Pro-Cyclical Emissions and Optimal Monetary Policy*

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Abstract

We study optimal monetary policy in an analytically tractable New Keynesian DSGE-model with pro-cyclical carbon emissions. The competitive equilibrium under flexible prices overreacts to productivity shocks relative to the efficient allocation. When prices are sticky, actual output increases by less than natural output: the relationship between actual and efficient output depends on the degree of emission pro-cyclicality and the severity of price stickiness. The real interest rate that monetary policy optimally tracks is distinct from the natural rate of interest, implying that divine coincidence is broken also in the presence of demand shocks. For central banks with a dual mandate, we characterize the optimal monetary policy response and show that it generally places a larger weight on output stabilization. However, even under optimal monetary policy, inflation and output gap are more volatile than in the baseline New Keynesian model without emissions.

Keywords: Optimal Monetary Policy, Output Gap, Central Bank Loss Function, Emission Externality, Phillips Curve

JEL Classification: D84, E31, Q58

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1 Introduction

There is now a broad consensus that the emission of greenhouse gases inflicts severe damages on the wider economy, both through short turn losses in air quality and through potentially disastrous long run consequences of climate change (Muller et al., 2011). Economic theory suggests that emission taxes are the best instrument to achieve the necessary emission reduction. It is becoming increasingly clear that financial regulators in general and central banks in particular can play at most a supporting role in addressing emission externalities related to climate change. First, conventional monetary policy instruments, such as short-term interest rates are naturally not well-suited to address long run issues (Nakov and Thomas, 2023). Second, even the unconventional central bank toolkit provides very limited potential to induce a sectoral re-allocation away from fossil fuels (see Giovanardi et al., 2023; Ferrari and Nispi Landi, 2023 among others). The attention of policymakers is therefore shifting towards the optimal response of monetary policy to climate change from an *adaptation* perspective, rather than a *mitigation* perspective.

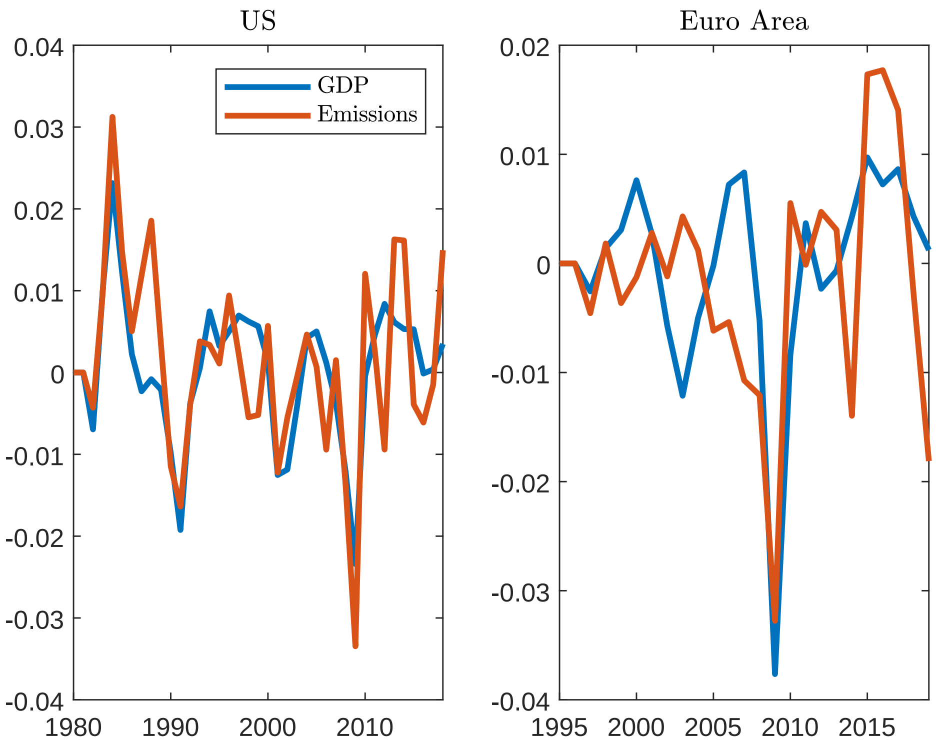
This paper offers a normative analysis of monetary policy in the presence of socially harmful emissions. We focus on the short run damages of emissions that have, to the best of our knowledge, not been studied in the context of optimal monetary policy. To that end, we augment a standard New Keynesian model by emission damages in the long run, related to climate change, and in the short run, which can be interpreted as air quality losses.[1] We assume that emission damages are higher during booms than in recessions. This assumption is consistent with the observation that emissions are highly pro-cyclical, both in the US and in the euro area, see Figure 1. Doda (2014) and Khan et al. (2019) provide additional evidence. If emissions are pro-cyclical, a Pigouvian emission tax that optimally addresses climate change, i.e. the long run consequences of emissions, does not implement the efficient allocation. Instead, the competitive equilibrium allocation *under flexible prices* implies an overreaction of output in response to productivity shocks, relative to the efficient allocation.

The relative over-reaction of output in the flexible price equilibrium interacts non-trivially with nominal rigidities and, hence, monetary policy. Consider a positive shock to TFP. Price rigidities prevent a large share of firms from reducing prices, such that the economy expands by less than it would do under flexible prices. Absent emission externalities, the central bank aims at closing the gap between the sticky price and flexible price output. We refer to this gap as the *natural output gap*. With pro-cyclical emissions, closing the natural output gap does not implement the efficient allocation. We refer to the difference between the output reaction under sticky prices minus the output reaction in the efficient allocation as the *welfare-relevant output gap*. Price stickiness attenuates the over-reaction of the flexible price equilibrium allocation vis-a-vis the welfare-relevant output gap.

We then show that pro-cyclical emissions also affect the competitive equilibrium, which is

[1] The environmental economics literature typically views the negative economic consequences of climate change through carbon emissions as only a subset of the overall adverse effects that the emission of polluting substances exerts on the wider economy. This includes negative health consequences, decreased timber and agriculture yields, depreciation of materials, and reductions of recreation services. See Muller et al. (2011) and the references therein. In contrast to climate change, these negative effects materialize very quickly in response to an increase in emission activities.

Figure 1: Emissions and GDP over Time



Notes: Data at annual frequency, detrended using a one-sided HP-filter with smoothing parameter 6.25.

described by a dynamic IS equation and the New Keynesian Phillips curve. The latter describes a macroeconomic relationship between the natural output gap and inflation. It is straightforward to establish that also this relationship is affected by pro-cyclical emissions. On the one hand, output expands by less than it would do without pro-cyclical emission damages which, as a by-product, also implies that the natural output gap is less volatile. On the other hand, it does not directly change firm's price setting behavior. Therefore, the Phillips curve steepens. In contrast, the dynamic IS equation is not directly affected by pro-cyclical emissions. When the central bank reaction function is held constant, pro-cyclical emissions imply a smaller volatility of inflation and the natural output gap.

Sign and volatility of the welfare-relevant output gap, however, are ambiguously affected by pro-cyclical emissions. Consider a positive shock to total factor productivity (TFP). It can be shown analytically that, in contrast to the baseline New Keynesian model, the Phillips curve steepens and is *shifted* downwards. This shift, which resembles the effects of a cost-push shock, will imply that the central bank is unable to achieve perfect stabilization of inflation and the welfare-relevant output gap: divine coincidence as defined by Blanchard and Gali (2007) is broken.[2] For a high degree of price stickiness, inefficiencies associated with firms being unable to reduce their prices dominate the welfare-relevant output gap. It is still negative, but of smaller sign than in the baseline New Keynesian model. In contrast, for a low degree of price stickiness, the emission externality dominates and the welfare-relevant output gap is positive. Consequently, the volatility of the welfare-relevant output gap is non-monotonic in the degree of price stickiness.

We incorporate this insight into an analytical characterization of optimal monetary policy along the lines of Clarida et al. (1999) and Woodford (2011). Our analysis is applicable for central banks with a *dual mandate* and proceeds in two steps. First, we discuss how the interaction between nominal rigidities and pro-cyclical emissions affects the central bank's objective function, which is derived from first principles. Using a second order approximation to welfare, it can be shown that previously discussed overreaction of output in competitive equilibrium relative to the efficient allocation implies a higher weight on output stabilization.

In a second step, we combine this insight with the modified New Keynesian Phillips curve to study whether the interaction between pro-cyclical emissions and nominal rigidities could *qualitatively* change the optimal reaction of monetary policy. While monetary policy would typically cut interest rates after a positive TFP shock, a sufficiently emission externality might render a tightening of monetary policy optimal.[3] We can show that this is never the case. Irrespective of the degree of price stickiness and the severity of short-run emission damages, the central bank always cuts interest rates by less in absolute terms after a positive TFP shock than it would to absent pro-cyclical emission damages. Consistent with Khan et al. (2003), the central banks' optimal policy problem is resolved heavily in favor of replicating the equilibrium allocation under flexible prices.

[2] Breaking divine coincidence requires frictions that go beyond nominal rigidities. Sims et al. (2023) obtain a similar result to ours in the presence of financial shocks.

[3] Such non-standard responses of optimal monetary policy have been documented in Khan et al. (2003).

Therefore, our analysis yields a clear prediction for the effects of pro-cyclical emissions on macroeconomic fluctuations, taking into account the *optimal response* of monetary policy. By breaking divine coincidence, pro-cyclical emissions imply that inflation and the welfare-relevant output gap are more volatile than in the baseline model, where the central bank can achieve perfect stabilization of inflation and the output gap at the same time. In a last step, we show numerically that our characterization of optimal monetary policy also carries over to a larger model with capital and investment adjustment costs.

By providing a simple analytical framework, our framework contributes to the growing discussion on welfare-relevant output gaps, which are not only relevant for monetary policy frameworks in all jurisdictions that provide their central bank with a dual mandate, but for all policies that take output gaps into account. Conditioning macroeconomic stabilization policies at business cycle frequencies on output gaps has to bear in mind that those output gaps need not be efficient from a welfare perspective.[4] In spirit of the analysis in Blanchard and Gali (2007), we have shown how the optimal monetary policy is affected by externalities originating in the real sector, which do not have a direct effect on nominal rigidities. Finally, it should be noted that our analysis of monetary policy under cyclical emissions is a *second best* solution to a welfare-maximization problem. If appropriate cyclical adjustments to emission taxes were in place, monetary policy can be conducted *as usual*.[5]

Related Literature This paper mainly draws from the E-DSGE literature, starting from the contribution by Heutel (2012). This literature studies the interaction between environmental policies and macroeconomic activity at a business cycle frequency, which makes them a suitable model class to study the relationship between environmental and monetary policies, see Annicchiarico et al. (2021) for a survey. Related to monetary policy, Annicchiarico and Di Dio (2015) study the role of nominal rigidities for the effectiveness of environmental policies. Faria et al. (2022) We contribute to the ongoing discussion on how to adapt monetary policy to climate change, see for example Hansen (2021). Currently, the literature is converging to the conclusion that monetary policy instruments can not play a decisive role in climate change *mitigation*, but that it might have to adapt to climate change and the implementation of climate policy through carbon pricing or taxation.[6]

We also contribute to a growing literature studying how monetary policy optimally adapts to climate change. McKibbin et al. (2020) provide an overview about potential interactions between

[4]On a conceptual level, our analysis also relates to the literature of optimal monetary policy in the presence of hysteresis effects. If such effects are present, it is not optimal to close the natural gap. In sharp contrast to a setting with emission externalities, however, optimal monetary policy is more expansionary in response to a positive TFP shock than in the baseline New Keynesian model, see Cerra et al. (2023) and the references therein.

[5]In Sims et al. (2023), the central bank has an additional policy instrument in the form of asset purchases to offset financial shocks and restore divine coincidence. It appears rather implausible from an institutional background that central bank policy instruments can be used in an appropriate way to address pro-cyclical emissions.

[6]Potential options to pursue climate policy objectives include the preferential treatment of green assets in central bank asset purchases and the collateral framework. Such policies are plagued by multiple shortcomings: first, the effect on the green investment share and environmental performance is quantitatively very small (Ferrari and Nispi Landi, 2023 and Giovanardi et al., 2023). Second, these policies might interact with financial stability objectives in non-trivial ways, which makes them qualitatively inferior to direct taxation (Giovanardi et al., 2023).

climate policy and monetary policy. In this strand of literature, our paper is most closely related to Muller (2021). Using a standard New-Keynesian framework, Muller (2021) proposes a natural interest rate taking time-varying pollution intensities into account. By tracking such a refined "green interest rate", monetary policy intertemporally re-allocates consumption from periods with high-pollution intensity to periods with a low-pollution intensity. Nakov and Thomas (2023) show that climate change, i.e. the long run consequences of emissions, only has a limited impact on the optimal conduct of monetary policy. Economides and Xepapadeas (2018) study optimal monetary policy when climate change is a propagation mechanism for TFP shocks, such that positive shocks have negative side effects through elevated damages from climate change in the future, and vice versa.

A series of papers discusses (optimal) monetary policy when inflation is (at least partially) driven by rising energy prices. In a New Keynesian model with an energy sector, Olovsson and Vestin (2023) show that targeting core inflation is welfare-optimal. The literature also recognizes that monetary policy might be affected by potentially inflationary effects of carbon taxation more generally. Konradt and Weder di Mauro (2023) provide empirical evidence, while Ferrari and Nispi Landi (2022) and Del Negro et al. (2023) study this channel through the lenses of New Keynesian models, which are conceptually similar to ours. However, we do not incorporate direct effects of carbon taxes on price rigidities.

Outline Our paper is structured as follows. Section 2 presents the emission-augmented New Keynesian model without capital. In Section 3, we characterize optimal monetary policy. Section 4 shows that our analytical results also carry over to a larger setting with capital, while Section 5 concludes.

2 A Simple E-NK Framework

We present the basic monetary policy trade-off in a New Keynesian model, augmented by an environmental friction. The model is composed of one representative household, monopolistically competitive firms, a fiscal authority, and the central bank. Emissions negatively affected the productivity of final good producers, but analytically similar results can be obtained by assuming that emissions exert a utility loss on households.

2.1 Households

The representative household saves using nominal deposits S_t that pay a one-period interest rate r_t^s, consumes the final consumption good c_t, and supplies labor n_t at the nominal wage W_t. The household also owns firms and receives their profits d_t^{firms}, expressed in real terms. The maximization problem is given by

$$\max_{\{c_t,n_t,S_t\}_{t=0}^{\infty}} \quad \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left(\frac{c_t^{1-\sigma}-1}{1-\sigma}-\frac{n_t^{1+\varphi}}{1+\varphi}\right)\right]$$
$$\text{s.t.} \quad P_t c_t + S_t = W_t n_t + (1+i_{t-1}^s)S_{t-1} + P_t d_t^{firms} \ .$$

The parameters σ and φ determine the inverse of, respectively, the intertemporal elasticity of substitution and the elasticity of labor supply. Solving this maximization problem yields a standard Euler equation and an intra-temporal labor supply condition

$$c_t^{-\sigma} = \beta r_t^s \mathbb{E}_t\left[\frac{c_{t+1}^{-\sigma}}{\Pi_{t+1}}\right] \ , \tag{1}$$

$$n_t^{\varphi} = w_t c_t^{-\sigma} \ . \tag{2}$$

Here, P_t is the price level, $w_t \equiv \frac{W_t}{P_t}$ is the real wage, and $\Pi_t \equiv \frac{P_t}{P_{t-1}}$ denotes gross inflation.

2.2 Firms

There is a mass-one continuum of monopolistic firms, indexed by i. Firm i hires labor $n_t(i)$ to produce the intermediate good $y_t(i)$ with the following technology:

$$y_t(i) = \Lambda_t A_t n_t(i) \ . \tag{3}$$

Pollution damage $\Lambda_t = \exp\left\{-\gamma\frac{y_t}{y}\right\}$ depends positively on production *and* on its cyclical component. The rest of the supply side coincides with the baseline New Keynesian model: monopolistic producers are not perfectly able to adjust their prices due to nominal rigidities, modeled as in Calvo (1983), with θ being the fraction of firms that is not allowed to change prices. The optimal price for a firm that is able to adjust prices is given by

$$p_t^* = \frac{1}{1-\tau_t^c}\frac{\epsilon}{\epsilon-1}\frac{\xi_{1,t}}{\xi_{2,t}} \ . \tag{4}$$

where τ_t^c is a *carbon tax* raised by the government and where

$$\xi_{1,t} = mc_t\, y_t + \beta\,\theta\,\mathbb{E}_t\left[\frac{c_{t+1}^{-\sigma}}{c_t^{-\sigma}}\pi_{t+1}^{\epsilon}\xi_{1,t+1}\right] \quad \text{and} \quad \xi_{2,t} = y_t + \beta\,\theta\,\mathbb{E}_t\left[\frac{c_{t+1}^{-\sigma}}{c_t^{-\sigma}}\pi_{t+1}^{\epsilon-1}\xi_{2,t+1}\right]$$

This nominal friction implies that monopolistic producers face time-varying real marginal costs, thus generating a relationship between inflation and real economic activity summarized in a New-Keynesian Phillips Curve. Total factor productivity A_t follows an AR(1) process in logs:

$$\log(A_t) = \rho_A \log(A_{t-1}) + \sigma_A\epsilon_t \ , \quad \text{where } \epsilon_t \sim N(0,1) \ . \tag{5}$$

2.3 Efficient Allocation and Competitive Equilibrium under Flexible Prices

This section characterizes the efficient allocation and competitive equilibrium of the simple E-NK model. For the remainder of this paper, we assume that the fiscal authority sets a constant labor subsidy, $\tau^n = \frac{1}{\epsilon} \Rightarrow (1-\tau^n)\mu = 1$, to eliminate the steady state distortion generated by monopolistic competition. We begin by characterizing the efficient output level y_t^e and natural output level y_t^n and their responses $\widehat{y}_t^n$ and $\widehat{y}_t^e$ to a technology shock a_t, expressed in deviations from steady state.

Proposition 1. The natural level y_t^n and efficient level y_t^e can be written as a function of the only state variable A_t:

$$(y_t^n)^{\sigma+\varphi} =(1-\tau_t^c)(A_t\Lambda_t)^{1+\varphi} \ . \tag{6}$$

$$(y_t^e)^{\sigma+\varphi} =\frac{(A_t\Lambda_t)^{1+\varphi}}{1+\gamma\frac{y_t}{y}} \ , \tag{7}$$

Their log-deviations around the deterministic steady state are given by:

$$\widehat{y}_t^n =\frac{(1+\varphi)a_t - \frac{\tau^c}{1-\tau^c}\widehat{\tau}_t^c}{\varphi+\gamma(1+\varphi)+\sigma} \tag{8}$$

$$\widehat{y}_t^e =\frac{1+\varphi}{\varphi+\gamma(1+\varphi)+\widetilde{\gamma}+\sigma}a_t \ , \tag{9}$$

where $\widetilde{\gamma} = \frac{\gamma}{1+\gamma}$. Proof: see Appendix A.1.

Combining (7) and (6), the ratio of natural and efficient output simplifies to

$$\left(\frac{y_t^n}{y_t^e}\right)^{\sigma+\varphi} = 1+\gamma\frac{y_t}{y}(1-\tau_t^c) > 1 \ \Leftrightarrow \ \tau_t^c < \frac{\gamma\frac{y_t}{y}}{1+\gamma\frac{y_t}{y}} \ .$$

Hence, absent emission taxes ($\tau_t^c = 0$), the natural level of output generally exceeds its efficient level. Furthermore, setting $\tau_t^c = \frac{\gamma\frac{y_t^n}{y}}{1+\gamma\frac{y_t}{y}}$ implements the efficient allocation.

However, even with a carbon tax implementing the efficient steady state output, emissions generate a dynamic (short-run) inefficiency. Specifically, with $\tau^c = \widetilde{\gamma}$ and $\widehat{\tau}^c = 0$, output in the competitive equilibrium $\widehat{y}_t^n$ over-reacts to technology shocks relative to the efficient allocation $\widehat{y}_t^e$, since $\frac{1+\varphi}{\varphi+\gamma(1+\varphi)+\frac{\gamma}{1+\gamma}+\sigma} < \frac{1+\varphi}{\varphi+\gamma(1+\varphi)+\sigma}$. Since this short run inefficiency is the key element of our analysis, we will often resort to the special case $\tau^c = \widetilde{\gamma}$ and $\widehat{\tau}^c = 0$ in the following characterization of monetary policy.

3 Monetary Policy with Pro-Cyclical Emissions

By making prices flexible, we have isolated the role of carbon emissions for the welfare relevant output gap $x_t^e \equiv \widehat{y}_t - \widehat{y}_t^e$ in relation to the natural output gap $x_t^n \equiv \widehat{y}_t - \widehat{y}_t^n$. An over-reaction of the natural economy in response to a TFP shock implies a positive welfare-relevant output gap. Nominal rigidities imply instead a under-reaction of the competitive equilibrium, relative to the flexible price case, that is a negative natural output gap. Whether the competitive equilibrium still overreacts relative to the efficient allocation, thus, depends on the relative strength of nominal rigidities and the pro-cyclicality of emissions. In Figure 2, we provide graphical intuition for the interaction between emission cyclicality and nominal rigidities.

Figure 2: IRF to TFP-Shock: The Role of Nominal Rigidities

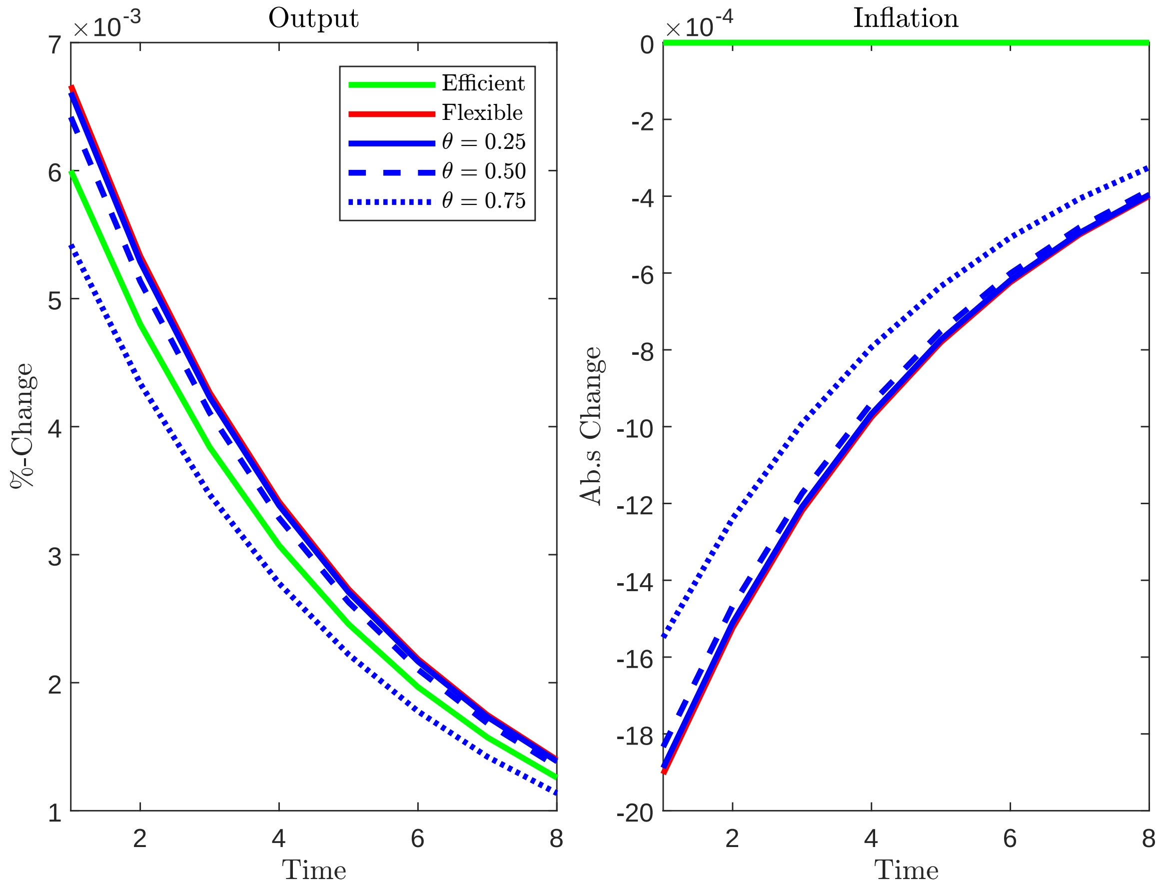


Figure 3: *Notes*: The results are generated by subjecting the model economy to a one standard deviation shock to TFP (5). We set $\rho_A = 0.95$ and $\sigma_A = 0.005$. The Taylor parameter is $\phi = 1.5$, for all other parameters, we refer to Section 4.

The more severe are nominal rigidities (the larger is θ), the lower is the over-reaction of output with respect to the efficient allocation, up to the point in which also the welfare relevant output gap turns negative. In Figure 2, this happens for a Calvo parameter between 0.5 and 0.75, i.e. for low, but still reasonable parts of the parameter space. It will turn out that the interaction of these two dynamic inefficiencies, nominal rigidities and pro-cyclical emissions, is non-trivial and has direct implications for the conduct of monetary policy. In the following, we first flesh this interactions out from a positive point of view, under a canonical representation of monetary policy based on a Taylor-type rule, and then characterize optimal monetary policy in closed-form. This is possible thanks to the extremely simplistic E-NK model that we consider in this Section. Section 4 then consider a larger and more realistic E-DSGE model and tests the quantitative relevance of the monetary policy implications from the E-NK model.

3.1 Equilibrium Effects of Pro-Cyclical Emissions

We first characterize these interactions using the standard representation of our simple model in terms of a dynamic IS curve and a New Keynesian Phillips curve, closed by a Taylor rule for the nominal interest rate r_t^s. Specifically, we show that short run welfare losses from emissions affect inflation and output volatility. For the sake of notation, we omit the hat-symbol from now on. All the variables are expressed in log-deviations from steady-state.

Proposition 2. The equilibrium conditions that characterize the economy with nominal rigidities simplify to the following two linear conditions in terms of log-deviations from the steady-state:

$$x_t^n = \mathbb{E}_t[x_{t+1}^n] - \frac{r_t^s - \mathbb{E}_t[\pi_{t+1}]}{\sigma} + \underbrace{\frac{1}{\zeta}\left[(1+\varphi)(a_{t+1}-a_t) - \frac{\tau^c}{1-\tau^c}(\tau_{t+1}^c - \tau_t^c)\right]}_{=-r_t^n/\sigma} \tag{10}$$

$$\pi_t = \zeta\kappa x_t^n + \beta\mathbb{E}_t[\pi_{t+1}] + \beta(1-\theta)\frac{\tau^c}{1-\tau^c}(\tau_t^c - \tau_{t+1}^c)\ . \tag{11}$$

Proof: see Appendix A.2.

Equation (10) is a dynamic IS curve: the (natural) output gap x_t^n positively depends on the expected output gap next period and negatively depends on the real interest rate gap, defined as the real interest rate, $r_t^s - E_t[\pi_{t+1}]$, minus the natural real interest rate, r_t^n. The natural interest rate is the real interest rate consistent with the natural level of output, which is in turn defined as the level of output consistent with flexible prices. The New Keynesian Phillips curve is given by (11). As usual, its slope depends on nominal rigidities, through the expression $\kappa = \frac{(1-\theta\beta)(1-\theta)}{\theta}$. Here, the slope is also affected by the auxiliary parameter:

$$\zeta \equiv \varphi + \gamma(1+\varphi) + \sigma\ . \tag{12}$$

Equation (12) shows that the emission externality affects the New Keynesian Phillips curve. The inflation response is determined by the share of firms that can reduce their price, which does not depend on the emission externality. At the same time, the short run emission externality dampens the effects of TFP shocks on the output gap. Thus, for a given output gap, inflation responds more strongly to TFP shocks if $\gamma > 0$. Pro-cyclical emissions steepen the Phillips curve.

Note that this does not imply that the emission externality is inflationary in equilibrium. To characterize the equilibrium impact, we close the simple E-NK model with a Taylor-type rule for the nominal interest rate:

$$r_t^s = \bar{r}^s + \pi_t^\phi\ , \tag{13}$$

where ϕ governs the response of the short-term nominal interest rates to inflation. We first keep the monetary policy reaction function constant and show how cyclical emissions affect price stability in the competitive equilibrium by iterating forward the Phillips curve.

Proposition 3. Under time-invariant emission taxes, the policy functions for output gap and inflation read

$$
\begin{aligned}
x_t^n =& \frac{\sigma}{\zeta} \cdot \frac{(1+\varphi)(1-\beta\rho_a)}{\sigma(1-\beta\rho_a)(1-\rho_a)+\zeta\kappa(\phi-\rho_a)} \cdot (\rho_a - 1)a_t \equiv \Theta_{xa}a_t \\
x_t^e =& \widetilde{\gamma}\frac{1+\varphi}{\zeta(\zeta+\widetilde{\gamma})} + \Theta_{xa}a_t \\
\pi_t =& \sigma\kappa \cdot \frac{1+\varphi}{\sigma(1-\beta\rho_a)(1-\rho_a)+\zeta\kappa(\phi-\rho_a)} \cdot (\rho_a - 1)a_t \equiv \Theta_{\pi a}a_t \ .
\end{aligned}
$$

Moreover, the variances of output gap and inflation are given by:

$$
Var[x_t^n] = \Theta_{xa}^2\sigma_A^2, \quad Var[\pi_t] = \Theta_{\pi a}^2\sigma_A^2 \ .
$$

Proof: By undetermined coefficients. Guess a linear policy function for $x_t^n = \Theta_{xa}a_t$ and $\pi_t = \Theta_{\pi a}a_t$, and impose equilibrium consistency in eq. (10), eq. (11), and eq. (13), together with $E_t[a_{t+1}] = \rho_a a_t$ and $\tau_t = 0$ to get:

$$
\begin{aligned}
&\Theta_{xa}a_t = \Theta_{xa}\rho_a a_t - \frac{\phi\Theta_{\pi a}a_t - \Theta_{\pi a}\rho_a a_t}{\sigma} + \frac{1}{\zeta}\Bigg[(1+\varphi)(\rho_a a_t - a_t)\Bigg] \\
&\Theta_{\pi a}a_t = \zeta\kappa\Theta_{xa}a_t + \beta\Theta_{\pi a}\rho_a a_t \ .
\end{aligned}
$$

For the guess to be correct, the last two equations have to hold for each $a_t \in \mathcal{R}$. Hence, imposing $a_t = 1$ and solving the system of the two equations into the two unknowns, $\Theta_{\pi a}$ and Θ_{xa} yields:

$$
\Theta_{xa} = \frac{\sigma}{\zeta} \cdot \frac{(1+\varphi)(1-\beta\rho_a)}{\sigma(1-\beta\rho_a)(1-\rho_a)+\zeta\kappa(\phi-\rho_a)} \cdot (\rho_a - 1) \tag{14}
$$

$$
\Theta_{\pi a} = \sigma\kappa \cdot \frac{1+\varphi}{\sigma(1-\beta\rho_a)(1-\rho_a)+\zeta\kappa(\phi-\rho_a)} \cdot (\rho_a - 1) \ . \tag{15}
$$

□

Figure 4: Policy functions and variances as functions of θ and γ

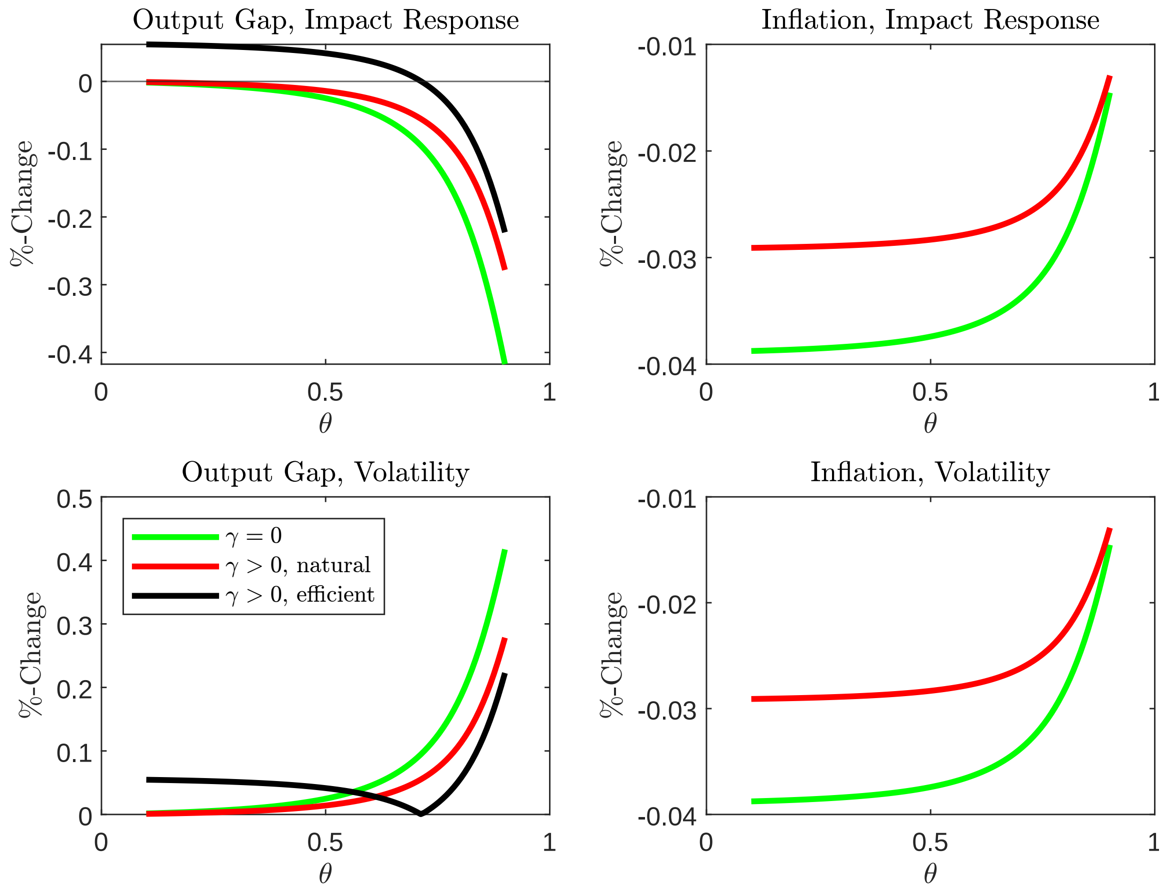


Figure 5: *Notes*: The results are generated by subjecting the model economy to a one standard deviation shock to TFP (5). We set $\rho_A = 0.95$ and $\sigma_A = 0.005$. The Taylor parameter is $\phi = 1.5$, for all other parameters, we refer to Section 4.

In Figure 5, we plot, in the first row, the impact response of inflation and output gap to a technology shock as a function of θ, both for the case of pro-cyclical emissions (green) and the baseline model (red). A larger θ means that prices are more rigid. We consider both the natural output gap x_t^n (red) and the welfare relevant output gap x_t^e (black), which coincide for the baseline model. In the second row, we plot the variances of both the output gaps and of inflation. While both the variances of the natural output gap and of inflation decrease in γ, the variance of the welfare-relevant output gap is non-monotonic in γ, suggesting again that the interaction of nominal rigidities and the emission externality generates non-trivial effects on the trade-off between inflation and output gap volatility which is at the core of optimal monetary policy. Next, we characterize optimal monetary policy by solving linear-quadratic minimization problem a la Benigno and Woodford (2005).

3.2 Monetary Policy Objective

To characterize optimal monetary policy, we first derive its objective function, which is based on the standard assumption of utilitarian welfare maximization and, thus, closely linked to the distinction between efficient and natural output gap described in Proposition 1. Since overproduction in the competitive equilibrium allocation, we follow Benigno and Woodford (2005) and consider the general case with $\Phi > 0$, i.e. the steady-state level of output and labor are not above their efficient levels.

Proposition 4. A second order approximation of the welfare function around the distorted steady state yields the following quadratic loss function:

$$\mathcal{W} = -\mathbb{E}_0\left[\sum_{t=0}^{\infty} \beta^t \frac{U_t - U}{U_c C}\right] \approx \mathbb{E}_0\left[\pi_t^2 + \omega_x (x_t^e)^2\right] + t.i.p.\ , \tag{16}$$

where

$$\omega_x = \frac{\kappa}{\epsilon} \cdot \frac{\zeta(\sigma-1) + \zeta(1+\Phi)(1+\varphi)(1+\gamma) - \Phi\left[(1+\gamma)^2(1+\varphi)^2 - (1-\sigma)^2\right]}{\left[\frac{\zeta(1+\Phi)}{1+\gamma} - \Phi(1+\varphi)\right]}\ . \tag{17}$$

Proof: see Appendix A.3.

Absent the emission externality, the weight on the output gap ω_x in the loss function collapses to the familiar expression

$$\omega_x = \frac{\kappa}{\epsilon}(\sigma + \varphi)\ .$$

where $\kappa = \frac{(1-\theta\beta)(1-\theta)}{\theta}$ is related to the share of firms that can adjust prices. With the emission externality, the weight on output stabilization contains the steady state wedge Φ between the marginal rate of substitution between consumption and labor and the efficient marginal product of labor. Specifically, we can use the optimality condition for labor from the planner problem (A.5) to express the labor market clearing condition as follows:

$$n^{\varphi} c^{\sigma} \equiv (1+\Phi) MPN^e = (1+\Phi)\frac{A\Lambda}{1+\gamma}\ .$$

This wedge can be expressed in terms of the emission externality and the tax:

$$\Phi = (1+\gamma)(1-\tau^c) - 1\ .$$

Note that this wedge vanishes if emission taxes eliminate the externality in steady state. From Proposition 4, we can derive two properties of the loss function.

Lemma 1. For any time-invariant carbon tax τ^c, the weight on output stabilization ω_x in the central bank objective is an increasing function of γ.

Lemma 2. As a special case of Lemma 1, with $\tau^c = \gamma$, the weight of the output gap (17) in the loss function reduces to

$$\omega_x = \frac{\kappa}{\epsilon}\big((\sigma-1) + (1+\varphi)(1+\gamma)\big)(1+\gamma) = \frac{\kappa}{\epsilon}\big(\sigma - 1 + 1 + \varphi + \gamma + \varphi\gamma\big)(1+\gamma) = \frac{\kappa}{\epsilon}\zeta(1+\gamma)\ .$$

The central bank places a higher weight on output stabilization if the externality is more severe. The intuition behind this is the dynamic inefficiency of the competitive equilibrium induced by the pollution externality. Production over-reacts to a technology shock, relative to the efficient allocation. The central bank then optimally takes this dynamic inefficiency into account by placing a higher weight on output stabilization.

3.3 Optimal Monetary Policy

Next, we characterize optimal monetary policy, by minimizing the loss function derived in Proposition 4 under time-invariant carbon taxes and with i.i.d. shocks to TFP. Under these assumptions, the policy problem under discretion can be solved for in closed form.

Proposition 5. If TFP shocks are i.i.d. and $\tau_t^c = 0$, optimal monetary policy is characterized by

$$\pi_t = -\frac{\omega_x \kappa \widetilde{\gamma}(1+\varphi)}{(\zeta+\widetilde{\gamma})(\zeta^2\kappa^2+\omega_x)} a_t \tag{18}$$

$$x_t^e = \frac{\zeta\kappa^2\widetilde{\gamma}(1+\varphi)}{(\zeta+\widetilde{\gamma})(\zeta^2\kappa^2+\omega_x)} a_t \tag{19}$$

$$r_t^e = r_t^n + \frac{\sigma\widetilde{\gamma}(1+\varphi)}{\zeta+\widetilde{\gamma}}\left(\frac{1}{\zeta} - \frac{\zeta\kappa^2}{\kappa^2\zeta^2+\omega_x}\right) a_t, \tag{20}$$

where r_t^n is the natural rate of interest in the model without an emission externality and $\widetilde{\gamma} = \frac{\gamma}{1+\gamma}$.
Proof: The natural output gap can be expressed in terms of the efficient output gap as follows

$$x_t^n = y_t - y_t^n = y_t - y_t^e + y_t^e - y_t^n = x_t^e + \left[\frac{1+\varphi}{\zeta+\widetilde{\gamma}} - \frac{1+\varphi}{\zeta}\right] a_t = x_t^e - \widetilde{\gamma}\frac{1+\varphi}{\zeta(\zeta+\widetilde{\gamma})} a_t\ .$$

Plugging the relationship between natural and efficient output gap into the Phillips curve, the central bank's problem reads:

$$\begin{aligned} &\min_{\pi_t, x_t^e} \quad \frac{1}{2}\mathbb{E}_0\left[\pi_t^2 + \omega_x (x_t^e)^2\right] \\ &\text{s.t.} \quad \pi_t = \zeta\kappa x_t^e - \kappa\widetilde{\gamma}\frac{1+\varphi}{\zeta+\widetilde{\gamma}} a_t + \beta\pi_{t+1} \end{aligned} \tag{21}$$

Taking FOCs and combining them we get the optimal monetary policy that summarizes the trade-off between the welfare-relevant output gap x_t^e and inflation π_t:

$$\pi_t = -\frac{\omega_x}{x_t^e}\zeta\kappa \tag{22}$$

Plugging the monetary policy rule into the Phillips curve, we get eq. (19) for x_t^e. Plugging the last two conditions into the IS curve and solving for the efficient policy rate r_t^e we get eq. (20). □

Proposition 5 is consistent with Muller (2021), who shows that a central bank tracking potential output has to takes into account cyclical pollution and should adjust the nominal interest rate accordingly. Divine coincidence is then broken, because of the presence of the emission adjustment term in eq. (20). Its sign depends on the expression $\frac{1}{\zeta} - \frac{\zeta\kappa^2}{\kappa^2\zeta^2+\omega_x}$. If the adjustment term is positive, the central bank decreases the policy rate by less in response to a positive TFP shock than it would in the standard New Keynesian model, where tracking the natural interest rate is optimal. Under our baseline case, with an steady-state efficient, but time-invariant emission tax, we can show that this term reduces to $\frac{1+\gamma}{\epsilon\zeta(\kappa\zeta+\frac{1+\gamma}{\epsilon})} > 0$ for every $\gamma > 0$.

Hence, the presence of pro-cyclical emissions in an otherwise standard New-Keynesian model generates dynamic inefficiency that interact with nominal rigidities in a non-trivial way so that divine coincidence is broken for a technology shock. In response to a positive TFP shock, the central bank finds it optimal to trade off some output gap at the expense of higher inflation. To do so, the optimal interest rate cut is smaller, in absolute terms, compared to the case where the central bank does not take into account the emission externality. We demonstrate how the optimal monetary policy trade-off is affected by pro-cyclical emissions for different degrees of the price rigidity θ. For very sticky prices, the central bank almost closes the welfare-relevant output gap, since the economy's overreaction to a TFP shock is relatively modest. As the right panel shows, the adjustment term between natural and efficient interest rate is very large in this case.

Figure 6: IRF to TFP-Shock: Optimal Monetary Policy

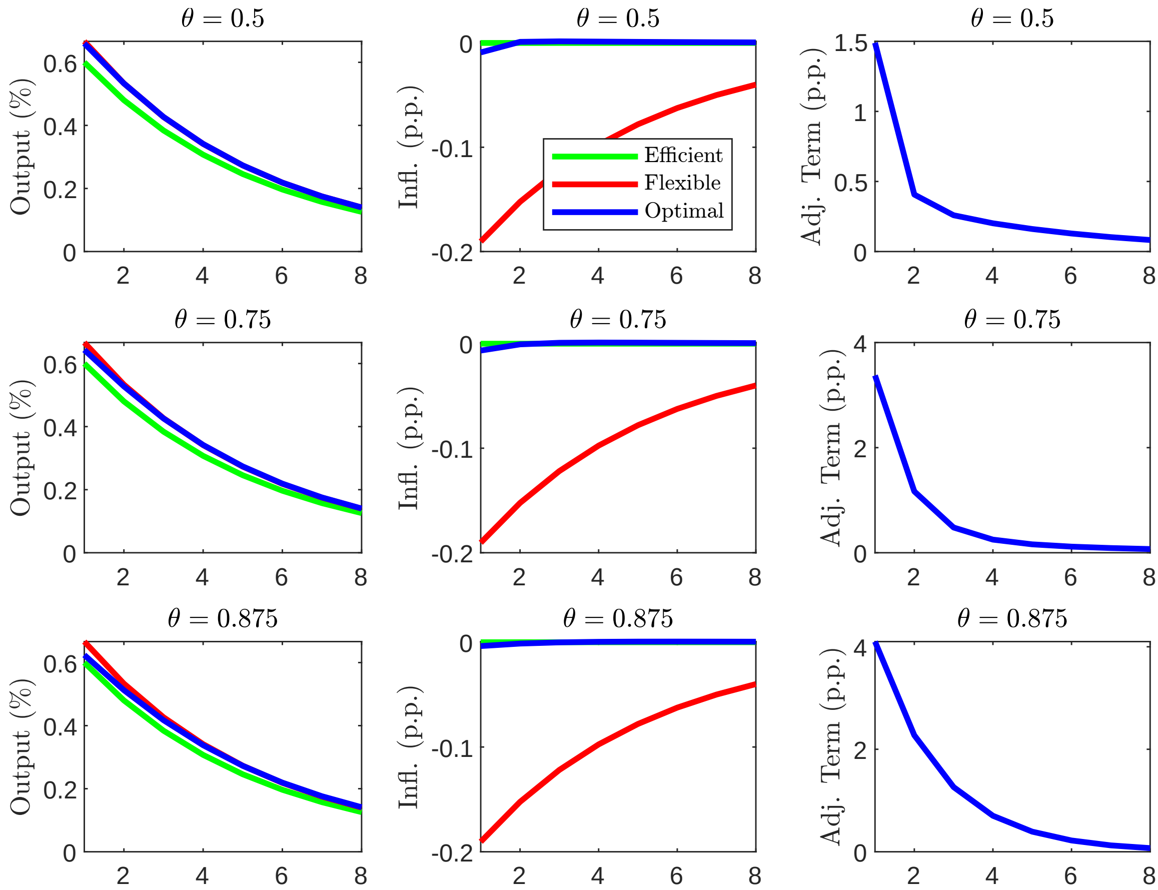
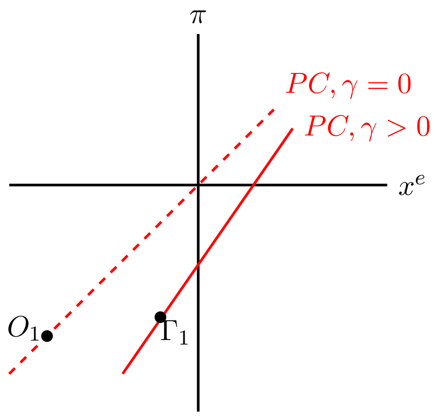


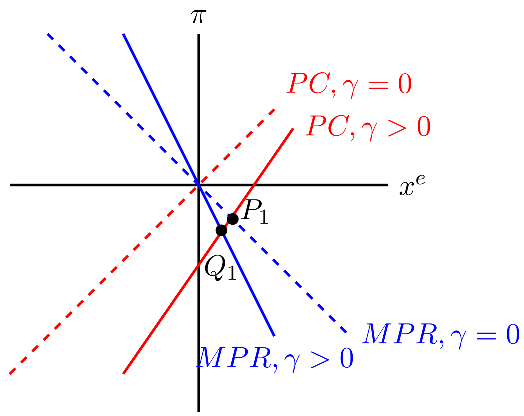
Figure 7: *Notes*: The results are generated by subjecting the model economy to a one standard deviation shock to TFP (5). We set $\rho_A = 0.95$ and $\sigma_A = 0.005$ and use a utilitarian welfare criterion. For the parameterization, we refer to Section 4.

Section 3.3 summarizes the effect of pro-cyclical emissions on macroeconomic outcomes using the canonical representation in a Phillips Curve - Monetary Policy Rule diagram. In the upper left panel, we show first how the Phillips curve is affected by pro-cyclical emissions. The dashed red line refers to the baseline New Keynesian model: marginal costs go down in response to the TFP shock. However, due to the nominal rigidity, not all firms are unable to reduce their prices. Holding the central banks' reaction function constant, this implies that inflation is negative. At the same time, output increases by less than its natural level, i.e. natural and efficient output gap, which coincide in the baseline model, are negative. This is represented by the point O_1.

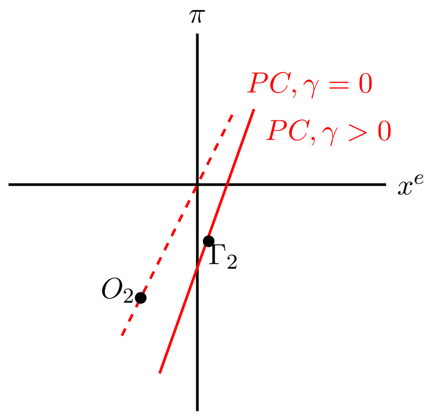
The solid line refers to the case with $\gamma > 0$. From (21), we see that a TFP shock induces both a downward shift and a steepening of the Phillips curve. If the central bank uses the same reaction function as in the economy without the emission externality, the inflation response is smaller. This follows directly from Proposition 3. Differentiating (15) with respect to γ, we see that the inflation response to a TFP shock is smaller in absolute terms for every γ. The sign of the welfare-relevant output gap is ambiguous and depends on the degree of nominal rigidities and the severity of emission damages, consistent with the upper left panel of Figure 5. When θ is high, only a small share of firms can adjust prices and the welfare relevant output gap x_t^e is still negative. This is summarized in the point Γ_1.



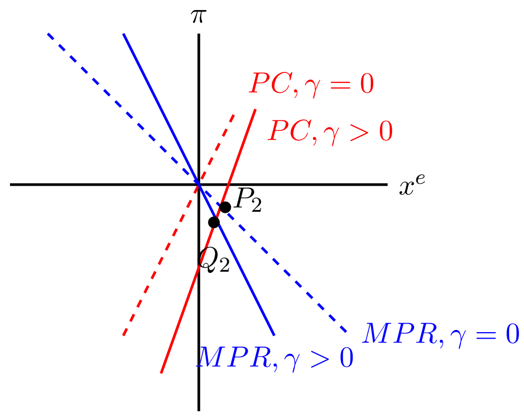
(a) Phillips curve, high θ



(b) Monetary policy rule, high θ



(c) Phillips curve, low θ



(d) Monetary policy rule, low θ

In the upper right panel, we add optimal monetary policy. In the baseline case, the central bank is able to implement first best by shrinking both output gap and inflation to zero, irrespective of their monetary policy rule. With pro-cyclical emissions, this is no longer possible. Divine co-incidence is broken and the central bank is unable to close the output gap and implement an inflation rate of zero at the same time. Instead, it selects and equilibrium by moving on the Phillips curve associated with $\gamma > 0$. Under the optimal monetary policy rule that does not take pro-cyclical emissions into account, the dashed blue line, this corresponds to the point P_1. From Proposition 4, we know that the central bank places a larger weight on output stabilization whenever $\gamma > 0$. Thus, the equilibrium response of output gap and inflation under optimal policy are characterized by P_1^*, where the solid blue line intersects the Phillips curve.

In the lower panel, we illustrate a comparative statics exercise with respect to the Calvo parameter. The Phillips curve is steeper if there is a larger share of price adjusters (a lower θ). When $\gamma > 0$, the steeper, downward shifted Phillips curve might imply a *positive* output gap in response to a TFP shock, consistent with the upper left panel of Figure 5, while the inflation response is still dampened. Once monetary policy is set optimally in the bottom right panel, the central bank faces a trade-off between output and inflation stabilization which is reminiscent of supply shocks. Again, with $\gamma > 0$, the trade-off is solved with a larger emphasis on output stabilization. Lastly, it is worth noting that, irrespective of the Calvo parameter θ, the volatility of inflation and output gap under optimal policy will be larger for $\gamma > 0$ due to the broken divine co-incidence.

4 Extended Model

In this section, we demonstrate that our analytical results derived in the simple setting also carry over to a more general model that includes capital and investment adjustment costs. We leave all other model ingredients unchanged.

Households The representative household holds capital K_t, consumes the final consumption good c_t, and supplies labor at the nominal wage, W_t. The household owns firms and receives a lump-sum transfer from the government T_t. The maximization problem is given by

$$\begin{aligned} \max_{\{c_t, n_t, S_t\}_{t=0}^{\infty}} \quad & \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1-\sigma} - \omega \frac{n_t^{1+\varphi}}{1+\varphi} \right) \right] \\ \text{s.t.} \quad & P_t c_t + S_t = W_t n_t + (1 + r_{t-1}^s) S_{t-1} + P_t (\Pi_t + T_t) \ . \end{aligned}$$

While φ determines the elasticity of labor supply and ω is a weighting parameter. Euler equation and intra-temporal labor supply condition are largely identical to the simplified model.

Final Good Firms Monopolistic producer i acquires the homogeneous intermediate good z_t, differentiates it into variety i and sells it to households at price p_t^z. Their production technology is linear, such that their marginal cost are simply given by $mc_t = p_t^z$ and the solution to their price setting problem coincides with eq. (4) in the simple model. Final good supply then depends on the price dispersion: $y_t = \Delta_t z_t$.

Intermediate Good Firms Perfectly competitive intermediate good firms invest in capital k_{t+1} and hire labor n_t to produce the homogeneous intermediate good z_t with the following technology:

$$z_t = A_t \Lambda_t k_t^{\alpha} n_t^{1-\alpha} \ . \tag{23}$$

The law of motion for capital is given by $k_{t+1} = (1-\delta) k_t + i_t$. Investment goods have to be purchased at price ψ_t from perfectly competitive investment good producers (described below). Denoting the intermediate good price by p_t^z, the first-order conditions associated with the profit maximization problem are given by

$$\begin{aligned} \frac{w_t}{p_t^z} &= (1-\alpha) \frac{z_t}{n_t} \ , \\ \psi_t &= \mathbb{E}_t \left[(1-\delta) \psi_{t+1} + p_t^z \alpha \frac{z_{t+1}}{k_{t+1}} \right] \ . \end{aligned}$$

Investment Good Firms A representative investment good firm acquires $\left(1 + \frac{\Psi_I}{2}(\frac{i_t}{i_{t-1}})\right)$ units of the final goods bundle into one unit of a homogeneous investment good, which they sell to intermediate good firms at price p_t^K. The profit maximization problem

$$\max_{\{i_s\}_{s=0}^{\infty}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \Lambda_{t,t+s} \left\{ p_{t+s}^K i_{t+s} - \left(1 + \frac{\Psi_I}{2} \left(\frac{i_{t+s}}{i_{t+s-1}} - 1 \right)^2 \right) i_{t+s} \right\} \right]$$

delivers an additional equilibrium condition for the investment good price:

$$p_t^K = 1 + \frac{\Psi_I}{2}\Big(\frac{i_t}{i_{t-1}} - 1\Big)^2 + \Psi_I\Big(\frac{i_t}{i_{t-1}} - 1\Big)\frac{i_t}{i_{t-1}} - \mathbb{E}_t \left[\Lambda_{t,t+1} \Psi_I \Big(\frac{i_{t+1}}{i_t} - 1\Big)\Big(\frac{i_{t+1}}{i_t}\Big)^2 \right] . \tag{24}$$

Market Clearing With capital and investment, output does not equal consumption. To maintain the equivalence between cap-and-trade schemes and carbon taxes, we make damages explicitly dependent on consumption:

$$\Lambda_t = \exp\left\{ -\gamma_1 \frac{c_t}{c} - \gamma_0 \right\} \tag{25}$$

To separately match the long run damages of carbon emissions and its cyclical component, we also add the auxiliary parameter γ_0, which pins down the optimal long run carbon tax. The goods market clearing condition now also includes investment:

$$y_t = c_t + i_t \left(1 + \frac{\psi_I}{2} (\frac{i_t}{i_{t-1}} - 1)^2 \right) . \tag{26}$$

The competitive equilibrium conditional on policy instruments (r_t^s, τ_t^c) is fully described by all agent's first-order conditions and budget constraints as well as the goods market clearing condition (26). The model can be closed by imposing policy rules for the nominal interest rate and the carbon tax.

Calibration The model is calibrated to standard values used in the New Keynesian DSGE literature. Households' risk aversion and discount factor are set to $\sigma = 1$ and $\beta = 0.995$. This discount factor implies an annual real rate of 2%. Furthermore, we set $\varphi = 1$ to obtain a Frisch elasticity of labor supply of one. The weight $\omega = 11$ in the household utility function is implies a steady state labor supply of 0.33.

The parameter γ governing the pollution cost of emissions is difficult to calibrate, since there is considerable uncertainty about measurement in the data. We follow the approach in Giovanardi et al. (2023) and set it to $\gamma = 0.1$ to target an emission damage of 10% of GDP, corresponding to the point estimate by Muller (2020). As customary in the literature, we set $\alpha = 1/3$ in the production function and the capital depreciation rate to $\delta = 0.025$. The investment adjustment cost parameter is set to $\Psi_I = 10$, following Coenen et al. (2023). The demand elasticity for final good varieties is fixed at $\epsilon = 6$, implying a 20% markup. As a baseline, we set the Calvo parameter to $\theta = 0.75$ although we will vary this parameter throughout the analysis. Lastly, the parameters governing exogenous TFP are set to $\rho_A = 0.8$ and $\sigma_A = 0.01$.

Optimal Monetary Policy As a final step, we numerically evaluate optimal policy in the extended model. Using the same parameters as in the simple model, we again compare the efficient (green), natural (red) and Ramsey-optimal (blue) response of output, inflation, and the adjustment term between efficient and natural rate of interest rate. Similar to the simple model (Figure 6), we observe that optimal monetary policy gets closer to the efficient level as θ increases. Furthermore, the initial response is larger in the extended model, such that the adjustment is also more drastic than in the small model, for every θ. It should be noted that the consequences of cyclical emissions for optimal monetary policy are sizeable, but not huge, which is in line with the analysis of Nakov and Thomas (2023) for the implications of long-run effects of climate change on the conduct of monetary policy.

Figure 9: IRF to TFP-Shock: Optimal Monetary Policy

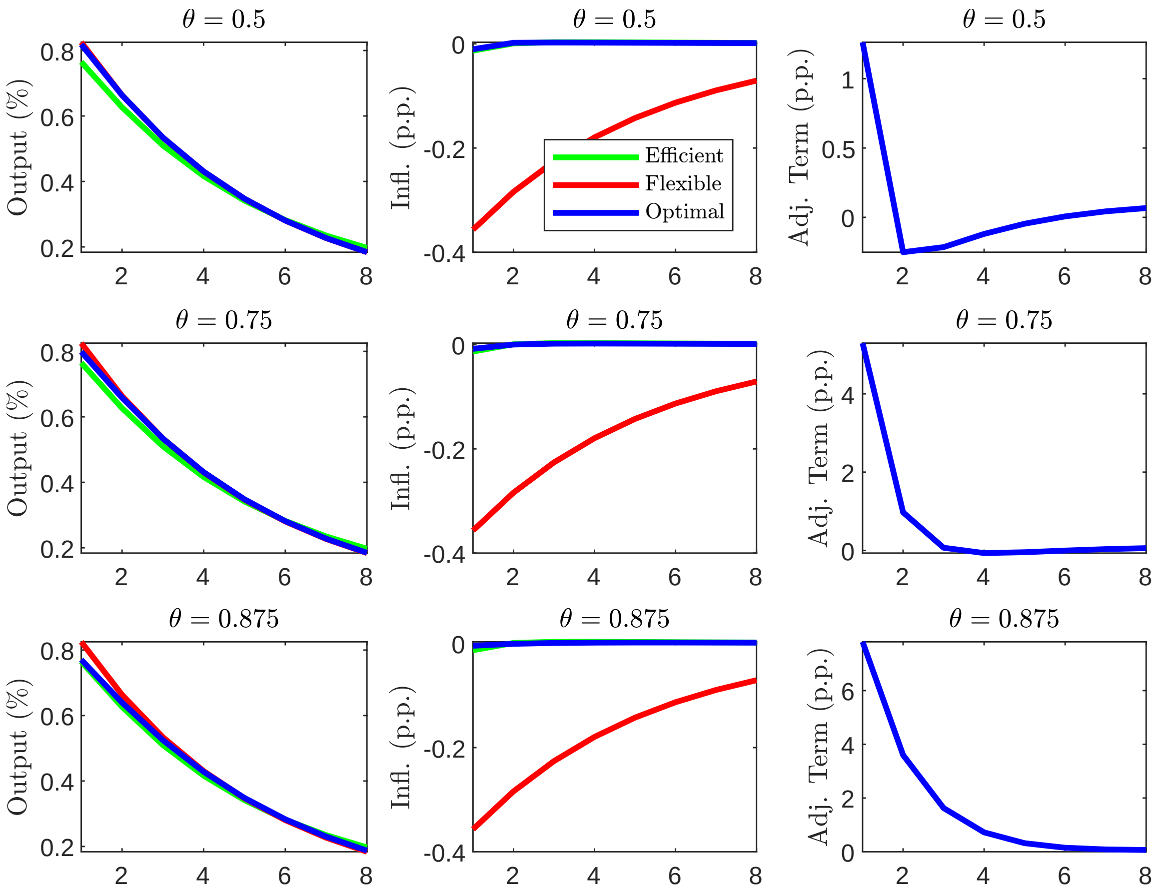


Figure 10: *Notes*: The results are generated by subjecting the model economy to a one standard deviation shock to TFP. We use a utilitarian welfare criterion, for the parameterization, we refer to Section 4.

5 Conclusion

In this paper, we explore the interactions between pro-cyclical emissions, nominal rigidities, and monetary policy. We show that cyclical emissions have implications for optimal monetary policy even when the long run (or trend-specific) costs of emissions are addressed optimally.

Specifically, the natural output gap is not efficient from a utilitarian welfare perspective, and neither is it optimal to track the natural rate of interest from the New Keynesian model. Divine coincidence is broken even for TFP shocks. We show that the central bank generally places a higher weight on output stabilization, to tackle this short-run inefficiency. This result also holds in a larger model with capital and investment adjustment costs.

There is evidence that emissions also have a direct effect on macroeconomic volatility and inflation through a disaster risk channel and associated swings in commodity prices. Disaster risk itself can also be a source of macroeconomic volatility. We abstract from these physical risk components, since they do not necessarily point to short-run inefficiencies that can reasonably addressed by short-run policies, such as nominal interest rates. Furthermore, carbon taxation can also induce inflation by increasing electricity and energy prices, which has been subject to recent discussion. Exploring the interactions between these additional channels, nominal rigidities, and monetary policy is left for future research.

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A Proofs

This section contains all proofs omitted in Section 3.

A.1 Proof of Proposition 1

The aggregate production function can be written $y_t = A_t \Lambda_t n_t$, while the goods market clearing condition is given by $y_t = c_t$.

Efficient Allocation The planner problem is

$$\max_{c_t, n_t, y_t, \Lambda_t, u_t} \quad \sum_t \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} - \frac{n_t^{1+\varphi}}{1+\varphi} \right] \quad \text{s.t.}$$

$$c_t = y_t \qquad (\lambda_t)$$

$$y_t = A_t \Lambda_t n_t \qquad (\mu_t)$$

$$\Lambda_t = \exp\left\{ -\gamma \frac{y_t}{y} \right\} \qquad (\nu_t)$$

Setting up the Lagrangian

$$\max_{c_t, n_t, y_t, \Lambda_t} \sum \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} - \frac{n_t^{1+\varphi}}{1+\varphi} + \lambda_t \Big(y_t - c_t \Big) + \mu_t \Big(A_t \Lambda_t n_t - y_t \Big) + \nu_t \Big(\exp\left\{ -\gamma \frac{y_t}{y} \right\} - \Lambda_t \Big) \right]$$

and taking FOCs yields

$$\lambda_t = c_t^{-\sigma} \tag{A.1}$$

$$\mu_t A_t \Lambda_t = n_t^{\varphi} \tag{A.2}$$

$$\lambda_t - \mu_t - \nu_t \frac{\gamma}{y} \Lambda_t = 0 \tag{A.3}$$

$$\mu_t A_t n_t = \nu_t \tag{A.4}$$

Combining (A.3) and (A.4):

$$\lambda_t - \mu_t - \mu_t A_t n_t \frac{\gamma \epsilon_t}{y} \Lambda_t = 0 \Leftrightarrow \mu_t = \frac{\lambda_t}{1 + A_t \Lambda_t n_t \frac{\gamma}{y}}$$

Plugging in (A.1) and (A.2), the efficient allocation is characterized by a socially optimal labor supply condition:

$$\frac{c_t^{-\sigma}}{1 + A_t \Lambda_t n_t \frac{\gamma}{y}} A_t \Lambda_t = n_t^{\varphi} \ ,$$

which implicitly defines the marginal product of labor as

$$MPN_t^e \equiv \frac{A_t \Lambda_t}{1 + \gamma \frac{y_t}{y}} \ . \tag{A.5}$$

The resource constraint is given by $c_t = y_t$. Hence, using the production technology $y_t = A_t\Lambda_t n_t$

$$\frac{y_t^{-\sigma}}{1 + A_t\Lambda_t n_t \frac{\gamma}{y}} A_t\Lambda_t = \frac{y_t^{\varphi}}{(A_t\Lambda_t)^{\varphi}}$$

Rearranging delivers eq. (7). Log-linearizing yields

$$\begin{aligned} & (\sigma+\varphi)\widehat{y}_t^e = (1+\varphi)a_t - (1+\varphi)\gamma\widehat{y}_t^e - \frac{\gamma}{1+\gamma}\widehat{y}_t^e \\ \Leftrightarrow \quad & \left[\sigma+\varphi+(1+\varphi)\gamma+\frac{\gamma}{1+\gamma}\right]\widehat{y}_t^e = (1+\varphi)a_t\ . \end{aligned} \tag{A.6}$$

Re-arranging for $\widehat{y}_t^e$, we arrive at eq. (9).

Competitive Equilibrium Next, we derive the natural level of output consistent with flexible prices and a labor subsidy $\tau^n = \frac{1}{\epsilon}$ that corrects for the steady state monopolistic distortion. The relevant equilibrium conditions are the aggregate production function, where Δ_t is the price dispersion

$$\Delta_t y_t = A_t\Lambda n_t\ , \tag{A.7}$$

and labor demand:

$$(1-\tau^n)w_t = mc_t A_t\Lambda_t\ . \tag{A.8}$$

Labor supply:

$$w_t = n_t^{\varphi} c_t^{\sigma}\ .$$

Goods market clearing requires

$$y_t = c_t\ .$$

Optimal price

$$p_t^* = \frac{\mu}{1-\tau_t^c}\frac{\xi_{1,t}}{\xi_{2,t}}\ , \tag{A.9}$$

where $\mu \equiv \frac{\epsilon}{\epsilon-1}$ and

$$\xi_{1,t} = mc_t y_t + \beta\theta\frac{c_{t+1}^{-\sigma}}{c_t^{-\sigma}}\pi_{t+1}^{\epsilon}\xi_{1,t+1}\ , \tag{A.10}$$

$$\xi_{2,t} = y_t + \beta\theta\frac{c_{t+1}^{-\sigma}}{c_t^{-\sigma}}\pi_{t+1}^{\epsilon-1}\xi_{2,t+1}\ . \tag{A.11}$$

Inflation is pinned down by

$$1 = (1-\theta)(p_t^*)^{1-\epsilon} + \theta\pi_t^{\epsilon-1} \ . \tag{A.12}$$

Price dispersion:

$$\Delta_t = (1-\theta)(p_t^*)^{-\epsilon} + \theta\pi_t^{\epsilon}\Delta_{t-1} \ . \tag{A.13}$$

If prices are flexible, then $\Delta_t = \pi_t = p_t^* = 1$, $\xi_{1,t} = mc_t y_t$, $\xi_{2,t} = y_t$, and $p_t^* = (1-\tau^n)\mu mc_t$. Hence:

$$\begin{aligned}
1 =& \frac{\mu}{1-\tau_t^c} mc_t = (1-\tau^n)\frac{\mu}{1-\tau_t^c}\frac{w_t}{A_t\Lambda_t} = \frac{n_t^{\varphi}c_t^{\sigma}}{(1-\tau_c^t)A_t\Lambda_t} \\
=& \frac{y_t^{\sigma+\varphi}}{(1-\tau_t^c)(A_t\Lambda_t)^{1+\varphi}}
\end{aligned}$$

where we used the fact that the labor subsidy appropriately corrects for the monopolistic distortion ($\tau^n = \frac{1}{\epsilon}$). Solving for y_t yields the natural output level (6). Log-linearizing around the deterministic steady state:

$$(\sigma+\varphi)\widehat{y}_t^n = (1+\varphi)\widehat{a}_t - (1+\varphi)\gamma\widehat{y}_t^n - \frac{\tau^c}{1-\tau^c}\widehat{\tau}_t^c \ .$$

Re-arranging for $\widehat{y}_t^n$ yields eq. (8) □

A.2 Proof of Proposition 2

Equilibrium Conditions The linearized equilibrium conditions are the following.
Optimal labor supply eq. (2):

$$\widehat{w}_t = \varphi\widehat{n}_t + \sigma\widehat{c}_t \ . \tag{A.14}$$

Euler equation eq. (1):

$$\sigma\widehat{c}_t = \sigma\widehat{c}_{t+1} - (r_t^s - \pi_{t+1}) \ . \tag{A.15}$$

Pollution:

$$\widehat{\Lambda}_t = -\gamma\widehat{y}_t$$

Production function eq. (A.7):

$$\widehat{\Delta}_t + \widehat{y}_t = a_t - \gamma\widehat{y}_t + \widehat{n}_t \tag{A.16}$$

Labor demand eq. (A.8):

$$\widehat{w}_t = \widehat{mc}_t + a_t - \gamma \widehat{y}_t \tag{A.17}$$

Optimal pricing eqs. (A.9), (A.10), and (A.11):

$$p_t^* = \frac{\tau^c}{1-\tau^c}\tau_t^c + \xi_{1t} - \xi_{2t} \tag{A.18}$$

$$\xi_{1,t} = (1-\theta\beta)mc_t + (1-\theta\beta)y_t - \theta\beta\sigma c_{t+1} + \theta\beta\sigma c_t + \epsilon\theta\beta\pi_{t+1} + \theta\beta\xi_{1,t+1} \tag{A.19}$$

$$\xi_{2,t} = (1-\theta\beta)y_t - \theta\beta\sigma c_{t+1} + \theta\beta\sigma c_t + (\epsilon-1)\theta\beta\pi_{t+1} + \theta\beta\xi_{2,t+1} \tag{A.20}$$

Inflation eq. (A.12):

$$0 = (1-\epsilon)(1-\theta)\widehat{p}_t^* + \theta(\epsilon-1)\widehat{\pi}_t \;\Leftrightarrow\; \widehat{p}_t^* = \frac{\theta}{1-\theta}\widehat{\pi}_t \tag{A.21}$$

Price dispersion eq. (A.13):

$$\widehat{\Delta}_t = -\epsilon(1-\theta)\widehat{p}_t^* + \theta\epsilon\widehat{\pi}_t + \theta\widehat{\Delta}_{t-1} \;\Leftrightarrow\; \widehat{\Delta}_t = \theta\widehat{\Delta}_{t-1} \Leftrightarrow\; \widehat{\Delta}_t = 0$$

Market clearing:

$$\widehat{c}_t = \widehat{y}_t$$

Natural output gap:

$$x_t^n = \widehat{y}_t - \widehat{y}_t^n = \widehat{y}_t - \frac{1}{\zeta}\left[(1+\varphi)\widehat{a}_t - \frac{\tau^c}{1-\tau^c}\widehat{\tau}_t^c\right]$$

Welfare relevant output gap:

$$x_t^e = \widehat{y}_t - \widehat{y}_t^e = \widehat{y}_t - \frac{1}{\zeta + \frac{\gamma}{1+\gamma}}\left[(1+\varphi)\widehat{a}_t\right]$$

Subtracting eq. (A.20) from eq. (A.19) we get:

$$\xi_{1t} - \xi_{2t} = (1-\theta\beta)mc_t + \theta\beta\pi_{t+1} + \theta\beta(\xi_{1,t+1} - \xi_{2,t+1})$$

Plugging this condition and eq. (A.21) into eq. (A.18) we get:

$$\frac{\theta}{1-\theta}\pi_t = \frac{\tau^c}{1-\tau^c}\tau_t^c + (1-\theta\beta)mc_t + \theta\beta\left(\pi_{t+1} + \frac{\theta}{1-\theta}\pi_{t+1} - \frac{\tau^c}{1-\tau^c}\tau_{t+1}^c\right) \Leftrightarrow \tag{A.22}$$

$$\Leftrightarrow \pi_t = \underbrace{\frac{(1-\theta\beta)(1-\theta)}{\theta}}_{\kappa} mc_t + \beta\pi_{t+1} + \frac{1-\theta}{\theta}\frac{\tau^c}{1-\tau^c}\left(\tau_t - \theta\beta\tau_{t+1}\right) \tag{A.23}$$

Now, combining eqs. (A.14), (A.16), and (A.17) we get:

$$
\begin{aligned}
mc_t &= w_t - a_t + \gamma y_t = \varphi n_t + \sigma c_t - a_t + \gamma y_t = \varphi(y_t - a_t + \gamma y_t) + \sigma y_t - a_t - \gamma y_t = \\
&= \underbrace{[\sigma + \varphi + (1+\varphi)\gamma]}_{=\zeta} y_t - (1+\varphi) a_t
\end{aligned}
$$

Plugging this condition into eq. (A.23):

$$
\begin{aligned}
\pi_t &= \kappa\zeta \Bigg[\underbrace{y_t - \frac{1+\varphi}{\zeta} a_t + \frac{1}{\zeta}\frac{\tau^c}{1-\tau^c}\tau_t}_{x_t^n} - \frac{1}{\zeta}\frac{\tau^c}{1-\tau^c}\tau_t \Bigg] + \beta\pi_{t+1} + \frac{1-\theta}{\theta}\frac{\tau^c}{1-\tau^c}\Bigg(\tau_t - \theta\beta\tau_{t+1}\Bigg) = \\
&= \kappa\zeta x_t^n + \beta\pi_{t+1} - \kappa\frac{\tau^c}{1-\tau^c}\tau_t + \frac{\kappa}{1-\theta\beta}\frac{\tau^c}{1-\tau^c}\tau_t - (1-\theta)\beta\frac{\tau^c}{1-\tau^c}\tau_{t+1} = \\
&= \kappa\zeta x_t^n + \beta\pi_{t+1} + (1-\theta)\beta\frac{\tau^c}{1-\tau_c}(\tau_t - \tau_{t+1}),
\end{aligned}
$$

which is eq. (11).

To get eq. (10), start from eq. (A.15) and impose market clearing to get:

$$
\begin{aligned}
& y_t = y_{t+1} - \frac{1}{\sigma}(r_t^s - \pi_{t+1}) \;\Leftrightarrow \\
& \widehat{y}_t - \frac{1}{\zeta}\left[(1+\varphi)a_t - \frac{\tau^c}{1-\tau^c}\tau_t\right] + \frac{1}{\zeta}\left[(1+\varphi)a_t - \frac{\tau^c}{1-\tau^c}\tau_t\right] = \\
& = \widehat{y}_{t+1} - \frac{1}{\zeta}\left[(1+\varphi)a_{t+1} - \frac{\tau^c}{1-\tau^c}\tau_{t+1}\right] + \frac{1}{\zeta}\left[(1+\varphi)a_{t+1} - \frac{\tau^c}{1-\tau^c}\tau_{t+1}\right] - \frac{1}{\sigma}(r_t^s - \pi_{t+1}) \;\Leftrightarrow \\
& x_t^n + \frac{1}{\zeta}\left[(1+\varphi)a_t - \frac{\tau^c}{1-\tau^c}\tau_t\right] = x_{t+1}^n + \frac{1}{\zeta}\left[(1+\varphi)a_{t+1} - \frac{\tau^c}{1-\tau^c}\tau_{t+1}\right] - \frac{1}{\sigma}(r_t^s - \pi_{t+1}) \;\Leftrightarrow \\
& x_t^n = x_{t+1}^n - \frac{1}{\sigma}(r_t^s - \pi_{t+1}) + \frac{1}{\zeta}\left[(1+\varphi)(a_{t+1} - a_t) - \frac{\tau^c}{1-\tau^c}(\tau_{t+1} - \tau_t)\right]
\end{aligned}
$$

□

A.3 Proof of Proposition 4

We can show that the wedge between efficient and natural level of output satisfies

$$
\Phi \equiv (y^e)^{\sigma+\varphi} - (y^n)^{\sigma+\varphi} = \frac{\Lambda^{1+\varphi}}{1+\gamma} - \frac{\Lambda^{1+\varphi}}{1+\tau^c} = \Lambda^{1+\varphi}\left(\frac{1}{1+\gamma} - \frac{1}{1+\tau^c}\right)
$$

For $\tau^c = \gamma$, we have $\Phi = 0$ and output is efficient in the steady state. We will consider the general case $\Phi < 0$.

Equilibrium Conditions The linearized equilibrium conditions are optimal labor supply:

$$
\widehat{w}_t = \varphi\widehat{n}_t + \sigma\widehat{c}_t\ .
$$

Euler equation:

$$-\sigma\widehat{c}_t = -\sigma\widehat{c}_{t+1} + r_t^s - \pi_{t+1} \ .$$

Production function:

$$\widehat{\Delta}_t + \widehat{y}_t = a_t - \gamma\widehat{y}_t + \widehat{n}_t$$

Labor demand:

$$\widehat{w}_t = \widehat{mc}_t - a_t + \gamma\widehat{y}_t$$

Optimal pricing:

$$\begin{aligned}
&p_t^* = x_{1t} - x_{2t} \\
&\xi_{1,t} = (1-\theta\beta)mc_t + (1-\theta\beta)y_t - \theta\beta\sigma c_{t+1} + \theta\beta\sigma c_t + \epsilon\theta\beta\pi_{t+1} + \theta\beta\xi_{1,t+1} \\
&\xi_{2,t} = (1-\theta\beta)y_t - \theta\beta\sigma c_{t+1} + \theta\beta\sigma c_t + (\epsilon-1)\theta\beta\pi_{t+1} + \theta\beta\xi_{2,t+1}
\end{aligned}$$

Market clearing:

$$\widehat{c}_t = \widehat{y}_t$$

Natural output gap:

$$\widehat{x}_t^n = \widehat{y}_t - \widehat{y}_t^n = \widehat{y}_t - \frac{(1+\varphi)}{\varphi+\sigma}a_t$$

Welfare relevant output gap:

$$\widehat{x}_t^e = \widehat{y}_t - \widehat{y}_t^e$$

Pollution:

$$\widehat{\Lambda}_t = -\gamma\widehat{y}_t$$

Loss Function Taking a second order approximation of the welfare function U_t:

$$U_t - U \approx c^{1-\sigma}\left\{\frac{c_t - c}{c} - \frac{\sigma}{2}\left(\frac{c_t - c}{c}\right)^2 - \frac{n^{1+\varphi}}{c^{1-\sigma}}\left[\frac{n_t - n}{n} + \frac{\varphi}{2}\left(\frac{n_t - n}{n}\right)^2\right]\right\}$$

yields

$$\frac{U_t - U}{U_c c} = \frac{U_t - U}{c^{1-\sigma}} \approx \frac{c_t - c}{c} - \frac{\sigma}{2}\left(\frac{c_t - c}{c}\right)^2 - \frac{n^{1+\varphi}}{c^{1-\sigma}}\left[\frac{n_t - n}{n} + \frac{\varphi}{2}\left(\frac{n_t - n}{n}\right)^2\right]$$

For a generic variable x, up to second order, $\frac{x_t - x}{x} = \widehat{x}_t + \frac{\widehat{x}_t^2}{2}$ with $\widehat{x} = \log x_t - \log x$. Also, the following condition holds:

$$\frac{n^{1+\varphi}}{c^{1-\sigma}} = n^{\varphi} c^{\sigma} \frac{n}{c} = \frac{A\Lambda}{1+\gamma}(1+\Phi)\frac{n}{c} = \frac{1+\Phi}{(1+\gamma)}$$

Hence:

$$\frac{U_t - U}{c^{1-\sigma}} \approx \widehat{c}_t + \frac{\widehat{c}_t^2}{2} - \frac{\sigma}{2}\widehat{c}_t^2 - \frac{1+\Phi}{1+\gamma}\left[\widehat{n}_t + \frac{\widehat{n}_t^2}{2} + \frac{\varphi}{2}\widehat{n}_t^2\right]$$

Plugging in the market clearing condition $\widehat{c}_t = \widehat{y}_t$ and the production function $\widehat{n}_t = \widehat{y}_t + \widehat{\Delta}_t - a_t - \widehat{\Lambda}_t = (1+\gamma)\widehat{y}_t + \widehat{\Delta}_t - a_t$:

$$\frac{U_t - U}{c^{1-\sigma}} \approx \widehat{y}_t + \frac{1-\sigma}{2}\widehat{y}_t^2 - \frac{1+\Phi}{1+\gamma}\left[(1+\gamma)\widehat{y}_t + \widehat{\Delta}_t - a_t + \frac{1+\varphi}{2}\left((1+\gamma)\widehat{y}_t + \widehat{\Delta}_t - a_t\right)^2\right]$$

Eliminating all terms independent of policy and of order higher than two:

$$\begin{aligned}
\frac{U_t - U}{c^{1-\sigma}} &\approx \widehat{y}_t + \frac{1-\sigma}{2}\widehat{y}_t^2 - \frac{1+\Phi}{1+\gamma}\left[(1+\gamma)\widehat{y}_t + \widehat{\Delta}_t + \frac{1+\varphi}{2}[(1+\gamma)^2\widehat{y}_t^2 - 2(1+\gamma)\widehat{y}_t a_t]\right] + t.i.p. \\
&\approx -\Phi\widehat{y}_t + \frac{1-\sigma}{2}\widehat{y}_t^2 - \frac{1+\Phi}{1+\gamma}\widehat{\Delta}_t - \frac{(1+\Phi)(1+\gamma)(1+\varphi)}{2}\widehat{y}_t^2 + (1+\Phi)(1+\varphi)\widehat{y}_t a_t + t.i.p.
\end{aligned}$$

Using the definition of ζ, the efficient output level can be written

$$\widehat{y}_t^e = \frac{1+\varphi}{\varphi + \gamma(1+\varphi) + \frac{\gamma}{1+\gamma} + \sigma} a_t = \frac{1+\varphi}{\zeta + \frac{\gamma}{1+\gamma}} a_t$$

Then, plugging in the definition of the output gap $\widehat{y}_t = \widehat{x}_t + \widehat{y}_t^e$:

$$\begin{aligned}
\frac{U_t - U}{c^{1-\sigma}} &\approx -\Phi\widehat{x}_t + \frac{1-\sigma}{2}(\widehat{x}_t^2 + 2\widehat{x}_t\widehat{y}_t^e) - \frac{1+\Phi}{1+\gamma}\widehat{\Delta}_t \\
&\quad - \frac{(1+\Phi)(1+\gamma)(1+\phi)}{2}(\widehat{x}_t^2 + 2\widehat{x}_t\widehat{y}_t^e) + (1+\Phi)(1+\varphi)\widehat{x}_t a_t + t.i.p. \\
&\approx -\Phi\widehat{x}_t - \frac{1}{2}\left\{(1+\Phi)(1+\gamma)(1+\varphi) + (\sigma-1)\right\}\widehat{x}_t^2 - \frac{1+\Phi}{1+\gamma}\widehat{\Delta}_t \\
&\quad - \left\{(1+\Phi)(1+\gamma)(1+\varphi) - (1-\sigma)\right\}\widehat{x}_t\widehat{y}_t^e + (1+\Phi)(1+\varphi)\widehat{x}_t a_t + t.i.p. \\
&\approx -\Phi\widehat{x}_t - \frac{1}{2}\left\{(1+\Phi)(1+\gamma)(1+\varphi) + (\sigma-1)\right\}\widehat{x}_t^2 - \frac{1+\Phi}{1+\gamma}\widehat{\Delta}_t \\
&\quad \underbrace{- (1+\varphi)\left\{\left[(1+\Phi)(1+\gamma)(1+\varphi) - (1-\sigma)\right]\frac{1}{\zeta + \frac{\gamma}{1+\gamma}} - (1+\Phi)\right\}}_{\equiv V_1}\widehat{x}_t a_t + t.i.p.
\end{aligned}$$

The coefficient V_1 in front of the interaction term $\widehat{x}_t a_t$ simplifies to:

$$
\begin{aligned}
V_1 &= -(1+\varphi)\left\{\left[(1+\Phi)(1+\gamma)(1+\varphi)-(1-\sigma)\right]\frac{1}{\zeta+\frac{\gamma}{1+\gamma}}-(1+\Phi)\right\}\\
&= -\frac{1+\varphi}{\zeta+\frac{\gamma}{1+\gamma}}\left\{(1+\Phi)(1+\gamma)(1+\varphi)+\sigma-1-(1+\Phi)\left(\zeta+\frac{\gamma}{1+\gamma}\right)\right\}\\
&= -\frac{(1+\varphi)}{\zeta+\frac{\gamma}{1+\gamma}}(1+\Phi)\left\{(1+\gamma)(1+\varphi)+\frac{\sigma-1}{1+\Phi}-\left(\zeta+\frac{\gamma}{1+\gamma}\right)\right\}\\
&= -\frac{(1+\varphi)}{\zeta+\frac{\gamma}{1+\gamma}}(1+\Phi)\left\{(\varphi+\gamma+\gamma\varphi)+\sigma-\sigma+\frac{\sigma-1}{1+\Phi}+1-\left(\zeta+\frac{\gamma}{1+\gamma}\right)\right\}\\
&= -\frac{(1+\varphi)}{\zeta+\frac{\gamma}{1+\gamma}}(1+\Phi)\left\{\zeta-\sigma+\frac{\sigma-1}{1+\Phi}+1-\left(\zeta+\frac{\gamma}{1+\gamma}\right)\right\}\\
&= -\frac{(1+\varphi)}{\zeta+\frac{\gamma}{1+\gamma}}(1+\Phi)\left\{-\sigma+\frac{\sigma-1}{1+\Phi}-\frac{\gamma}{1+\gamma}\right\}\\
&= -\frac{1+\varphi}{\zeta+\frac{\gamma}{1+\gamma}}(1+\Phi)\left\{\frac{-\sigma\Phi-1}{1+\Phi}-\frac{\gamma}{1+\gamma}\right\}
\end{aligned}
$$

Hence:

$$
\begin{aligned}
\frac{U_t-U}{c^{1-\sigma}} &\approx -\Phi\widehat{x}_t-\frac{1}{2}\left\{(1+\Phi)(1+\gamma)(1+\varphi)+(\sigma-1)\right\}\widehat{x}_t^2-\frac{1+\Phi}{1+\gamma}\widehat{\Delta}_t\\
&\quad-\frac{(1+\varphi)}{\zeta+\frac{\gamma}{1+\gamma}}(1+\Phi)\left\{\frac{-\sigma\Phi-1}{1+\Phi}+\left(1-\frac{\gamma}{1+\gamma}\right)\right\}\widehat{x}_t a_t\\
&\approx -\Phi\widehat{x}_t-\frac{1}{2}\left\{(1+\Phi)(1+\gamma)(1+\varphi)+(\sigma-1)\right\}\widehat{x}_t^2-\frac{1+\Phi}{1+\gamma}\widehat{\Delta}_t\\
&\quad-\frac{(1+\varphi)}{\zeta+\frac{\gamma}{1+\gamma}}\left\{-\Phi(\sigma-1)-(1+\Phi)\frac{\gamma}{1+\gamma}\right\}\widehat{x}_t a_t
\end{aligned}
$$

We are then ready to evaluate the loss function:

$$
\begin{aligned}
\mathcal{L}\equiv-\mathcal{W}\approx\mathbb{E}_0\Bigg[\sum_{t=0}^{\infty}\beta^t\Bigg\{&\frac{1}{2}\Big((1+\Phi)(1+\gamma)(1+\varphi)+(\sigma-1)\Big)\widehat{x}_t^2+\frac{1+\Phi}{1+\gamma}\widehat{\Delta}_t+\\
&\frac{(1+\phi)}{\zeta+\frac{\gamma}{1+\gamma}}\left[-\Phi(\sigma-1)-(1+\Phi)\left(\frac{\gamma}{1+\gamma}\right)\right]\widehat{x}_t a_t+\Phi\widehat{x}_t\Bigg\}\Bigg]
\end{aligned}
$$

The discounted sum of log price dispersion is given by $\sum_{t=0}^{\infty}\beta^t\widehat{\Delta}_t\approx\frac{\epsilon}{2\kappa}\sum_{t=0}^{\infty}\beta^t\pi_t^2$, with $\kappa=\frac{(1-\theta)(1-\theta\beta)}{\beta}$ governing the slope of the NKPC. Therefore:

$$
\begin{aligned}
\mathcal{L}\approx\mathbb{E}_0\Bigg[\sum_{t=0}^{\infty}\beta^t\Bigg\{&\frac{1}{2}\Big((1+\Phi)(1+\gamma)(1+\varphi)+(\sigma-1)\Big)\widehat{x}_t^2+\frac{1+\Phi}{1+\gamma}\frac{\epsilon}{2\kappa}\pi_t^2\\
&+\frac{(1+\phi)}{\zeta+\frac{\gamma}{1+\gamma}}\left[-\Phi(\sigma-1)-(1+\Phi)\frac{\gamma}{1+\gamma}\right]\widehat{x}_t a_t+\Phi\widehat{x}_t\Bigg\}\Bigg]
\end{aligned}
$$

Now one can show that a second order approximation of the optimal pricing condition leads to the following (extended) NKPC:

$$\pi_t + \frac{\epsilon - 1}{2(1-\theta)}\pi_t^2 + \frac{1-\theta\beta}{2}G_t\pi_t = \kappa\left[\widehat{x}_{1t} - \widehat{x}_{2t} + \frac{1}{2}(\widehat{x}_{1t}^2 - \widehat{x}_{2t}^2)\right] + \beta\pi_{t+1} + \beta\frac{1-\theta\beta}{2}G_{t+1}\pi_{t+1} + \beta\frac{\epsilon-1}{2(1-\theta)}\pi_{t+1}^2 + \beta\frac{\epsilon}{2}\pi_{t+1}^2 \ , \tag{A.24}$$

where $\widehat{x}_{1t} \equiv mc_t - \sigma\widehat{c}_t + \widehat{y}_t$ and $\widehat{x}_{2t} \equiv \widehat{y}_t - \sigma\widehat{c}_t$, and:

$$G_t = \sum_{\tau=t}^{\infty}(\theta\beta)^{\tau-t}(x_{1,t,\tau} + x_{2,t,\tau}) \ ,$$

where $\widehat{x}_{1,t,\tau} \equiv \widehat{x}_{1\tau} + \epsilon\sum_{s=t+1}^{\tau}\pi_s$ and $\widehat{x}_{1,t,\tau} \equiv \widehat{x}_{1\tau} + (\epsilon-1)\sum_{s=t+1}^{\tau}\pi_s$. Defining $Y_t \equiv \pi_t + \frac{\epsilon-1}{2(1-\theta}\pi_t^2 + \frac{1-\theta\beta}{2}G_t\pi_t + \frac{\epsilon}{2}\pi_t^2$, eq. (A.24) can be rewritten as:

$$Y_t = \kappa\left[\widehat{x}_{1t} - \widehat{x}_{2t} + \frac{1}{2}(\widehat{x}_{1t}^2 - \widehat{x}_{2t}^2)\right] + \beta\frac{\epsilon}{2}\pi_t^2 + \beta Y_{t+1}$$

Hence:

$$Y_0 = \kappa\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\widehat{x}_{1t} - \widehat{x}_{2t} + \frac{1}{2}(\widehat{x}_{1t}^2 - \widehat{x}_{2t}^2)\right\}\right] + \frac{\epsilon}{2}\sum_{t=0}^{\infty}\beta^t\pi_t^2 \tag{A.25}$$

The difference between the Calvo terms reduces to the marginal costs $\widehat{mc}_t$, which, using households labor-supply condition and the production technology, can be expressed as

$$\begin{aligned}\widehat{x}_{1t} - \widehat{x}_{2t} = \widehat{mc}_t &= \widehat{w}_t - a_t + \gamma\widehat{y}_t \\ &= \varphi\widehat{n}_t + \sigma\widehat{c}_t - a_t + \gamma\widehat{y}_t \\ &= \varphi\left(\widehat{\Delta}_t + \widehat{y}_t - a_t + \gamma\widehat{y}_t\right) + \sigma\widehat{c}_t - a_t + \gamma\widehat{y}_t\end{aligned}$$

Hence:

$$\widehat{x}_{1t} - \widehat{x}_{2t} = \varphi\widehat{\Delta}_t + \left(\varphi + \gamma(1+\varphi) + \sigma\right)\widehat{y}_t - (1+\varphi)a_t - \widehat{x}_{1t} - \widehat{x}_{2t} = \varphi\widehat{\Delta}_t + \zeta y_t - (1+\varphi)a_t$$

Ignoring higher-order terms and terms independent of policy, we can then rewrite Y_0 as :

$$Y_0 \approx \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t y_t\right] + \frac{1}{2\zeta}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\frac{\epsilon(1+\varphi)}{\kappa}\pi_t^2 + \left[\left(1+\varphi+\gamma(1+\varphi)\right)^2 - (1-\sigma)^2\right]\widehat{y}_t^2 - 2(1+\varphi)(1+\varphi+\gamma(1+\varphi))\widehat{y}_t a_t\right\}\right]$$

Since V_0 is given we then have:

$$\mathbb{E}_0\Bigg[\sum_{t=0}^{\infty}\beta^t y_t\Bigg] \approx -\frac{1}{2\zeta}\mathbb{E}_0\Bigg[\sum_{t=0}^{\infty}\beta^t\Bigg\{\frac{\epsilon(1+\varphi)}{\kappa}\pi_t^2 + \big[(1+\varphi+\gamma(1+\varphi))^2-(1-\sigma)^2\big]\widehat{y}_t^2 - 2(1+\varphi)(1+\varphi+\gamma(1+\varphi))\widehat{y}_t a_t\Bigg\}\Bigg] + t.i.p.$$

Rewriting in terms of the output gap $\widehat{x}_t^e$ we get:

$$\mathbb{E}_0\Bigg[\sum_{t=0}^{\infty}\beta^t \widehat{x}_t\Bigg] \approx X_1 + X_2 + X_3 + t.i.p.$$

where

$$X_1 = -\frac{1}{2\zeta}\mathbb{E}_0\Bigg[\sum_{t=0}^{\infty}\beta^t\Bigg\{\frac{\epsilon(1+\varphi)}{\kappa}\pi_t^2\Bigg\}\Bigg]$$

$$X_2 = -\frac{1}{2\zeta}\mathbb{E}_0\Bigg[\sum_{t=0}^{\infty}\beta^t\Bigg\{\big[(1+\varphi+\gamma(1+\varphi))^2-(1-\sigma)^2\big](\widehat{x}_t^e)^2\Bigg\}\Bigg]$$

$$X_3 = -\frac{1}{\zeta}\mathbb{E}_0\Bigg[\sum_{t=0}^{\infty}\beta^t\Bigg\{\big[(1+\varphi+\gamma(1+\varphi))^2-(1-\sigma)^2\big]\widehat{x}_t^e\widehat{y}_t^e - (1+\varphi)(1+\varphi+\gamma(1+\varphi))\widehat{x}_t^e a_t\Bigg\}\Bigg]$$

Using $\widehat{y}_t^e = \frac{1+\varphi}{\zeta+\frac{\gamma}{1+\gamma}}a_t$ and $\zeta = \sigma+\varphi+\gamma(1+\varphi)$ into X_3 to get:

$$X_3 = -\frac{1+\varphi}{\zeta\Big(\zeta+\frac{\gamma}{1+\gamma}\Big)}\mathbb{E}_0\Bigg[\sum_{t=0}^{\infty}\beta^t\Bigg\{\zeta\Big(1-\sigma\Big) - \zeta\frac{\gamma}{1+\gamma} - \frac{\gamma}{1+\gamma}\Big(1-\sigma\Big)\Bigg\}\Bigg]\widehat{x}_t^e a_t$$