I. Криволинейные интегралы.

Формула Грина.

№10.1(4)

4) отрезок с концами $(0;-2)$ и $(4;0)$ $\Rightarrow y=\frac{1}{2}x-2$

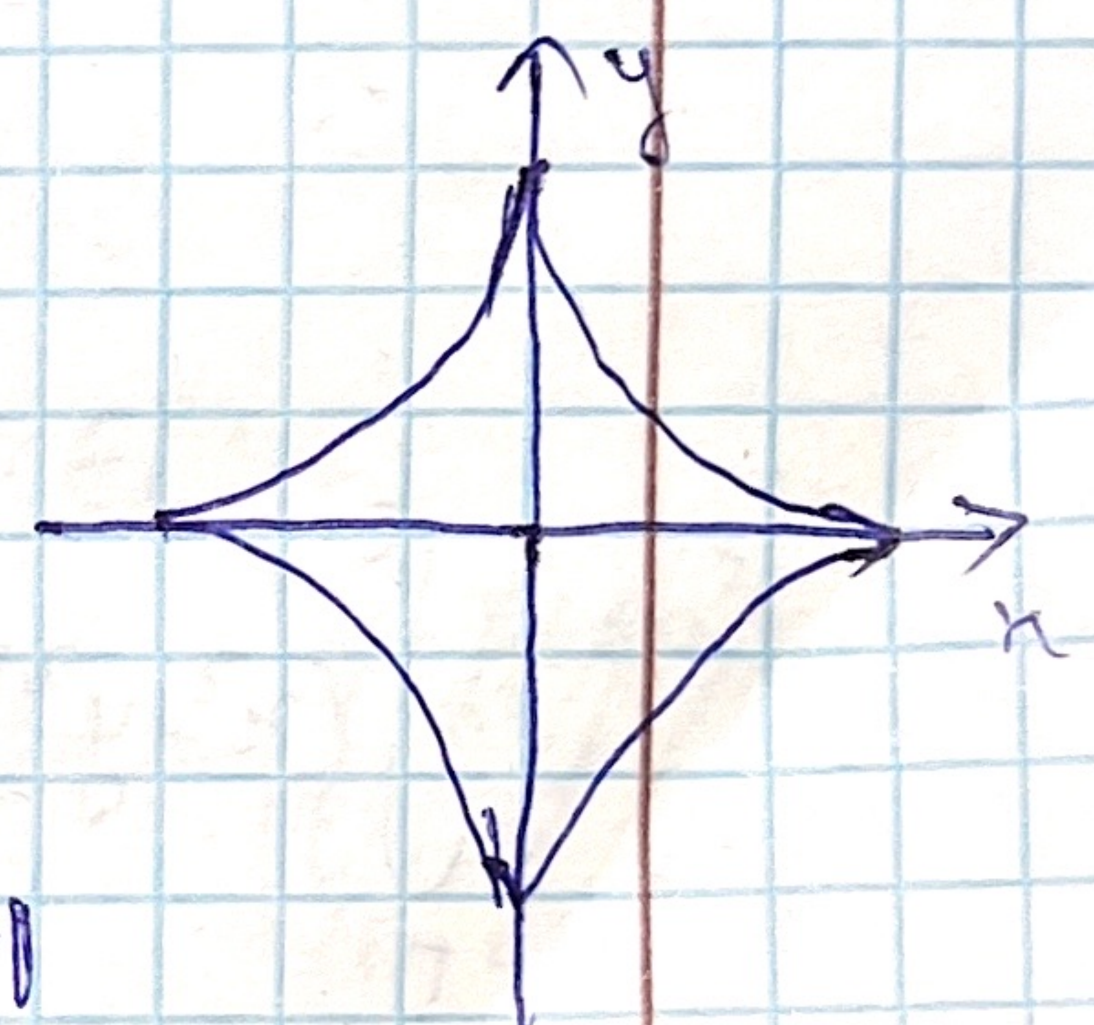
$$x=t$$
$$y=\frac{1}{2}t-2, \quad t\in[0;4]$$

$$|\overline{r}'(t)|=\sqrt{1+\frac{1}{4}}=\frac{\sqrt{5}}{4}$$

$$\int\limits_{\Gamma}\frac{ds}{y-x}=\int\limits_0^4\frac{dt}{-\frac{1}{2}t-2}\cdot\frac{\sqrt{5}}{4}=-\frac{\sqrt{5}}{2}\int\limits_0^4\frac{dt}{t+4}=-\frac{\sqrt{5}}{2}\cdot\ln 4=\underline{-\sqrt{5}\ln 2}$$

№10.9

$$x=a\cos^3 t$$
$$y=a\cancel{\cos}\sin^3 t, \quad \cancel{t\in}\ t\in[0;2\pi], \text{ т.к. астроида}$$



$$|\overline{r}'(t)|=\sqrt{a^2(9\cos^4 t\sin^2 t+9\cos^2 t\sin^4 t)}=3a|\sin t\cos t|$$

при $t\in[0;\frac{\pi}{2}]$: $|\overline{r}'(t)|=3a\sin t\cos t$

$$\int\limits_{\Gamma}(x^{\frac{4}{3}}+y^{\frac{4}{3}})ds=4a^{\frac{4}{3}}\int\limits_0^{\frac{\pi}{2}}(\cos^4 t+\sin^4 t)\,3a\sin t\cos t\,dt=$$

$$=\cancel{12}\,6a^{\frac{7}{3}}\int\limits_0^{\frac{\pi}{2}}(1-2\sin^2 t\cos^2 t)\sin 2t\,dt=6a^{\frac{7}{3}}\int\limits_0^{\frac{\pi}{2}}\left(1-\frac{\sin^2 2t}{2}\right)\sin 2t\,dt=$$

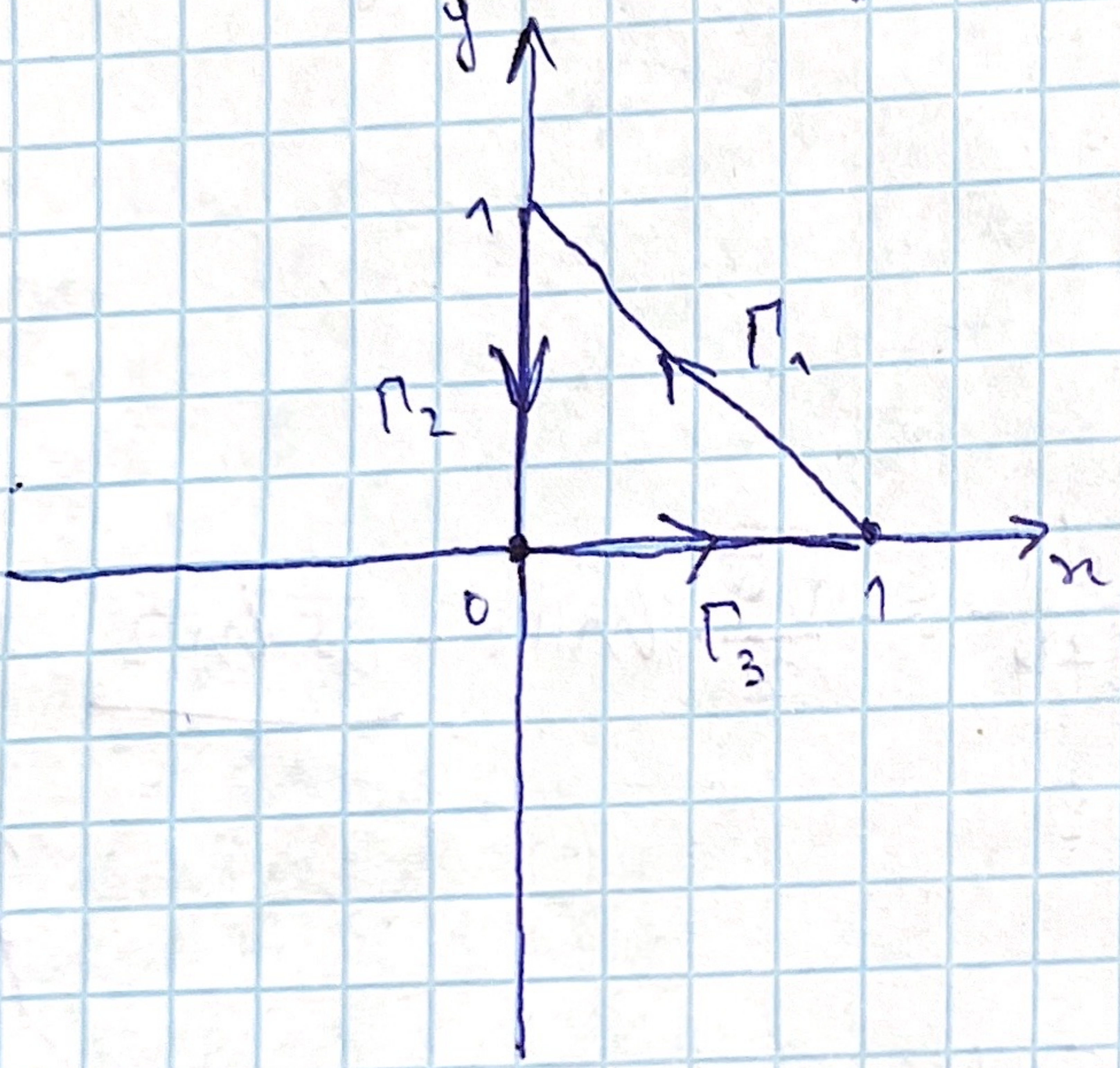
$$=\cancel{6a}\,\cancel{3a\int}\,3a^{\frac{7}{3}}(-\cos 2t)\Big|_0^{\frac{\pi}{2}}-3a^{\frac{7}{3}}\int\limits_0^{\frac{\pi}{2}}\sin^3 2t\,dt=6a^{\frac{7}{3}}-$$

$6a^{\frac{7}{3}}+$

$= 3a^{\frac{7}{3}}\left(\frac{1}{4}\int\limits_0^{\frac{\pi}{2}} \sin 6t\,dt + \frac{3}{4}\int\limits_0^{\frac{\pi}{2}} \sin 2t\,dt\right) = 3a^{\frac{7}{3}}\cdot\frac{1}{24}(-\cos 6t)\Big|_0^{\frac{\pi}{2}} + $

$+ \frac{9a^{\frac{7}{3}}}{8}(-\cos 2t)\Big|_0^{\frac{\pi}{2}} = 6a^{\frac{7}{3}} + \frac{1}{4}a^{\frac{7}{3}} + \frac{9}{8}a^{\frac{7}{3}} = \underline{4a^{\frac{7}{3}}}$

№10.29(4)



~~$0 \le x \le 1$~~

~~$0 \le y \le 1$~~

~~$x = \cos t$~~, ~~$t \in [0; \frac{\pi}{2}]$~~

~~$y = \sin t$~~

$\int\limits_{\Gamma} (x^2+y^2)\,dx + (x^2-y^2)\,dy = \int\limits_{\Gamma_1} (x^2+y^2)\,dx + (x^2-y^2)\,dy + \int\limits_{\Gamma_2} \ldots + \int\limits_{\Gamma_3} \ldots$

$= \cancel{\int\limits_0^{\frac{\pi}{2}} (-\sin t)\,dt + \cos 2t \cos t)}\,dt$

$\Gamma_1:\ y = -x + 1,\ 0 \le x \le 1$

$x = t$

$y = -t + 1$, $t \in [0; 1]$

$$\int_{\Gamma_1} = \int_0^1 (2t^2-2t+1-t^2+t^2-2t+1)\,dt = 2\int_0^1 (t-1)^2\,dt = \frac{2}{3}(t-1)^3\Big|_0^1 = -\frac{2}{3}$$

Γ_2: ~~$y=0$~~ $y\in[0;1]$; $x=0$

$$\int_{\Gamma_2} = -\int_0^1 (-y^2)\,dy = \frac{y^3}{3}\Big|_0^1 = \frac{1}{3}$$

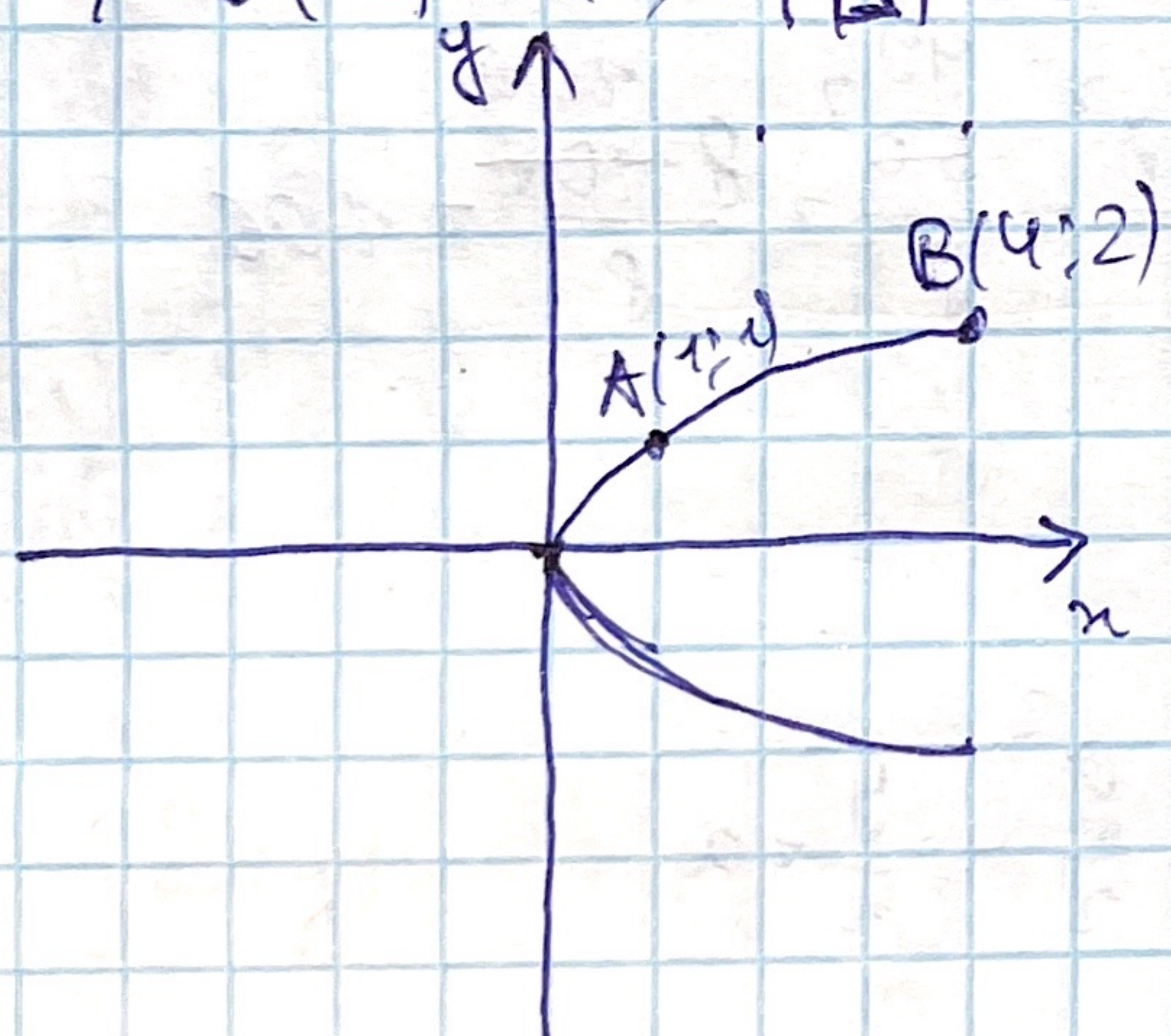
Γ_3: $y=0$; $x\in[0;1]$

$$\int_{\Gamma_3} = \int_0^1 x^2\,dx = \frac{1}{3}$$

$$\Rightarrow \oint_\Gamma = \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3} = 0.$$

№10.85(4)

4) $\rho(A)=1$; $\rho(B)=2$ ~~$x=t;\ y=t$~~ $y=t$; $x=t^2$, $t\in[1;2]$



$$\int_\Gamma \rho(x;y)\,ds = \int_1^2 t\sqrt{1+4t^2}\,dt = \cancel{\left| \begin{array}{l} 1+4t=a^2 \\ t=\frac{a^2-1}{4} \\ dt=\frac{a}{2}da \end{array} \right.}$$

$$\cancel{= \int_1^2 \sqrt{t^2+2t^2}\,dt}$$

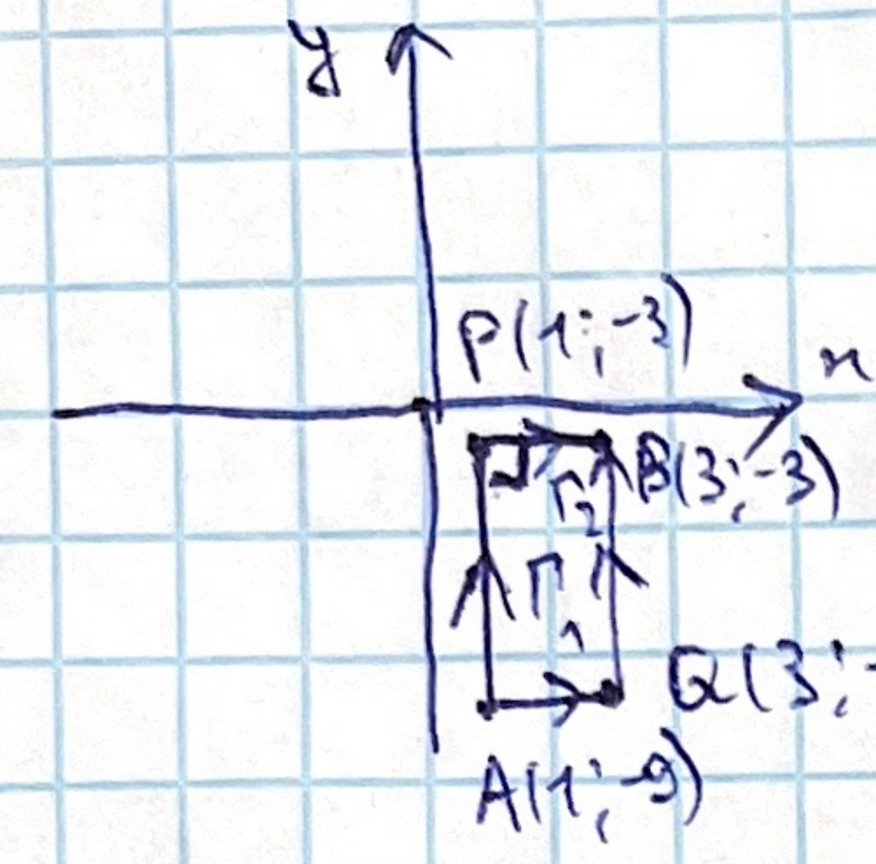
$$\cancel{= \int_{\sqrt{3}}^{\sqrt{5}} a\cdot\frac{a^2-1}{4}\,da = \frac{1}{2}\cdot\frac{a^4}{4}\Big|_{\sqrt{3}}^{\sqrt{5}} - \frac{1}{2}\cdot\frac{a^2}{2}\Big|_{\sqrt{3}}^{\sqrt{5}} =}$$

$$\cancel{= \frac{1}{8}\cdot 16 - \frac{1}{4}\cdot 2 = \frac{3}{2}}$$

$$= \frac{1}{2}\int_1^2 \sqrt{1+4t^2}\,dt^2 = \frac{1}{8}\int_1^2 \sqrt{1+4t^2}\,d(4t^2) = \cancel{\frac{1}{12}\int_1^2} \frac{1}{12}(1+4t^2)^{\frac{3}{2}}\Big|_1^2 = \frac{1}{12}(17\sqrt{17}-5\sqrt{5})$$

№110(1; 2)

1)



$$\int_{\Gamma}(\bar{F};d\bar{r}) = \int_{\Gamma_1}(\bar{F};d\bar{r}) + \int_{\Gamma_2}(\bar{F};d\bar{r})$$

Γ_1: $x=1$; $y\in[-9;-3]$

$$\int_{\Gamma_1} = \int_{-9}^{-3}(2+y)\,dy = \frac{(y+2)^2}{2}\Big|_{-9}^{-3} = \frac{1}{2} - \frac{49}{2} = -24$$

Γ_2: $x\in[1;3]$; $y=-3$

$$\int_{\Gamma_2} = \int_{1}^{3}(4x+15)\,dx = \frac{(4x+15)^2}{8}\Big|_{1}^{3} = \frac{27^2}{8} - \frac{19^2}{8} = 46$$

$$\Rightarrow \int_{\Gamma}(\bar{F};d\bar{r}) = \underline{22}$$

2) Γ_1: $x\in[1;3]$; $y=-9$

$$\int_{\Gamma_1} = \int_{1}^{3}(4x+45)\,dx = \frac{(4x+45)^2}{8}\Big|_{1}^{3} = \frac{57^2}{8} - \frac{\cancel{50}^2}{8} = \frac{8\cdot\cancel{107}}{8} = 106$$

Γ_2: $x=3$; $y\in[-9;-3]$

$$\int_{\Gamma_2} = \int_{-9}^{-3}(6+y)\,dy = \frac{(6+y)^2}{2}\Big|_{-9}^{-3} = \frac{9^2}{2} - \frac{9^2}{2} = 0$$

$$\Rightarrow \int_{\Gamma} = \int_{\Gamma_1} = \underline{106}$$

№10.46

$P = 2xy - y$; $Q = x^2$

$\frac{\partial P}{\partial y} = 2x - 1$; $\frac{\partial Q}{\partial x} = 2x$

$$\Rightarrow \int\limits_{\Gamma} (2xy - y)dx + x^2 dy = \iint\limits_{S} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx dy = \iint\limits_{S} \underbrace{dx dy}_{\text{площадь}} = \underline{\pi ab}$$

№10.100

Всё по ф-ле Грина:

$$\int\limits_{\partial G} \overset{Q}{x} dy = \iint\limits_{\text{int}\, G} dx dy = S$$

$$-\int\limits_{\partial G} \overset{P}{y} dx = \iint\limits_{\text{int}\, G} dx dy = S$$

$$\frac{1}{2}\int\limits_{\partial G} \overset{Q}{x} dy - \overset{P}{y} dx = \frac{1}{2}\iint\limits_{\text{int}\, G} (1+1)\, dx dy = \iint\limits_{\text{int}\, G} dx dy = S$$

№10.59

$$\int\limits_{\Gamma} 2xy dx + x^2 dy = \int\limits_{\Gamma} d(x^2 y) = \int\limits_{0}^{-4} dt = \underline{-4}$$

№T1.

$$\int\limits_{\gamma} \frac{x dy - y dx}{x^2 + y^2} =$$

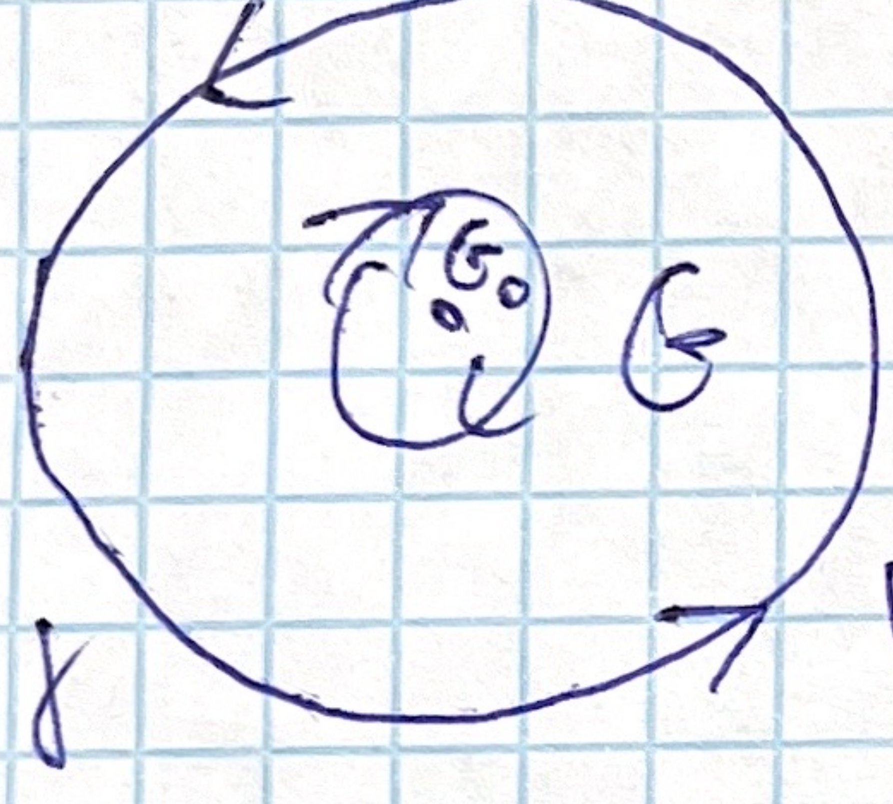
1) $\partial G = \gamma$

$P = -\frac{y}{x^2+y^2}$; $\frac{\partial P}{\partial y} = -\frac{x^2+y^2-2y^2}{x^2+y^2} = -\frac{x^2-y^2}{x^2+y^2}$

$$Q = \frac{x}{x^2+y^2}; \quad \frac{\partial Q}{\partial x} = \frac{x^2+y^2-2x^2}{x^2+y^2} = -\frac{x^2-y^2}{x^2+y^2}$$

$$\Rightarrow \iint_{\text{in } G} \left(-\frac{x^2-y^2}{x^2+y^2} + \frac{x^2-y^2}{x^2+y^2}\right) dx\,dy = 0$$

2) $\partial G = \gamma$



Пусть G_0 - область, содерж. $(0;0)$

$G_1 = G \setminus G_0$

$$0 = \int_{\partial G_1} = \int_{\partial G} + \int_{\partial G_0}$$ (когда G и G_0 ориент. в разные стор)

$$\Rightarrow \int_{\partial G} = \int_{\partial G_0}$$ (когда ориент. в одну сторону)

$\Rightarrow$ $x = a\cos t$, $y = a\sin t$ $\Rightarrow$ $$\int_{\gamma} = \int_{\partial G} = \int_{\partial G_0} = \int_0^{2\pi} \frac{a^2\cos^2 t + a^2\sin^2 t}{a^2}\,dt = 2\pi$$

II. Поверхностные интегралы

№ 9.37.

$~~S = \iint_G \sqrt{1+}~~$ $\quad z = \sqrt{x^2+y^2} = f(x;y) \Rightarrow$

$$\Rightarrow S = \iint_G \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\,dx\,dy$$

$x^2+y^2=2x \Rightarrow (x-1)^2+y^2=1$ – окр-ть с ц. в $(1;0)$ и $R=1$

$$\frac{\partial f}{\partial x}=\frac{2x}{2\sqrt{x^2+y^2}}=\frac{x}{\sqrt{x^2+y^2}};\ \frac{\partial f}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}$$

площадь G

$$\Rightarrow S=\iint\limits_G \sqrt{1+1}\,dxdy=\sqrt{2}\iint\limits_G dxdy=\sqrt{2}\,\pi$$

№9.51.

$x=(b+a\cos\psi)\cos\varphi$ Для тора: $0\le\varphi\le 2\pi;\ 0\le\psi\le 2\pi$

$y=(b+a\cos\psi)\sin\varphi$

$z=a\sin\psi,\ 0<a\le b$

$$\cancel{[\bar r'_x;\bar r'_y]}=[\bar r'_\varphi;\bar r'_\psi]=\begin{vmatrix}\bar i & \bar j & \bar k\\ -\sin\varphi(b+a\cos\psi) & (b+a\cos\psi)\cos\varphi & 0\\ -a\sin\psi\cos\varphi & -a\sin\psi\sin\varphi & a\cos\psi\end{vmatrix}=$$

$$=\left(a\sin\psi\sin^2\varphi(b+a\cos\psi)+a\sin\psi\cos^2\varphi(b+a\cos\psi)\right)\bar k+$$

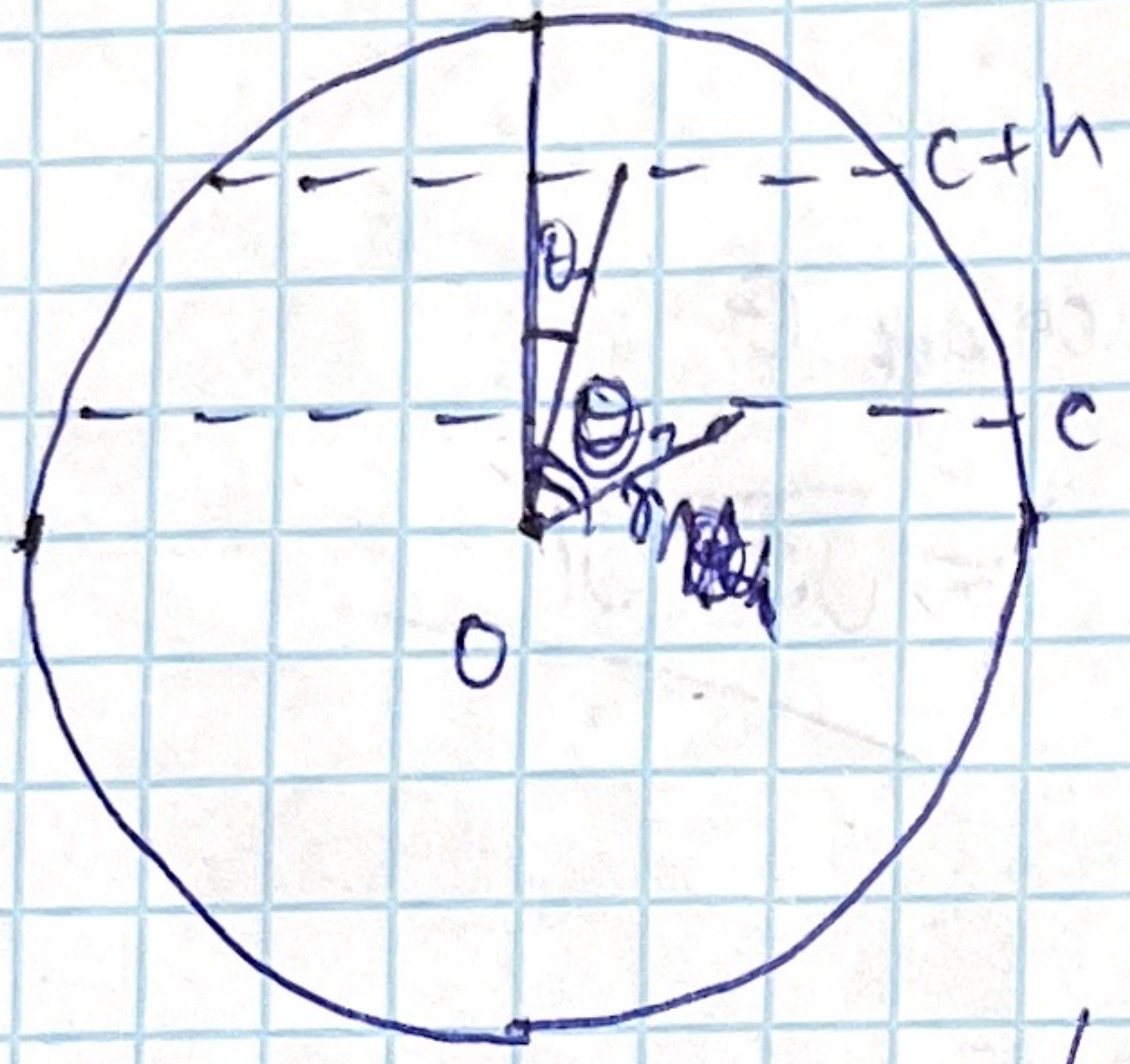
$$+\cancel{(\sin\varphi(b+a\cos\psi)\cdot a\cos\psi)}\ a\sin\varphi\cos\psi(b+a\cos\psi)\,\bar j+$$

$$+a\cos\psi\cos\varphi(b+a\cos\psi)\,\bar i$$

$$\Rightarrow |[\bar r'_\varphi;\bar r'_\psi]|=\sqrt{a^2\sin^2\psi(b+a\cos\psi)^2+a^2\sin^2\varphi\cos^2\psi(b+a\cos\psi)^2+a^2\cos^2\psi\cos^2\varphi(b+a\cos\psi)^2}=a\sqrt{\sin^2\psi(b+a\cos\psi)^2+\cos^2\psi(b+a\cos\psi)^2}=a(b+a\cos\psi)$$

$$\Rightarrow S=\iint\limits_G a(b+a\cos\psi)\,d\varphi d\psi=\int_0^{2\pi}d\varphi\int_0^{2\pi}a(b+a\cos\psi)\,d\psi=4\pi^2ab$$

№T2.



$x = R\sin\theta\cos\varphi$

$y = R\sin\theta\sin\varphi$, $0 \le \varphi \le 2\pi$, $0 \le \theta \le \pi$

$z = R\cos\theta$

$[\bar r'_\varphi ; \bar r'_\theta] = \cancel{\begin{vmatrix} \bar i & \bar j & \bar k \\ -R\sin\theta & & \end{vmatrix}}$

$$= \begin{vmatrix} \bar i & \bar j & \bar k \\ -R\sin\theta\sin\varphi & R\sin\theta\cos\varphi & 0 \\ R\cos\theta\cos\varphi & R\cos\theta\sin\varphi & -R\sin\theta \end{vmatrix} =$$

$$= -R^2\sin^2\theta\cos\varphi\,\bar i - R^2\sin^2\theta\sin\varphi\,\bar j - \bar k\cdot(R^2\sin\theta\cos\theta\cancel{\sin^2\varphi}$$

$$\cancel{+R^2\sin\theta\cos\theta\cos^2\varphi})$$

$$\Rightarrow |[\bar r'_\varphi ; \bar r'_\theta]| = R^2\sqrt{\sin^4\theta + \sin^2\theta\cos^2\theta} = R^2\sin\theta$$

$$S = \iint_S dS = \iint_G R^2\sin\theta\, d\varphi\, d\theta = \int_0^{2\pi} d\varphi \int_{\theta_1}^{\theta_2} R^2\sin\theta\, d\theta =$$

$$= 2\pi R^2\cdot(-\cos\theta)\Big|_{\theta_1}^{\theta_2} \ominus$$

Заметим, что ~~$\cos\theta_2 = \sqrt{1-\sin^2\theta_2} = \sqrt{1-\frac{(c+h)^2}{R^2}}$~~

$\cos\theta_2 = \sin(\frac{\pi}{2} - \theta_2) = \frac{c+h}{R}$; $\cos\theta_1 = \frac{c+h}{R}$

$$\ominus\ 2\pi R^2\cdot\frac{h}{R} = 2\pi Rh$$

№11.1(1)

1) $z = 1 - \frac{x}{4} - \frac{y}{2} \geq 0 \Rightarrow y \leq -\frac{x}{2} + 2,\ x \leq 4$ из $x+2y+4z=4$

$\Rightarrow x + y + z = 1 + \frac{3}{4}x + \frac{1}{2}y$

$$[\bar r_x'; \bar r_y'] = \begin{vmatrix} \bar i & \bar j & \bar k \\ 1 & 0 & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{2} \end{vmatrix} = \frac{1}{4}\bar i + \frac{1}{2}\bar j + \bar k$$

$$\Rightarrow |[\bar r_x'; \bar r_y']| = \sqrt{1 + \frac{1}{16} + \frac{1}{4}} = \frac{\sqrt{21}}{4}$$

$$\Rightarrow \iint_S (x+y+z)\,dS = \frac{\sqrt{21}}{4}\int_0^4 dx \int_0^{-\frac{x}{2}+2} \left(1 + \frac{3x}{4} + \frac{y}{2}\right) dy =$$

$$= \frac{\sqrt{21}}{4}\int_0^4 dx \left(\left(1 + \frac{3x}{4}\right) y \Big|_0^{-\frac{x}{2}+2} + \frac{y^2}{4}\Big|_0^{-\frac{x}{2}+2} \right) =$$

$$= \frac{\sqrt{21}}{4}\int_0^4 dx \left(\left(1 + \frac{3x}{4}\right)\left(2 - \frac{x}{2}\right) + \frac{1}{4}\left(-\frac{x}{2}+2\right)^2 \right) =$$

$$= \frac{\sqrt{21}}{4}\int_0^4 dx \left(2 - \frac{x}{2} + \frac{3x}{2} - \frac{3x^2}{8} + \frac{x^2}{16} - \frac{x}{2} + 1 \right) =$$

$$\frac{\sqrt{21}}{4}\int_0^4 \left(-\frac{5x^2}{16} + \frac{x}{2} + 3 \right) dx = \frac{\sqrt{21}}{64}\left(-\frac{5x^3}{3}\Big|_0^4 + 4x^2\Big|_0^4 + 48x\Big|_0^4 \right) =$$

$$= \frac{\sqrt{21}}{64}\left(-\frac{5\cdot 64}{3} + 64 + 3\cdot 64 \right) = \frac{7\sqrt{21}}{3}$$

№11.17(1)

1) $x = R\cos\varphi\cos\psi$

$y = R\sin\varphi\cos\psi$ $\qquad G = \left\{0 \le \varphi \le \frac{\pi}{2},\ 0 \le \psi \le \frac{\pi}{2}\right\}$

$z = R\sin\psi$

$$M = \iint\limits_S ds = \iint\limits_G R^2\cos\psi\,d\varphi\,d\psi = R^2\int\limits_0^{\frac{\pi}{2}} d\varphi\int\limits_0^{\frac{\pi}{2}}\cos\psi\,d\psi = \frac{\pi R^2}{2}$$

$$x_c = \frac{1}{M}\iint\limits_S x\,ds = \frac{R^3}{M}\int\limits_0^{\frac{\pi}{2}}\cos\varphi\,d\varphi\int\limits_0^{\frac{\pi}{2}}\cos^2\psi\,d\psi = \frac{2}{\pi R^2}\cdot\frac{\pi R^3}{4} = \frac{R}{2}$$

Из симметрии $x_c = y_c = z_c = \frac{R}{2}$

№11.37(1)

1) $x = a\cos\varphi\cos\psi$

$y = b\sin\varphi\cos\psi$

$z = c\sin\psi$

$$\iint\limits_S yz\,dz\,dx = \iint\limits_G \begin{vmatrix} 0 & \begin{matrix} bc\sin\varphi\cos\varphi\cos\psi \\ \cancel{abc\cos\varphi\sin\varphi\cos^2\psi} \end{matrix} & 0 \\ -a\sin\varphi\cos\psi & b\cos\varphi\cos\psi & 0 \\ -a\cos\varphi\sin\psi & -b\sin\varphi\sin\psi & c\cos\psi \end{vmatrix} yz\,d\varphi\,d\psi =$$

$$= abc^2\int\limits_0^{2\pi}\sin^2\varphi\,d\varphi\int\limits_0^{\frac{\pi}{2}}\sin\psi\cos^3\psi\,d\psi = abc^2\pi\int\limits_0^{\frac{\pi}{2}}\cos^3\psi\,d\cos\psi = \frac{1}{4}\pi abc^2$$

№ 11.38.

$x = \frac{1}{\sqrt{2}} \quad x = \rho$

$y = \rho\cos\varphi$

$z = \rho\sin\varphi$

$0 \le \varphi \le 2\pi;\ 0 \le \rho \le H$

$$\iint_S (2x^2+y^2+z^2)dydz = \int_0^{2\pi} d\varphi \int_0^H \begin{vmatrix} 2\rho^2+\rho^2 & 0 & 0 \\ 0 & -\rho\sin\varphi & \rho\cos\varphi \\ 1 & \cos\varphi & \sin\varphi \end{vmatrix} d\rho =$$

$$= \int_0^{2\pi} d\varphi \int_0^H 3\rho^2(-\rho)d\rho = 2\pi(-3)\int_0^H \rho^3 d\rho = \cancel{6\pi} - \frac{3}{2}\pi H^4$$

№ 11.42

$x = \rho\cos\varphi$

$y = \rho\sin\varphi \quad , \rho \le 1, \ \varphi \in [0; 2\pi]$

$z = \rho$

$$\iint_S x^6 dydz + y^4 dzdx + z^2 dxdy = -\int_0^{2\pi} d\varphi \int_0^1 \begin{vmatrix} \rho^6\cos^6\varphi & \rho^4\sin^4\varphi & \rho^2 \\ \cos\varphi & \sin\varphi & 1 \\ -\rho\sin\varphi & \rho\cos\varphi & 0 \end{vmatrix} d\rho =$$

$$= -\int_0^{2\pi} d\varphi \int_0^1 (-\rho^5\sin^5\varphi + \rho^4 - \rho^7\cos^7\varphi)\rho\, d\rho = \int_{2\pi}^0 \Big(-\underbrace{\frac{\sin^5\varphi}{6}}_{=0} - \underbrace{\frac{\cos^7\varphi}{7}}_{=0} + \frac{1}{6}\Big) d\varphi =$$

$$= -\frac{\pi}{3}$$

III. Элементы теории поля

№ 3.44(1)

$\text{grad} f = (12x^3+y,\ 3y^2+x)^T$

$\text{grad} f(M) = (14, 13)^T$

$\bar{e} = \frac{\overline{AM}}{AM} = \left(-\frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}}\right)^T$

$\frac{\partial f}{\partial \bar{e}} = (\text{grad} f(M), \bar{e}) = -\frac{1}{\sqrt{2}}$

№ 3.47(1)

$\text{grad} f = (y^2-12x^3y^5\ \ 2xy-15x^4y^4)^T$

$\text{grad} f(M) = (-11\ -13)^T$

$\bar{e} = (\cos\alpha\ \sin\alpha)^T \Rightarrow \frac{\partial f}{\partial \bar{e}} = (\bar{e}, \text{grad} f) \le |\text{grad} f|$

$|\text{grad} f(M)| = \sqrt{11^2+13^2} = \sqrt{290}$

№ 12.13.

$\text{grad} f(u) = \frac{\partial f(u)}{\partial x}\bar{i} + \frac{\partial f(u)}{\partial y}\bar{j} + \frac{\partial f(u)}{\partial z}\bar{k} =$

$= \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}\bar{i} + \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}\bar{j} + \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z}\bar{k} = f'(u)\,\text{grad}\,u$

№ 12.19

$f'(r) = \frac{\partial f}{\partial r}$ $\cancel{f(x,y,z) = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial z}}$

$\cancel{= \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}}\quad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}\quad \frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}}$

$\cancel{\overline{F} = \frac{1}{\sqrt{x^2+y^2+z^2}}\,\overline{r}}$

$\cancel{\nabla f(r) = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x}\overline{i} + \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial y}\overline{j} + \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial z}\overline{k}} = f'(r)$

$\nabla f(r) = \operatorname{grad} f(r) = f'(r)\operatorname{grad} r = f'(r)\cdot\frac{1}{\sqrt{x^2+y^2+z^2}}(x\overline{i}+y\overline{j}+z\overline{k}) =$

$= f'(r)\cdot\frac{\overline{r}}{r}$

№12.15 (3, 5, 6)

$\cancel{3)\ \operatorname{grad} u = \operatorname{grad}\frac{1}{r} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} +}$

3) $\operatorname{grad} u(r) = u'(r)\cdot\frac{\overline{r}}{r} = -\frac{\overline{r}}{r^3}$

5) $\cancel{\operatorname{grad}(\overline{a};\overline{r}) = \operatorname{grad}\overline{a}\,(\overline{a};\nabla)\overline{r} + (\overline{r};\nabla)\overline{a} + [\overline{a};\operatorname{rot}\overline{r}] + [\overline{r};\operatorname{rot}\overline{a}] = a_x\frac{\partial r}{\partial x}\overline{i} + a_y\frac{\partial r}{\partial y}\overline{j} + a_z\frac{\partial r}{\partial z}\overline{k} + [\overline{a};\nabla;\overline{r}] =}$

$\cancel{= \left(\overline{a};\frac{\overline{r}}{r}\right) + \nabla(\overline{a};\overline{r}) - \overline{r}(\overline{a};\nabla)} = \cancel{(\overline{a};\nabla)\overline{r}} + \nabla(\overline{a};\overline{r}) - \overline{r}(\overline{a};\nabla) =$

$= \nabla(\overline{a};\overline{r})$

5) $\cancel{\operatorname{grad}}\, u = (a_x x\ \ a_y y\ \ a_z z)^T \Rightarrow \operatorname{grad} u = \overline{a}$

6) $u = (\overline{a}, \overline{b}, \overline{r}) = (\overline{c}, \overline{r})$

$\operatorname{grad}(\overline{c};\overline{r}) = \overline{c} = [\overline{a};\overline{b}]$

№12.37(2)

$$2)\ \operatorname{div}(u\bar a) = (\nabla; u\bar a) = \cancel{\nabla u a_x} + \frac{\partial(ua_x)}{\partial x}\bar i + \frac{\partial(ua_y)}{\partial y}\bar j +$$

$$+\frac{\partial(ua_z)}{\partial z}\bar k = (u'_x a_x + ua'_x)\bar i + (u'_y a_y + ua'_y)\bar j +$$

$$+(u'_z a_z + ua'_z)\bar k = u'_x a_x \bar i + u'_y a_y \bar j + u'_z a_z \bar k +$$

$$+ua'_x\bar i + ua'_y\bar j + ua'_z\bar k = (\operatorname{grad}u, \bar a) + u\operatorname{div}\bar a$$

$$\operatorname{div}(u\bar a) = (\nabla; u\bar a) = (\nabla_u; u\bar a) + (\nabla_{\bar a}; u\bar a) =$$

$$= (\operatorname{grad}u; \bar a) + u\operatorname{div}\bar a.$$

№12.39

$$\cancel{\operatorname{div}\operatorname{grad}u = \operatorname{div}\begin{vmatrix} \bar i & \bar j & \bar k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u' & & \end{vmatrix}}$$

$$\operatorname{div}\operatorname{grad}u = \operatorname{div}\left(\frac{\partial u}{\partial x}\bar i + \frac{\partial u}{\partial y}\bar j + \frac{\partial u}{\partial z}\bar k\right) =$$

$$= \frac{\partial^2 u}{\partial x^2}\bar i + \frac{\partial^2 u}{\partial y^2}\bar j + \frac{\partial^2 u}{\partial z^2}\bar k = \Delta u$$

№12.40(1)

$$\cancel{1)\ \operatorname{div}(u\operatorname{grad}u) = \operatorname{div}(u\nabla u) = \operatorname{div}\left(u\frac{\partial u}{\partial x}\bar i + u\frac{\partial u}{\partial y}\bar j + u\frac{\partial u}{\partial z}\bar k\right) = (\nabla; u\nabla u)}$$

$$1)\ \operatorname{div}(u\operatorname{grad}u) = \cancel{d}\ (\nabla; u\operatorname{grad}u) =$$

$$= (\nabla_{gr}; u\operatorname{grad}u) + (\nabla_u; u\operatorname{grad}u) =$$

$= u\Delta u + (\mathrm{grad}\, u ; \mathrm{grad}\, u) = u\Delta u + |\mathrm{grad}\, u|^2 =$

$= (\nabla u)^2 + u\Delta u$

№ 12.41(3; 6; 7)

~~3) $\mathrm{div}\,\mathrm{grad}\, r^2 = \Delta u = \Delta r$~~ ⊖

$\frac{\partial r}{\partial x}$ $\cancel{= \frac{x}{\sqrt{x^2+y^2+z^2}}}$; $\frac{\partial^2 r}{\partial x}$ =

3) $\mathrm{div}\, r\bar{c} = (\mathrm{grad}\, r ; \bar{c}) + r\,\mathrm{div}\,\overset{=0}{\bar{c}}$ ⊖ $=$

$= \frac{1}{r}(x\bar{i} + y\bar{j} + z\bar{k} ; \bar{c}) = \frac{(\bar{r};\bar{c})}{r}$

6) $\mathrm{div}(f(r)\bar{c}) = f(r)\,\mathrm{div}\,\overset{=0}{\bar{c}} + (\mathrm{grad}\, f(r) ; \bar{c}) = f'(r)(\mathrm{grad}\, r ; \bar{c}) =$

$= f'(r)\frac{(\bar{r};\bar{c})}{r}$

rot $\bar{r}$ =

7) $\mathrm{div}[\bar{c};\bar{r}] = (\bar{r};\mathrm{rot}\,\overset{=0}{\bar{c}}) - (\bar{c};\mathrm{rot}\,\bar{r})$ ⊖

$\mathrm{rot}\,\bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)\bar{i} + \ldots$ остальные тоже 0, т.к. независимые переменные

$\Rightarrow$ ⊖ $\cancel{=}$ 0

№ 49(3; 5; 6)

3) $\operatorname{rot}(u\bar{a}) = [\nabla; u\bar{a}] = [\nabla_u; u\bar{a}] + [\nabla_{\bar{a}}; u\bar{a}] =$

$= [\operatorname{grad} u; \bar{a}] + u \operatorname{rot}\bar{a}$

5) $\operatorname{rot}[\bar{a}; \bar{b}] = [\nabla; \bar{a}; \bar{b}] = \cancel{\bar{a}(\nabla; \bar{b}) - \bar{c}(\nabla; \bar{a})}$

$= [\nabla_a; \bar{a}; \bar{b}] + [\nabla_b; \bar{a}; \bar{b}] = \bar{a}(\nabla_a; \bar{b}) - \bar{b}(\nabla_a; \bar{a}) +$

$+ \bar{a}(\nabla_b; \bar{b}) - \bar{b}(\nabla_b; \bar{b}) = \bar{a}\operatorname{div}\bar{b} - \bar{b}\operatorname{div}\bar{a} + (\bar{b}; \nabla)\bar{a} - (\bar{a}; \nabla)\bar{b}$

$\cancel{6)\ \operatorname{div}[\bar{a}; \bar{b}] = (\nabla; [\bar{a}; \bar{b}]) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}}$

6) $\operatorname{div}[\bar{a}; \bar{b}] = (\nabla_a; \bar{a}; \bar{b}) + (\nabla_b; \bar{a}; \bar{b}) =$

$= (\bar{b}; \nabla_a; \bar{a}) - (\bar{a}; \nabla_b; \bar{b}) = (\bar{b}; \operatorname{rot}\bar{a}) - (\bar{a}; \operatorname{rot}\bar{b})$

№ 12.50(5)

из 12.19

5) $\operatorname{rot}(u(r)\bar{r}) = u(r)\underbrace{\operatorname{rot}\bar{r}}_{=0\ (\text{из } 12.41(7))} + [\operatorname{grad} u(r); \bar{a}] =$

$= u'(r)[\cancel{\operatorname{grad} r}; \bar{a}] = \frac{u'(r)}{r}[\bar{r}; \bar{a}]$ ($\frac{\bar{r}}{r}$)

№ 12.54(2)

2) $\operatorname{rot}[\bar{r}; [\bar{c}; \bar{r}]] = \operatorname{rot}(\bar{c}\underbrace{(\bar{r}; \bar{r})}_{r^2} - \bar{r}(\bar{r}; \bar{c})) =$

$= \cancel{[\nabla; -\bar{r}(\bar{r}; \bar{c})] = (\bar{r}; \bar{c}) \operatorname{rot}(-\bar{r})}\ \oplus$

$\cancel{\operatorname{rot}(-\bar{r}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x & -y & -z \end{vmatrix}}$

$= \dots$

$[\nabla; \bar{c}r^2 - \bar{r}(\bar{r};\bar{c})] = [\nabla r^2; \bar{c}] - [\nabla(\bar{r};\bar{c}); \bar{r}] =$

$= 2[\bar{r};\bar{c}] + [\bar{r};\bar{c}] = 3[\bar{r};\bar{c}]$

IV. Формулы Гаусса-Остроградского и Стокса.

№11.45(2;3)

2) $\bar{a} = (5x+y, 0, z)$

$\operatorname{div}\bar{a} = 6$

← объём эллипсоида

$$\iint\limits_S (5x+y)\,dy\,dz + z\,dx\,dy = \iiint\limits_G 6\,dx\,dy\,dz = 6\iiint\limits_G dx\,dy\,dz =$$

$$= 6\cdot\frac{4}{3}\pi\cdot 2\cdot 3\cdot 1 = 48\pi$$

3) $\iint = \quad x = r\cos\varphi\cos\psi, \quad y = r\sin\varphi\cos\psi, \quad z = r\sin\psi, \quad 1<r<2$

$$\iint\limits_S = 6\iiint\limits_G dx\,dy\,dz = 6\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\psi\,d\psi \int_1^2 r^2\,dr =$$

$$= 12\pi\cdot 2\cdot\frac{r^3}{3}\Big|_1^2 = 56\pi$$

№11.52(3)

3) $\overline{a}=(x^2\ \ y^2\ \ z^2)$

$div\,\overline{a}=(2x\ \ 2y\ \ 2z)$

$$\iint\limits_S = -\iiint\limits_G 2(x+y+z)\,dx\,dy\,dz \ominus$$

$\varphi\in[0;2\pi]$

$x=r\cos\varphi\ ;y=r\sin\varphi;\ z=h\le H\ ,h\in[0;H]$

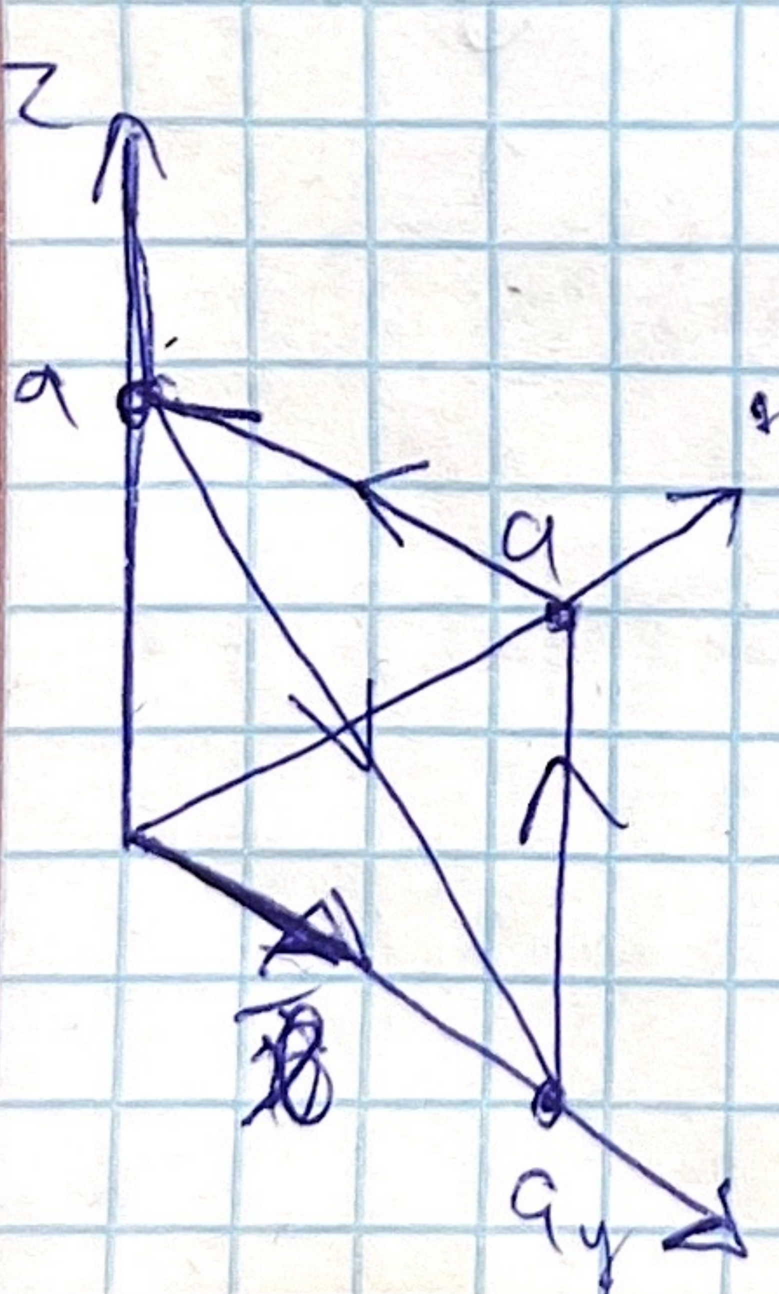
$$\cancel{\ominus -2\int_0^{2\pi}\sin\varphi\cos\varphi\,d\varphi\int_0^H dh\int_0^h r^3dr = -2\cdot\frac{\sin^2\varphi}{2}\Big|_0^{2\pi}}$$

$$\ominus -2\int_0^{2\pi}d\varphi\int_0^H dh\int_0^h (r\cos\varphi+r\sin\varphi+h)\,r\,dr =$$

$$= -2\int_0^{2\pi}d\varphi\int_0^H dh\left(\frac{r^3}{3}\cos\varphi\Big|_0^h+\frac{r^3}{3}\sin\varphi\Big|_0^h+h\frac{r^2}{2}\Big|_0^h\right) =$$

$$= -2\int_0^{2\pi}d\varphi\int_0^H\Big(\frac{h^3}{3}\underbrace{(\cos\varphi+\sin\varphi)}_{\to 0}+\frac{h^3}{2}\Big)dh = -2\int_0^{2\pi}\frac{H^4}{8}d\varphi = -\frac{\pi H^4}{2}$$

№11.62



$\overline{\nu}=\frac{1}{\sqrt{3}}(1\ \ 1\ \ 1)^T$

$\cos(\widehat{\overline{\nu},\overline{b}}) = \frac{1}{\sqrt{3}} > 0$

$$\operatorname{rot}\bar{a} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2 \end{vmatrix} = -2z\bar{i} - 2x\bar{j} - 2y\bar{k}$$

$$\int_\Gamma = \iint_S (\operatorname{rot}\bar{a}; \bar{\nu})\, ds = -\frac{2}{\sqrt{3}} \iint_S (\overset{a}{x+y+z})\, ds =$$

ур-ие

$$= -\frac{2a}{\sqrt{3}} \iint_S ds = -\frac{2a}{\sqrt{3}} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \cdot (\sqrt{2}a)^2 = \underline{-a^3}$$

№ 11.63

$$\operatorname{rot}\bar{a} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & z \end{vmatrix} = \bar{0}$$

Как и в Т1 $G_1 = G \setminus G_0$

$$\iint_{\partial G_0 \downarrow} + \iint_{L\uparrow} = \iint_{\partial G_1 \uparrow} = \iint_S (\operatorname{rot}\bar{a}; d\bar{S})$$

$$\operatorname{rot}\bar{a} = 0 \Rightarrow \iint_S (\operatorname{rot}\bar{a}; d\bar{S}) = 0 \Rightarrow \iint_{L\uparrow} = \iint_{\partial G_0 \uparrow}$$

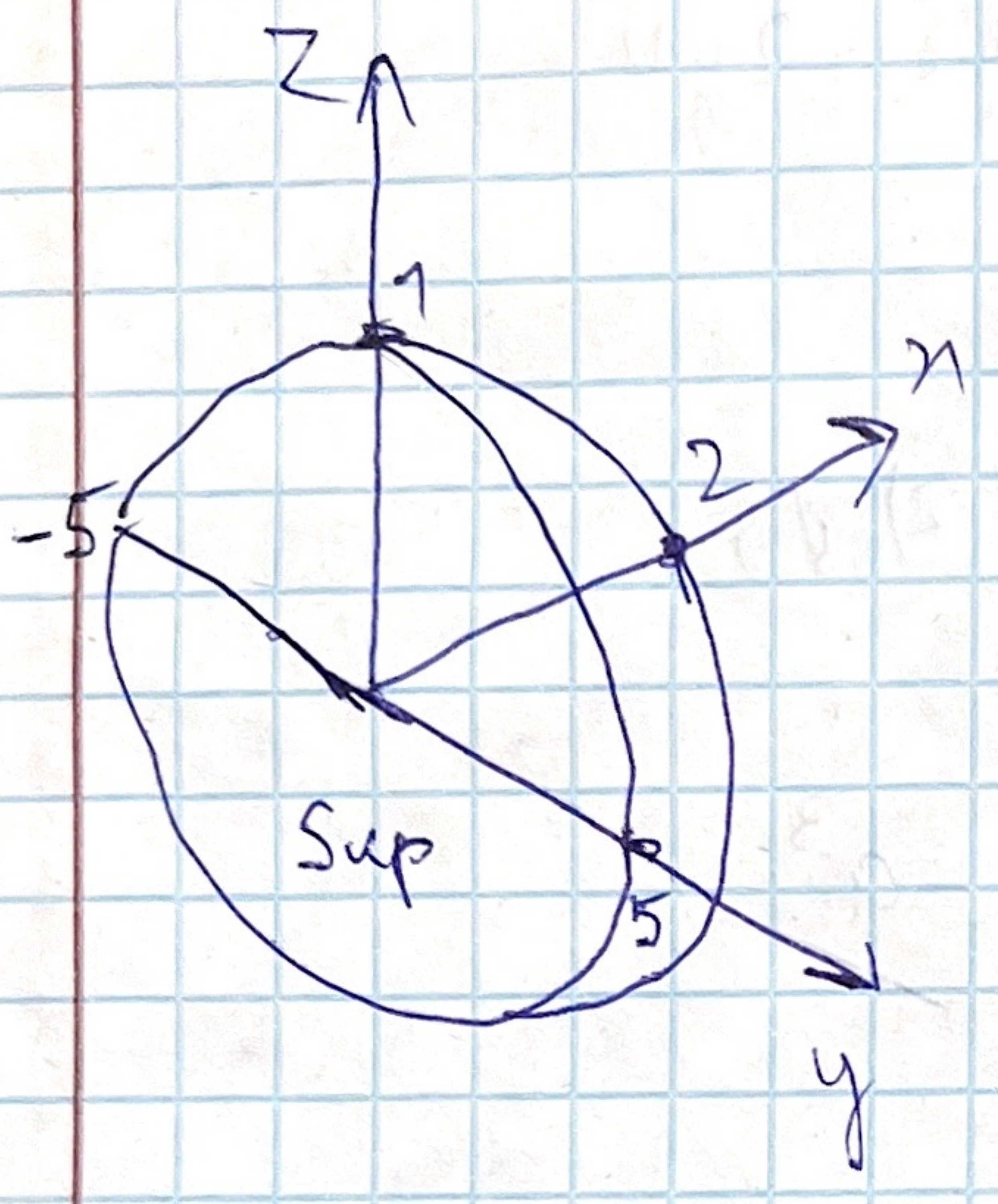
$x = R\cos\varphi$

$y = R\sin\varphi$

$z = -R(\cos\varphi + \sin\varphi)$

$$\Rightarrow \int_0^{2\pi} (1 + \underbrace{R^2\cos 2\varphi}_{=0})\, d\varphi = \underline{2\pi}$$

№3.



$S = S_{up} + S_{пов}$ $\quad z^2 + \frac{y^2}{5} = 1$

$S_{up}: x = 0;\ \cancel{\frac{x^2}{4}} + \frac{y^2}{25} = \cancel{1}$

$\iint_{S_{up}} 3y\,dy\,dz = \left| \begin{matrix} y = 5r\sin\varphi \\ z = r\cos\varphi \\ |J| = 5r \end{matrix} \right| =$

$= \iint_{S_{up}} 15r\sin\varphi \cdot 5r \cdot dr\,d\varphi =$

$= \int_0^{2\pi} \underbrace{\sin\varphi\,d\varphi}_{0} \int_0^1 65r^2\,dr = 0$

$\overline{a} = (2x + 3y \quad x + y + z \quad x + 2y + 3z)^T$, $\operatorname{div}\overline{a} = 2 + 1 + 3 = 6$

$x = 2r\cos\varphi\cos\psi$

$y = 5r\sin\varphi\cos\psi$ $\quad \Rightarrow \iint_S = 6\iiint_G dx\,dy\,dz =$

$z = r\sin\psi$

$= 60\int_0^{2\pi} d\varphi \int_0^{\pi/2} \cos\psi\,d\psi \int_0^1 r^2\,dr = 40\pi$

$\Rightarrow S_{вн} = 40\pi$

№12.68(4)

$\operatorname{div} \bar{r} = 3$

$$\iint_S (\bar{a}; \bar{n})\, dS = \iint_S (\bar{a}; d\bar{S}) = \frac{1}{R^3} \iiint_G \operatorname{div} \bar{R}\, dx\, dy\, dz = \frac{1}{R^3} 3V_G = \frac{3}{R^3} \cdot \frac{4}{3} \pi R^3 = \underline{4\pi}$$

№12.94(4)

$\bar{a} = (y \quad -x \quad z)^T$

$$\operatorname{rot} \bar{a} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & z \end{vmatrix} = (0 \quad 0 \quad -2)^T$$

$$\int_\Gamma (\bar{a}; d\bar{r}) = \iint_S (\operatorname{rot} \bar{a}; d\bar{S}) = -2 \iint_S dS = \underline{-4\pi}$$

№12.104(1;2)

$\operatorname{rot} \bar{a} = \bar{0}$

1) $G = \{x > 0\}$ – пов. односв. $\Rightarrow$ да

2) $G = \{x \neq 0, y \neq 0\}$

$$\operatorname{rot} \bar{a} = \nabla \times \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

$$\int_\Gamma (\bar{a}; d\bar{r}) = \frac{2I}{R^2} \iint_S (\operatorname{rot} \bar{a}, d\bar{S}) = \frac{2I}{R^2} \cdot 2\pi R^2 = 4\pi I \neq 0 \Rightarrow \underline{\text{непот}}.$$

№12.112(1)

$G = R \setminus \{0\}$ – не явл. объемно односв.

$\operatorname{rot} \bar{a} = \bar{0}$, G – пов. односв. $\Rightarrow$ $\underline{\text{потенц}}$.

$$\operatorname{div} \bar{a} = \frac{3}{r^3} - \frac{3}{r^4} \cdot r = 0$$

1. $r>0$ - об. одноc $\Rightarrow$ соленоид.

2. $r\geq 0$ - не явл. об. одноc

$$\iint\limits_S \left(\frac{\bar r}{r^3}, d\bar S\right) = \frac{1}{R^3}\iint\limits_S (\bar r, d\bar S) = \frac{1}{R^3}\iiint\limits_G \operatorname{div}\bar r \, dx\,dy\,dz =$$

$$= \frac{3}{R^3}\cdot\frac{4}{3}\pi R^3 = 4\pi \neq 0 \Rightarrow$$ не соленоид