Conversion of random variables to other distributions

- Programming languages only have in-built functions for specific probability distributions, eg. uniform or normal distribution
- Variable transformations can be used to convert random variables sampled from a certain distribution to those sampled from another distribution

$$X \sim U(0,1) \xrightarrow[\boxed{Y = f(X)}]{?} Y \sim \mathrm{Exp}(\lambda)$$

Given a random variable X with pdf $f(x)$ and cdf $F(x)$,
let $Y = g(X)$ and $g(y)$ denote the pdf of Y and $G(y)$

denote the cdf of Y. Let $r^{-1}(y)$ denote the inverse of $r(x)$. Then:

* If r^{-1} is an increasing function:

cdf: $G(y) = F[r^{-1}(y)]$

pdf: $g(y) = f(r^{-1}(y)) \cdot \frac{d}{dy}(r^{-1}(y))$

* If r^{-1} is a decreasing function

cdf: $G(y) = 1 - F[r^{-1}(y)]$

pdf: $g(y) = -f(r^{-1}(y)) \cdot \frac{d}{dy}(r^{-1}(y))$

Examples of pairs of inverse functions:

$*\quad r(x) = e^x \;;\quad r^{-1}(x) = \ln(x)$

$r^{-1}(r(x)) = \ln(e^x) = \boxed{x}$

$*\quad r(x) = \sin(x) \;;\quad r^{-1}(x) = \sin^{-1}(x)$

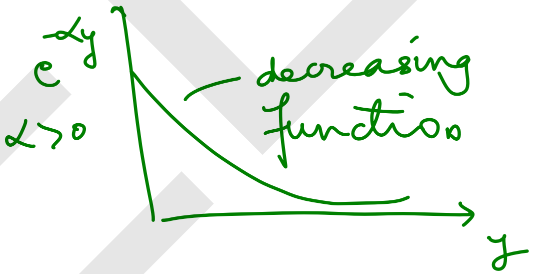
$r^{-1}(r(x)) = \sin^{-1}(\sin x) = \boxed{x}$

Example: Let $\boxed{X \sim U(0,1)}$. Find the distribution followed by $y = \frac{1}{\alpha}\ln\left(\frac{1}{x}\right)$, $\alpha > 0$.

pdf of X: $f(x) = \begin{cases} 1; & 0 \le x \le 1 \\ 0; & \text{otherwise} \end{cases}$

$$r(x) = \frac{1}{\alpha}\ln\left(\frac{1}{x}\right) = y \Rightarrow \ln\left(\frac{1}{x}\right) = \alpha y \Rightarrow \frac{1}{x} = e^{\alpha y}$$

$$\boxed{r^{-1}(y) = x = e^{-\alpha y}}$$



pdf: $g(y) = -f(r^{-1}(y)) \cdot \frac{d}{dy}(r^{-1}(y))$

$$= -f(e^{-\alpha y}) \cdot \frac{d}{dy}(e^{-\alpha y})$$

$$= -f(e^{-\alpha y}) \times -\alpha e^{-\alpha y}$$

$$= \boxed{\alpha e^{-\alpha y} f(e^{-\alpha y})}$$

$$f(x) = \begin{cases} 1 ; & 0 \le x \le 1 \\ 0 ; & \text{otherwise} \end{cases}$$

$$f(e^{-\alpha y}) = \begin{cases} 1 ; & \boxed{0 \leq e^{-\alpha y} \leq 1} \\ 0 ; & \text{otherwise} \end{cases} \longrightarrow \begin{aligned} \log(0) &\leq -\alpha y \leq \log(1) \\ -\infty &< -\alpha y \leq 0 \\ \infty &> \alpha y \geq 0 \\ \infty &> y \geq 0 \end{aligned}$$

$$g(y) = \alpha e^{-\alpha y} \times \begin{cases} 1 ; & 0 \leq y < \infty \\ 0 ; & \text{otherwise} \end{cases}$$

$$\boxed{g(y) = \begin{cases} \alpha e^{-\alpha y} ; & 0 \leq y < \infty \\ 0 ; & \text{otherwise} \end{cases}}$$

→ pdf of a random variable Y that is exponentially distributed

$Y \sim \text{Exp}(\alpha)$ ✓

✓ $X \sim U(0,1) \xrightarrow{\;y = \frac{1}{\lambda}\ln\left(\frac{1}{x}\right)\;} Y \sim \text{Exp}(\alpha)$ ✓