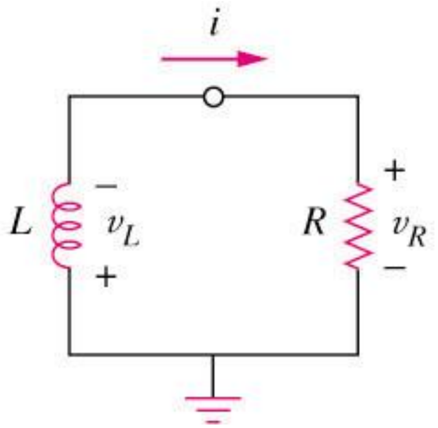

Lecture 11: First-Order Circuits. RL Circuit

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First-Order Circuits. RL Circuit

1 Introduction

2 The Source-Free RL Circuit

3 Time constant

4 The power and energy in RL Circuits

5 Problem solving

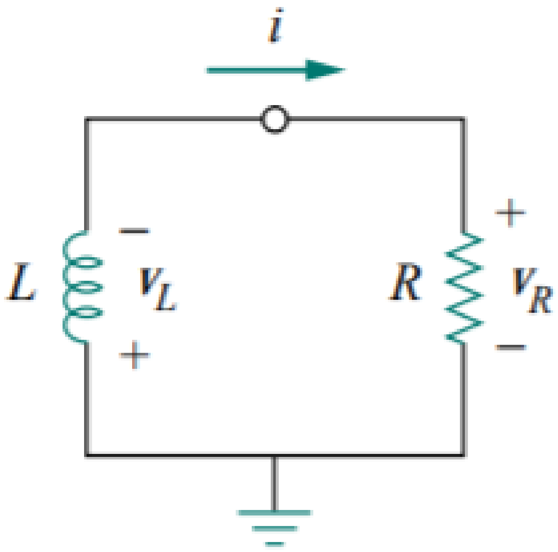
1 Introduction

Consider the series connection of a resistor and an inductor, as shown in figure. Our goal is to determine the circuit response, which we will assume to be the current i(t) through the inductor. We select the inductor current as the response in order to take advantage of the idea that the inductor current cannot change instantaneously. At t = 0, we assume that the inductor has an initial current I_0, or

$$i(0) = I_0$$

with the corresponding energy stored in the inductor as

$$w(0) = \frac{1}{2} L I_0^2$$



A source-free RL circuit.

2 The Source-Free RL Circuit

Applying KVL around the loop in Fig

$$v_L + v_R = 0$$

But $v_L = L\,di/dt$ and $v_R = iR$. Thus,

$$L\frac{di}{dt} + Ri = 0$$

or

$$\frac{di}{dt} + \frac{R}{L}i = 0$$

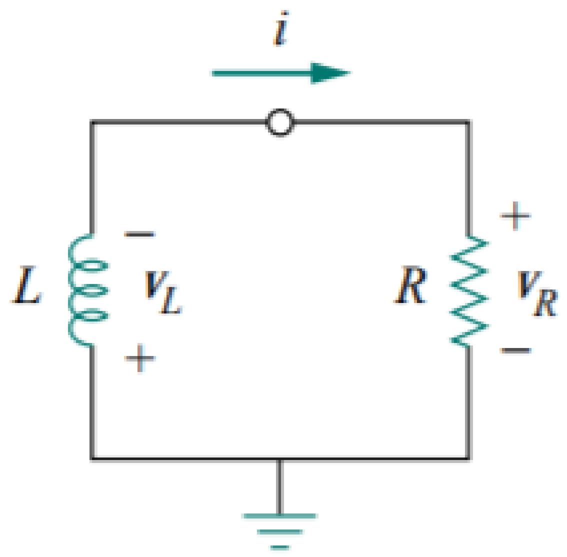
Rearranging terms and integrating gives

$$\int_{I_0}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L}dt$$

$$\ln i\Big|_{I_0}^{i(t)} = -\frac{Rt}{L}\Big|_0^t \quad \Longrightarrow \quad \ln i(t) - \ln I_0 = -\frac{Rt}{L} + 0$$

or

$$\ln\frac{i(t)}{I_0} = -\frac{Rt}{L}$$



A source-free RL circuit.

2 The Source-Free RL Circuit (1)

Rearranging terms and integrating gives

$$\int_{I_0}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L} dt$$

$$\ln i \Big|_{I_0}^{i(t)} = -\frac{Rt}{L}\Big|_0^t \quad \Longrightarrow \quad \ln i(t) - \ln I_0 = -\frac{Rt}{L} + 0$$

or

$$\ln \frac{i(t)}{I_0} = -\frac{Rt}{L}$$

Taking the powers of e, we have

$$i(t) = I_0 e^{-Rt/L}$$

This shows that the natural response of the RL circuit is an exponential decay of the initial current. The current response is shown in figure

3 Time constant

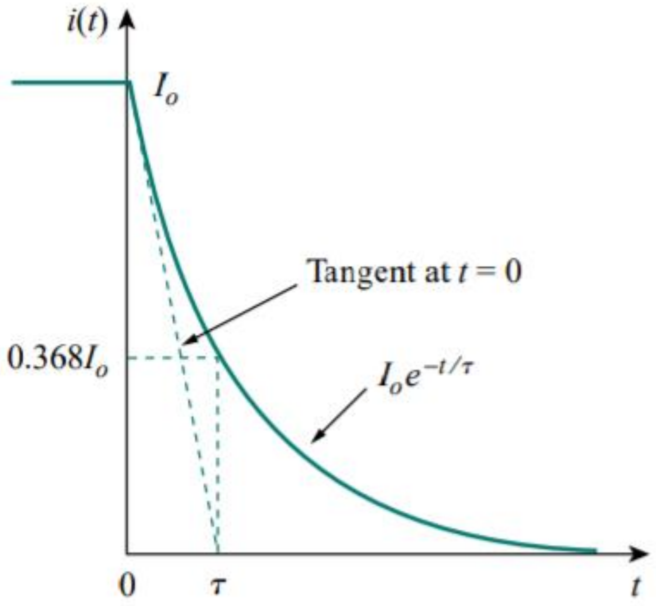
The current response is shown in figure
The smaller the time constant τ of a circuit, the faster the rate of decay of the response. The larger the time constant, the slower the rate of decay of the response. At any rate, the response decays to less than 1 percent of its initial value (i.e., reaches steady state) after 5τ
.

It is evident from that the time constant for the RL circuit is

$$\boxed{\tau = \frac{L}{R}}$$

with τ again having the unit of seconds. Thus,

$$\boxed{i(t) = I_0 e^{-t/\tau}}$$



The current response of the RL circuit.

4 The power and energy

With the current, we can find the voltage across the resistor as

$$v_R(t) = iR = I_0 R e^{-t/\tau}$$

The power dissipated in the resistor is

$$p = v_R i = I_0^2 R e^{-2t/\tau}$$

The energy absorbed by the resistor is

$$w_R(t) = \int_0^t p\,dt = \int_0^t I_0^2 R e^{-2t/\tau}\,dt = -\frac{1}{2}\tau I_0^2 R e^{-2t/\tau}\Big|_0^t, \quad \tau = \frac{L}{R}$$

or

$$w_R(t) = \frac{1}{2} L I_0^2 (1 - e^{-2t/\tau})$$

Note that as $t \to \infty$, $w_R(\infty) \to \frac{1}{2} L I_0^2$, which is the same as $w_L(0)$, the initial energy stored in the inductor as in $w(0) = \frac{1}{2} L I_0^2$. Again, the energy initially stored in the inductor is eventually dissipated in the resistor.

In summary

When a circuit has a single inductor and several resistors and dependent sources, the Thevenin equivalent can be found at the terminals of the inductor to form a simple RL circuit. Also, one can use Thevenin's theorem when several inductors can be combined to form a single equivalent inductor.

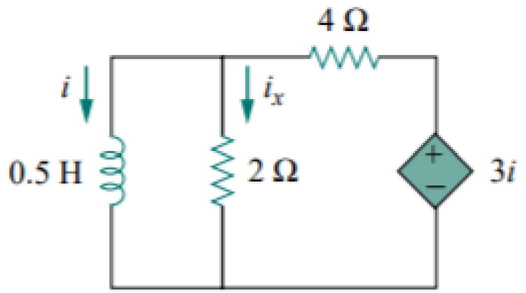
The Key to Working with a Source-free RL Circuit is to Find:

1. The initial current $i(0) = I_0$ through the inductor.
2. The time constant τ of the circuit.

With the two items, we obtain the response as the inductor current $i_L(t) = i(t) = i(0)e^{-t/\tau}$. Once we determine the inductor current i_L, other variables (inductor voltage v_L, resistor voltage v_R, and resistor current i_R) can be obtained. Note that in general, R in Eq. $\tau = \frac{L}{R}$ is the Thevenin resistance at the terminals of the inductor.

5 Problem (1)

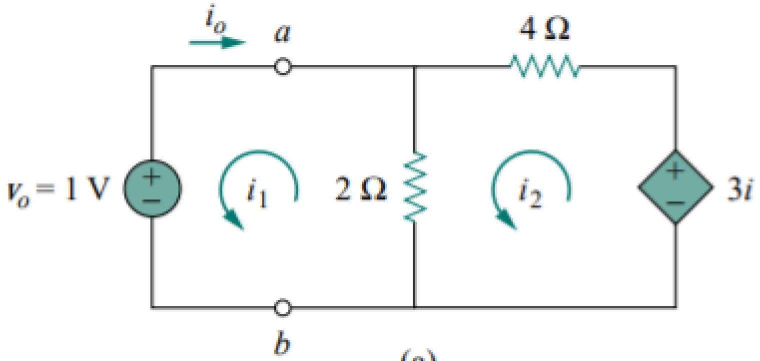
Assuming that $i(0) = 10$ A, calculate $i(t)$ and $i_x(t)$ in the circuit in Fig.



Applying KVL to the two loops results in

$$2(i_1 - i_2) + 1 = 0 \quad \Longrightarrow \quad i_1 - i_2 = -\frac{1}{2}$$

$$6i_2 - 2i_1 - 3i_1 = 0 \quad \Longrightarrow \quad i_2 = \frac{5}{6}i_1$$



(a)

$$i_1 = -3 \text{ A}, \qquad i_o = -i_1 = 3 \text{ A}$$

Hence,

$$R_{eq} = R_{Th} = \frac{v_o}{i_o} = \frac{1}{3}\ \Omega$$

The time constant is

$$\tau = \frac{L}{R_{eq}} = \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2} \text{ s}$$

Thus, the current through the inductor is

$$i(t) = i(0)e^{-t/\tau} = 10e^{-(2/3)t} \text{ A}, \qquad t > 0$$

5 Problem (1.1)

$$\frac{1}{2}\frac{di_1}{dt} + 2(i_1 - i_2) = 0$$

or

$$\frac{di_1}{dt} + 4i_1 - 4i_2 = 0$$

For loop 2,

$$6i_2 - 2i_1 - 3i_1 = 0 \quad \Longrightarrow \quad i_2 = \frac{5}{6}i_1$$

Substituting Eq. (7.3.4) into Eq. (7.3.3) gives

$$\frac{di_1}{dt} + \frac{2}{3}i_1 = 0$$

Rearranging terms,

$$\frac{di_1}{i_1} = -\frac{2}{3}dt$$

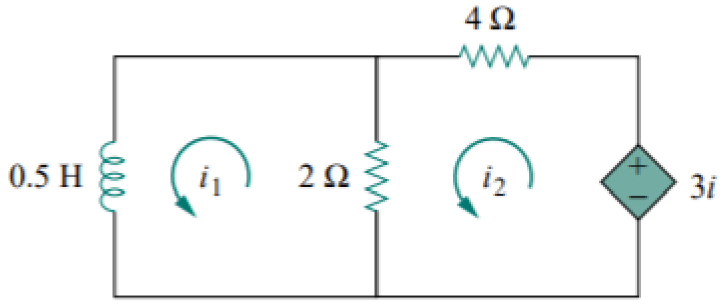
Since $i_1 = i$, we may replace i_1 with i and integrate:

$$\ln i\Big|_{i(0)}^{i(t)} = -\frac{2}{3}t\Big|_0^t$$

or

$$\ln\frac{i(t)}{i(0)} = -\frac{2}{3}t$$

We may directly apply KVL to the circuit



(b)

Taking the powers of e, we finally obtain

$$i(t) = i(0)e^{-(2/3)t} = 10e^{-(2/3)t} \text{ A}, \qquad t > 0$$

which is the same as by Method 1.

The voltage across the inductor is

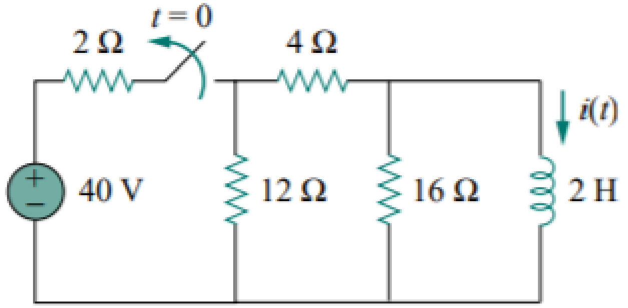
$$v = L\frac{di}{dt} = 0.5(10)\left(-\frac{2}{3}\right)e^{-(2/3)t} = -\frac{10}{3}e^{-(2/3)t} \text{ V}$$

Since the inductor and the 2-Ω resistor are in parallel,

$$i_x(t) = \frac{v}{2} = -1.667e^{-(2/3)t} \text{ A}, \qquad t > 0$$

5 Problem (2)

The switch in the circuit has been closed for a long time. At t = 0, the switch is opened. Calculate i(t) for t > 0.



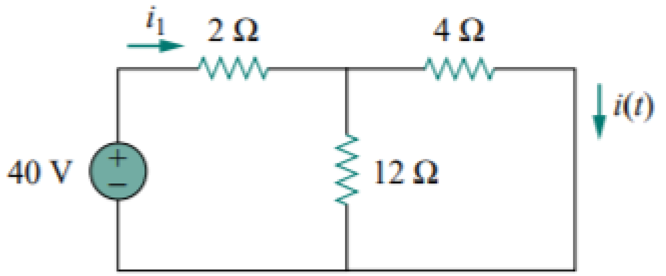
$$\frac{4 \times 12}{4 + 12} = 3\ \Omega$$

Hence,

$$i_1 = \frac{40}{2 + 3} = 8\text{ A}$$

We obtain $i(t)$ from i_1 in Fig. 7.17(a) using current division, by writing

$$i(t) = \frac{12}{12 + 4} i_1 = 6\text{ A}, \qquad t < 0$$



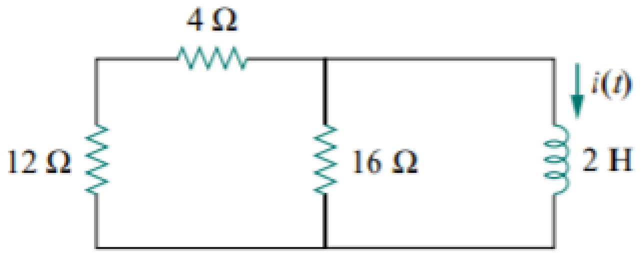
(a)

Since the current through an inductor cannot change instantaneously,

$$i(0) = i(0^-) = 6\text{ A}$$

When t > 0, the switch is open and the voltage source is disconnected. We now have the RL circuit in Fig. (b). Combining the resistors, we have

$$R_{\text{eq}} = (12 + 4) \parallel 16 = 8\ \Omega$$



(b)

The time constant is

$$\tau = \frac{L}{R_{\text{eq}}} = \frac{2}{8} = \frac{1}{4}\text{ s}$$

Thus,

$$i(t) = i(0)e^{-t/\tau} = 6e^{-4t}\text{ A}$$

5 Problem (3)

In the circuit shown in Fig., find io, vo, and i for all time, assuming that the switch was open for a long time.

It is better to first find the inductor current i and then obtain other quantities from it.
For t < 0, the switch is open. Since the inductor acts like a short circuit to dc, the 6-Ohm resistor is short-circuited, so that we have the circuit shown in Fig(a). Hence, io = 0, and

$$i(t) = \frac{10}{2+3} = 2\text{ A}, \qquad t < 0$$

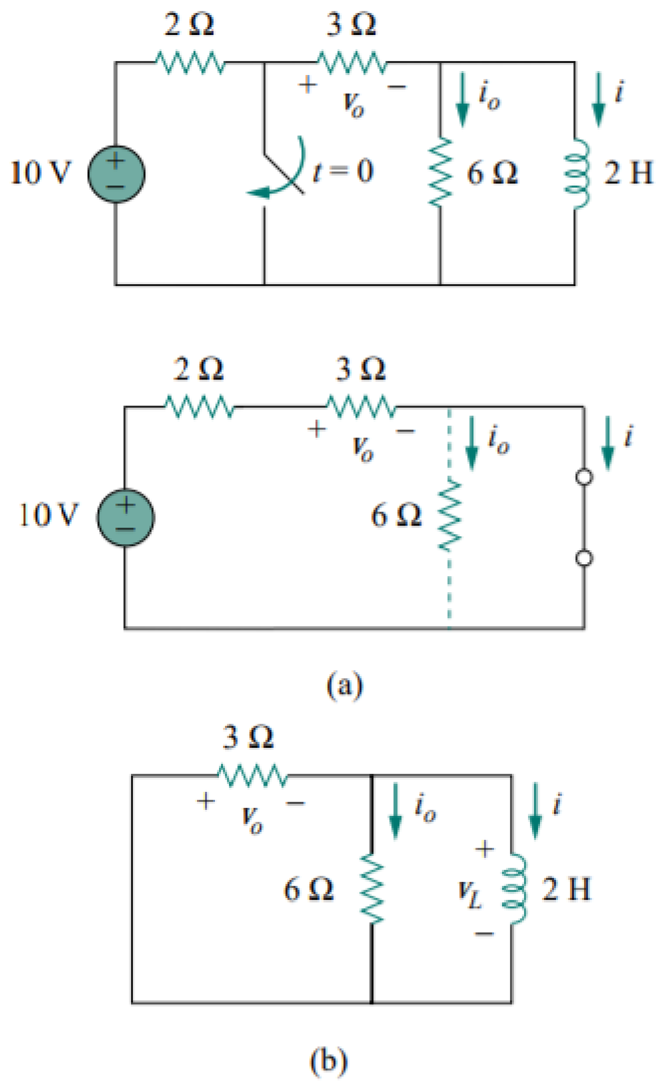
$$v_o(t) = 3i(t) = 6\text{ V}, \qquad t < 0$$

Thus, i(0) = 2. For t > 0, the switch is closed, so that the voltage source is short-circuited. We now have a source-free RL circuit as shown in Fig(b). At the inductor terminals,

$$R_{\text{Th}} = 3 \parallel 6 = 2\ \Omega$$

so that the time constant is

$$\tau = \frac{L}{R_{\text{Th}}} = 1\text{ s}$$



The circuit for: (a) t < 0, (b) t > 0.

5 Problem (3.1)

Hence,

$$i(t) = i(0)e^{-t/\tau} = 2e^{-t}\ \text{A}, \qquad t > 0$$

Since the inductor is in parallel with the 6-Ω and 3-Ω resistors,

$$v_o(t) = -v_L = -L\frac{di}{dt} = -2(-2e^{-t}) = 4e^{-t}\ \text{V}, \qquad t > 0$$

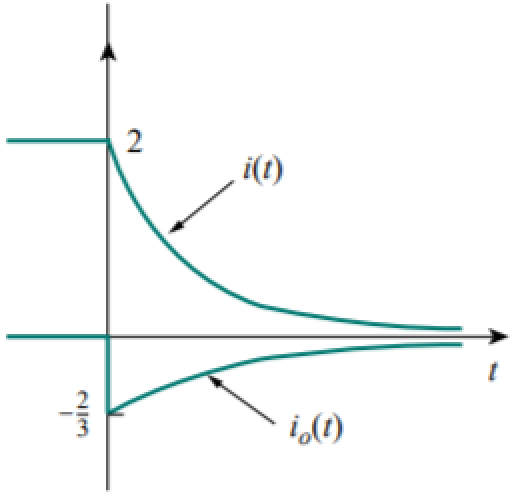
and

$$i_o(t) = \frac{v_L}{6} = -\frac{2}{3}e^{-t}\ \text{A}, \qquad t > 0$$

Thus, for all time,

$$i_o(t) = \begin{cases} 0\ \text{A}, & t < 0 \\ -\frac{2}{3}e^{-t}\ \text{A}, & t > 0 \end{cases}, \qquad v_o(t) = \begin{cases} 6\ \text{V}, & t < 0 \\ 4e^{-t}\ \text{V}, & t > 0 \end{cases}$$

$$i(t) = \begin{cases} 2\ \text{A}, & t < 0 \\ 2e^{-t}\ \text{A}, & t \geq 0 \end{cases}$$



A plot of i and i_0.

We notice that the inductor current is continuous at t = 0, while the current through the 6-Ohm resistor drops from 0 to −2/3 at t = 0, and the voltage across the 3-Ohm resistor drops from 6 to 4 at t = 0. We also notice that the time constant is the same regardless of what the output is defined to be.