y direction

Result:

- The low-energy states show a remarkable concentration of their probability density $|\psi|^2$ at the edges.
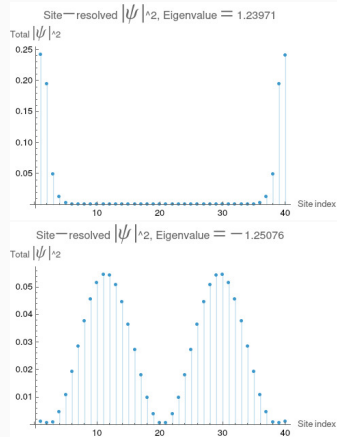

Figure: The resolved site $|\psi|^2$ shows a clear localization at the edges of the system(with 40 sites)

Conclusion: edge states in 4-Band Model

- The 4-band model provides demonstration of the bulk-boundary correspondence.
- We are investigating a framework to explain and generalize these results.
- **Outlook:**
- Edge states can potentially be harnessed as robust channels for **quantum information processing**.

Outlook

A. Bouhon, R.-J. Slager, T. Bzdušek.
Wilson loop approach to fragile topology of split elementary band representations.
Physical Review B 100, 125149 (2019).

D. Vanderbilt.
Berry Phases in Electronic Structure Theory.
Cambridge University Press (2018).

Thank You!

Questions and Discussion