Case Study # 1: 1D Transient Heat Diffusion

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1 Problem Description

The problem of 1D unsteady heat diffusion in a slab of unit length with a zero initial temperature and both ends maintained at a unit temperature can be described by:

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \text{ subject to } \begin{cases} T(x,0^-) = 0 & \text{for } 0 \le x \le 1 \\ T(0,t) = T(1,t) = 1 & \text{for } t > 0 \end{cases} \tag{1}$$

and has the well known analytical solution:

$$T^*(x,t) = 1 - \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} \sin[(2k-1)\pi x] \star \exp[-(2k-1)^2\pi^2 t]. \tag{2}$$

2 Solution Algorithms

The Taylor-series (TS) method can be used on this equation to derive a finite difference approximation to the PDE. Applying the definition of the derivative,

$$f'(x) \approx \frac{f(x+\varepsilon) - f(x)}{\varepsilon} \tag{3}$$

to Eqn. (1) yields:

$$\begin{aligned} \frac{\partial T}{\partial t} &= \frac{T(x,t+\Delta t) - T(x,t)}{\Delta t} \\ &= \frac{T_i^{k+1} - T_i^k}{\Delta t} \end{aligned} \tag{4}$$

From the definition of the Taylor series,

$$f(x+\varepsilon) = f(x) + \varepsilon f'(x) + \frac{\varepsilon^2}{2} f''(x) + \ldots \tag{5}$$

which, when applied to T_i^{k+1} and T_i^k gives:

$$T_{i+1} = T_i + \Delta x \frac{\partial T_i}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 T_i}{\partial x^2} + O(\Delta x^3) \tag{6}$$

and

$$T_{i-1} = T_i - \Delta x \frac{\partial T_i}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 T_i}{\partial x^2} - O(\Delta x^3) \tag{7}$$

Adding Eqn. (6) and Eqn. (7) yields:

$$T_{i+1} + T_{i-1} = 2T_i + \Delta x^2 \frac{\partial^2 T_i}{\partial x^2} + O(\Delta x^4) \tag{8}$$

we can also combine the approximation for the second order term from Eqn. (1) to find

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1} + 2T_i + T_{i-1}}{\Delta x^2} + O(\Delta x^4), \tag{9}$$

which we can combine withe the above equations to form

$$\frac{T_i^{k+1} - T_i^k}{\Delta t} \approx \frac{T_{i+1}^k - 2T_i^k + T_{i-1}^k}{\Delta x^2} \tag{10}$$

This result can be arranged to form both Forward-Time, Centered-Space (FTCS) explicit and implicit schemes. Rearranging, the explicit scheme is:

$$T_i^{k+1} = sT_{i+1}^k + (1-2s)T_i^k + sT_i^k \tag{11}$$

and the implicit scheme is

$$T_i^k = -sT_{i+1}^{k+1} + (1+2s)T_i^{k+1} - sT_{i-1}^k \tag{12}$$

where

$$s = \frac{\alpha \Delta t}{\Delta x^2} \tag{13}$$

and α is the thermal diffusivity of the material.

s	t	Implicit RMS	Explicit RMS
1/6	0.03	7.27E-4	9.25E-3
1/6	0.06	6.99E-5	1.19E-2
1/6	0.09	1.98E-4	1.31E-2
0.25	0.03	4.70E-4	1.38E-2
0.25	0.06	1.27E-4	1.90E-2
0.25	0.09	1.96E-4	2.02E-2
0.5	0.03	7.92E-4	3.66E-2
0.5	0.06	3.20E-4	4.71E-2
0.5	0.09	4.10E-4	4.63E-2
0.75	0.03	1.13E-3	8.13E-2
0.75	0.06	5.17E-4	8.83E-2
0.75	0.09	6.22E-4	7.38E-2

Table 1. RMS results from the numerical simulations compared to the analytic solution

3 Results

A Python script was used to obtain results for a 21 point mesh (N=21), and the Root Mean Square error,

$$\text{RMS} = \frac{1}{N}\sqrt{\sum_{i=1}^{N}[T_i^n - T^*(x_i,t_n)]^2} \tag{14}$$

was obtained for $s(=\Delta t/\Delta x^2)$ = 1/6, 0.25, 0.5, and 0.75, at t = 0.03, 0.06, and 0.09.

The RMS between the implicit and analytic solutions and the explicit and analytic solutions are shown in Table 3. The RMS tended to grow as a function of s, and shrink as a function of t. Additionally, the RMS for the explicit solution tended to be two orders of magnitude larger than the RMS for the implicit solution.

4 Discussions

Shankle chicken tail, fatback short ribs meatball pancetta ball tip sirloin short loin. Pork tongue pork belly pork loin beef ribs. Shank turkey pork belly pork loin ham hock ball tip leberkas meatloaf chuck ground round filet mignon kielbasa sirloin turducken tri-tip. Pancetta brisket sirloin beef ribs spare ribs, swine bacon ham hock. Ham kielbasa corned beef turkey turducken. Kevin biltong pork, tenderloin chuck pig ball tip filet mignon.

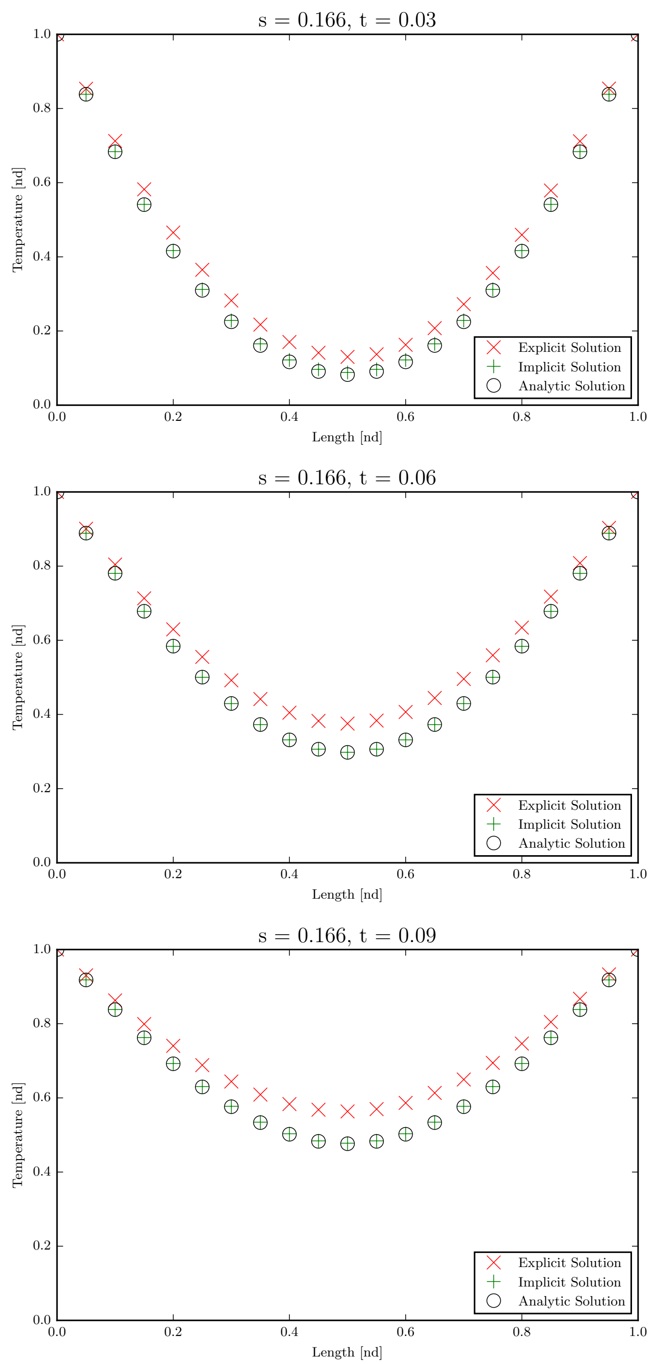

Fig. 1. Results for $s = 1/6$

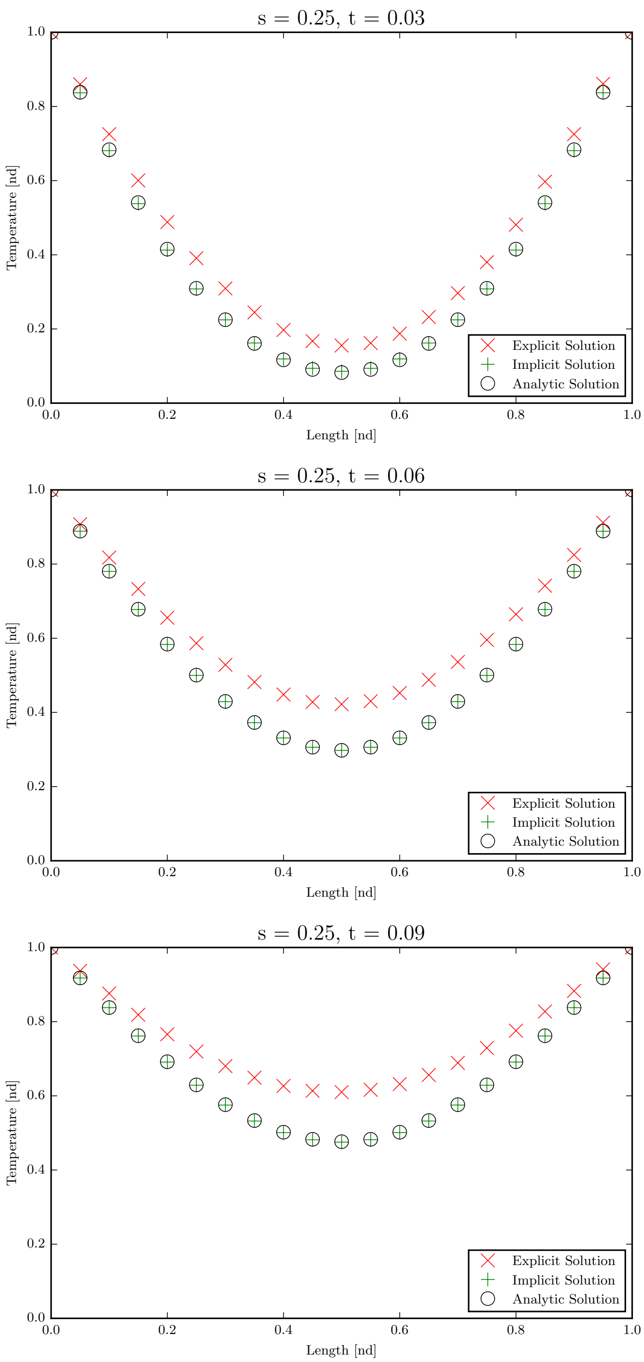

Fig. 2. Results for $s = 0.25$

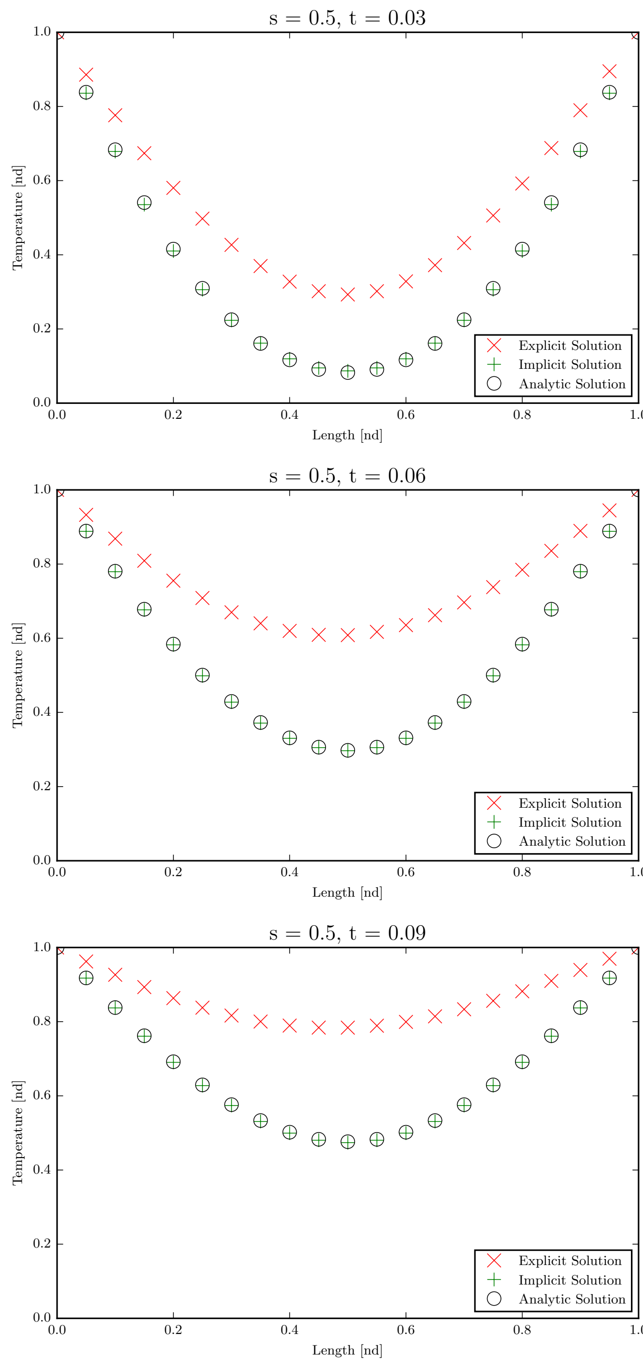

Fig. 3. Results for $s = 0.5$

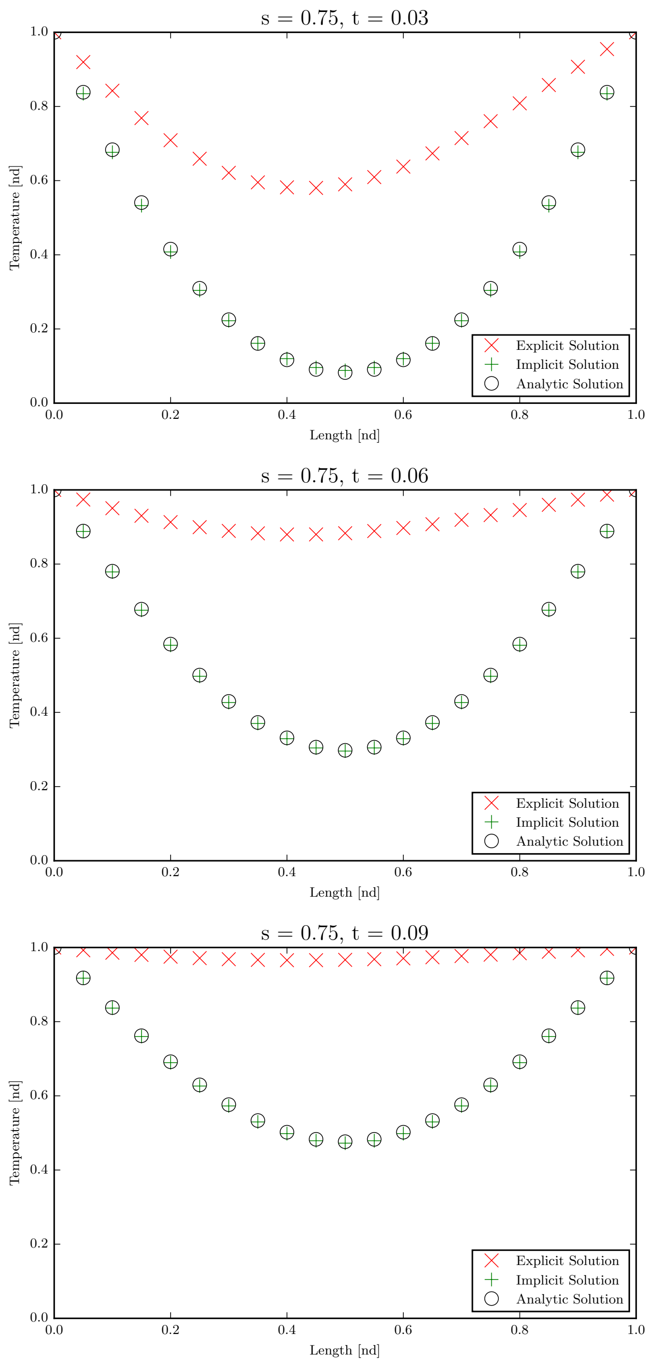

Fig. 4. Results for $s = 0.75$