Case Study # 3: Structural Analysis: Perforated Plate in Tension

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1 Problem Description

This case study involves a linear-elastic, steady-state stress analysis on a square plate with a circular hole at its center. The plate dimensions are: side length $x = 4$m and radius $R = 0.5$m. It is loaded with a uniform traction of $\sigma = 10$kPa over its left and right faces as can be seen in Figure 1.

A mesh sensitivity study is performd for a plate with side length $x = 4$m, and an effect of the plate length study is performed for plates of length $x = 3$m, 4m, 5m, and 100m. The stress normal to the vertical plane of symmetry is calculated for each case, and the results are compared to the analytical solution:

$$(\sigma_{xx})_{x=0} \begin{cases} \sigma(1 + \frac{R^2}{2y^2} + \frac{3R^4}{2y^4}) & \text{for} \quad |y| \geq R \\ 0 & \text{for} \quad |y| < R \end{cases}. \tag{1}$$

Similarly, the stress normal to the horizontal plane of symmetry can also be computed and compared to the analytical solution:

$$(\sigma_{yy})_{y=0} \begin{cases} \sigma(\frac{R^2}{2x^2} - \frac{3R^4}{2x^4}) & \text{for} \quad |x| \geq R \\ 0 & \text{for} \quad |x| < R \end{cases}. \tag{2}$$

A Python script was created to automatically generate the configuration files, calculate the resulting steady-state stress through the plate using OpenFOAM [1], and plot the results for both the sensitivity and plate length studies. This script is included in the Appendix.

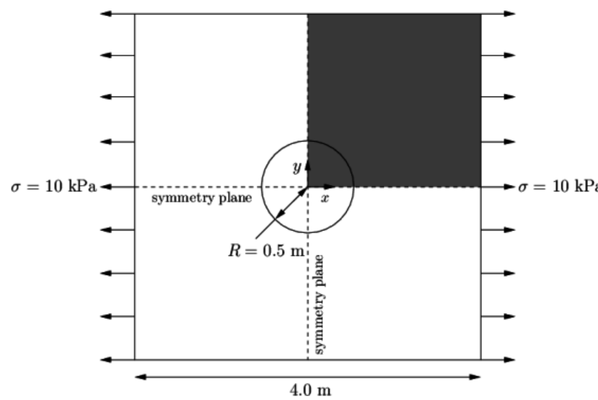

Fig. 1. Geometry of the plate with a hole [2]

2 Numerical Solution Approach

Two symmetry planes can be identified for this geometry and therefore the solution domain need only cover a quarter of the geometry, shown by the shaded area in Figure 1. The quarter plate is then broken into five blocks of varying sizes, as can be seen in Figure 2. These blocks have a multiple of the characteristic number of points, n, in the x_i and y_i directions. Blocks 0 and 1 consist of n by n points, block 2 consists of $2n$ by n points, block 3 consists of $2n$ by $2n$ points, and block 4 consists of n by $2n$ points. The mesh is generated with OpenFOAM's 'blockMesh' command, and the resulting mesh for $n = 10$ can be seen in Figure 3.

The mesh sensitivity study looks at meshes resulting from $n = 10, 100$, and 1000 on a plate width of $x = 4$m. The effect of plate length study looks at plate lengths of $x = 3$m, 4m, 5m, and 100m with a mesh of $n = 10$. Once the meshes have been generated, OpenFOAM's 'solidDisplacementFoam' solver runs the simulation, and σ_{xx} is calculated and sampled by the OpenFOAM commands 'foamCalc components sigma' and 'sample'. The 'solidDisplacementFoam' solver is a transient segregated finite-volume solver for linear-elastic, small-strain deformation of a solid body, and is well suited to solve this problem.

3 Results Discussion

The computational result for normal stress along the vertical and horizontal symmetries can be compared to the an-

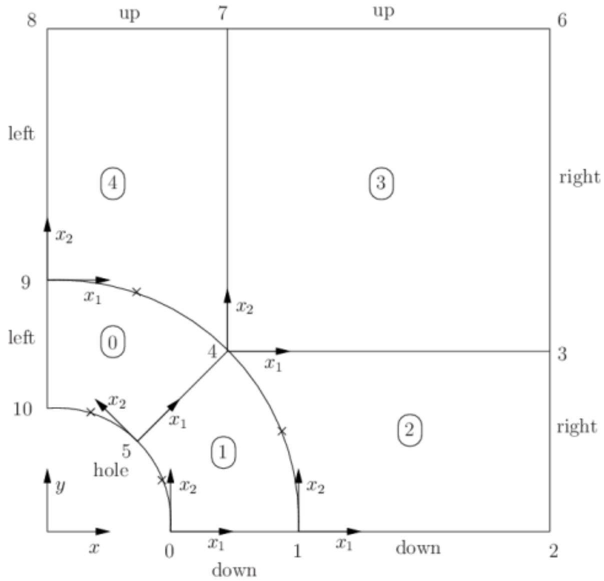

Fig. 2. Block structure of the mesh for the plate with a hole [2]

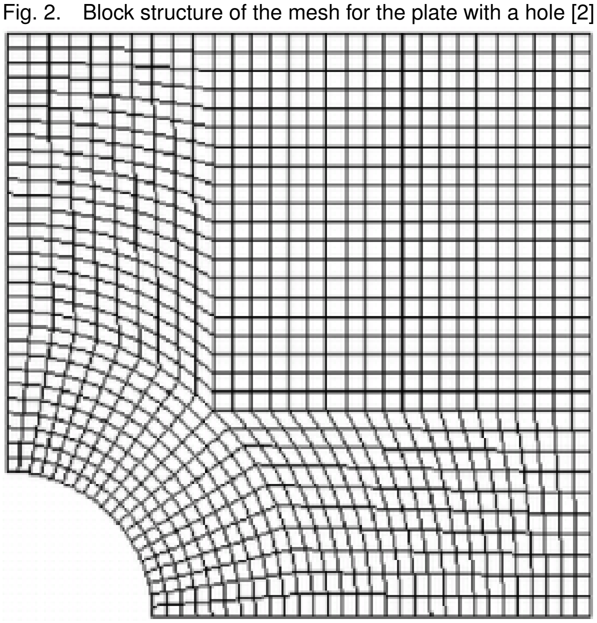

Fig. 3. Mesh of the hole in a plate problem with $n = 10$ [2]

alyticals result above, Equations 1 and 2. The Root Mean Square error,

$$RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}[\sigma_i - \sigma_i^*]^2}, \tag{3}$$

and the Normalized Root Mean Square error,

$$NRMS = \frac{RMSE}{max(\sigma^*) - min(\sigma^*)}, \tag{4}$$

can be calculated. Here σ_i is the computational result for the the stress for each point along the boundary, σ_i^* is the analytical solution, and N is the number of points along the boundary. The *NRMS* for each case is expressed as a percentage, where lower values indicate a result closer to the analytic solution. For the remainder of this paper $NRMS_{xx}$ will note the *NRMS* error along the vertical plane of symmetry, while $NRMS_{yy}$ will note the *NRMS* error along the horizontal plane of symmetry.

3.1 Results for the base case

The base case described in the OpenFOAM tutorial involves a plate of width $x = 4$ with a mesh resulting from $n = 10$ [2]. The *NRMS* from the analytical solution for this case is approximately 9.41% for the vertical symmetry, and 18.36% for the horizontal symmetry. The results for the normal stress along the symmetries and their deviation from the analytical solution for this case can be seen in Figure 4.

A quick look at the error plots suggest that the stress along the vertical symmetry is overestimated, while the stress along the horizontal symmetry is underestimated. These results suggest that a square plate with a relatively low number of cells can fairly accurately reproduce the analytical solution of an infinitely wide plate for the vertical symmetry, but fails to do well in estimating the stress along the horizontal symmetry. Increasing the mesh and the length of the plate could lead to results that closer approximate this analytic solution.

3.2 Mesh sensitivity

The mesh sensitivity study looks at meshes resulting from $n = 10, 100$, and 1000 on a plate width of $x = 4$m. Little change is seen *NRMS* for either symmetry, suggesting only a weak dependence on the mesh size. See the plots in Figure 5 for complete results for this case. Looking at a plate of width $x = 100$m, however, shows a dramatic decrease in error in the vertical symmetry for larger mesh sizes. This is due to the sparse nature of the meshes generated from $n = 10$. Having only 30 grid points over a 100m plate is not significant, and results in a spacing of about 1.65m, worse than using only 2 grid points over the initial 4m case. The 300 grid points resulting from a $n = 100$ leads to a spacing of about .16m and leads to much better results—a $NRMS_{yy}$ of only 5.96% compared to 31.86%.

Based on these results, meshes resulting from $n = 100$ are the minimum recommended meshes for this type of study. Meshes resulting from $n = 1000$ will be used for the effect of plate length study. Full results for these cases can be seen in Table 1.

3.3 Effect of the plate length

The effect of plate length study looks at plate lengths of $x = 3$m, 4m, 5m, and 100m with a mesh of $n = 1000$. The *NRMS* for both the vertical and horizontal symmetries decrease rapidly with increases in the plate length, and both reach values of around 4% for the case of $x = 100$m. The $NRMS_{yy}$ is more dramatically changed by the increase in plate length, which makes sense as it is along the direction being directly affected by varying the plate length. Full results for these cases can be seen in Table 2.

$x(m)$	n	$NRMS_{xx}$	$NRMS_{yy}$
4	10	9.41%	18.36%
4	100	9.72%	18.77%
4	1000	9.01%	17.43%
100	10	3.10%	31.86%
100	100	4.24%	5.96%
100	1000	4.06%	4.19%

Table 1. *NRMS* results from the numerical simulations for the mesh sensitivity study

$x(m)$	n	$NRMS_{xx}$	$NRMS_{yy}$
3	1000	14.58%	28.29%
4	1000	9.01%	17.43%
5	1000	6.04%	8.82%
100	1000	4.06%	4.19%

Table 2. *NRMS* results from the numerical simulations for the effect of plate length study

4 Conclusion

OpenFOAM was used to investigate the mesh sensitivity and the effects of plate length between the analytical solution for an infinitely wide plate and the computational results for stress on a square plate with a circular hole at its center. Increasing the mesh density to create a grid spacing of less than .2m led to acceptable values for the *NRMS* along both symmetries, while larger grid spacing led to generally poor results. The analytical solutions for the stress along the vertical and horizontal symmetries for the infinitely wide plate compared quite well to computational results for a plate of width $x = 100$m with a mesh resulting from $n = 1000$, leading to *NRMS* values of around 4%. A plate of $x = 5$m with the same mesh also compared quite well, leading to a $NRMS_{xx}$ of around 6% and a $NRMS_{yy}$ of around 9%.

References

[1] Jasak, H., Jemcov, A., and Tukovic, Z., 2007. "Openfoam: A c++ library for complex physics simulations". In International workshop on coupled methods in numerical dynamics, Vol. 1000, pp. 1–20.

[2] OpenFOAM_Foundation, 2014. Stress analysis of a plate with a hole. http://www.openfoam.org/docs/user/plateHole.php.

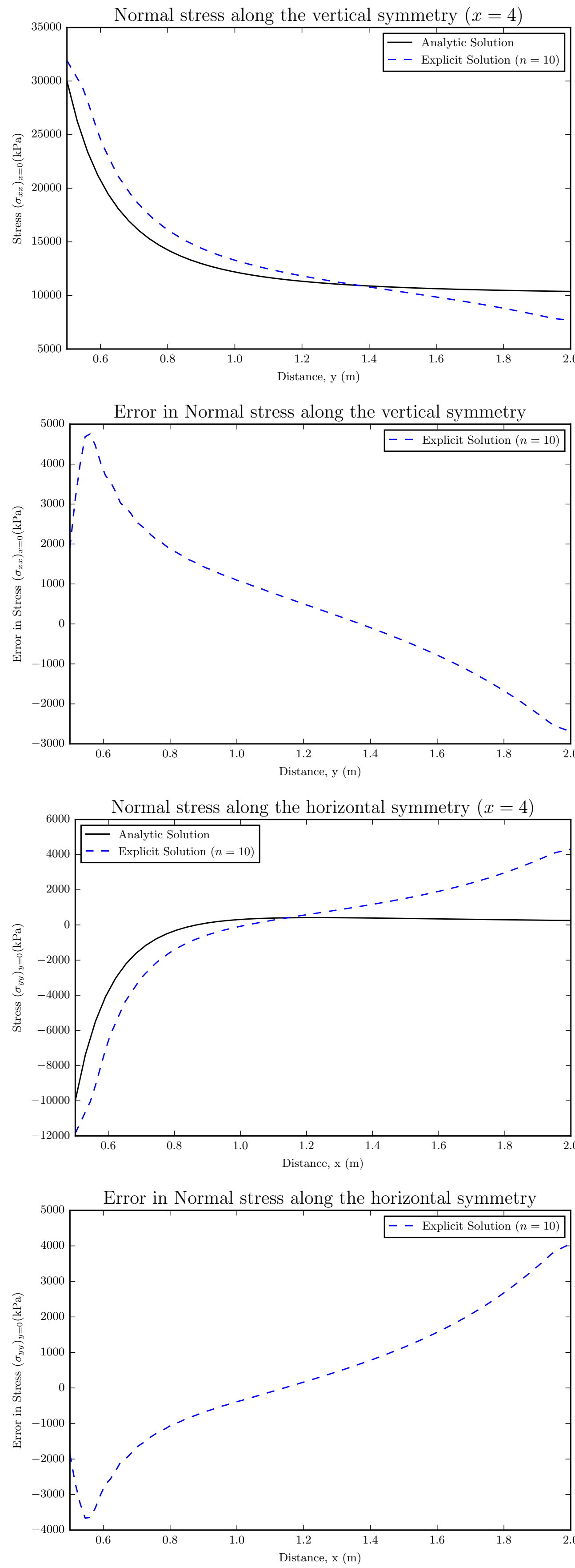

Fig. 4. Results of base case

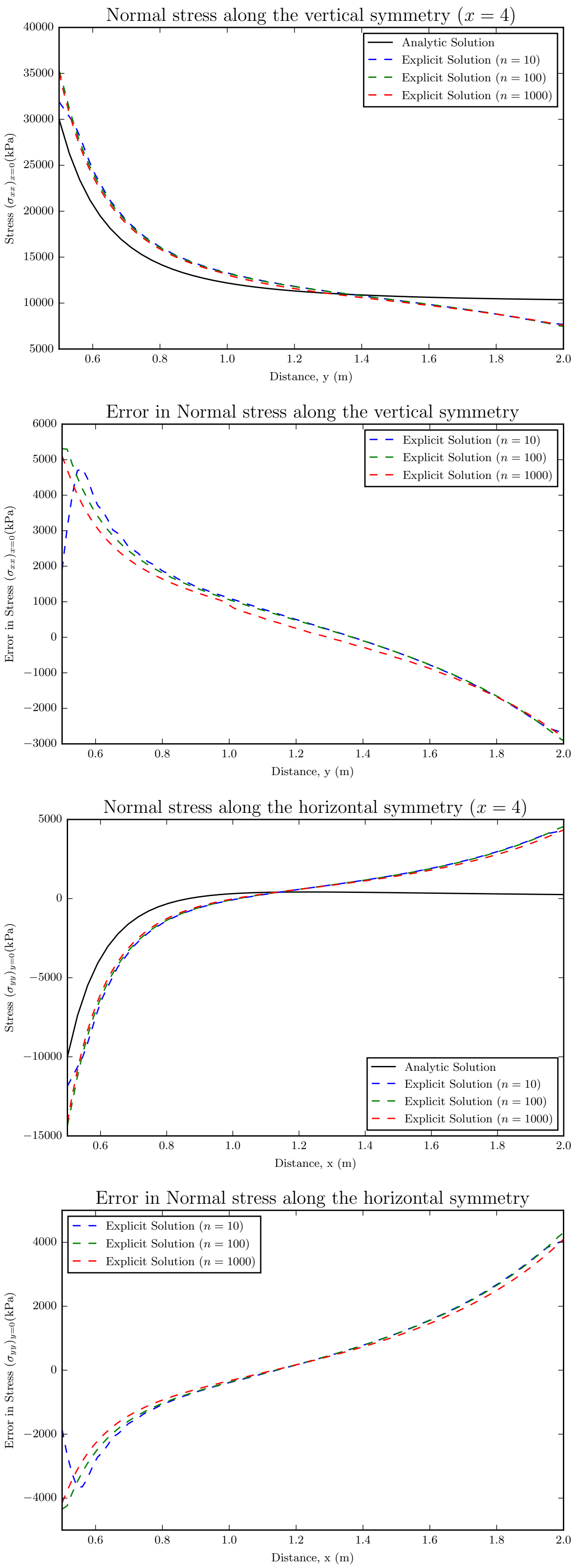

Fig. 5. Results of mesh sensitivity study

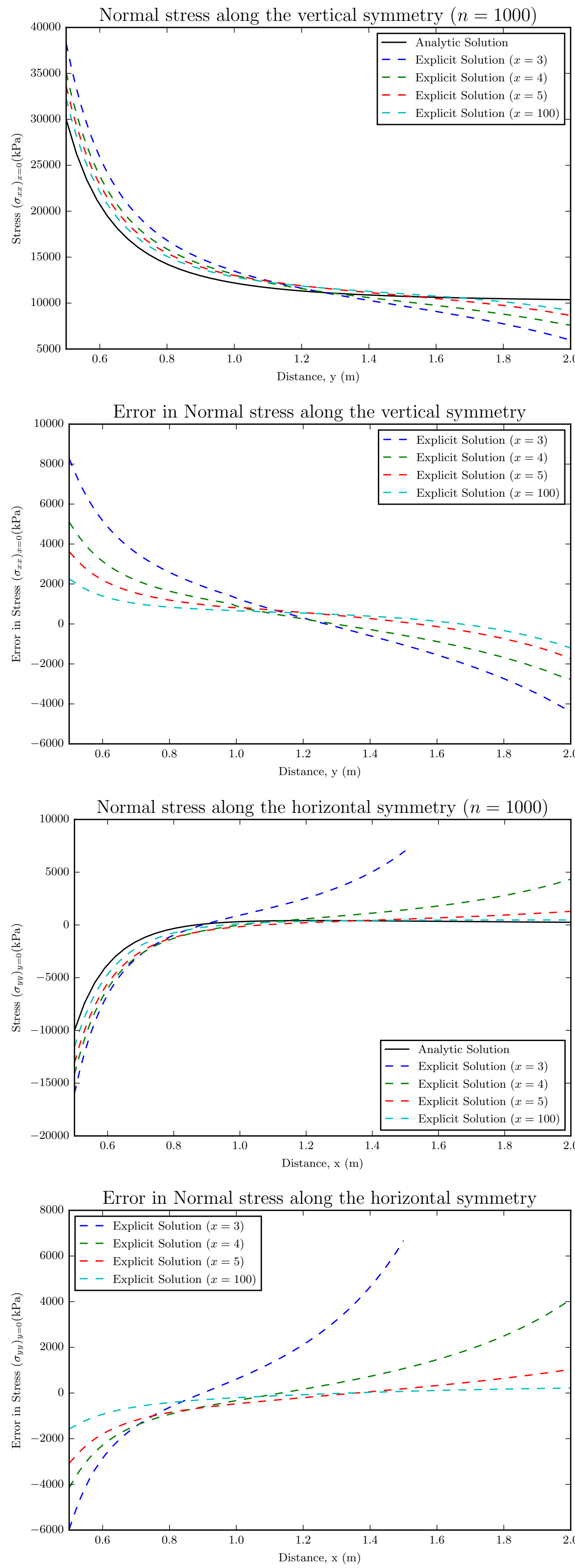

Fig. 6. Results of plate length study

Appendix A: Python Code

```python
import subprocess
import numpy as np
import matplotlib.pyplot as plt
import os


# Configure figures for production
WIDTH = 495.0  # width of one column
FACTOR = 1.0   # the fraction of the width the figure should occupy
fig_width_pt  = WIDTH * FACTOR

inches_per_pt = 1.0 / 72.27
golden_ratio  = (np.sqrt(5) - 1.0) / 2.0      # because it looks good
fig_width_in  = fig_width_pt * inches_per_pt  # figure width in inches
fig_height_in = fig_width_in * golden_ratio   # figure height in inches
fig_dims      = [fig_width_in, fig_height_in] # fig dims as a list


def subprocess_cmd(command):
    process = subprocess.Popen(command, stdout=subprocess.PIPE, shell=True)
    proc_stdout = process.communicate()[0].strip()
    # print proc_stdout


def generate_folders(widths, meshes):
    for width, mesh in zip(widths, meshes):
        run = "Run" + str(width) + '-' + str(mesh)
        if not os.path.exists(run):
            command = "cp -rf base/ " + run + "/; "
            subprocess_cmd(command)

    print ('Folders generated.')


def create_config_file(width, mesh):
    config = '''
    /*--------------------------------*- C++ -*----------------------------------*\
    | =========                 |                                                 |
    | \\      /  F ield         | OpenFOAM: The Open Source CFD Toolbox           |
    |  \\    /   O peration     | Version:  2.3.0                                 |
    |   \\  /    A nd           | Web:      www.OpenFOAM.org                      |
    |    \\/     M anipulation  |                                                 |
    \*---------------------------------------------------------------------------*/
    FoamFile
    {
        version     2.0;
        format      ascii;
        class       dictionary;
        object      blockMeshDict;
    }
    // * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * //

    convertToMeters 1;


    vertices
    (
        (0.5 0 0)
        (1 0 0)
        (''' + str(width) + ''' 0 0)
        (''' + str(width) + ''' 0.707107 0)
        (0.707107 0.707107 0)
        (0.353553 0.353553 0)
        (''' + str(width) + ''' 2 0)
        (0.707107 2 0)
        (0 2 0)
        (0 1 0)
```

```
        (0 0.5 0)
        (0.5 0 0.5)
        (1 0 0.5)
        (''' + str(width) + ''' 0 0.5)
        (''' + str(width) + ''' 0.707107 0.5)
        (0.707107 0.707107 0.5)
        (0.353553 0.353553 0.5)
        (''' + str(width) + ''' 2 0.5)
        (0.707107 2 0.5)
        (0 2 0.5)
        (0 1 0.5)
        (0 0.5 0.5)
    );

    blocks
    (
        hex (5 4 9 10 16 15 20 21) (''' + str(mesh) + ' ' + str(mesh) + ''' 1) simpleGrading (1 1 1)
        hex (0 1 4 5 11 12 15 16) (''' + str(mesh) + ' ' + str(mesh) + ''' 1) simpleGrading (1 1 1)
        hex (1 2 3 4 12 13 14 15) (''' + str(mesh * 2) + ' ' + str(mesh) + ''' 1) simpleGrading (1 1 1)
        hex (4 3 6 7 15 14 17 18) (''' + str(mesh * 2) + ' ' + str(mesh * 2) + ''' 1) simpleGrading (1 1 1)
        hex (9 4 7 8 20 15 18 19) (''' + str(mesh) + ' ' + str(mesh * 2) + ''' 1) simpleGrading (1 1 1)
    );

    edges
    (
        arc 0 5 (0.469846 0.17101 0)
        arc 5 10 (0.17101 0.469846 0)
        arc 1 4 (0.939693 0.34202 0)
        arc 4 9 (0.34202 0.939693 0)
        arc 11 16 (0.469846 0.17101 0.5)
        arc 16 21 (0.17101 0.469846 0.5)
        arc 12 15 (0.939693 0.34202 0.5)
        arc 15 20 (0.34202 0.939693 0.5)
    );

    boundary
    (
        left
        {
            type symmetryPlane;
            faces
            (
                (8 9 20 19)
                (9 10 21 20)
            );
        }
        right
        {
            type patch;
            faces
            (
                (2 3 14 13)
                (3 6 17 14)
            );
        }
        down
        {
            type symmetryPlane;
            faces
            (
                (0 1 12 11)
                (1 2 13 12)
            );
        }
        up
        {
            type patch;
            faces
            (
```

```
                (7 8 19 18)
                (6 7 18 17)
            );
        }
        hole
        {
            type patch;
            faces
            (
                (10 5 16 21)
                (5 0 11 16)
            );
        }
        frontAndBack
        {
            type empty;
            faces
            (
                (10 9 4 5)
                (5 4 1 0)
                (1 4 3 2)
                (4 7 6 3)
                (4 9 8 7)
                (21 16 15 20)
                (16 11 12 15)
                (12 13 14 15)
                (15 14 17 18)
                (15 18 19 20)
            );
        }
    );

    mergePatchPairs
    (
    );

    // ************************************************************************* //

    '''

    return config


def update_dimensions(widths, meshes):
    for width, mesh in zip(widths, meshes):
        run = "Run" + str(width) + '-' + str(mesh)
        path = run + '/constant/polyMesh/blockMeshDict'
        with open(path, 'w') as config_file:
            config_file.write(create_config_file(width, mesh))

    print ('Config generated.')


def run_simulations(widths, meshes):
    for width, mesh in zip(widths, meshes):
        run = "Run" + str(width) + '-' + str(mesh)
        if not os.path.exists(run + '/100/'):
            print(run + ' running now.')
            command = "hdiutil attach -quiet -mountpoint $HOME/OpenFOAM OpenFOAM.sparsebundle; "
            command += "sleep 1; "
            command += "source $HOME/OpenFOAM/OpenFOAM-2.3.0/etc/bashrc; "
            command += "cd " + run + "; "
            command += "blockMesh; "
            command += "solidDisplacementFoam > log; "
            command += "foamCalc components sigma; "
            command += "sample"
            subprocess_cmd(command)
        print(run + ' complete.')
```

```python
    print('Simulations complete.')


def sigma_xx(x):
    return 1E4*(1+(0.125/(x**2))+(0.09375/(x**4)))


def sigma_yy(x):
    return 1E4*((0.125/(x**2))-(0.09375/(x**4)))


def plot_xx(widths, meshes):
    # Format plot
    plt.figure(figsize=fig_dims)
    plt.xlabel('Distance, y (m)')
    plt.ylabel('Stress ($\sigma_{xx}$)$_{x=0}$(kPa)')
    title = 'Normal stress along the vertical symmetry'
    x = np.linspace(0.5, 2)
    sigmaxx = sigma_xx(x)

    plt.plot(x, sigmaxx, '-k', label='Analytic Solution')
    plt.xlim(0.5, 2)

    for width, mesh in zip(widths, meshes):
        path = "Run" + str(width) + '-' + str(mesh) + '/postProcessing/sets/100/leftPatch_sigmaxx_sigmayy.xy'
        data = np.loadtxt(path)

        if widths.count(widths[0]) == len(widths):
            label = 'Explicit Solution ($n=' + str(int(mesh)) + '$)'
        else:
            label = 'Explicit Solution ($x=' + str(int(2*width)) + '$)'
        plt.plot(data[:, 0], data[:, 1], '--', markersize=5, label=label)

    if widths.count(widths[0]) == len(widths):
        title += ' ($x=' + str(int(2*width)) + '$)'
    else:
        title += ' ($n=' + str(int(mesh)) + '$)'

    plt.title(title)
    plt.legend(loc='best')

    # Save plots
    save_name = 'result-x-' + str(widths) + str(meshes) + '.pdf'
    try:
        os.mkdir('figures')
    except Exception:
        pass

    plt.savefig('figures/' + save_name, bbox_inches='tight')
    plt.clf()


def plot_xx_err(widths, meshes):
    # Format plot
    plt.figure(figsize=fig_dims)
    plt.xlabel('Distance, y (m)')
    plt.ylabel('Error in Stress ($\sigma_{xx}$)$_{x=0}$(kPa)')
    plt.title('Error in Normal stress along the vertical symmetry')
    plt.xlim(0.5, 2)

    for width, mesh in zip(widths, meshes):
        path = "Run" + str(width) + '-' + str(mesh) + '/postProcessing/sets/100/leftPatch_sigmaxx_sigmayy.xy'
        data = np.loadtxt(path)

        if widths.count(widths[0]) == len(widths):
            label = 'Explicit Solution ($n=' + str(int(mesh)) + '$)'
        else:
            label = 'Explicit Solution ($x=' + str(int(2*width)) + '$)'

```

```
        x = data[:, 0]
        sigmaxx = sigma_xx(x)
        err = data[:, 1] - sigmaxx

        RMS = np.sqrt(np.mean(np.square(err)))/(max(sigmaxx) - min(sigmaxx))
        print('x err', width, mesh, '{0:.3e}'.format(RMS))

        plt.plot(x, err, '--', markersize=5, label=label)

    plt.legend(loc='best')

    # Save plots
    save_name = 'error-x-' + str(widths) + str(meshes) + '.pdf'
    try:
        os.mkdir('figures')
    except Exception:
        pass

    plt.savefig('figures/' + save_name, bbox_inches='tight')
    plt.clf()


def plot_yy(widths, meshes):
    # Format plot
    plt.figure(figsize=fig_dims)
    plt.xlabel('Distance, x (m)')
    plt.ylabel('Stress ($\sigma_{yy}$)$_{y=0}$(kPa)')
    title = 'Normal stress along the horizontal symmetry'
    y = np.linspace(0.5, 2)
    sigmayy = sigma_yy(y)

    plt.plot(y, sigmayy, '-k', label='Analytic Solution')
    plt.xlim(0.5, 2)

    for width, mesh in zip(widths, meshes):
        path = "Run" + str(width) + '-' + str(mesh) + '/postProcessing/sets/100/downPatch_sigmaxx_sigmayy.xy'
        data = np.loadtxt(path)

        if widths.count(widths[0]) == len(widths):
            label = 'Explicit Solution ($n=' + str(int(mesh)) + '$)'
        else:
            label = 'Explicit Solution ($x=' + str(int(2*width)) + '$)'
        plt.plot(data[:, 0], data[:, 2], '--', markersize=5, label=label)

    if widths.count(widths[0]) == len(widths):
        title += ' ($x=' + str(int(2*width)) + '$)'
    else:
        title += ' ($n=' + str(int(mesh)) + '$)'

    plt.title(title)
    plt.legend(loc='best')

    # Save plots
    save_name = 'result-y-' + str(widths) + str(meshes) + '.pdf'
    try:
        os.mkdir('figures')
    except Exception:
        pass

    plt.savefig('figures/' + save_name, bbox_inches='tight')
    plt.clf()


def plot_yy_err(widths, meshes):
    # Format plot
    plt.figure(figsize=fig_dims)
    plt.xlabel('Distance, x (m)')
    plt.ylabel('Error in Stress ($\sigma_{yy}$)$_{y=0}$(kPa)')
    plt.title('Error in Normal stress along the horizontal symmetry')
```

```python
    plt.xlim(0.5, 2)

    for width, mesh in zip(widths, meshes):
        path = "Run" + str(width) + '-' + str(mesh) + '/postProcessing/sets/100/downPatch_sigmaxx_sigmayy.xy'
        data = np.loadtxt(path)

        if widths.count(widths[0]) == len(widths):
            label = 'Explicit Solution ($n=' + str(int(mesh)) + '$)'
        else:
            label = 'Explicit Solution ($x=' + str(int(2*width)) + '$)'

        y = data[:, 0]
        sigmayy = sigma_yy(y)
        err = data[:, 2] - sigmayy

        RMS = np.sqrt(np.mean(np.square(err)))/(max(sigmayy) - min(sigmayy))
        print('y err', width, mesh, '{0:.3e}'.format(RMS))

        plt.plot(y, err, '--', markersize=5, label=label)

    plt.legend(loc='best')

    # Save plots
    save_name = 'error-y-' + str(widths) + str(meshes) + '.pdf'
    try:
        os.mkdir('figures')
    except Exception:
        pass

    plt.savefig('figures/' + save_name, bbox_inches='tight')
    plt.clf()


def generate_plots(widths, meshes):
    plot_xx(widths, meshes)
    plot_xx_err(widths, meshes)
    plot_yy(widths, meshes)
    plot_yy_err(widths, meshes)

    print('Plots generated.')


def main(widths, meshes):
    print('Running widths ' + str(widths) + ' with meshes ' + str(meshes) + '.')
    generate_folders(widths, meshes)
    update_dimensions(widths, meshes)
    run_simulations(widths, meshes)
    generate_plots(widths, meshes)
    print('Done!')

if __name__ == "__main__":
    # Base case
    widths = [ 2]
    meshes = [10]
    main(widths, meshes)

    # Increasing mesh resolution
    widths = [2,    2,    2, 50,   50,   50]
    meshes = [10, 100, 1000, 10, 100, 1000]
    main(widths, meshes)


    # Changing the plate size with better meshes
    widths = [1.5, 2, 2.5, 50]
    meshes = [1000 for _ in widths]
    main(widths, meshes)
```