Диффуры.

I. Ур-ия с постоянными коэффициентами

№8.3.

Хар ур-ие: $\lambda^2+3\lambda+2=0$, $\lambda_1=-1$; $\lambda_2=-2$

$\Rightarrow y(x)=C_1e^{-x}+C_2e^{-2x}$

№8.7.

Хар. ур-ие: $\lambda^2 \cancel{+} -6\lambda+18=0$

~~$D=-36 \Rightarrow \lambda_{1,2}=+3\pm 3i \Rightarrow y(x)=C_1e^{+3x}\sin 3x+C_2e^{+3x}\cos 6x$~~

$D=-36 \Rightarrow \lambda_{1,2}=3\pm 3i \Rightarrow y(x)=e^{3x}(C_1\sin 3x+C_2\cos 3x)$

№8.12.

$\lambda^2-6\lambda+9=0 \Rightarrow \lambda_1=3$ – кратность 2.

$\Rightarrow y(x)=C_1e^{3x}+C_2xe^{3x}$

№8.23.

$\lambda^4-\lambda^3+2\lambda=0$

$$\left[\begin{array}{l}\lambda_1=0\\ \lambda^3-\lambda^2+2=0\end{array}\right.$$

$$\begin{array}{r|l}
\lambda^3-\lambda^2+2 & \lambda+1\\
\lambda^3+\lambda^2 & \lambda^2-2\lambda+2\\
\hline
-2\lambda^2+0 & \\
-2\lambda^2-2\lambda & \\
\hline
2\lambda+2 & \\
2\lambda+2 & \\
\hline
0 &
\end{array}$$

$\lambda_1=0$, $\lambda_2=-1$, $\lambda_{3,4}=1\pm i$

$\Rightarrow y(x)=C_1+C_2e^{-x}+e^{x}(C_3\cos x+C_4\sin x)$

№8.31.

$$\lambda^4+6\lambda^3+12\lambda^2+8\lambda=0$$

$$\lambda(\lambda+2)^3=0$$

$$\Rightarrow \underline{y(x)=C_1+e^{-2x}(C_2+C_3x+C_4x^2)}$$

№8.35.

$$\lambda^4+8\lambda^2+16=0$$

$$(\lambda^2+4)^2=0$$

$$(\lambda+2i)^2(\lambda-2i)^2=0$$

$\lambda_1=-2i$ – кр. 2

$\lambda_2=2i$ – кр. 2.

$$\Rightarrow \underline{y(x)=(C_1+C_2x)\sin 2x+(C_3+C_4x)\cos 2x}$$

№8.47.

$$\lambda^2+4=0 \Rightarrow \lambda_{1,2}=\pm 2i$$

$$y_o(x)=C_1\sin 2x+C_2\cos 2x$$

$$y_ч(x)=y_1(x)+y_2(x)$$

$$\cancel{y''+4y=4xe^{-2x}}$$

$$\cancel{y_1(x)=axe^{-bx}}$$

$$\cancel{(ae^{-bx}-abxe^{-bx})'+4axe^{-bx}=4xe^{-2x}}$$

$$\cancel{-abae^{-bx}-abxe^{-bx}+ab^2xe^{-bx}+4axe^{-bx}=4x^{-2x}}$$

$$\Rightarrow \begin{cases} \cancel{-a\theta a = \cdot a b} \\ \cancel{b=2} \\ \cancel{4a = ab^2} \end{cases} \Rightarrow a=1;\ b=2 \Rightarrow$$

$\cancel{\Rightarrow y_1(x) = -x e^{-2x}}$

$\cancel{y'' + 4y = -\sin 2x}$

$\cancel{y_2(x) =}$

$y'' + 4y = 4x e^{-2x}$

$y_1(x) = x^s (ax+b) e^{-2x}$

-2 – не явл. корнем хар. ур-ия $\Rightarrow y_1(x) = (ax+b)e^{-2x}$

$(ae^{-2x} - 2(ax+b)e^{-2x})' + 4(ax+b)e^{-2x} = 4xe^{-2x}$

$-2ae^{-2x} - 2ae^{-2x} + 4(ax+b)e^{-2x} + 4(ax+b)e^{-2x} = 4xe^{-2x}$

$\Rightarrow a = \frac{1}{2};\ -2 + 8b = 0 \Rightarrow b = \frac{1}{4}$

$\Rightarrow y_1(x) = \frac{1}{4}(2x+1)e^{-2x}$

y_2 $y'' + 4y = -\sin 2x$

$y_2(x) = x^s(A\sin 2x + B\cancel{\sin}^{\cos} 2x)$

$2i$ – ~~не~~ корень $\Rightarrow y_2(x) = (A\sin 2x + B\cos 2x)x$

$\cancel{-4A\sin 2x - 4B\cos 2x + 4A\sin 2x + 4B\cos 2x = -\sin 2x}$

$\Rightarrow (A\sin 2x + B\cos 2x + 2Ax\cancel{\sin}^{\cos} 2x - 2Bx\sin 2x)' + 4x(A\sin 2x + B\cos 2x) =$

$= 2A\cancel{\cos 2x} - 2B\sin 2x + 2A\cancel{\cos 2x} - 4Ax\sin 2x - 2B\sin 2x - 4Bx\cos 2x +$

$+ 4xA\sin 2x + 4xB\cos 2x = -\sin 2x \Rightarrow$ $\cancel{B=0\ A=B}$ $A=0$

$B = \frac{1}{4}$

$\Rightarrow y_2(x) = \frac{1}{4} x \cos 2x$

$\Rightarrow y(x) = y_0(x) + y_1(x) + y_2(x) =$

$= C_1 \sin 2x + C_2 \cos 2x + \frac{1}{4}(2x+1)e^{-2x} + \frac{1}{4} x \cos 2x$

№ 8.56.

$\lambda^2 + 4\lambda + 4 = 0$

$\lambda_1 = -2$ – кр. 2

$\Rightarrow y_0(x) = C_1 e^{-2x} + C_2 x e^{-2x}$

$y_ч(x) = A e^{-2x} \cdot x^2$

$\cancel{4Ae^{-2x} - 8Ae^{-2x} +}$

$(2Axe^{-2x} - 2Ax^2e^{-2x})' + 8Axe^{-2x} - 8Ax^2e^{-2x} + 4Ae^{-2x} \cdot x^2 =$

$\cancel{= 2Ae^{-2x} - 2Ae^{-2x} + 4Axe^{-2x} + 4Ae^{-2x} - 8Axe^{-2x} + 4Axe^{-2x} = 2e^{-2x}}$

$= 2Ae^{-2x} - \cancel{4Axe^{-2x}} - \cancel{4Axe^{-2x}} + \cancel{4Ax^2e^{-2x}} + \cancel{8Axe^{-2x}} - \cancel{8Ax^2e^{-2x}} +$

$+ \cancel{4Ae^{-2x}x^2} = 2e^{-2x} \Rightarrow A = 1.$

$\Rightarrow y_ч(x) = x^2 e^{-2x}$

$\cancel{\Rightarrow y(x) = C_1 + e^{-2x}(x^2 + C_2 x)}$

$\Rightarrow y(x) = e^{-2x}(C_1 + C_2 x + x^2)$

№8.107.

$y^{IV}-2y''+2y'=20\sin^2\frac{x}{2}=10-10\cos x$

$\lambda^3-2\lambda^2+2\lambda=0$

$\lambda_1=0$

$\lambda_{2,3}=1\pm i$ $\Rightarrow y_0(x)=C_1+(C_2 x\sin x+C_3\cos x)e^x$

$y'''-2y''+2y'=10$

$y_1(x)=ax+b \Rightarrow a=5 \Rightarrow y_1(x)=5x$

$y^{IV}-2y''+2y'=-10\cos x$

$y_2(x)=A\cos x+B\sin x$

$\cos x:\ -B+2A+2B=-10$

$\sin x:\ A+2B-2A=0$ $\Rightarrow$ $A=-4$, $B=-2$ $\Rightarrow y_2(x)=-4\cos x-2\sin x$

$\Rightarrow \underline{y(x)=C_1+e^x(C_2\sin x+C_3\cos x)+5x-4\cos x-2\sin x}$

№8.131.

$y^{IV}-y''-2y=12\sin 3x\cos 2x-6(e^{-2x}+\sin 5x)$

$\lambda^4-\lambda^2-2=0$

$(\lambda^2+1)(\lambda^2-2)=0 \Rightarrow \lambda_{1,2}=\pm i,\ \lambda_{3,4}=\pm\sqrt{2}$

$\Rightarrow y_0(x)=C_1\cos x+C_2\sin x+C_3e^{-x\sqrt{2}}+C_4e^{x\sqrt{2}}$

Преобр пр-ую часть: $12\sin 3x\cos 2x=12\cdot\frac{1}{2}(\sin 5x+\sin x)=$

$=6\sin 5x+6\sin x \Rightarrow$ пр. часть: $6\sin x-6e^{-2x}$

$y^{(4)} - y'' - 2y = 6\sin x$

$y_1(x) = ax\cos x$

$y_1' = a\cos x - ax\sin x$

$y_1'' = -a\sin x - a\sin x - ax\cos x$

$y_1''' = -2a\cos x - y_1'$

$y_1^{(4)} = 2a\sin x - y_1'''$

$\sin x: \ 2a + 4a = 6$

$\Rightarrow a = 1 \Rightarrow y_1 = x\cos x$

$\cos x: \ 2ax - 2ax = 0$

$y^{(4)} - y^{(3)} - 2y = -6e^{-2x}$

$y_2(x) = ae^{-2x}$

$16a - 4a - 2a = -6 \Rightarrow a = -\frac{3}{5} \Rightarrow y_2(x) = -\frac{3}{5}e^{-2x}$

$\Rightarrow \underline{y(x) = C_1\cos x + C_2\sin x + C_3e^{-x\sqrt{2}} + C_4e^{x\sqrt{2}} - \frac{3}{5}e^{-2x} + x\cos x}$

№8.153

$y'' - 3y' + 2y = \frac{1}{1+e^x}$

$\lambda^2 - 3\lambda + 2 = 0$

$\lambda_1 = 1; \lambda_2 = 2$

$y_0(x) = C_1e^x + C_2e^{2x}$

$$\begin{cases} C_1'e^x + C_2'e^{2x} = 0 \\ C_1'e^x + 2C_2'e^{2x} = \frac{1}{1+e^x} \end{cases} \Rightarrow C_1' = -C_2'e^x$$

$$C_2' = \frac{1}{e^{2x}+e^{3x}} \Rightarrow C_2 = \int \frac{dx}{e^{2x}+e^{3x}} = \left| \begin{matrix} e^x = t \\ dt = t\,dx \end{matrix} \right| =$$

$$= \int \frac{dt}{t^3+t^4} = \cancel{\int t^{-3}(1+t)^{-1}\,dt} \ominus$$

$$\cancel{\frac{At^2}{t^3} + \frac{1}{t+1}} \quad \frac{At^2+Bt+C}{t^3} + \frac{D}{t+1} = \frac{At^3+At^2+Bt^2+Bt+Ct+C+Dt^3}{t^3+t^4}$$

$$\Rightarrow \cancel{A = C = -B} \quad A = -D = -B = -C = -1$$

$$\frac{-t^2+t+1}{t^3} + \frac{1}{t+1}$$

$$C_1' = -\frac{1}{e^{2x}+e^x}$$

$$\Rightarrow C_1 = -\int \frac{dx}{e^x(e^x+1)} = \left| \begin{matrix} e^x = t \\ dt = t\,dx \end{matrix} \right| = -\int \frac{dt}{t^3+t^2} = \int \left(\frac{1}{t} - \frac{1}{t^2} - \frac{1}{t+1} \right) dt =$$

$$= -\ln(e^x+1) + x + e^{-x} + \tilde{C}_1$$

$$C_2 = \int \frac{dx}{e^{2x}(e^x+1)} = \left| \begin{matrix} e^x = t \\ dt = t\,dx \end{matrix} \right| = \int \frac{dt}{t^4+t^3} = \int \left(\frac{1}{t} - \frac{1}{t^2} + \frac{1}{t^3} - \frac{1}{t+1} \right) dt =$$

$$= x + e^{-x} - \frac{1}{2} e^{-2x} - \ln(1+e^x) + \tilde{C}_2$$

$$\Rightarrow y(x) = C_1 e^x + C_2 e^{2x} + (e^x + e^{2x})[x - \ln(1+e^x)] + e^x + \frac{1}{2}.$$

№593

$x^2 y'' - x y' + y = 8x^3$

$x = e^t, \ t = \ln|x|$

$\cancel{y'' - y' + y = y'' - y' e}$

$\cancel{y' = y'_t}$ $\quad y'_t = y'_x \cdot e^t \Rightarrow y' = y'_t \cdot e^{-t}$

$y''_{tt} = y' \cdot e^t + y''_{xx} e^{2t} = y'_t + y''_{xx} e^{2t} \Rightarrow y''_{xx} = (y''_{tt} - y'_t) e^{-2t}$

$e^{2t} \cdot (y''_{tt} - y'_t) e^{-2t} - e^t \cdot y'_t \cdot e^{-t} + y = 8e^{3t}$

$y''_{tt} - 2y'_t + y = 8e^{3t}$

$\lambda^2 - 2\lambda + 1 = 0$

$\lambda_1 = 1$, кр-ть 2.

$y_0(t) = C_1 e^t + C_2 t e^t$

$y_r(t) = a e^{3t}$

$9ae^{3t} - 6ae^{3t} + ae^{3t} = 8e^{3t} \Rightarrow a = 2$

$\Rightarrow y(t) = C_1 e^t + C_2 t e^t + \cancel{2a}\, 2e^{3t}$

$\cancel{y(x) = C_1 + C_2 \ln x}$

$y(x) = C_1 x + C_2 x \ln|x| + 2x^3$

$y(x) = x(C_1 + C_2 \ln|x|) + 2x^3$

№598.

$x^2y'' - 2y = \sin \ln x \Rightarrow x>0$

$x = e^t \Rightarrow t = \ln x$

$y'_x = y'_t \cdot e^{-t}; \; y''_{xx} = e^{-2t}(y''_{tt} - y'_t)$

$\Rightarrow y''_{tt} - y'_t - 2y = \sin t$

$\lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda_1 = -1; \lambda_2 = 2 \Rightarrow y_0(t) = C_1 e^{-t} + C_2 e^{2t}$

$y_1(t) = a\cos t + b\sin t$

$\cos t: -a - b - 2a = 0$

$\sin t: -b + a - 2b = 1$

$\Rightarrow y_1(t) = 0{,}1\cos t - 0{,}3\sin t$

$\Rightarrow y(x) = C_1 x^{-1} + C_2 x^2 + 0{,}1\cos \ln x - 0{,}3\sin \ln x$

№613.

$y_1 = x^2 e^x$

$\lambda = 1$, кр-ть 3 $\Rightarrow (\lambda - 1)^3 = 0 \Rightarrow y''' - 3y'' + 3y' - y = 0$

№615.

$y_1 = x\sin x$

$\lambda = \pm i$, кр-ть 2 $\Rightarrow (\lambda^2 + 1)^2 = 0 \Rightarrow y^{IV} + 2y'' + y = 0$

№617

$y_1 = xe^x, \; y_2 = e^{-x}$

$\lambda_1 = 1$, кр-ть 2 $\lambda_2 = -1$, кр-ть 1 $\Rightarrow (\lambda - 1)^2(\lambda + 1) = 0 \Rightarrow y''' - y'' - y' + y = 0$

№T1.

$y'' - ay' + 2y = e^x \cos x$

$\lambda^2 - a\lambda + 2 = 0 \Rightarrow \lambda_{1,2} = \frac{a}{2} \pm \frac{\sqrt{a^2-8}}{2}$

1. $a^2 > 8 \Rightarrow y_0 = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$

2. $a^2 = 8 \Rightarrow y_0 = e^{\sqrt{2}}(C_1 x + C_2)$

3. $a^2 < 8 \Rightarrow y_0 = e^{\frac{a}{2}x}\left(C_1 \cos \frac{\sqrt{8-a^2}}{2} + C_2 \sin \frac{\sqrt{8-a^2}}{2}\right)$

$a = 2$: $y_r(x) = x e^x (b\cos x + d\sin x)$, $a \neq 2$: $y_r(x) = e^x(b\cos x + d\sin x)$

$y(x) = y_0(x) + y_r(x)$

II. Линейные системы с постоянными коэффициентами.

№11.1

$$\begin{cases} \dot{x} = -5x - 6y \\ \dot{y} = 8x + 9y \end{cases}$$

$$\begin{vmatrix} -5-\lambda & -6 \\ 8 & 9-\lambda \end{vmatrix} = (\lambda+5)(\lambda-9) + 48 = \lambda^2 - 4\lambda - 45 + 48 = 0$$

$\Rightarrow$ с.з.: $\lambda_1 = 1$; $\lambda_2 = 3$

$\lambda_1 = 1$: $\left(\begin{array}{cc|c} -6 & -6 & 0 \\ 8 & 8 & 0 \end{array}\right) \sim (1 \quad 1 \mid 0) \Rightarrow \overline{e}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2 = 3$:

$$\left(\begin{array}{cc|c} -8 & -6 & 0 \\ 8 & 6 & 0 \end{array}\right) \sim (4 \quad 3 \mid 0) \Rightarrow \bar{e}_2 = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t} + C_2 \begin{pmatrix} 3 \\ -4 \end{pmatrix} e^{3t}$$

№11.5.

$$\begin{cases} \dot{x} = -5x - 4y \\ \dot{y} = 10x + 7y \end{cases}$$

$$\begin{vmatrix} -5-\lambda & -4 \\ 10 & 7-\lambda \end{vmatrix} = (\lambda+5)(\lambda-7) + 40 = \lambda^2 - 2\lambda + 5 = 0 \Rightarrow$$

$$\Rightarrow \cancel{\lambda_1 = 5,\ \lambda_2 = -3} \quad \lambda_{1,2} = 1 \pm 2i$$

$\cancel{\lambda_2 = 5:}$

$\lambda_1 = 1+2i$:

$$\left(\begin{array}{cc|c} -6-2i & -4 & 0 \\ 10 & 6-2i & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 3+i & 2 & 0 \\ 5 & 3-i & 0 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cc|c} 3+i & 2 & 0 \\ 0 & \frac{(3-i)(3+i)}{5} - 2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 3+i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \bar{e}_1 = \begin{pmatrix} -2 \\ 3+i \end{pmatrix}$$

$$\Rightarrow \bar{e}_1 e^{\lambda_1 t} = e^{t}\left(\begin{pmatrix} -2 \\ 3+i \end{pmatrix} \cos 2t + i \begin{pmatrix} -2 \\ 3+i \end{pmatrix} \sin 2t\right) =$$

$$= e^{t}\left(\begin{pmatrix} -2 \\ 3+i \end{pmatrix} \cos 2t + \begin{pmatrix} -2i \\ 3i-1 \end{pmatrix} \sin 2t\right) = e^{t}\left(\begin{pmatrix} -2\cos 2t \\ 3\cos 2t - \sin 2t \end{pmatrix} + \begin{pmatrix} -2i \sin 2t \\ 3i \sin 2t + i\cos 2t \end{pmatrix}\right)$$

$$\Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t}\left(C_1\begin{pmatrix} -2\cos 2t \\ 3\cos 2t-\sin 2t \end{pmatrix} + C_2\begin{pmatrix} -2\sin 2t \\ 3\sin 2t+\cos 2t \end{pmatrix}\right)$$

№11.12.

$$\begin{cases} \dot{x} = -5x+4y \\ \dot{y} = -x-y \end{cases}$$

$$\begin{vmatrix} -5-\lambda & 4 \\ -1 & -1-\lambda \end{vmatrix} = (\lambda+5)(\lambda+1)+4 = \lambda^2+6\lambda+9=0$$

$$\Rightarrow \lambda_1 = -3,\ \text{кр-ть } 2.$$

$$\left(\begin{array}{cc|c} -2 & 4 & 0 \\ -1 & 2 & 0 \end{array}\right) \sim (1\ \ -2\ |\ 0) \Rightarrow \bar{e}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

ищем присоед. вектор:

$$\left(\begin{array}{cc|c} 1 & 4 & 2 \\ -1 & 5 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 4 & 2 \\ 0 & 3 & 1 \end{array}\right) \Rightarrow \bar{e}_2 = ($$

$$\ddot{x} = -5\dot{x}+4\dot{y} = -5\dot{x}-4x-4y = -5\dot{x}-4x-\dot{x}-5x = -6\dot{x}-9x$$

$$\Rightarrow \ddot{x}+6\dot{x}+9x=0$$

$$\lambda^2+6\lambda+9=0$$

$$\Rightarrow \lambda_{1,2}=-3,\ \text{кр-ть } 2$$

$$\Rightarrow x(t) = C_1e^{-3t}+C_2te^{-3t} = e^{-3t}(C_1+C_2t)$$

$$4y = e^{-3t}(-3C_1-3C_2t+C_2+5C_1+5C_2t) = e^{-3t}(2C_1+C_2+2C_2t)$$

$$\Rightarrow \begin{pmatrix} x(t)\\ y(t)\end{pmatrix} = C_1 e^{-3t}\begin{pmatrix}2\\1\end{pmatrix} + C_2 e^{-3t}\left[t\begin{pmatrix}2\\1\end{pmatrix}+\begin{pmatrix}1\\1\end{pmatrix}\right]$$

№ 11.23.

$$\begin{cases} \dot{x} = x+2y-z \\ \dot{y} = 9x-6y+3z \\ \dot{z} = 20x-20y+10z \end{cases}$$

$$\begin{vmatrix} 1-\lambda & 2 & -1 \\ 9 & -6-\lambda & 3 \\ 20 & -20 & 10-\lambda \end{vmatrix} = (1-\lambda)((\lambda+6)(\lambda-10)+60) - 2(9(10-\lambda)-60) - (-180+(\lambda+6)\cdot 20) =$$

$$= (1-\lambda)(\lambda^2-4\lambda-60+60) - 2(30-9\lambda) + 180 - 20\lambda - 120 =$$

$$= -\lambda^3 + \lambda^2 - 4\lambda + 4\lambda^2 - 2\lambda = 0$$

$$\lambda^3 - 5\lambda^2 + 6\lambda = 0 \Rightarrow \lambda_1 = 0;\ \lambda_2 = 2;\ \lambda_3 = 3$$

$\lambda_1 = 0$:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 9 & -6 & 3 & 0 \\ 20 & -20 & 10 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 3 & -2 & 1 & 0 \\ 2 & -2 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -8 & 4 & 0 \\ 0 & -6 & 3 & 0 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right) \Rightarrow \bar{e}_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$\lambda_2 = 2$:

$$\begin{pmatrix} -1 & 2 & -1 \\ 9 & -8 & 3 \\ 20 & -20 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 \\ 9 & -8 & 3 \\ 5 & -5 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & 5 & -3 \\ 0 & 5 & -3 \end{pmatrix} \sim \left(\begin{array}{ccc|c} 5 & 0 & -1 & 0 \\ 0 & 5 & -3 & 0 \end{array}\right) \Rightarrow \bar{e}_2 = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

$$\lambda_3=3:\ \begin{pmatrix}-2 & 2 & -1\\ 9 & -9 & 3\\ 20 & -20 & 7\end{pmatrix}\sim\begin{pmatrix}-2 & 2 & -1\\ 3 & -3 & 1\\ 20 & -20 & 7\end{pmatrix}\sim\begin{pmatrix}1 & -1 & 0\\ 3 & -3 & 1\\ 20 & -20 & 7\end{pmatrix}\sim$$

$$\sim\cancel{\begin{pmatrix}1 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}}\begin{pmatrix}1 & -1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 7\end{pmatrix}\sim\begin{pmatrix}1 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}\Rightarrow \vec{e}_3=\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=C_1\begin{pmatrix}0\\1\\2\end{pmatrix}+C_2e^{2t}\begin{pmatrix}1\\3\\5\end{pmatrix}+C_3e^{3t}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

№11.31

$$\begin{cases}\dot{x}=x+2y+2z\\ \dot{y}=2x+y+2z\\ \dot{z}=2x+2y+z\end{cases}$$

$$\begin{vmatrix}1-\lambda & 2 & 2\\ 2 & 1-\lambda & 2\\ 2 & 2 & 1-\lambda\end{vmatrix}=(1-\lambda)((1-\lambda)^2-4)-2(2(1-\lambda)-4)+2(4-2(1-\lambda))=$$

$$=(1-\lambda)^3-4(1-\lambda)+8(\lambda+1)=1-3\lambda+3\lambda^2-\lambda^3-$$

$$-4+4\lambda+8\lambda+8=-\lambda^3+3\lambda^2+9\lambda+5=-\lambda^3-\lambda^2+4\lambda^2+9\lambda+5=$$

$$=-(\lambda+1)^2(\lambda-5)=0$$

$\lambda_{1,2}=-1;\ \lambda_3=5$

кр-ть 2.

$\lambda_{1,2}=-1:$

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \Rightarrow \bar{e}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}; \bar{e}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$\lambda_3 = 5:$

$$\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \sim \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \Rightarrow \bar{e}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = C_1 e^{-t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + C_3 e^{5t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

№11.46.

$$\begin{cases} \dot{x} = 7x - 4y + z \\ \dot{y} = 7x - 3y + z \\ \dot{z} = 4x - 2y + 2z \end{cases}$$

$$\begin{vmatrix} 7-\lambda & -4 & 1 \\ 7 & -3-\lambda & 1 \\ 4 & -2 & 2-\lambda \end{vmatrix} = (7-\lambda)((\lambda+3)(\lambda-2)+2) + 4(7(2-\lambda)-4) - -14 + 4(3+\lambda) = 0 \Leftrightarrow \lambda_1 = 2; \lambda_{2,3} = 2 \pm i$$

$\lambda_1 = 2:$

$$\begin{pmatrix} 5 & -4 & 1 \\ 7 & -5 & 1 \\ 4 & -2 & 0 \end{pmatrix} \sim \begin{pmatrix} -2 & 1 & 0 \\ 7 & -5 & 1 \\ 2 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \bar{e}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$\lambda_2 = 2+i$

$$\begin{pmatrix} 5-i & -4 & 1 \\ 7 & -(5+i) & 1 \\ 4 & -2 & -i \end{pmatrix} \sim \begin{pmatrix} 5-i & -4 & 1 \\ 0 & -(5+i)+\frac{28}{5-i} & 1-\frac{7}{5-i} \\ 0 & -2+\frac{16}{5-i} & -i-\frac{4}{5-i} \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 5-i & -4 & 1 \\ 0 & \frac{2}{\cancel{5+i}} & \frac{-2-i}{\cancel{5+i}} \\ 0 & 2i+6 & -5i-5 \end{pmatrix} \sim \begin{pmatrix} 5-i & -4 & 1 \\ 0 & 2 & -2-i \\ 0 & 0 & \frac{-10i-10}{2i+6}+2+i \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 5-i & -4 & 1 \\ 0 & 2 & -2-i \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 5-i & -4 & 1 \\ 0 & 2 & -2-i \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 5-i & 0 & \cancel{5+2i}\;\; -3-2i \\ 0 & 1 & -1-\frac{i}{2} \end{pmatrix} \Rightarrow \bar{e}_2 = \begin{pmatrix} 1+i \\ 2+i \\ 2 \end{pmatrix}$$

$$\cancel{\Rightarrow \bar{e}_3 = \begin{pmatrix} 1+i \\ 2+i \\ 2 \end{pmatrix}}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{2t}\left[C_1\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + C_2\begin{pmatrix} \cos t - \sin t \\ 2\cos t - \sin t \\ 2\cos t \end{pmatrix} + C_3\begin{pmatrix} \cos t + \sin t \\ \cos t + 2\sin t \\ 2\sin t \end{pmatrix}\right]$$

№ 11.68.

$$\begin{cases} \dot{x} = -2x + y - z \\ \dot{y} = -6x - 4y + 3z \\ \dot{z} = -2x + 2y - 3z \end{cases}$$

$$\begin{vmatrix} -2-\lambda & 1 & -1 \\ -6 & -4-\lambda & 3 \\ -2 & 2 & -3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda_1 = -1;\ \lambda_{2,3} = -4, \text{ кр-ть } 2.$$

$\lambda_1 = 1:$

$$\begin{pmatrix} -1 & 1 & -1 \\ -6 & -3 & 3 \\ -2 & 2 & -2 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & -1 \\ -2 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \Rightarrow \bar{e}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$\lambda_{2,3} = -4:$

$$\begin{pmatrix} 2 & 1 & -1 \\ -6 & 0 & 3 \\ -2 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & -1 \\ -2 & 0 & 1 \end{pmatrix} \Rightarrow,\ k = 2;\ m = 1 \Rightarrow \text{ищем реш. в виде:}$$

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} at + b \\ ct + d \\ et + f \end{pmatrix} e^{-4t}$$

$$\begin{cases} -4ate^{-4t} + ae^{-4t} - 4be^{-4t} = -2(at+b) + ct + d - et - f \\ -4ct + c - 4d = -6(at+b) - 4(ct+d) + 3(et+f) \\ -4et + e - 4f = -2(at+b) + 2(ct+d) - 3(et+f) \end{cases}$$

$$\Rightarrow \begin{cases} -4a = -2a + c - e \\ -4c = -6a - 4c + 3e \\ -4e = -2a + 2c - 3e \end{cases} \Rightarrow e = 2a \Rightarrow c = 0$$

$$\Rightarrow \begin{cases} -2at+a-2b=ct+d-et-f \\ c=-6(at+b)+3(et+f) \\ -et+e-f=-2(at+b)+2(ct+d) \end{cases} \Rightarrow \begin{cases} c=-2a+e \\ a-2b=d-f \\ c=-6b+3f \\ -6a+3e=0 \\ e-f=-2b+2d \\ -e=-2a+2c \end{cases}$$

$$\Rightarrow \begin{cases} c=0 \\ e=2a=2C_2 \\ f=2b=2C_3 \\ d=a-2b+f=C_2-2C_3+2C_3=C_2 \end{cases}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} C_2t+C_3 \\ C_2 \\ 2C_2t+2C_3 \end{pmatrix} e^{-4t}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = C_1e^{-t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + C_3e^{-4t}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + C_2e^{-4t}\left[t\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right]$$

№ 11.79.

$$\begin{cases} \dot{x}=4x-y+z \\ \dot{y}=-2x+3y-z \\ \dot{z}=-5x+4y-z \end{cases}$$

$$\begin{vmatrix} 4-\lambda & -1 & 1 \\ -2 & 3-\lambda & -1 \\ -5 & 4 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda_{1,2,3}=2, \text{кр-ть } 3$$

$$\begin{pmatrix} 2 & -1 & 1 \\ -2 & 1 & -1 \\ -5 & 4 & -3 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 3 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{1}{2} & 1 \\ 0 & 1 & -\frac{1}{3} \end{pmatrix} \sim \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & -1 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \overline{e}_1 = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

Строим жорданову цепочку:

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & -1 \\ -2 & 1 & -1 & 1 \\ -5 & 4 & -3 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & -1 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 3 & 0 & 1 & -1 \\ 3 & 3 & 0 & 0 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 3 & 0 & 1 & -1 \\ 0 & 3 & -1 & 1 \end{array}\right) \Rightarrow \overline{e}_2 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & -1 \\ -2 & 1 & -1 & 1 \\ -5 & 4 & -3 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & -1 & 1 & -1 \\ 1 & 1 & 0 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 3 & 0 & 1 & -2 \\ 3 & 3 & 0 & -3 \end{array}\right) \Rightarrow \overline{e}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{2t}\left(c_1 \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} + c_2\left(t\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}\right) + c_3\left(\frac{t^2}{2}\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right)\right)$$

№11.88.

$$\begin{cases} \dot{x} = 9x - 6y - 2z \\ \dot{y} = 18x - 12y - 3z \\ \dot{z} = 18x - 9y - 6z \end{cases}$$

$$\begin{vmatrix} 9-\lambda & -6 & -2 \\ 18 & -12-\lambda & -3 \\ 18 & -9 & -6-\lambda \end{vmatrix} = 0 \Rightarrow \lambda_{1,2,3} = -3$$

$\lambda = -3$:

$$\begin{pmatrix} 12 & -6 & -2 \\ 18 & -9 & -3 \\ 18 & -9 & -3 \end{pmatrix} \sim \begin{pmatrix} 6 & -3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \overline{e}_1 = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}; \ \overline{e}_2 = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$

присоед к $\overline{e}_2$: $\left(\begin{array}{ccc|c} 12 & -6 & -2 & 2 \\ 18 & -9 & -3 & 3 \\ 18 & -9 & -3 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 6 & -3 & -1 & 1 \end{array}\right) \Rightarrow$

$$\Rightarrow \overline{e}_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{-3t}\left(C_1\begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} + C_2\begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} + C_3\left(t\begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}\right)\right)$$

№11.154.

$$\begin{cases} \dot{x} = x - 2y - 2te^t \\ \dot{y} = 5x - y - (2t+6)e^t \end{cases}$$

$$\begin{vmatrix} 1-\lambda & -2 \\ 5 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda_{1,2} = \pm 3i$$

$$\lambda_1 = 3i: \begin{cases} (1-3i)x_0 - 2y_0 = 0 \\ 5x_0 - (1+3i)y_0 = 0 \end{cases} \Rightarrow x_0 = \frac{2}{1-3i}y_0 \Rightarrow \overline{e}_1 = \begin{pmatrix} 2 \\ 1-3i \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_0 = 2C_1\cos 3t + 2C_2\sin 3t \\ y_0 = C_1(\cos 3t + 3\sin 3t) + C_2(\sin 3t - 3\cos 3t) \end{cases}$$

частное решение:

$$\begin{cases} x_1 = (at+b)e^t \\ y_1 = (ct+d)e^t \end{cases}$$

$$\Rightarrow \begin{cases} a = a - 2c - 2 \\ c = 5a - c - 2 \\ a + b = b - 2d \\ c + d = 5b - d - 6 \end{cases} \Rightarrow \begin{cases} x_1 = e^t \\ y_1 = -te^t \end{cases}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} 2\cos 3t \\ \cos 3t + 3\sin 3t \end{pmatrix} + C_2 \begin{pmatrix} 2\sin 3t \\ \sin 3t - 3\cos 3t \end{pmatrix} + e^t \begin{pmatrix} 1 \\ -t \end{pmatrix}$$

№11.159.

$$\begin{cases} \dot{x} = -3x - 4y + 4z + \sin t + \cos t \quad (1) \\ \dot{y} = 3x + 4y - 5z - \sin t - \cos t \quad (2) \\ \dot{z} = x + y - 2z \quad (3) \end{cases}$$

$$\begin{vmatrix} -3-\lambda & -4 & -4 \\ 3 & 4-\lambda & -5 \\ 1 & 1 & -2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda_{1,2} = -1;\ \lambda_3 = 1$$

$$\lambda_{1,2}=-1:\ \begin{pmatrix}-2 & -4 & 4\\ 3 & 5 & -5\\ 1 & 1 & -1\end{pmatrix}\sim\begin{pmatrix}1 & 2 & -2\\ 0 & 0 & 0\\ 1 & 1 & -1\end{pmatrix}\sim$$

$$\sim\begin{pmatrix}1 & 2 & -2\\ 0 & -1 & 1\end{pmatrix}\sim\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -1\end{pmatrix}\Rightarrow \overline{e_1}=\begin{pmatrix}0\\1\\1\end{pmatrix}$$

Найдем присоед:

$$\left(\begin{array}{ccc|c}-2 & -4 & 4 & 0\\ 3 & 5 & -5 & 1\\ 1 & 1 & -1 & 1\end{array}\right)\sim\left(\begin{array}{ccc|c}1 & 2 & -2 & 0\\ 0 & -1 & 1 & 1\end{array}\right)\sim\left(\begin{array}{ccc|c}1 & 0 & 0 & 2\\ 0 & 1 & -1 & -1\end{array}\right)\Rightarrow$$

$$\overline{e_2}=\begin{pmatrix}2\\-1\\0\end{pmatrix}$$

$$\lambda_3=1:\ \begin{pmatrix}-4 & -4 & 4\\ 3 & 3 & -5\\ 1 & 1 & -3\end{pmatrix}\sim\begin{pmatrix}1 & 1 & -1\\ 0 & 0 & 2\\ 1 & 1 & -3\end{pmatrix}\Rightarrow \overline{e_3}=\begin{pmatrix}-1\\1\\0\end{pmatrix}$$

Частн. реш:

$$\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}=\begin{pmatrix}a\sin t+b\cos t\\ c\sin t+d\cos t\\ e\sin t+f\cos t\end{pmatrix}$$

$(1)+(2)$: $\dot{x}_1+\dot{y}_1=-z_1$ и $z_1=0 \Rightarrow \dot{y}_1=-\dot{x}_1$

$$\Rightarrow \begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}=\begin{pmatrix}-\cos t\\ \cos t\\ 0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = C_1 e^{t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + C_3 e^{-t} \left[t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right] + \begin{pmatrix} -\cos t \\ 4\cos t \\ 0 \end{pmatrix}$$

№11.183

$$\begin{cases} \dot{x} = 3x + 2y - \dfrac{1}{1+e^{-t}} \\ \dot{y} = -3x - 2y - \dfrac{1}{1+e^{-t}} \end{cases}$$

$$\begin{vmatrix} 3-\lambda & 2 \\ -3 & -2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda_1 = 0,\ \lambda_2 = 1$$

$$\begin{pmatrix} 3 & 2 \\ -3 & -2 \end{pmatrix} \Rightarrow \bar{e}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 \\ -3 & -3 \end{pmatrix} \Rightarrow \bar{e}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ -3 \end{pmatrix} + C_2 e^{t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{cases} x_0 = 2C_1 - C_2 e^{t} \\ y_0 = -3C_1 + C_2 e^{t} \end{cases}$$

Если подставим в систему, то получим:

$$\begin{cases} C_1'(t) + C_2'(t) e^{t} = -\dfrac{1}{1+e^{-t}} \\ -3C_1'(t) + -2C_2'(t) e^{t} = -\dfrac{2}{1+e^{-t}} \end{cases} \Rightarrow \begin{cases} C_1'(t) = \dfrac{4}{1+e^{-t}} \\ C_2'(t) = -\dfrac{5e^{-t}}{1+e^{-t}} \end{cases}$$

$$\Rightarrow \begin{array}{l} c_1(t) = 4\ln(e^{t}+1) + \tilde{C}_1 \\ c_2(t) = 5\ln(e^{-t}+1) + \tilde{C}_2 \end{array}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} + C_2 e^{t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 4\ln(1+e^{t}) + 5e^{t}\ln(e^{-t}+1) \\ -6\ln(1+e^{t}) - 5e^{t}\ln(e^{-t}+1) \end{pmatrix}$$

III. Матричная экспонента.

№11.117.

$$\begin{cases} \dot{x} = 2x + y \\ \dot{y} = x + 2y \end{cases}$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0;\ \lambda_1 = 1,\ \lambda_2 = 3$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow \bar{e}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow \bar{e}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow A' = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \cancel{E + B + C,\ B = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix},\ B^2 = \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix}}$$

$$\cancel{,\ B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ C = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}},\ B^2 = C^2 = 0$$

$$\Rightarrow e^{A't} = \begin{pmatrix} e^{t} & 0 \\ 0 & e^{3t} \end{pmatrix}$$

матр. перехода:

$$S=\begin{pmatrix}1&1\\-1&1\end{pmatrix};$$

$$\left(\begin{array}{cc|cc}1&1&1&0\\-1&1&0&1\end{array}\right)\sim\left(\begin{array}{cc|cc}1&1&1&0\\0&2&1&1\end{array}\right)\sim\left(\begin{array}{cc|cc}1&0&\frac{1}{2}&-\frac{1}{2}\\0&1&\frac{1}{2}&\frac{1}{2}\end{array}\right)\Rightarrow$$

$$\Rightarrow S^{-1}=\frac{1}{2}\begin{pmatrix}1&-1\\1&1\end{pmatrix}$$

$$e^{At}=Se^{A't}S^{-1}=\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}e^{t}&0\\0&e^{3t}\end{pmatrix}\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}e^{t}&e^{3t}\\-e^{t}&e^{3t}\end{pmatrix}\begin{pmatrix}1&-1\\1&1\end{pmatrix}=$$

$$=\frac{1}{2}\begin{pmatrix}e^{t}+e^{3t}&-e^{t}+e^{3t}\\-e^{t}+e^{3t}&e^{t}+e^{3t}\end{pmatrix}\Rightarrow\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}e^{3t}+e^{t}&e^{3t}-e^{t}\\e^{3t}-e^{t}&e^{3t}+e^{t}\end{pmatrix}\begin{pmatrix}C_1\\C_2\end{pmatrix}$$

№11.124.

$$\begin{cases}\dot{x}=2x-y\\\dot{y}=x+4y\end{cases}$$

$$\begin{vmatrix}2-\lambda&-1\\1&4-\lambda\end{vmatrix}=0\Rightarrow\lambda_{1,2}=3.$$

$$\begin{pmatrix}-1&-1\\1&1\end{pmatrix}\sim\begin{pmatrix}1&1\end{pmatrix}\Rightarrow\overline{e}_1=\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\left(\begin{array}{cc|c}-1&-1&1\\1&1&-1\end{array}\right)\sim\left(\begin{array}{cc|c}1&1&-1\end{array}\right)\Rightarrow\overline{e}_2=\begin{pmatrix}-1\\0\end{pmatrix}$$

$A' = \begin{pmatrix}3 & 1\\ 0 & 3\end{pmatrix}; \quad e^{A't} = \begin{pmatrix}e^{3t} & te^{t}\\ 0 & e^{3t}\end{pmatrix}$

$S = \begin{pmatrix}1 & -1\\ -1 & 0\end{pmatrix}; \quad \left(\begin{array}{cc|cc}1 & -1 & 1 & 0\\ -1 & 0 & 0 & 1\end{array}\right) \sim \left(\begin{array}{cc|cc}1 & -1 & 1 & 0\\ 0 & -1 & 1 & 1\end{array}\right) \sim$

$\sim \left(\begin{array}{cc|cc}1 & 0 & 0 & -1\\ 0 & -1 & 1 & 1\end{array}\right) \sim \left(\begin{array}{cc|cc}1 & 0 & 0 & -1\\ 0 & 1 & -1 & -1\end{array}\right) \Rightarrow S^{-1} = \begin{pmatrix}0 & -1\\ -1 & -1\end{pmatrix}$

$e^{At} = Se^{A't}S^{-1} = \begin{pmatrix}1 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}e^{3t} & e^{t}\\ 0 & e^{3t}\end{pmatrix}\begin{pmatrix}0 & -1\\ -1 & -1\end{pmatrix} =$

$= \begin{pmatrix}e^{3t} & e^{t} - e^{3t}\\ -e^{3t} & -e^{t}\end{pmatrix}\begin{pmatrix}0 & -1\\ -1 & -1\end{pmatrix} = \begin{pmatrix}e^{3t} - e^{t} & -e^{3t} - e^{t}\\ e^{t} & e^{3t} + e^{t}\end{pmatrix}$

$A' = \begin{pmatrix}3 & 1\\ 0 & 3\end{pmatrix}; \quad S = \begin{pmatrix}1 & -1\\ -1 & 0\end{pmatrix}; \quad S^{-1} = \begin{pmatrix}0 & -1\\ -1 & -1\end{pmatrix}$

$e^{A't} = \begin{pmatrix}e^{3t} & te^{3t}\\ 0 & e^{3t}\end{pmatrix}$

$e^{At}\bar{C} = \begin{pmatrix}1 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}e^{3t} & te^{3t}\\ 0 & e^{3t}\end{pmatrix}\begin{pmatrix}0 & -1\\ -1 & -1\end{pmatrix}\begin{pmatrix}C_1\\ C_2\end{pmatrix} =$

$= e^{3t}\begin{pmatrix}1-t & -t\\ t & 1+t\end{pmatrix}\begin{pmatrix}C_1\\ C_2\end{pmatrix}$ (выч-ия выше зачеркнуты)

№ 11.128

$$\begin{cases} \dot{x} = 2x - 3y \\ \dot{y} = 3x + 2y \end{cases}$$

$$\begin{vmatrix} 2-\lambda & -3 \\ 3 & 2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda_{1,2} = 2 \pm 3i$$

$\lambda = 2 - 3i$:

$$\begin{pmatrix} 3i & -3 \\ 3 & 3i \end{pmatrix} \sim \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \sim \begin{pmatrix} -1 & -i \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & i \end{pmatrix} \Rightarrow \bar{e}_1 = \begin{pmatrix} i \\ -1 \end{pmatrix}$$

$$\bar{e}_2 = \overline{\bar{e}_1} = \begin{pmatrix} -i \\ -1 \end{pmatrix}$$

$$S = \begin{pmatrix} i & -i \\ -1 & -1 \end{pmatrix}; \left(\begin{matrix} i & -i \\ -1 & -1 \end{matrix}\middle|\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right) \sim \left(\begin{matrix} -1 & 1 \\ 0 & -2 \end{matrix}\middle|\begin{matrix} i & 0 \\ -i & 1 \end{matrix}\right) \sim$$

$$\sim \left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\middle|\begin{matrix} -\frac{i}{2} & -\frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{matrix}\right) \Rightarrow S^{-1} = \frac{1}{2}\begin{pmatrix} -i & -1 \\ i & 1 \end{pmatrix}$$

$$\Rightarrow e^{At}\bar{c} = S e^{A't} S^{-1} \bar{c} = \begin{pmatrix} i & -i \\ -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} e^{(2-3i)t} & 0 \\ 0 & e^{(2+3i)t} \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} -i & -1 \\ i & 1 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = e^{2t}\begin{pmatrix} \cos 3t & -\sin 3t \\ \sin 3t & \cos 3t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

при $t = 0$:

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t}\begin{pmatrix} \cos 3t & -\sin 3t \\ \sin 3t & \cos 3t \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

№ T2

$$\begin{vmatrix} -\lambda & 0 & -a \\ 0 & -\lambda & a \\ a & a & -\lambda \end{vmatrix} = -\lambda^3 = 0 \Rightarrow \lambda_{1,2,3} = 0, \text{кр-ть } 3$$

$$\begin{pmatrix} 0 & 0 & -a \\ 0 & 0 & a \\ a & a & 0 \end{pmatrix} \sim \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \sim \begin{array}{c} \begin{array}{ccc} x & z & y \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array} \Rightarrow \bar{e}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

присоед $\bar{e}_1$: $\left(\begin{array}{ccc|c} 0 & 0 & -a & 1 \\ 0 & 0 & a & -1 \\ a & a & 0 & 0 \end{array}\right) \sim \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 0 & 0 & 1 & -\frac{1}{a} \\ 1 & 1 & 0 & 0 \end{array}\right) \end{array} \sim$

$$\sim \begin{array}{c} \begin{array}{ccc} z & y & x \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{a} \\ 0 & 1 & 1 & 0 \end{array}\right) \end{array} \Rightarrow \bar{e}_2 = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{a} \end{pmatrix}$$

присоед $\bar{e}_2$: $\bar{e}_3 = \begin{pmatrix} -\frac{1}{a^2} \\ 0 \\ 0 \end{pmatrix}$

$$S = \begin{pmatrix} 1 & 0 & -\frac{1}{a^2} \\ -1 & 0 & 0 \\ 0 & -\frac{1}{a} & 0 \end{pmatrix} \Rightarrow S^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -a \\ -a^2 & -a^2 & 0 \end{pmatrix}$$

$$\cancel{e^{A't}\bar{C} = S e^{A't} S^{-1} \bar{C} = \begin{pmatrix} 1 & 0 & -\frac{1}{a^2} \\ -1 & 0 & 0 \\ 0 & -\frac{1}{a} & 0 \end{pmatrix} \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}}$$

$$A' = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A'^3 = 0$$

$$\cancel{e^{A't} = E + \sum_{u=1}^{\infty} \frac{A'^u t^u}{u!} = E +}$$

$$e^{A't} = e^{\lambda t} E \cdot \left(E + A't + A'^2 \frac{t^2}{2}\right) = \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow e^{At}\bar{c} = S e^{A't} S^{-1}\bar{c} = \begin{pmatrix} 1 & 0 & -\frac{1}{a^2} \\ -1 & 0 & 0 \\ 0 & -\frac{1}{a} & 0 \end{pmatrix}\begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -a \\ -a^2 & -a^2 & 0 \end{pmatrix}\bar{x}_0$$

$$\Rightarrow \bar{x} = \begin{pmatrix} 1-\frac{a^2t^2}{2} & -\frac{a^2t^2}{2} & -at \\ \frac{a^2t^2}{2} & 1+\frac{a^2t^2}{2} & at \\ at^2 & at^2 & 1 \end{pmatrix}\bar{x}_0$$

№73

Исходя из матрицы A':

$\lambda_{1,2,3} = 2$, кр-ть 3, $\bar{h}_1$ - собств. в.; $\bar{h}_2, \bar{h}_3$ - присоед.

$\lambda_{4,5} = 3$, кр-ть 2 $\bar{h}_3, \bar{h}_4$ - собств. в.

$$\Rightarrow \dot{\bar{x}} = A\bar{x} - e^{At}\cdot\bar{c} = H\cdot e^{A't}\cdot H^{-1}\cdot\bar{c},\ H = (\bar{h}_1^{\uparrow}\dots\bar{h}_5^{\uparrow})$$

б)

$$e^{A'} = \begin{pmatrix} e^{2t} & te^{2t} & \frac{t^2}{2}e^{2t} & 0 & 0 \\ 0 & e^{2t} & te^{2t} & 0 & 0 \\ 0 & 0 & e^{2t} & 0 & 0 \\ 0 & 0 & 0 & e^{3t} & 0 \\ 0 & 0 & 0 & 0 & e^{3t} \end{pmatrix}$$

IV. Операционный метод.

№8.172

$y'' - 3y' + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$

$p^2F - 1 - 3pF + 2F = \frac{1}{p+1}$; $F(p^2 - 3p + 2) = \frac{1}{p+1} + 1 \Rightarrow$

$$\Rightarrow F = \frac{p+2}{(p+1)(p-1)(p-2)} = \frac{A}{p+1} + \frac{B}{p-1} + \frac{C}{p-2} \Rightarrow \underline{y = \frac{1}{6}e^{-t} - \frac{3}{2}e^{t} + \frac{4}{3}e^{2t}}$$

№8.182.

$y'' + 4y = 4(\cos 2t + \sin 2t),\ y(0)=0,\ y'(0)=1$

$p^2F - 1 + 4F = \frac{4p}{p^2+4} + \frac{8}{p^2+4} \Rightarrow F = \frac{4p}{(p^2+4)^2} + \frac{8}{(p^2+4)^2} + \frac{1}{p^2+4}$

$\frac{8}{(p^2+4)^2} + \frac{1}{p^2+4} = \frac{-p^2+4+4+p^2}{(p^2+4)^2} + \frac{1}{p^2+4} = -\frac{p^2-4}{(p^2+4)^2} + \frac{2}{p^2+4}$

$\Rightarrow y = (1+t)\sin 2t - t\cos 2t$

№11.179.

$$\begin{cases} \dot{x} = x + y + e^{2t} \\ \dot{y} = -2x + 4y + e^{2t} \end{cases}$$

$y(0) = 2,\ x(0) = 1$

$x(t) = F(p),\ y(t) = G(p)$

$$\begin{cases} pF - 1 = F + G + \frac{1}{p-2} \\ pG - 2 = -2F + 4G + \frac{1}{p-2} \end{cases} \quad \begin{cases} G = (p-1)F - 1 - \frac{1}{p-2} \\ (p-4)(p-1)F - (p-4) - \frac{p-4}{p-2} - 2 = -2F + \frac{1}{p-2} \end{cases}$$

$F(p-2)(p-3) = p - 2 + \frac{p-3}{p-2}$

$F = \frac{1}{p-3} + \frac{1}{(p-2)^2},\ G = \frac{p-1}{p-3} + \frac{p-1}{(p-2)^2} - 1 - \frac{1}{p+2} = \frac{2}{p-3} + \frac{1}{(p-2)^2}$

$$\Rightarrow \begin{cases} x = e^{3t} + te^{2t} \\ y = 2e^{3t} + te^{2t} \end{cases}$$

№ 11.194

$$\begin{cases} \dot{x} = 4x + 5y + 4 \\ \dot{y} = -4x - 4y + 4t \end{cases}$$

$x(0) = 0,\ y(0) = 3$

$x(t) = F(p),\ y(t) = G(p)$

$$\begin{cases} pF = 4F + 5G + \frac{4}{p} \\ pG - 3 = -4F - 4G + \frac{4}{p^2} \end{cases} \Rightarrow \begin{cases} G = \frac{1}{5}\left[F(p-4) - \frac{4}{p}\right] \\ \frac{p+4}{5}\left[F(p-4) - \frac{4}{p}\right] - 3 = -4F + \frac{4}{p^2} \end{cases}$$

$$F(p^2+4) = 4 \cdot \frac{p+4}{p} + 15 + \frac{20}{p^2}$$

$$F = 4 \cdot \frac{p+4}{p(p^2+4)} + \frac{15}{p^2+4} + \frac{20}{p^2(p^2+4)} = \frac{A}{p} + \frac{B}{p^2} + \frac{Cp+D}{p^2+4}$$

$$\Rightarrow \quad F = \frac{4}{p} + \frac{5}{p^2} - 4\frac{p}{p^2+4} + 7 \cdot \frac{2}{p^2+4}$$

$$G = -\frac{3}{p} - \frac{4}{p^2} + \frac{6p}{p^2+4} - 4 \cdot \frac{2}{p^2+4}$$

$$\Rightarrow \begin{cases} x = 4 + 5t - 4\cos 2t + 7\sin 2t \\ y = -3 - 4t + 6\cos 2t - 4\sin 2t \end{cases}$$