Pro-Cyclical Emissions and Optimal Monetary Policy*

Francesco Giovanardi[†]   Matthias Kaldorf[‡]

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Abstract

We study optimal monetary policy in an analytically tractable New Keynesian DSGE-model with pro-cyclical carbon emissions. The competitive equilibrium under flexible prices overreacts to productivity shocks relative to the efficient allocation. When prices are sticky, actual output increases by less than natural output: the relationship between actual and efficient output depends on the pro-cyclicality of emissions and the severity of price stickiness. The real interest rate that monetary policy optimally tracks is distinct from the natural rate of interest, implying that divine coincidence is broken also in the presence of productivity shocks. For central banks with a dual mandate, we characterize the optimal monetary policy response and show that it generally places a larger weight on output stabilization. While monetary policy can implement the flexible price allocation, it is not optimal to simultaneously stabilize inflation and the natural output gap. At the optimum, inflation is more volatile than in the baseline New Keynesian model without socially harmful emissions.

Keywords: Optimal Monetary Policy, Output Gap, Central Bank Loss Function, Emission Externality, Phillips Curve

JEL Classification: E31, E58, Q58

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[†]Prometeia & University of Cologne. Email: francesco.giovanardi@prometeia.com.

[‡]Deutsche Bundesbank, Research Centre. Wilhelm-Epstein-Str. 14, 60431 Frankfurt am Main, Germany. Email: matthias.kaldorf@bundesbank.de (Corresponding Author).

1 Introduction

There is now a broad consensus that the emission of greenhouse gases inflicts severe damages on the wider economy. Economic theory suggests that Pigouvian emission taxes are the best instrument to address such an emission externality. It is becoming increasingly clear that financial regulators in general and central banks in particular can play at most a supporting role in addressing emission externalities related to climate change. First, conventional monetary policy instruments, such as short-term interest rates are naturally not well-suited to address long run issues (Nakov and Thomas 2023). Second, even the unconventional central bank toolkit provides very limited potential to induce a sectoral re-allocation away from fossil fuels (see Giovanardi et al. 2023 or Ferrari and Nispi Landi 2023 among others). The attention of policymakers is therefore shifting towards the optimal response of monetary policy to climate change from an *adaptation* perspective, rather than a *mitigation* perspective.

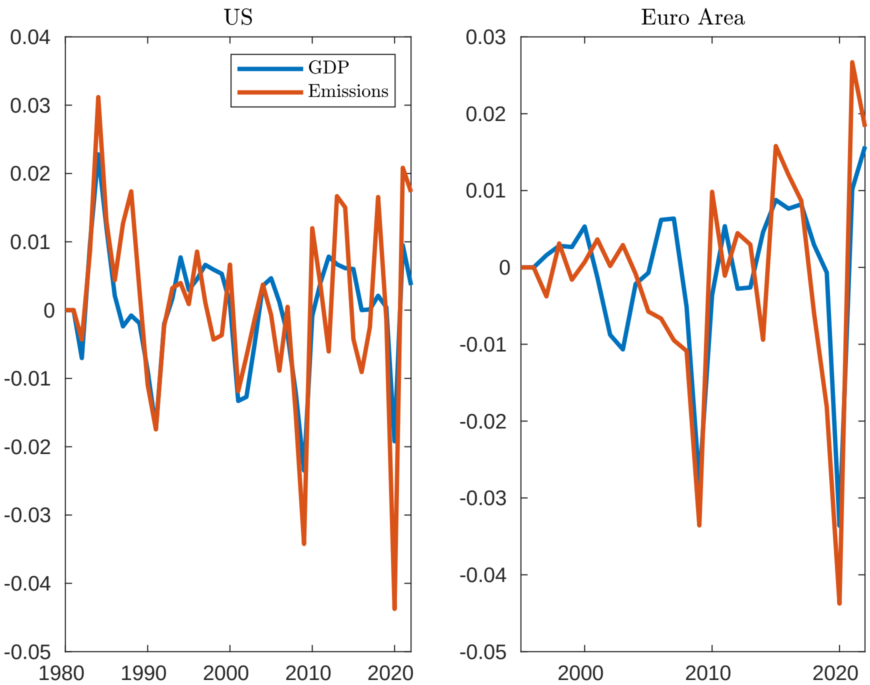
This paper offers a normative analysis of monetary policy in the presence of socially harmful emissions. To that end, we augment a standard New Keynesian model by emission damages. Under standard assumptions on the damage function, damages are higher during booms than in recessions. This assumption is consistent with the observation that emissions are highly pro-cyclical, both in the US and in the euro area, see Figure 1 for the case of carbon dioxide. Doda (2014) and Khan et al. (2019) provide additional evidence. If emissions are pro-cyclical, a Pigouvian emission tax that addresses the emission externality in the long run but ignores cyclical damages does *not* implement the efficient allocation. Instead, the competitive equilibrium allocation under flexible prices implies an overreaction of output in response to productivity shocks, relative to the efficient allocation.[1]

The relative over-reaction of output in the flexible price equilibrium interacts non-trivially with nominal rigidities and, hence, monetary policy. Consider a positive shock to TFP. Price rigidities prevent a fraction of firms from reducing prices, such that the economy expands by less than it would do under flexible prices. Absent emission externalities, the central bank aims at closing the gap between the sticky price and flexible price output. We refer to this gap as the *natural output gap*. With pro-cyclical emissions, closing the natural output gap does not implement the efficient allocation. We refer to the difference between the output reaction under sticky prices minus the output reaction in the efficient allocation as the *welfare-relevant output gap*. Price stickiness attenuates the over-reaction of the flexible price equilibrium allocation vis-a-vis the welfare-relevant output gap.

We then show that pro-cyclical emissions also affect the competitive equilib-

[1] Formally, the efficient allocation is implemented by a time-varying tax. Under a time-invariant tax, optimal monetary policy addresses two dynamic inefficiencies with only one instrument - the nominal interest rate - and will not be able to offset both inefficiencies at once.

Figure 1: Carbon Emissions and GDP over Time



Notes: Data at annual frequency, detrended using a one-sided HP-filter with smoothing parameter 6.25. The full-sample correlations are 0.78 for the US and 0.77 for the Euro Area.

rium, which is described by a dynamic IS equation and the New Keynesian Phillips curve. The latter describes a macroeconomic relationship between the natural output gap and inflation. It is straightforward to establish that also this relationship is affected by pro-cyclical emissions. On the one hand, output expands by less than it would do without pro-cyclical emission damages which, as a by-product, also implies that the natural output gap is less volatile. On the other hand, it does not directly change firm's price setting behavior. Therefore, the Phillips curve steepens. In contrast, the dynamic IS equation is not directly affected by pro-cyclical emissions. When the central bank reaction function is held constant, pro-cyclical emissions imply a smaller volatility of inflation and the natural output gap.

Sign and volatility of the welfare-relevant output gap, however, are ambiguously affected by pro-cyclical emissions. Consider a positive shock to total factor productivity (TFP). It can be shown analytically that, in contrast to the baseline New Keynesian model, the Phillips curve steepens and is *shifted* downwards. This inflation shifter, which resembles a cost-push shock, implies that the central bank is unable to achieve perfect stabilization of inflation and the welfare-relevant output gap: divine coincidence as defined by Blanchard and Gali (2007) is broken.[2] For a high degree of price stickiness, inefficiencies associated with firms being unable to reduce their prices dominate the welfare-relevant output gap. It is still negative, but of smaller sign than in the baseline New Keynesian model. In contrast, for a low degree of price stickiness, the emission externality dominates and the welfare-relevant output gap is positive. Consequently, the volatility of the welfare-relevant output gap is non-monotonic in the degree of price stickiness.

We incorporate this insight into an analytical characterization of optimal monetary policy along the lines of Clarida, Galí, and Gertler (1999) and Woodford (2011). Our analysis is applicable for central banks with a *dual mandate* and proceeds in two steps. First, we discuss how the interaction between nominal rigidities and pro-cyclical emissions affects the central bank's objective function, which is derived from first principles. Using a second order approximation to welfare, it can be shown that previously discussed overreaction of output in competitive equilibrium relative to the efficient allocation implies a higher weight on output stabilization.

In a second step, we combine this insight with the modified New Keynesian Phillips curve to study whether the interaction between pro-cyclical emissions and

[2]Breaking divine coincidence in the presence of productivity shocks requires frictions that go beyond nominal rigidities. For example, Faia (2009) shows that search frictions on the labor market render the flexible price allocation infeasible. In contrast, the flexible price allocation is implementable in our framework, but it is not optimal to do so. Bernardino, Isabel, and Teles (2003) demonstrate that in an economy with cash-in-advance constraints, it is not optimal to fully stabilize prices and output gaps, which is conceptually similar to our results. Sims, Wu, and Zhang (2023) discuss the role of financial shocks as inflation shifters in the New Keynesian Phillips curve, which also break divine coincidence.

nominal rigidities could *qualitatively* change the optimal reaction of monetary policy. While monetary policy would typically cut interest rates after a positive TFP shock, a sufficiently emission externality might render a tightening of monetary policy optimal.[3] We can show that this is never the case. Irrespective of the degree of price stickiness and the severity of short run emission damages, the central bank always cuts interest rates by less in absolute terms after a positive TFP shock than it would to absent pro-cyclical emission damages. Consistent with Khan, King, and Wolman (2003), the central banks' optimal policy problem is resolved heavily in favor of replicating the equilibrium allocation under flexible prices.

Therefore, our analysis yields a clear prediction for the effects of pro-cyclical emissions on macroeconomic fluctuations, taking into account the *optimal response* of monetary policy. By breaking divine coincidence, pro-cyclical emissions imply that inflation and the welfare-relevant output gap are more volatile than in the baseline model, where the central bank can achieve perfect stabilization of inflation and the output gap at the same time. In a last step, we show numerically that our characterization of optimal monetary policy also carries over to a larger model with capital and investment adjustment costs.

By providing a simple analytical framework, our framework contributes to the growing discussion on welfare-relevant output gaps, which are not only relevant for monetary policy frameworks in all jurisdictions that provide their central bank with a dual mandate, but for all policies that take output gaps into account. Conditioning macroeconomic stabilization policies at business cycle frequencies on output gaps has to bear in mind that those output gaps need not be efficient from a welfare perspective.[4] In spirit of the analysis in Blanchard and Gali (2007), we have shown how the optimal monetary policy is affected by externalities originating in the real sector, which do not have a direct effect on nominal rigidities. Finally, it should be noted that our analysis of monetary policy under cyclical emissions is a *second best* solution to a welfare-maximization problem. If appropriate cyclical adjustments to emission taxes were in place, monetary policy could be conducted *as usual*.[5]

[3] Such non-standard responses of optimal monetary policy have been documented in Khan, King, and Wolman (2003).

[4] On a conceptual level, our analysis also relates to the literature of optimal monetary policy in the presence of hysteresis effects. If such effects are present, it is not optimal to close the natural gap. In sharp contrast to a setting with emission externalities, however, optimal monetary policy is more expansionary in response to a positive TFP shock than in the baseline New Keynesian model, see Cerra, Fatás, and Saxena (2023) and the references therein.

[5] In Sims, Wu, and Zhang (2023), the central bank has an additional policy instrument in the form of asset purchases to offset financial shocks and restore divine coincidence. It appears rather implausible from an institutional background that central bank policy instruments can be used in an appropriate way to address pro-cyclical emissions.

Related Literature This paper mainly draws from the E-DSGE literature, starting from the contribution by Heutel (2012). This literature studies the interaction between environmental policies and macroeconomic activity at business cycle frequencies, which makes them a suitable model class to study the relationship between environmental and monetary policies, see Annicchiarico et al. (2021) for a survey. Related to monetary policy, Annicchiarico and Di Dio (2015) study the role of nominal rigidities for the effectiveness of environmental policies. Faria, McAdam, and Viscolani (2022) discuss the neutrality of monetary policy under different monetary frictions, such as cash-in-advance or money-in-the-utility function.

We primarily contribute to a growing literature studying how monetary policy optimally adapts to climate change. McKibbin et al. (2020) provide an overview about potential interactions between climate policy and monetary policy. For a general discussion of these interactions, we also refer to Hansen (2021). In this strand of literature, our paper is most closely related to Muller (2023). Using the New-Keynesian framework, Muller (2023) proposes a natural interest rate taking time-varying pollution intensities into account. By tracking such a refined "green interest rate", monetary policy intertemporally re-allocates consumption from periods with high-pollution intensity to periods with a low-pollution intensity. Nakov and Thomas (2023) show that climate change, i.e. the long run consequences of emissions, only has a limited impact on the optimal conduct of monetary policy. Economides and Xepapadeas (2018) study optimal monetary policy when climate change is an additional propagation channel for TFP shocks, such that positive shocks have negative side effects through elevated damages from climate change.

A series of papers discusses (optimal) monetary policy when inflation is (at least partially) driven by rising energy prices. In a New Keynesian model with an energy sector, Olovsson and Vestin (2023) show that targeting core inflation is welfare-optimal. The literature also recognizes that monetary policy might be affected by potentially inflationary effects of carbon taxation more generally. Konradt and Mauro (2023) provide empirical evidence, while Ferrari and Nispi Landi (2022) and Del Negro, Di Giovanni, and Dogra (2023) study this channel through the lenses of New Keynesian models, which are conceptually similar to ours. However, we do not incorporate direct effects of carbon taxes on price rigidities.

Outline Our paper is structured as follows. Section 2 presents the emission-augmented New Keynesian model without capital. In Section 3, we characterize optimal monetary policy. Section 4 shows that our analytical results also carry over to a larger setting with capital, while Section 5 concludes.

2 A Simple E-NK Framework

We present the basic monetary policy trade-off in an otherwise standard New Keynesian model, augmented by socially harmful emissions. As a first step, we characterize the efficient allocation and the competitive equilibrium of the model. There is a representative household, monopolistically competitive firms, a fiscal authority, and the central bank. Emissions negatively affect the productivity of final good producers.[6]

2.1 Households

The representative household saves using nominal deposits S_t that pay a one-period interest rate r_t^s, consumes the final consumption good c_t, and supplies labor n_t at the nominal wage W_t. The household also owns firms and receives their profits d_t^{firms}, expressed in real terms. The maximization problem is given by

$$\max_{\{c_t, n_t, S_t\}_{t=0}^{\infty}} \quad \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1-\sigma} - \frac{n_t^{1+\varphi}}{1+\varphi} \right) \right]$$
$$\text{s.t.} \quad P_t c_t + S_t = W_t n_t + (1 + r_{t-1}^s) S_{t-1} + P_t d_t^{firms} \ .$$

The parameters σ and φ determine the inverse of, respectively, the intertemporal elasticity of substitution and the elasticity of labor supply. Solving this maximization problem yields a standard Euler equation and an intra-temporal labor supply condition

$$c_t^{-\sigma} = \beta r_t^s \mathbb{E}_t \left[\frac{c_{t+1}^{-\sigma}}{\Pi_{t+1}} \right] \ , \tag{1}$$
$$n_t^{\varphi} = w_t c_t^{-\sigma} \ . \tag{2}$$

Here, P_t is the price level, $w_t \equiv \frac{W_t}{P_t}$ is the real wage, and $\Pi_t \equiv \frac{P_t}{P_{t-1}}$ denotes gross inflation.

2.2 Firms: Technology

There is a mass-one continuum of monopolistic firms, indexed by i. Firm i hires labor $n_t(i)$ to produce the intermediate good $y_t(i)$ with the following technology:

$$y_t(i) = \Lambda_t A_t n_t(i) \ . \tag{3}$$

[6] Analytically similar results can be obtained by assuming that emissions exert a utility loss on households.

While A_t is an exogenous productivity shock, pollution damage $\Lambda_t = \exp\{-\Gamma y_t\}$ endogenously reduces productivity since $\frac{\partial \Lambda_t}{\partial y_t} < 0$. Our analysis abstracts from technological change or abatement effort at the firm level such that we can assume that emissions are proportional to production and that damages in turn depend positively on emissions. As we shall see, optimal emission taxes are pro-cyclical in this setup, as in Golosov et al. (2014). Crucially, whenever emission taxes are not responsive to the business-cycle, the model features a dynamic inefficiency that affects the optimal conduct of monetary policy.[7]

Before characterizing nominal rigidities, some remarks on the damage function are in order. Emission damages can also be interpreted to go beyond damages from climate change. The environmental economics literature typically views climate change related damages as only a subset of the overall adverse effects that the emission of polluting substances exerts on the wider economy. This includes negative health consequences through air quality losses, decreased timber and agriculture yields, depreciation of materials, and reductions of recreation services. For details, we refer to Muller, Mendelsohn, and Nordhaus (2011) and the references therein. In contrast to climate change, these negative effects materialize very quickly in response to a cyclical increase in economic activity but also depreciate faster. More generally, our analysis is also applicable to the pro-cyclical depletion of other renewable resources, such as water, soil or fishing grounds.

These alternative interpretations will of course have different quantitative implications for the optimal conduct of monetary policy. Specifically, the elasticity of emission damages with respect to current output, the depreciation rate of polluting substances, and the recovery rate of renewable resources drives the wedge between efficient and natural level of output. Notably, our qualitative characterization of optimal monetary policy carries over to these situations as well.

2.3 Firms: Nominal Rigidities

The rest of the supply side coincides with the baseline New Keynesian model: monopolistic producers are not perfectly able to adjust their prices due to nominal rigidities, modeled as in Calvo (1983), with θ being the fraction of firms that is not allowed to change prices. The optimal price for a firm that is able to adjust prices is given by

$$p_t^* = \frac{1}{1-\tau_t^c}\frac{\epsilon}{\epsilon-1}\frac{\xi_{1,t}}{\xi_{2,t}} \,. \tag{4}$$

[7]Note that our analysis is cast in a stationary model. If climate policy is instead modeled in terms of a transition towards higher taxes, optimal taxes should still be above (below) trend during a boom (recession). As long as the carbon taxes do not deviate from their trend in response to business cycle fluctuations, the dynamic inefficiency arises.

where τ_t^c is a *carbon tax* raised by the government and where

$$\xi_{1,t} = mc_t\, y_t + \beta\,\theta\,\mathbb{E}_t\left[\frac{c_{t+1}^{-\sigma}}{c_t^{-\sigma}}\pi_{t+1}^{\epsilon}\xi_{1,t+1}\right] \quad \text{and} \quad \xi_{2,t} = y_t + \beta\,\theta\,\mathbb{E}_t\left[\frac{c_{t+1}^{-\sigma}}{c_t^{-\sigma}}\pi_{t+1}^{\epsilon-1}\xi_{2,t+1}\right]$$

This nominal friction implies that monopolistic producers face time-varying real marginal costs, thus generating a relationship between inflation and real economic activity summarized in a New-Keynesian Phillips Curve. Total factor productivity A_t follows an AR(1) process in logs:

$$\log(A_t) = \rho_A \log(A_{t-1}) + \sigma_A \epsilon_t\ , \quad \text{where } \epsilon_t \sim N(0,1)\ . \tag{5}$$

2.4 Efficient Allocation and Competitive Equilibrium under Flexible Prices

Having described the model's ingredients, we now characterize the efficient allocation and competitive equilibrium of the simple E-NK model. Since our analysis builds on linearizing equilibrium conditions around the deterministic steady state, we perform a change of variables and define the steady state output adjusted damage parameter $\gamma \equiv \frac{\Gamma}{y}$.

For the remainder of this paper, we assume that the fiscal authority sets a constant labor subsidy, $\tau^n = \frac{1}{\epsilon} \Rightarrow (1-\tau^n)\mu = 1$, to eliminate the steady state distortion generated by monopolistic competition. We begin by characterizing the efficient output level y_t^e and natural output level y_t^n and their responses $\widehat{y}_t^n$ and $\widehat{y}_t^e$ to a technology shock a_t, expressed in deviations from steady state.

Proposition 1. The natural level y_t^n and efficient level y_t^e can be written as a function of the only state variable A_t:

$$(y_t^n)^{\sigma+\varphi} = (1-\tau_t^c)(A_t\Lambda_t)^{1+\varphi}\ . \tag{6}$$

$$(y_t^e)^{\sigma+\varphi} = \frac{(A_t\Lambda_t)^{1+\varphi}}{1+\gamma\frac{y_t}{y}}\ , \tag{7}$$

Their log-deviations around the deterministic steady state are given by:

$$\widehat{y}_t^n = \frac{(1+\varphi)a_t - \frac{\tau^c}{1-\tau^c}\widehat{\tau}_t^c}{\varphi+\gamma(1+\varphi)+\sigma} \tag{8}$$

$$\widehat{y}_t^e = \frac{1+\varphi}{\varphi+\gamma(1+\varphi)+\widetilde{\gamma}+\sigma}a_t\ , \tag{9}$$

where $\widetilde{\gamma} = \frac{\gamma}{1+\gamma}$. Proof: see Appendix A.1.

Combining (7) and (6), the ratio of natural and efficient output simplifies to

$$\left(\frac{y_t^n}{y_t^e}\right)^{\sigma+\varphi} = 1 + \gamma\frac{y_t}{y}(1-\tau_t^c) > 1 \;\Leftrightarrow\; \tau_t^c < \frac{\gamma\frac{y_t}{y}}{1+\gamma\frac{y_t}{y}}\,.$$

Hence, absent emission taxes ($\tau_t^c = 0$), the natural level of output generally exceeds its efficient level. Furthermore, setting $\tau_t^c = \frac{\gamma\frac{y_t^n}{y}}{1+\gamma\frac{y_t}{y}}$ implements the efficient allocation.

However, even with a carbon tax implementing the efficient steady state output, emissions still generate a dynamic inefficiency. Specifically, with $\tau^c = \widetilde{\gamma}$ and $\widehat{\tau}^c = 0$, output in the competitive equilibrium $\widehat{y}_t^n$ over-reacts to technology shocks relative to the efficient allocation $\widehat{y}_t^e$, since $\frac{1+\varphi}{\varphi+\gamma(1+\varphi)+\frac{\gamma}{1+\gamma}+\sigma} < \frac{1+\varphi}{\varphi+\gamma(1+\varphi)+\sigma}$. Since this inefficiency is the key element of our analysis, we will often resort to the special case $\tau^c = \widetilde{\gamma}$ and $\widehat{\tau}^c = 0$ in the following characterization of monetary policy.

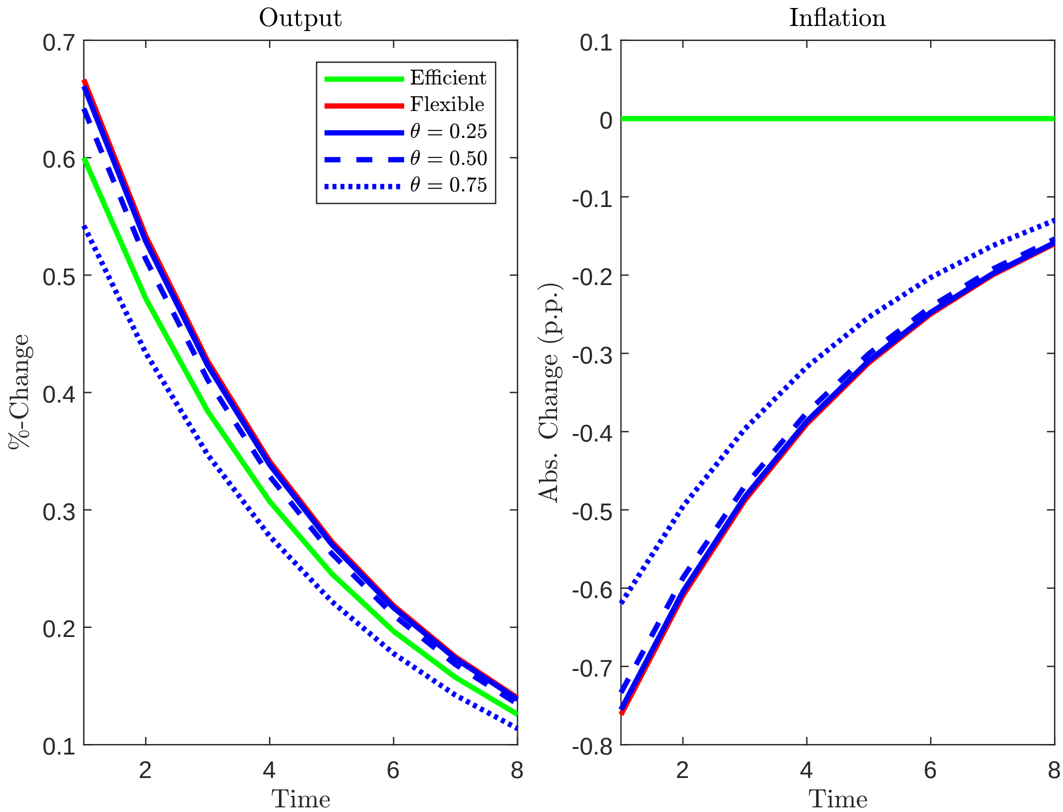
3 Monetary Policy with Pro-Cyclical Emissions

By making prices flexible, we have isolated the role of carbon emissions for the welfare relevant output gap $x_t^e \equiv \widehat{y}_t - \widehat{y}_t^e$ in relation to the natural output gap $x_t^n \equiv \widehat{y}_t - \widehat{y}_t^n$. An over-reaction of the flexible-price economy in response to a TFP shock implies a positive welfare-relevant output gap. This section presents the interactions between nominal rigidities and dynamically inefficient output expansions in the flexible-price economy.

Nominal rigidities imply instead a under-reaction of the competitive equilibrium, relative to the flexible price case, that is a negative natural output gap. Whether the competitive equilibrium still overreacts relative to the efficient allocation, thus, depends on the relative strength of nominal rigidities and the pro-cyclicality of emissions. In Figure 2, we provide graphical intuition for the interaction between emission cyclicality and nominal rigidities.

The more severe are nominal rigidities (the larger is θ), the lower is the over-reaction of output with respect to the efficient allocation, up to the point in which also the welfare relevant output gap turns negative. In Figure 2, this happens for a Calvo parameter between 0.5 and 0.75, i.e. for low, but still reasonable parts of the parameter space. It will turn out that the interaction of these two dynamic inefficiencies, nominal rigidities and pro-cyclical emissions, is non-trivial and has direct implications for the conduct of monetary policy. In the following, we first flesh this interactions out from a positive point of view, under a canonical representation of monetary policy based on a Taylor-type rule, and then characterize optimal monetary policy in closed-form. This is possible thanks to the extremely simplistic E-NK model that we consider in this section. Section 4 applies the

Figure 2: IRF to TFP-Shock: The Role of Nominal Rigidities



Notes: The results are generated by subjecting the model economy to a one standard deviation shock to TFP (5). We set $\rho_A = 0.95$ and $\sigma_A = 0.005$. The Taylor parameter is $\phi = 1.5$, for all other parameters, we refer to Section 4.

quantitative relevance of the monetary policy implications from the simple E-NK model to a larger model with carbon emissions and capital accumulation.

3.1 Equilibrium Effects of Pro-Cyclical Emissions

We first characterize these interactions using the standard representation of our simple model in terms of a dynamic IS curve and a New Keynesian Phillips curve, closed by a Taylor rule for the nominal interest rate r_t^s. Specifically, we show that emission damages affect inflation and output volatility. For the sake of notation, we omit the hat-symbol from now on. All the variables are expressed in log-deviations from steady-state.

Proposition 2. The equilibrium conditions for the economy with nominal rigidities simplify to the following two linear conditions in terms of log-deviations from the steady-state:

$$x_t^n = \mathbb{E}_t[x_{t+1}^n] - \frac{r_t^s - \mathbb{E}_t[\pi_{t+1}]}{\sigma} + \underbrace{\frac{1}{\zeta}\left[(1+\varphi)(a_{t+1} - a_t) - \frac{\tau^c}{1-\tau^c}(\tau_{t+1}^c - \tau_t^c)\right]}_{=r_t^n/\sigma} \tag{10}$$

$$\pi_t = \zeta\kappa x_t^n + \beta\mathbb{E}_t[\pi_{t+1}] + \beta(1-\theta)\frac{\tau^c}{1-\tau^c}(\tau_t^c - \tau_{t+1}^c)\ . \tag{11}$$

Proof: see Appendix A.2.

Equation (10) is a dynamic IS curve: the (natural) output gap x_t^n positively depends on the expected output gap next period and negatively depends on the real interest rate gap, defined as the real interest rate, $r_t^s - E_t[\pi_{t+1}]$, minus the natural real interest rate, r_t^n. The natural interest rate is the real interest rate consistent with the natural level of output, which is in turn defined as the level of output consistent with flexible prices. The New Keynesian Phillips curve is given by (11). As usual, its slope depends on nominal rigidities, through the expression $\kappa = \frac{(1-\theta\beta)(1-\theta)}{\theta}$. Here, the slope is also affected by the auxiliary parameter:

$$\zeta \equiv \varphi + \gamma(1+\varphi) + \sigma\ . \tag{12}$$

Equation (12) shows that the emission externality affects the New Keynesian Phillips curve. The inflation response is determined by the share of firms that can reduce their price, which does not depend on the emission externality. At the same time, the emission externality dampens the effects of TFP shocks on the output gap. Thus, for a given output gap, inflation responds more strongly to TFP shocks if $\gamma > 0$. Pro-cyclical emissions steepen the Phillips curve.

Note that this does not imply that the emission externality is inflationary in equilibrium. To characterize the equilibrium impact, we close the simple E-NK model with a Taylor-type rule for the nominal interest rate:

$$r_t^s = \overline{r}^s + \pi_t^\phi \ , \tag{13}$$

where ϕ governs the response of the short-term nominal interest rates to inflation. We first keep the monetary policy reaction function constant and show how cyclical emissions affect price stability in the competitive equilibrium by iterating forward the Phillips curve.

Proposition 3. Under time-invariant emission taxes, the policy functions for output gap and inflation read

$$\begin{aligned}
x_t^n =& \frac{\sigma}{\zeta} \cdot \frac{(1+\varphi)(1-\beta\rho_a)}{\sigma(1-\beta\rho_a)(1-\rho_a)+\zeta\kappa(\phi-\rho_a)} \cdot (\rho_a - 1)a_t \equiv \Theta_{xa}a_t \\
x_t^e =& \widetilde{\gamma}\frac{1+\varphi}{\zeta(\zeta+\widetilde{\gamma})} + \Theta_{xa}a_t \\
\pi_t =& \sigma\kappa \cdot \frac{1+\varphi}{\sigma(1-\beta\rho_a)(1-\rho_a)+\zeta\kappa(\phi-\rho_a)} \cdot (\rho_a - 1)a_t \equiv \Theta_{\pi a}a_t \ .
\end{aligned}$$

Moreover, the variances of output gap and inflation are given by:

$$Var[x_t^n] = \Theta_{xa}^2\sigma_A^2, \quad Var[\pi_t] = \Theta_{\pi a}^2\sigma_A^2 \ .$$

Proof: By undetermined coefficients. Guess a linear policy function for $x_t^n = \Theta_{xa}a_t$ and $\pi_t = \Theta_{\pi a}a_t$, and impose equilibrium consistency in equation (10), equation (11), and equation (13), together with $E_t[a_{t+1}] = \rho_a a_t$ and $\tau_t = 0$ to get:

$$\begin{aligned}
\Theta_{xa}a_t &= \Theta_{xa}\rho_a a_t - \frac{\phi\Theta_{\pi a}a_t - \Theta_{\pi a}\rho_a a_t}{\sigma} + \frac{1}{\zeta}\Bigg[(1+\varphi)(\rho_a a_t - a_t)\Bigg] \\
\Theta_{\pi a}a_t &= \zeta\kappa\Theta_{xa}a_t + \beta\Theta_{\pi a}\rho_a a_t \ .
\end{aligned}$$

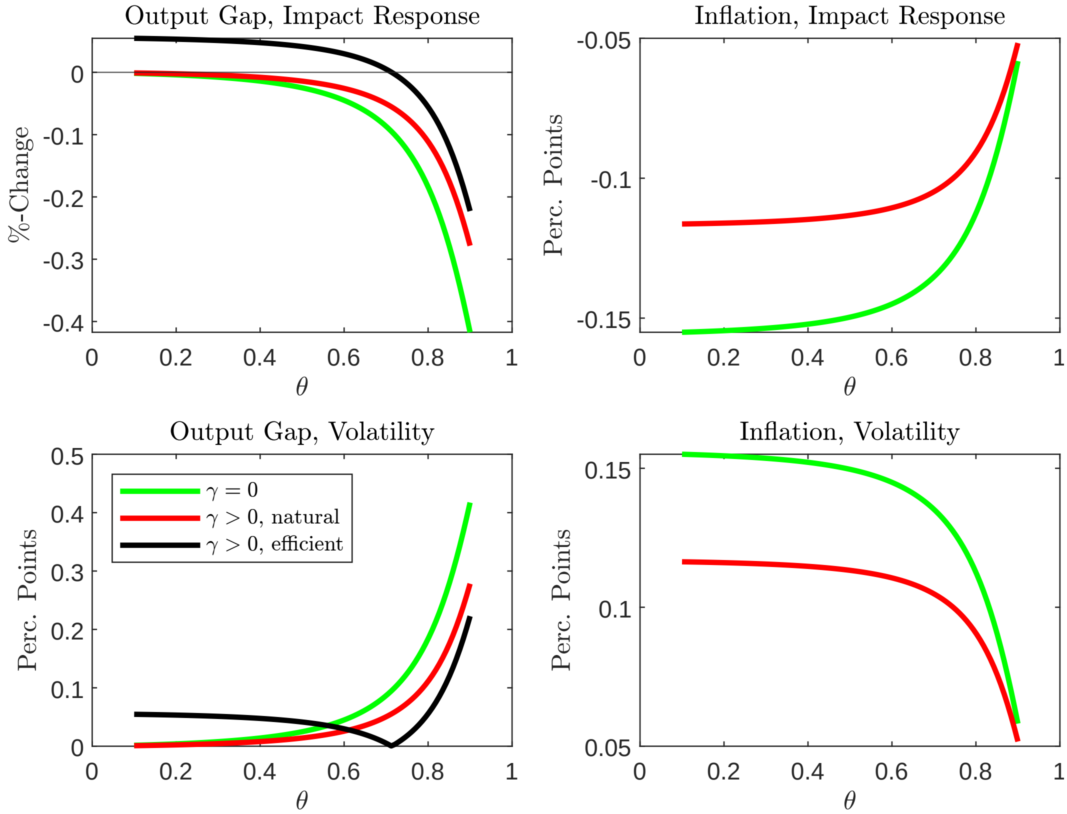
For the guess to be correct, the last two equations have to hold for each $a_t \in \mathcal{R}$. Hence, imposing $a_t = 1$ and solving the system of the two equations into the two unknowns, $\Theta_{\pi a}$ and Θ_{xa} yields:

$$\Theta_{xa} = \frac{\sigma}{\zeta} \cdot \frac{(1+\varphi)(1-\beta\rho_a)}{\sigma(1-\beta\rho_a)(1-\rho_a)+\zeta\kappa(\phi-\rho_a)} \cdot (\rho_a - 1) \tag{14}$$

$$\Theta_{\pi a} = \sigma\kappa \cdot \frac{1+\varphi}{\sigma(1-\beta\rho_a)(1-\rho_a)+\zeta\kappa(\phi-\rho_a)} \cdot (\rho_a - 1) \ . \tag{15}$$

□

Figure 3: Policy functions and variances as functions of θ and γ



Notes: The results are generated by subjecting the model economy to a one standard deviation shock to TFP (5). We set $\rho_A = 0.95$ and $\sigma_A = 0.005$. The Taylor parameter is $\phi = 1.5$, for all other parameters, we refer to Section 4.

In Figure 3, we plot, in the first row, the impact response of inflation and output gap to a technology shock as a function of θ, both for the case of pro-cyclical emissions (green) and the baseline model (red). A larger θ means that prices are more rigid. We consider both the natural output gap x_t^n (red) and the welfare relevant output gap x_t^e (black), which coincide for the baseline model. In the second row, we plot the variances of both the output gaps and of inflation. While both the variances of the natural output gap and of inflation decrease in γ, the variance of the welfare-relevant output gap is non-monotonic in γ, suggesting again that the interaction of nominal rigidities and the emission externality generates non-trivial effects on the trade-off between inflation and output gap volatility which is at the core of optimal monetary policy. Next, we characterize optimal monetary policy by solving linear-quadratic minimization problem a la Benigno and Woodford (2005).

3.2 Monetary Policy Objective

To characterize optimal monetary policy, we first derive its objective function, which is based on the standard assumption of utilitarian welfare maximization and, thus, closely linked to the distinction between efficient and natural output gap described in Proposition 1. Since over-production in the competitive equilibrium allocation, we follow Benigno and Woodford (2005) and consider the general case with $\Phi > 0$, i.e. the steady-state level of output and labor are not above their efficient levels.

Proposition 4. A second order approximation of the welfare function around the distorted steady state yields the following quadratic loss function:

$$\mathcal{W} = -\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t \frac{U_t - U}{U_c C}\right] \approx \mathbb{E}_0\left[\pi_t^2 + \omega_x (x_t^e)^2\right] + t.i.p. \; , \tag{16}$$

where

$$\omega_x = \frac{\kappa}{\epsilon} \cdot \frac{\zeta(\sigma-1) + \zeta(1+\Phi)(1+\varphi)(1+\gamma) - \Phi\left[(1+\gamma)^2(1+\varphi)^2 - (1-\sigma)^2\right]}{\left[\frac{\zeta(1+\Phi)}{1+\gamma} - \Phi(1+\varphi)\right]} \; . \tag{17}$$

Proof: see Appendix A.3.

Absent the emission externality, the weight on the output gap ω_x in the loss function collapses to the familiar expression

$$\omega_x = \frac{\kappa}{\epsilon}(\sigma + \varphi) \; .$$

where $\kappa = \frac{(1-\theta\beta)(1-\theta)}{\theta}$ is related to the share of firms that can adjust prices. A low κ is associated with a large share of firms unable to adjust their prices, i.e. with severe nominal rigidities. With the emission externality, the weight on output stabilization contains the steady state wedge Φ between the marginal rate of substitution between consumption and labor and the efficient marginal product of labor. Specifically, we can use the optimality condition for labor from the planner problem (A.5) to express the labor market clearing condition as follows:

$$n^{\varphi}c^{\sigma} \equiv (1+\Phi)MPN^e = (1+\Phi)\frac{A\Lambda}{1+\gamma} \; .$$

This wedge can be expressed in terms of the emission externality and the tax:

$$\Phi = (1+\gamma)(1-\tau^c) - 1 \; .$$

Note that this wedge vanishes if emission taxes eliminate the externality in steady state. From Proposition 4, we can derive two properties of the loss function.

Lemma 1. For any time-invariant carbon tax τ^c, the weight on output stabilization ω_x in the central bank objective is an increasing function of γ.

Lemma 2. As a special case of Lemma 1, with $\tau^c = \gamma$, the weight of the output gap (17) in the loss function reduces to

$$\omega_x = \frac{\kappa}{\epsilon}\big((\sigma-1)+(1+\varphi)(1+\gamma)\big)(1+\gamma) = \frac{\kappa}{\epsilon}\big(\sigma-1+1+\varphi+\gamma+\varphi\gamma\big)(1+\gamma) = \frac{\kappa}{\epsilon}\zeta(1+\gamma)\ .$$

The central bank places a higher weight on output stabilization if the externality is more severe. The intuition behind this is the dynamic inefficiency of the competitive equilibrium induced by the pollution externality. Production overreacts to a technology shock, relative to the efficient allocation. The central bank then optimally takes this dynamic inefficiency into account by placing a higher weight on output stabilization.

3.3 Optimal Monetary Policy

Next, we characterize optimal monetary policy, by minimizing the loss function derived in proposition 4 under time-invariant carbon taxes and with i.i.d. shocks to TFP. Under these assumptions, the policy problem under discretion can be solved for in closed form.

Proposition 5. If TFP shocks are i.i.d. and $\tau_t^c = 0$, optimal monetary policy is characterized by

$$\pi_t = -\frac{\omega_x\kappa\widetilde{\gamma}(1+\varphi)}{(\zeta+\widetilde{\gamma})(\zeta^2\kappa^2+\omega_x)}a_t \tag{18}$$

$$x_t^e = \frac{\zeta\kappa^2\widetilde{\gamma}(1+\varphi)}{(\zeta+\widetilde{\gamma})(\zeta^2\kappa^2+\omega_x)}a_t \tag{19}$$

$$r_t^e = r_t^n + \frac{\sigma\widetilde{\gamma}(1+\varphi)}{\zeta+\widetilde{\gamma}}\left(\frac{1}{\zeta} - \frac{\zeta\kappa^2}{\kappa^2\zeta^2+\omega_x}\right)a_t, \tag{20}$$

where r_t^n is the natural rate of interest in the model without an emission externality and $\widetilde{\gamma} = \frac{\gamma}{1+\gamma}$.
Proof: The natural output gap can be expressed in terms of the efficient output gap as follows

$$x_t^n = y_t - y_t^n = y_t - y_t^e + y_t^e - y_t^n = x_t^e + \left[\frac{1+\varphi}{\zeta+\widetilde{\gamma}} - \frac{1+\varphi}{\zeta}\right]a_t = x_t^e - \widetilde{\gamma}\frac{1+\varphi}{\zeta(\zeta+\widetilde{\gamma})}a_t\ .$$

Plugging the relationship between natural and efficient output gap into the Phillips curve, the central bank's problem reads:

$$\begin{aligned} \min_{\pi_t, x_t^e} \quad & \frac{1}{2}\mathbb{E}_0\left[\pi_t^2 + \omega_x (x_t^e)^2\right] \\ \text{s.t.} \quad & \pi_t = \zeta\kappa x_t^e - \kappa\widetilde{\gamma}\frac{1+\varphi}{\zeta+\widetilde{\gamma}}a_t + \beta\pi_{t+1} \end{aligned} \tag{21}$$

Taking FOCs and combining them we get the optimal monetary policy that summarizes the trade-off between the welfare-relevant output gap x_t^e and inflation π_t:

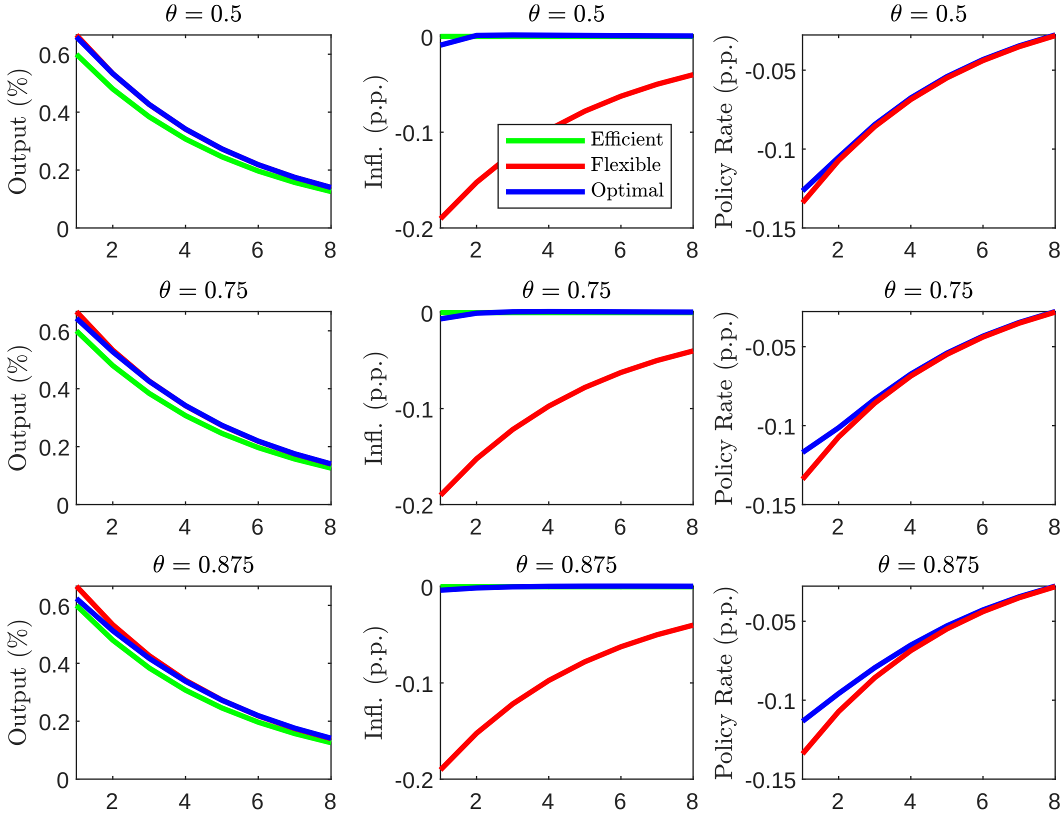
$$\pi_t = -\frac{\omega_x}{x_t^e}\zeta\kappa \tag{22}$$

Plugging the monetary policy rule into the Phillips curve, we get equation (19) for x_t^e. Using (21) and (22) in the IS curve and solving for the efficient policy rate r_t^e we get equation (20). $\square$

Proposition 5 is consistent with Muller (2023), who shows that a central bank tracking potential output has to takes into account cyclical pollution and should adjust the nominal interest rate accordingly. Divine coincidence is then broken, because of the presence of the emission adjustment term in equation (20). Its sign depends on the expression $\frac{1}{\zeta} - \frac{\zeta\kappa^2}{\kappa^2\zeta^2+\omega_x}$. If the adjustment term is positive, the central bank decreases the policy rate by less in response to a positive TFP shock than it would in the standard New Keynesian model, where tracking the natural interest rate is optimal. Under our baseline case, with an steady-state efficient, but time-invariant emission tax, we can show that this term reduces to $\frac{1+\gamma}{\epsilon\zeta(\kappa\zeta+\frac{1+\gamma}{\epsilon})} > 0$ for every $\gamma > 0$. Hence, the presence of pro-cyclical emissions in an otherwise standard New-Keynesian model generates a dynamic inefficiency that interacts with nominal rigidities in a non-trivial way so that divine coincidence is broken for a technology shock. In response to a positive TFP shock, the central bank finds it optimal to trade off some output gap at the expense of higher inflation. To do so, the optimal interest rate cut is smaller, in absolute terms, compared to the case where the central bank does not take into account the emission externality.

We demonstrate how the optimal monetary policy trade-off is affected by procyclical emissions for different degrees of the price rigidity θ. For very stick prices, the central bank almost closes the welfare-relevant output gap, since the economy's overreaction to a TFP shock is modest. Put differently, emission externalities are a relatively less relevant friction if θ is large. Since prices do hardly respond in this case, the central bank is also able to stabilize inflation very well. As the right panel shows, the adjustment term between natural and efficient interest rate is very large in this case. Formally, this follows from the small κ in (20), which increases the magnitude of the adjustment term.

Figure 4: IRF to TFP-Shock: Optimal Monetary Policy



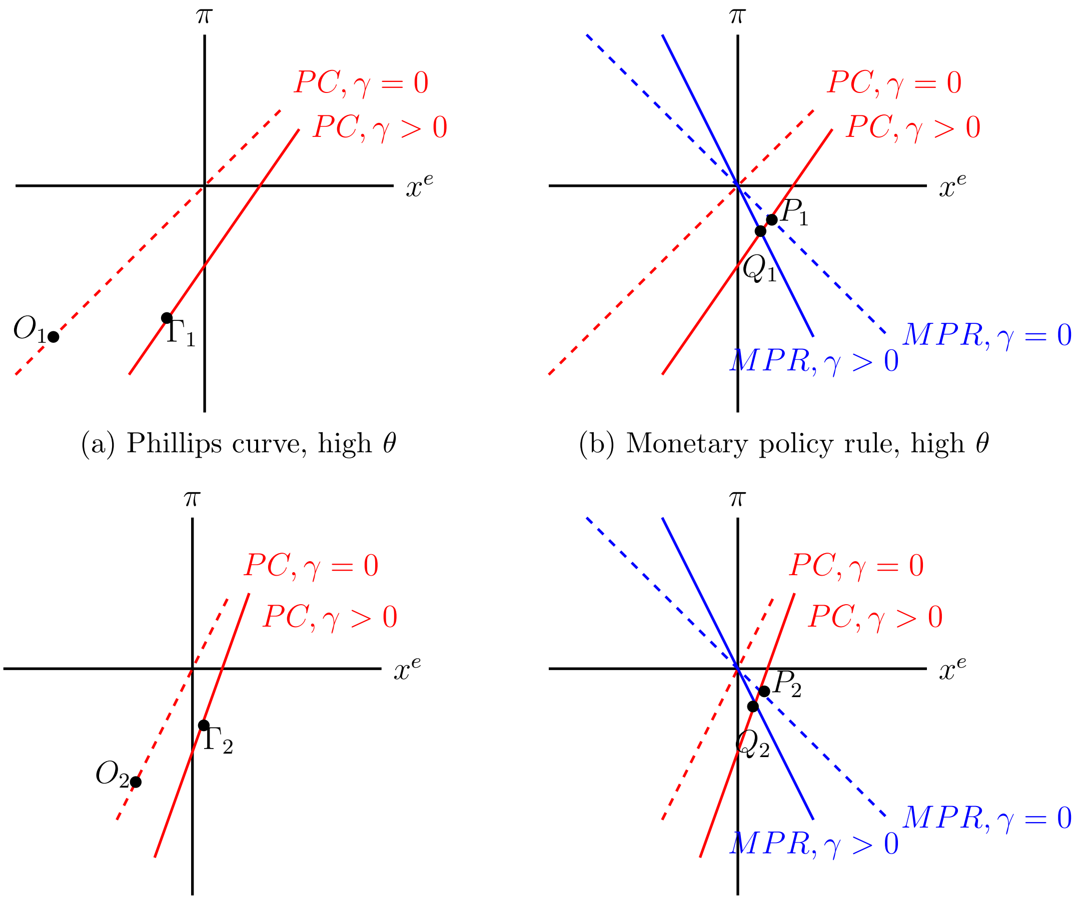
Notes: The results are generated by subjecting the simple E-NK model economy to a one standard deviation shock to TFP (5). We set $\rho_A = 0.95$ and $\sigma_A = 0.005$. For the parameterization, we refer to Section 4.

Figure 5 summarizes the effect of pro-cyclical emissions on macroeconomic outcomes using the canonical representation in a Phillips Curve - Monetary Policy Rule diagram. In the upper left panel, we show first how the Phillips curve is affected by pro-cyclical emissions. The dashed red line refers to the baseline New Keynesian model: marginal costs go down in response to the TFP shock. However, due to the nominal rigidity, not all firms are unable to reduce their prices. Holding the central banks' reaction function constant, this implies that inflation is negative. At the same time, output increases by less than its natural level, i.e. natural and efficient output gap, which coincide in the baseline model, are negative. This is represented by the point O_1.

The solid line refers to the case with $\gamma > 0$. From (21), we see that a TFP shock induces both a downward shift and a steepening of the Phillips curve. If the central bank uses the same reaction function as in the economy without the emission externality, the inflation response is smaller. This follows directly from

Proposition 3. Differentiating (15) with respect to γ, we see that the inflation response to a TFP shock is smaller in absolute terms for every γ. The sign of the welfare-relevant output gap is ambiguous and depends on the degree of nominal rigidities and the severity of emission damages, consistent with the upper left panel of Figure 3. When θ is high, only a small share of firms can adjust prices and the welfare relevant output gap x_t^e is still negative. This is summarized in the point Γ_1.

Figure 5: Phillips Curve - Monetary Policy Rule diagram



(a) Phillips curve, high θ (b) Monetary policy rule, high θ

(c) Phillips curve, low θ (d) Monetary policy rule, low θ

In the upper right panel, we add optimal monetary policy. In the baseline case, the central bank is able to implement first best by shrinking both output gap and inflation to zero, irrespective of their monetary policy rule. With pro-cyclical emissions, this is no longer possible. Divine coincidence is broken and the central bank is unable to close the output gap and implement an inflation rate of zero at the same time. Instead, it selects an equilibrium by moving on the Phillips curve associated with $\gamma > 0$. Under the optimal monetary policy rule that does not take pro-cyclical emissions into account, the dashed blue line, this corresponds to the point P_1. From Proposition 4, we know that the central bank places a larger weight on output stabilization whenever $\gamma > 0$. Thus, the equilibrium response of output gap and inflation under optimal policy are characterized by P_1^*, where the solid blue line intersects the Phillips curve.

In the lower panel, we illustrate a comparative statics exercise with respect to the Calvo parameter. The Phillips curve is steeper if there is a larger share of price adjusters (a lower θ). When $\gamma > 0$, the steeper, downward shifted Phillips curve might imply a *positive* output gap in response to a TFP shock, consistent with the upper left panel of Figure 3, while the inflation response is still dampened. Once monetary policy is set optimally in the bottom right panel, the central bank faces a trade-off between output and inflation stabilization which is reminiscent of supply shocks. Again, with $\gamma > 0$, the trade-off is solved with a larger emphasis on output stabilization. Lastly, it is worth noting that, irrespective of the Calvo parameter θ, the volatility of inflation and output gap under optimal policy will be larger for $\gamma > 0$ due to the broken divine coincidence.

4 Extended Model

In this section, we demonstrate that our analytical results derived in the simple setting also carry over to a more general model that includes capital and investment adjustment costs. We leave all other model ingredients unchanged.

Households The representative household holds capital K_t, consumes the final consumption good c_t, and supplies labor at the nominal wage, W_t. The household owns firms and receives a lump-sum transfer from the government T_t. The maximization problem is given by

$$\max_{\{c_t, n_t, S_t\}_{t=0}^{\infty}} \quad \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1-\sigma} - \omega \frac{n_t^{1+\varphi}}{1+\varphi} \right) \right]$$

$$\text{s.t.} \quad P_t c_t + S_t = W_t n_t + (1 + r_{t-1}^s) S_{t-1} + P_t(\Pi_t + T_t) \ .$$

While φ determines the elasticity of labor supply and ω is a weighting parameter. Euler equation and intra-temporal labor supply condition are largely identical to the simplified model.

Final Good Firms Monopolistic producer i acquires the homogeneous intermediate good z_t, differentiates it into variety i and sells it to households at price p_t^z. Their production technology is linear, such that their marginal cost are simply given by $mc_t = p_t^z$ and the solution to their price setting problem coincides with equation (4) in the simple model. Final good supply then depends on the price dispersion: $y_t = \Delta_t z_t$.

Intermediate Good Firms Perfectly competitive intermediate good firms invest in capital k_{t+1} which depreciates at rate δ_K and hire labor n_t to produce the homogeneous intermediate good z_t with the following technology:

$$z_t = A_t \Lambda_t k_t^{\alpha} n_t^{1-\alpha} \ . \tag{23}$$

The law of motion for capital is given by $k_{t+1} = (1-\delta_K)k_t + i_t$. Investment goods have to be purchased at price ψ_t from perfectly competitive investment good producers (described below). Denoting the intermediate good price by p_t^z, the first-order conditions associated with the profit maximization problem are given by

$$\frac{w_t}{p_t^z} = (1-\alpha)\frac{z_t}{n_t} \ ,$$
$$\psi_t = \mathbb{E}_t\left[(1-\delta_K)\psi_{t+1} + p_t^z \alpha \frac{z_{t+1}}{k_{t+1}}\right] \ .$$

Investment Good Firms A representative investment good firm acquires $\big(1 + \frac{\Psi_I}{2}(\frac{i_t}{i_{t-1}})\big)$ units of the final goods bundle into one unit of a homogeneous investment good, which they sell to intermediate good firms at price p_t^K. The profit maximization problem

$$\max_{\{i_s\}_{s=0}^{\infty}} \ \mathbb{E}_0\left[\sum_{t=0}^{\infty}\Lambda_{t,t+s}\left\{p_{t+s}^K i_{t+s} - \left(1+\frac{\Psi_I}{2}\left(\frac{i_{t+s}}{i_{t+s-1}}-1\right)^2\right) i_{t+s}\right\}\right]$$

delivers an additional equilibrium condition for the investment good price:

$$p_t^K = 1 + \frac{\Psi_I}{2}\Big(\frac{i_t}{i_{t-1}}-1\Big)^2 + \Psi_I\Big(\frac{i_t}{i_{t-1}}-1\Big)\frac{i_t}{i_{t-1}} - \mathbb{E}_t\left[\Lambda_{t,t+1}\Psi_I\Big(\frac{i_{t+1}}{i_t}-1\Big)\Big(\frac{i_{t+1}}{i_t}\Big)^2\right] \ . \tag{24}$$

Market Clearing Since we apply our qualitative results to the case of carbon emissions, we make damages dependent on the stock of carbon. Specifically, we assume that emissions are proportional to output and depreciate at a constant rate. Then, atmospheric carbon accumulates according to $E_t = \delta_E E_{t-1} + y_t$. Damages are specified as

$$\Lambda_t = \exp\{-\Gamma E_t\} \tag{25}$$

he goods market clearing condition now also includes investment:

$$y_t = c_t + i_t\Big(1 + \frac{\psi_I}{2}\big(\frac{i_t}{i_{t-1}} - 1\big)^2\Big)\ . \tag{26}$$

The competitive equilibrium conditional on policy instruments (r_t^s, τ_t^c) is fully described by all agent's first-order conditions and budget constraints as well as the goods market clearing condition (26). The model can be closed by imposing policy rules for the nominal interest rate and the carbon tax.

Calibration The model is calibrated to standard values used in the New Keynesian DSGE literature. Households' risk aversion and discount factor are set to $\sigma = 1$ and $\beta = 0.995$. This discount factor implies an annual real rate of 2%. Furthermore, we set $\varphi = 1$ to obtain a Frisch elasticity of labor supply of one. The weight $\omega = 11$ in the household utility function is implies a steady state labor supply of 0.33.

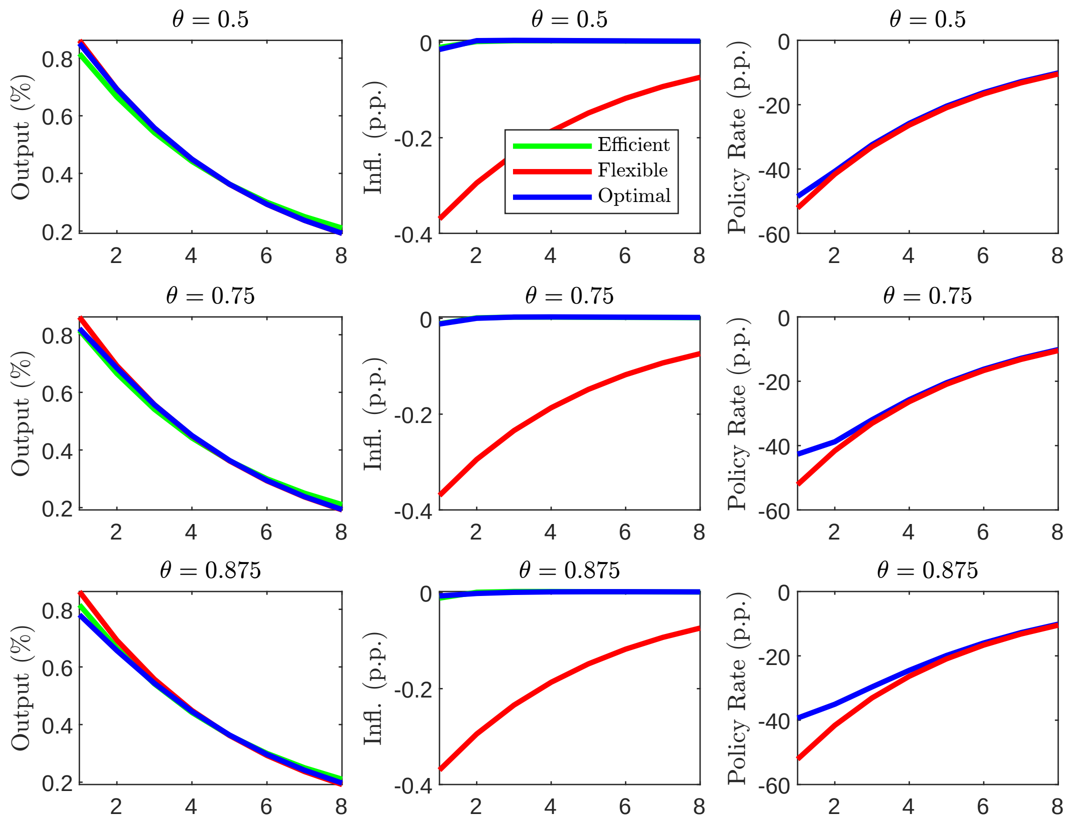
Regarding the emission externality, we use a narrow interpretation as carbon emissions in this section, which affects the choices of the emission damage and decay parameters, respectively. The parameter Γ governing the pollution cost of emissions is difficult to calibrate, since there is considerable uncertainty about measurement in the data. We follow the approach in Heutel (2012) and set the decay rate of pollution to $\delta_E = 0.9979$, while we choose a damage parameter of Γ = 5E-05 to target a proportional output loss of 2.5% of GDP each period absent climate policy. In the model, this corresponds to $\Lambda = 0.975$. Under this parameterization, the optimal long run tax τ_c is given by 0.12.

As customary in the literature, we set $\alpha = 1/3$ in the production function and the capital depreciation rate to $\delta_K = 0.025$. The investment adjustment cost parameter is set to $\Psi_I = 10$, following Coenen, Lozej, and Priftis (2023). The demand elasticity for final good varieties is fixed at $\epsilon = 6$, implying a 20% markup. As a baseline, we set the Calvo parameter to $\theta = 0.75$ although we will vary this parameter throughout the analysis. Lastly, the parameters governing exogenous TFP are set to $\rho_A = 0.8$ and $\sigma_A = 0.01$.

Optimal Monetary Policy As a final step, we numerically evaluate optimal policy in the extended model. Using the same parameters as in the simple model, we again compare the efficient (green), natural (red) and Ramsey-optimal (blue) response of output, inflation, and the adjustment term between efficient and natural rate of interest rate. Similar to the simple model (Figure 4), we observe that optimal monetary policy gets closer to the efficient output gap as θ increases. For a high Calvo parameter ($\theta = 0.875$), the initial output response under optimal monetary policy is even slightly smaller than in the efficient allocation, while it lies between efficient and flexible price output after three quarters.

Finally, it should be noted that the consequences of cyclical emissions for optimal monetary policy are sizable, but not huge: for a Calvo parameter of $\theta = 0.75$ the optimal interest rate reduction in response to a positive TFP shock is around 50 basis points compared to around 40 basis points in the economy without pro-cyclical emissions. This result is in line with the analysis of Nakov and Thomas (2023) for the implications of long-run effects of climate change on the conduct of monetary policy.

Figure 6: IRF to TFP-Shock: Optimal Monetary Policy



Notes: The results are generated by subjecting the model with capital to a one standard deviation shock to TFP.

5 Conclusion

In this paper, we explore the interactions between pro-cyclical emissions, nominal rigidities, and monetary policy. We show that cyclical emissions have implications for optimal monetary policy even when the long run (or trend-specific) costs of emissions are addressed optimally. Specifically, the natural output gap is not efficient from a utilitarian welfare perspective, and neither is it optimal to track the natural rate of interest from the New Keynesian model. Divine coincidence is broken even for TFP shocks. We show that the central bank generally places a higher weight on output stabilization, to tackle this dynamic inefficiency. This result also holds in a larger model with capital and investment adjustment costs.

There is evidence that emissions also have a direct effect on macroeconomic volatility and inflation through a disaster risk channel and associated swings in commodity prices. Disaster risk itself can also be a source of macroeconomic volatility, from which we abstract in our analysis. Furthermore, carbon taxation can also induce inflation by increasing electricity and energy prices, which has been subject to recent discussion. Exploring the interactions between these additional channels, nominal rigidities, and monetary policy is left for future research.

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A Proofs

This section contains all proofs omitted in Section 3.

A.1 Proof of proposition 1

The aggregate production function can be written $y_t = A_t \Lambda_t n_t$, while the goods market clearing condition is given by $y_t = c_t$.

Efficient Allocation The planner problem is

$$\max_{c_t, n_t, y_t, \Lambda_t, u_t} \quad \sum_t \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} - \frac{n_t^{1+\varphi}}{1+\varphi} \right] \quad \text{s.t.}$$

$$c_t = y_t \qquad (\lambda_t)$$

$$y_t = A_t \Lambda_t n_t \qquad (\mu_t)$$

$$\Lambda_t = \exp\left\{ -\gamma \frac{y_t}{y} \right\} \qquad (\nu_t)$$

Setting up the Lagrangian

$$\max_{c_t, n_t, y_t, \Lambda_t} \sum \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} - \frac{n_t^{1+\varphi}}{1+\varphi} + \lambda_t \Big(y_t - c_t \Big) + \mu_t \Big(A_t \Lambda_t n_t - y_t \Big) + \nu_t \Big(\exp\left\{ -\gamma \frac{y_t}{y} \right\} - \Lambda_t \Big) \right]$$

and taking FOCs yields

$$\lambda_t = c_t^{-\sigma} \tag{A.1}$$

$$\mu_t A_t \Lambda_t = n_t^{\varphi} \tag{A.2}$$

$$\lambda_t - \mu_t - \nu_t \frac{\gamma}{y} \Lambda_t = 0 \tag{A.3}$$

$$\mu_t A_t n_t = \nu_t \tag{A.4}$$

Combining (A.3) and (A.4):

$$\lambda_t - \mu_t - \mu_t A_t n_t \frac{\gamma \epsilon_t}{y} \Lambda_t = 0 \Leftrightarrow \mu_t = \frac{\lambda_t}{1 + A_t \Lambda_t n_t \frac{\gamma}{y}}$$

Plugging in (A.1) and (A.2), the efficient allocation is characterized by a socially optimal labor supply condition:

$$\frac{c_t^{-\sigma}}{1 + A_t \Lambda_t n_t \frac{\gamma}{y}} A_t \Lambda_t = n_t^{\varphi} \ ,$$

which implicitly defines the marginal product of labor as

$$MPN_t^e \equiv \frac{A_t\Lambda_t}{1+\gamma\frac{y_t}{y}} \ . \tag{A.5}$$

The resource constraint is given by $c_t = y_t$. Hence, using the production technology $y_t = A_t\Lambda_t n_t$

$$\frac{y_t^{-\sigma}}{1+A_t\Lambda_t n_t\frac{\gamma}{y}}A_t\Lambda_t = \frac{y_t^{\varphi}}{(A_t\Lambda_t)^{\varphi}}$$

Rearranging delivers equation (7). Log-linearizing yields

$$\begin{aligned} &(\sigma+\varphi)\widehat{y}_t^e = (1+\varphi)a_t - (1+\varphi)\gamma\widehat{y}_t^e - \frac{\gamma}{1+\gamma}\widehat{y}_t^e \\ \Leftrightarrow\quad &\left[\sigma+\varphi+(1+\varphi)\gamma+\frac{\gamma}{1+\gamma}\right]\widehat{y}_t^e = (1+\varphi)a_t \ . \end{aligned} \tag{A.6}$$

Re-arranging for $\widehat{y}_t^e$, we arrive at equation (9).

Competitive Equilibrium Next, we derive the natural level of output consistent with flexible prices and a labor subsidy $\tau^n = \frac{1}{\epsilon}$ that corrects for the steady state monopolistic distortion. The relevant equilibrium conditions are the aggregate production function, where Δ_t is the price dispersion

$$\Delta_t y_t = A_t\Lambda n_t \ , \tag{A.7}$$

and labor demand:

$$(1-\tau^n)w_t = mc_t A_t\Lambda_t \ . \tag{A.8}$$

Labor supply:

$$w_t = n_t^{\varphi}c_t^{\sigma} \ .$$

Goods market clearing requires

$$y_t = c_t \ .$$

Optimal price

$$p_t^* = \frac{\mu}{1-\tau_t^c}\frac{\xi_{1,t}}{\xi_{2,t}} \ , \tag{A.9}$$

where $\mu \equiv \frac{\epsilon}{\epsilon-1}$ and

$$\xi_{1,t} = mc_t y_t + \beta\theta \frac{c_{t+1}^{-\sigma}}{c_t^{-\sigma}} \pi_{t+1}^{\epsilon} \xi_{1,t+1} \ , \tag{A.10}$$

$$\xi_{2,t} = y_t + \beta\theta \frac{c_{t+1}^{-\sigma}}{c_t^{-\sigma}} \pi_{t+1}^{\epsilon-1} \xi_{2,t+1} \ . \tag{A.11}$$

Inflation is pinned down by

$$1 = (1-\theta)(p_t^*)^{1-\epsilon} + \theta \pi_t^{\epsilon-1} \ . \tag{A.12}$$

Price dispersion:

$$\Delta_t = (1-\theta)(p_t^*)^{-\epsilon} + \theta \pi_t^{\epsilon} \Delta_{t-1} \ . \tag{A.13}$$

If prices are flexible, then $\Delta_t = \pi_t = p_t^* = 1$, $\xi_{1,t} = mc_t y_t$, $\xi_{2,t} = y_t$, and $p_t^* = (1-\tau^n)\mu mc_t$. Hence:

$$\begin{aligned} 1 =& \frac{\mu}{1-\tau_t^c} mc_t = (1-\tau^n) \frac{\mu}{1-\tau_t^c} \frac{w_t}{A_t \Lambda_t} = \frac{n_t^{\varphi} c_t^{\sigma}}{(1-\tau_c^t) A_t \Lambda_t} \\ =& \frac{y_t^{\sigma+\varphi}}{(1-\tau_t^c)(A_t \Lambda_t)^{1+\varphi}} \end{aligned}$$

where we used the fact that the labor subsidy appropriately corrects for the monopolistic distortion ($\tau^n = \frac{1}{\epsilon}$). Solving for y_t yields the natural output level (6). Log-linearizing around the deterministic steady state:

$$(\sigma+\varphi)\widehat{y}_t^n = (1+\varphi)\widehat{a}_t - (1+\varphi)\gamma \widehat{y}_t^n - \frac{\tau^c}{1-\tau^c} \widehat{\tau}_t^c \ .$$

Re-arranging for $\widehat{y}_t^n$ yields equation (8) □

A.2 Proof of proposition 2

Equilibrium Conditions The linearized equilibrium conditions are the following.
Optimal labor supply equation (2):

$$\widehat{w}_t = \varphi \widehat{n}_t + \sigma \widehat{c}_t \ . \tag{A.14}$$

Euler equation equation (1):

$$\sigma \widehat{c}_t = \sigma \widehat{c}_{t+1} - (r_t^s - \pi_{t+1}) \ . \tag{A.15}$$

Pollution:

$$\widehat{\Lambda}_t = -\gamma \widehat{y}_t$$

Production function equation (A.7):

$$\widehat{\Delta}_t + \widehat{y}_t = a_t - \gamma \widehat{y}_t + \widehat{n}_t \tag{A.16}$$

Labor demand equation (A.8):

$$\widehat{w}_t = \widehat{mc}_t + a_t - \gamma \widehat{y}_t \tag{A.17}$$

Optimal pricing eqs. (A.9), (A.10), and (A.11):

$$p_t^* = \frac{\tau^c}{1-\tau^c}\tau_t^c + \xi_{1t} - \xi_{2t} \tag{A.18}$$
$$\xi_{1,t} = (1-\theta\beta)mc_t + (1-\theta\beta)y_t - \theta\beta\sigma c_{t+1} + \theta\beta\sigma c_t + \epsilon\theta\beta\pi_{t+1} + \theta\beta\xi_{1,t+1} \tag{A.19}$$
$$\xi_{2,t} = (1-\theta\beta)y_t - \theta\beta\sigma c_{t+1} + \theta\beta\sigma c_t + (\epsilon-1)\theta\beta\pi_{t+1} + \theta\beta\xi_{2,t+1} \tag{A.20}$$

Inflation equation (A.12):

$$0 = (1-\epsilon)(1-\theta)\widehat{p}_t^* + \theta(\epsilon-1)\widehat{\pi}_t \;\Leftrightarrow\; \widehat{p}_t^* = \frac{\theta}{1-\theta}\widehat{\pi}_t \tag{A.21}$$

Price dispersion equation (A.13):

$$\widehat{\Delta}_t = -\epsilon(1-\theta)\widehat{p}_t^* + \theta\epsilon\widehat{\pi}_t + \theta\widehat{\Delta}_{t-1} \;\Leftrightarrow\; \widehat{\Delta}_t = \theta\widehat{\Delta}_{t-1} \Leftrightarrow\; \widehat{\Delta}_t = 0$$

Market clearing:

$$\widehat{c}_t = \widehat{y}_t$$

Natural output gap:

$$x_t^n = \widehat{y}_t - \widehat{y}_t^n = \widehat{y}_t - \frac{1}{\zeta}\left[(1+\varphi)\widehat{a}_t - \frac{\tau^c}{1-\tau^c}\widehat{\tau}_t^c\right]$$

Welfare relevant output gap:

$$x_t^e = \widehat{y}_t - \widehat{y}_t^e = \widehat{y}_t - \frac{1}{\zeta + \frac{\gamma}{1+\gamma}}\left[(1+\varphi)\widehat{a}_t\right]$$

Subtracting equation (A.20) from equation (A.19) we get:

$$\xi_{1t} - \xi_{2t} = (1-\theta\beta)mc_t + \theta\beta\pi_{t+1} + \theta\beta(\xi_{1,t+1} - \xi_{2,t+1})$$

Plugging this condition and equation (A.21) into equation (A.18) we get:

$$\frac{\theta}{1-\theta}\pi_t = \frac{\tau^c}{1-\tau^c}\tau_t^c + (1-\theta\beta)mc_t + \theta\beta\left(\pi_{t+1} + \frac{\theta}{1-\theta}\pi_{t+1} - \frac{\tau^c}{1-\tau^c}\tau_{t+1}^c\right) \Leftrightarrow \tag{A.22}$$

$$\Leftrightarrow \pi_t = \underbrace{\frac{(1-\theta\beta)(1-\theta)}{\theta}}_{\kappa} mc_t + \beta\pi_{t+1} + \frac{1-\theta}{\theta}\frac{\tau^c}{1-\tau^c}\left(\tau_t - \theta\beta\tau_{t+1}\right) \tag{A.23}$$

Now, combining eqs. (A.14), (A.16), and (A.17) we get:

$$\begin{aligned} mc_t &= w_t - a_t + \gamma y_t = \varphi n_t + \sigma c_t - a_t + \gamma y_t = \varphi(y_t - a_t + \gamma y_t) + \sigma y_t - a_t - \gamma y_t = \\ &= [\underbrace{\sigma + \varphi + (1+\varphi)\gamma}_{=\zeta}]y_t - (1+\varphi)a_t \end{aligned}$$

Plugging this condition into equation (A.23):

$$\begin{aligned} \pi_t &= \kappa\zeta\left[\underbrace{y_t - \frac{1+\varphi}{\zeta}a_t + \frac{1}{\zeta}\frac{\tau^c}{1-\tau^c}\tau_t}_{x_t^n} - \frac{1}{\zeta}\frac{\tau^c}{1-\tau^c}\tau_t\right] + \beta\pi_{t+1} + \frac{1-\theta}{\theta}\frac{\tau^c}{1-\tau^c}\left(\tau_t - \theta\beta\tau_{t+1}\right) = \\ &= \kappa\zeta x_t^n + \beta\pi_{t+1} - \kappa\frac{\tau^c}{1-\tau^c}\tau_t + \frac{\kappa}{1-\theta\beta}\frac{\tau^c}{1-\tau^c}\tau_t - (1-\theta)\beta\frac{\tau^c}{1-\tau^c}\tau_{t+1} = \\ &= \kappa\zeta x_t^n + \beta\pi_{t+1} + (1-\theta)\beta\frac{\tau^c}{1-\tau_c}(\tau_t - \tau_{t+1}), \end{aligned}$$

which is equation (11).

To get equation (10), start from equation (A.15) and impose market clearing to get:

$$\begin{aligned} & y_t = y_{t+1} - \frac{1}{\sigma}(r_t^s - \pi_{t+1}) \Leftrightarrow \\ & \widehat{y}_t - \frac{1}{\zeta}\left[(1+\varphi)a_t - \frac{\tau^c}{1-\tau^c}\tau_t\right] + \frac{1}{\zeta}\left[(1+\varphi)a_t - \frac{\tau^c}{1-\tau^c}\tau_t\right] = \\ & = \widehat{y}_{t+1} - \frac{1}{\zeta}\left[(1+\varphi)a_{t+1} - \frac{\tau^c}{1-\tau^c}\tau_{t+1}\right] + \frac{1}{\zeta}\left[(1+\varphi)a_{t+1} - \frac{\tau^c}{1-\tau^c}\tau_{t+1}\right] - \frac{1}{\sigma}(r_t^s - \pi_{t+1}) \Leftrightarrow \\ & x_t^n + \frac{1}{\zeta}\left[(1+\varphi)a_t - \frac{\tau^c}{1-\tau^c}\tau_t\right] = x_{t+1}^n + \frac{1}{\zeta}\left[(1+\varphi)a_{t+1} - \frac{\tau^c}{1-\tau^c}\tau_{t+1}\right] - \frac{1}{\sigma}(r_t^s - \pi_{t+1}) \Leftrightarrow \\ & x_t^n = x_{t+1}^n - \frac{1}{\sigma}(r_t^s - \pi_{t+1}) + \frac{1}{\zeta}\left[(1+\varphi)(a_{t+1} - a_t) - \frac{\tau^c}{1-\tau^c}(\tau_{t+1} - \tau_t)\right] \end{aligned}$$

□

A.3 Proof of proposition 4

We can show that the wedge between efficient and natural level of output satisfies

$$\Phi \equiv (y^e)^{\sigma+\varphi} - (y^n)^{\sigma+\varphi} = \frac{\Lambda^{1+\varphi}}{1+\gamma} - \frac{\Lambda^{1+\varphi}}{1+\tau^c} = \Lambda^{1+\varphi}\left(\frac{1}{1+\gamma} - \frac{1}{1+\tau^c}\right)$$

For $\tau^c = \gamma$, we have $\Phi = 0$ and output is efficient in the steady state. We will consider the general case $\Phi < 0$.

Equilibrium Conditions The linearized equilibrium conditions are optimal labor supply:

$$\widehat{w}_t = \varphi\widehat{n}_t + \sigma\widehat{c}_t \ .$$

Euler equation:

$$-\sigma\widehat{c}_t = -\sigma\widehat{c}_{t+1} + r_t^s - \pi_{t+1} \ .$$

Production function:

$$\widehat{\Delta}_t + \widehat{y}_t = a_t - \gamma\widehat{y}_t + \widehat{n}_t$$

Labor demand:

$$\widehat{w}_t = \widehat{mc}_t - a_t + \gamma\widehat{y}_t$$

Optimal pricing:

$$p_t^* = x_{1t} - x_{2t}$$
$$\xi_{1,t} = (1-\theta\beta)mc_t + (1-\theta\beta)y_t - \theta\beta\sigma c_{t+1} + \theta\beta\sigma c_t + \epsilon\theta\beta\pi_{t+1} + \theta\beta\xi_{1,t+1}$$
$$\xi_{2,t} = (1-\theta\beta)y_t - \theta\beta\sigma c_{t+1} + \theta\beta\sigma c_t + (\epsilon-1)\theta\beta\pi_{t+1} + \theta\beta\xi_{2,t+1}$$

Market clearing:

$$\widehat{c}_t = \widehat{y}_t$$

Natural output gap:

$$\widehat{x}_t^n = \widehat{y}_t - \widehat{y}_t^n = \widehat{y}_t - \frac{(1+\varphi)}{\varphi+\sigma}a_t$$

Welfare relevant output gap:

$$\widehat{x}_t^e = \widehat{y}_t - \widehat{y}_t^e$$

Pollution:

$$\widehat{\Lambda}_t = -\gamma\widehat{y}_t$$

Loss Function Taking a second order approximation of the welfare function U_t:

$$U_t - U \approx c^{1-\sigma}\left\{\frac{c_t - c}{c} - \frac{\sigma}{2}\left(\frac{c_t - c}{c}\right)^2 - \frac{n^{1+\varphi}}{c^{1-\sigma}}\left[\frac{n_t - n}{n} + \frac{\varphi}{2}\left(\frac{n_t - n}{n}\right)^2\right]\right\}$$

yields

$$\frac{U_t - U}{U_c c} = \frac{U_t - U}{c^{1-\sigma}} \approx \frac{c_t - c}{c} - \frac{\sigma}{2}\left(\frac{c_t - c}{c}\right)^2 - \frac{n^{1+\varphi}}{c^{1-\sigma}}\left[\frac{n_t - n}{n} + \frac{\varphi}{2}\left(\frac{n_t - n}{n}\right)^2\right]$$

For a generic variable x, up to second order, $\frac{x_t - x}{x} = \widehat{x}_t + \frac{\widehat{x}_t^2}{2}$ with $\widehat{x} = \log x_t - \log x$. Also, the following condition holds:

$$\frac{n^{1+\varphi}}{c^{1-\sigma}} = n^{\varphi} c^{\sigma} \frac{n}{c} = \frac{A\Lambda}{1+\gamma}(1+\Phi)\frac{n}{c} = \frac{1+\Phi}{(1+\gamma)}$$

Hence:

$$\frac{U_t - U}{c^{1-\sigma}} \approx \widehat{c}_t + \frac{\widehat{c}_t^2}{2} - \frac{\sigma}{2}\widehat{c}_t^2 - \frac{1+\Phi}{1+\gamma}\left[\widehat{n}_t + \frac{\widehat{n}_t^2}{2} + \frac{\varphi}{2}\widehat{n}_t^2\right]$$

Plugging in the market clearing condition $\widehat{c}_t = \widehat{y}_t$ and the production function $\widehat{n}_t = \widehat{y}_t + \widehat{\Delta}_t - a_t - \widehat{\Lambda}_t = (1+\gamma)\widehat{y}_t + \widehat{\Delta}_t - a_t$:

$$\frac{U_t - U}{c^{1-\sigma}} \approx \widehat{y}_t + \frac{1-\sigma}{2}\widehat{y}_t^2 - \frac{1+\Phi}{1+\gamma}\left[(1+\gamma)\widehat{y}_t + \widehat{\Delta}_t - a_t + \frac{1+\varphi}{2}\left((1+\gamma)\widehat{y}_t + \widehat{\Delta}_t - a_t\right)^2\right]$$

Eliminating all terms independent of policy and of order higher than two:

$$\begin{aligned}\frac{U_t - U}{c^{1-\sigma}} &\approx \widehat{y}_t + \frac{1-\sigma}{2}\widehat{y}_t^2 - \frac{1+\Phi}{1+\gamma}\left[(1+\gamma)\widehat{y}_t + \widehat{\Delta}_t + \frac{1+\varphi}{2}[(1+\gamma)^2\widehat{y}_t^2 - 2(1+\gamma)\widehat{y}_t a_t]\right] + t.i.p. \\ &\approx -\Phi\widehat{y}_t + \frac{1-\sigma}{2}\widehat{y}_t^2 - \frac{1+\Phi}{1+\gamma}\widehat{\Delta}_t - \frac{(1+\Phi)(1+\gamma)(1+\varphi)}{2}\widehat{y}_t^2 + (1+\Phi)(1+\varphi)\widehat{y}_t a_t + t.i.p.\end{aligned}$$

Using the definition of ζ, the efficient output level can be written

$$\widehat{y}_t^e = \frac{1+\varphi}{\varphi + \gamma(1+\varphi) + \frac{\gamma}{1+\gamma} + \sigma} a_t = \frac{1+\varphi}{\zeta + \frac{\gamma}{1+\gamma}} a_t$$

Then, plugging in the definition of the output gap $\widehat{y}_t = \widehat{x}_t + \widehat{y}_t^e$:

$$\begin{aligned}
\frac{U_t - U}{c^{1-\sigma}} &\approx -\Phi\widehat{x}_t + \frac{1-\sigma}{2}(\widehat{x}_t^2 + 2\widehat{x}_t\widehat{y}_t^e) - \frac{1+\Phi}{1+\gamma}\widehat{\Delta}_t \\
&\quad - \frac{(1+\Phi)(1+\gamma)(1+\phi)}{2}(\widehat{x}_t^2 + 2\widehat{x}_t\widehat{y}_t^e) + (1+\Phi)(1+\varphi)\widehat{x}_t a_t + t.i.p. \\
&\approx -\Phi\widehat{x}_t - \frac{1}{2}\Big\{(1+\Phi)(1+\gamma)(1+\varphi) + (\sigma-1)\Big\}\widehat{x}_t^2 - \frac{1+\Phi}{1+\gamma}\widehat{\Delta}_t \\
&\quad - \Big\{(1+\Phi)(1+\gamma)(1+\varphi) - (1-\sigma)\Big\}\widehat{x}_t\widehat{y}_t^e + (1+\Phi)(1+\varphi)\widehat{x}_t a_t + t.i.p. \\
&\approx -\Phi\widehat{x}_t - \frac{1}{2}\Big\{(1+\Phi)(1+\gamma)(1+\varphi) + (\sigma-1)\Big\}\widehat{x}_t^2 - \frac{1+\Phi}{1+\gamma}\widehat{\Delta}_t \\
&\quad \underbrace{- (1+\varphi)\left\{\Big[(1+\Phi)(1+\gamma)(1+\varphi) - (1-\sigma)\Big]\frac{1}{\zeta + \frac{\gamma}{1+\gamma}} - (1+\Phi)\right\}}_{\equiv V_1}\widehat{x}_t a_t + t.i.p.
\end{aligned}$$

The coefficient V_1 in front of the interaction term $\widehat{x}_t a_t$ simplifies to:

$$\begin{aligned}
V_1 &= -(1+\varphi)\left\{\Big[(1+\Phi)(1+\gamma)(1+\varphi) - (1-\sigma)\Big]\frac{1}{\zeta + \frac{\gamma}{1+\gamma}} - (1+\Phi)\right\} \\
&= -\frac{1+\varphi}{\zeta + \frac{\gamma}{1+\gamma}}\left\{(1+\Phi)(1+\gamma)(1+\varphi) + \sigma - 1 - (1+\Phi)\left(\zeta + \frac{\gamma}{1+\gamma}\right)\right\} \\
&= -\frac{(1+\varphi)}{\zeta + \frac{\gamma}{1+\gamma}}(1+\Phi)\left\{(1+\gamma)(1+\varphi) + \frac{\sigma-1}{1+\Phi} - \left(\zeta + \frac{\gamma}{1+\gamma}\right)\right\} \\
&= -\frac{(1+\varphi)}{\zeta + \frac{\gamma}{1+\gamma}}(1+\Phi)\left\{(\varphi + \gamma + \gamma\varphi) + \sigma - \sigma + \frac{\sigma-1}{1+\Phi} + 1 - \left(\zeta + \frac{\gamma}{1+\gamma}\right)\right\} \\
&= -\frac{(1+\varphi)}{\zeta + \frac{\gamma}{1+\gamma}}(1+\Phi)\left\{\zeta - \sigma + \frac{\sigma-1}{1+\Phi} + 1 - \left(\zeta + \frac{\gamma}{1+\gamma}\right)\right\} \\
&= -\frac{(1+\varphi)}{\zeta + \frac{\gamma}{1+\gamma}}(1+\Phi)\left\{-\sigma + \frac{\sigma-1}{1+\Phi} - \frac{\gamma}{1+\gamma}\right\} \\
&= -\frac{1+\varphi}{\zeta + \frac{\gamma}{1+\gamma}}(1+\Phi)\left\{\frac{-\sigma\Phi - 1}{1+\Phi} - \frac{\gamma}{1+\gamma}\right\}
\end{aligned}$$

Hence:

$$\frac{U_t - U}{c^{1-\sigma}} \approx -\Phi\widehat{x}_t - \frac{1}{2}\Big\{(1+\Phi)(1+\gamma)(1+\varphi) + (\sigma-1)\Big\}\widehat{x}_t^2 - \frac{1+\Phi}{1+\gamma}\widehat{\Delta}_t$$
$$- \frac{(1+\varphi)}{\zeta + \frac{\gamma}{1+\gamma}}(1+\Phi)\Big\{\frac{-\sigma\Phi - 1}{1+\Phi} + \Big(1 - \frac{\gamma}{1+\gamma}\Big)\Big\}\widehat{x}_t a_t$$
$$\approx -\Phi\widehat{x}_t - \frac{1}{2}\Big\{(1+\Phi)(1+\gamma)(1+\varphi) + (\sigma-1)\Big\}\widehat{x}_t^2 - \frac{1+\Phi}{1+\gamma}\widehat{\Delta}_t$$
$$- \frac{(1+\varphi)}{\zeta + \frac{\gamma}{1+\gamma}}\Big\{-\Phi(\sigma-1) - (1+\Phi)\frac{\gamma}{1+\gamma}\Big\}\widehat{x}_t a_t$$

We are then ready to evaluate the loss function:

$$\mathcal{L} \equiv -\mathcal{W} \approx \mathbb{E}_0\Bigg[\sum_{t=0}^{\infty}\beta^t\Bigg\{\frac{1}{2}\Big((1+\Phi)(1+\gamma)(1+\varphi) + (\sigma-1)\Big)\widehat{x}_t^2 + \frac{1+\Phi}{1+\gamma}\widehat{\Delta}_t +$$
$$\frac{(1+\phi)}{\zeta + \frac{\gamma}{1+\gamma}}\Big[-\Phi(\sigma-1) - (1+\Phi)\Big(\frac{\gamma}{1+\gamma}\Big)\Big]\widehat{x}_t a_t + \Phi\widehat{x}_t\Bigg\}\Bigg]$$

The discounted sum of log price dispersion is given by $\sum_{t=0}^{\infty}\beta^t\widehat{\Delta}_t \approx \frac{\epsilon}{2\kappa}\sum_{t=0}^{\infty}\beta^t\pi_t^2$, with $\kappa = \frac{(1-\theta)(1-\theta\beta)}{\beta}$ governing the slope of the NKPC. Therefore:

$$\mathcal{L} \approx \mathbb{E}_0\Bigg[\sum_{t=0}^{\infty}\beta^t\Bigg\{\frac{1}{2}\Big((1+\Phi)(1+\gamma)(1+\varphi) + (\sigma-1)\Big)\widehat{x}_t^2 + \frac{1+\Phi}{1+\gamma}\frac{\epsilon}{2\kappa}\pi_t^2$$
$$+ \frac{(1+\phi)}{\zeta + \frac{\gamma}{1+\gamma}}\Big[-\Phi(\sigma-1) - (1+\Phi)\frac{\gamma}{1+\gamma}\Big]\widehat{x}_t a_t + \Phi\widehat{x}_t\Bigg\}\Bigg]$$

Now one can show that a second order approximation of the optimal pricing condition leads to the following (extended) NKPC:

$$\pi_t + \frac{\epsilon-1}{2(1-\theta)}\pi_t^2 + \frac{1-\theta\beta}{2}G_t\pi_t = \kappa\Big[\widehat{x}_{1t} - \widehat{x}_{2t} + \frac{1}{2}(\widehat{x}_{1t}^2 - \widehat{x}_{2t}^2)\Big] + \beta\pi_{t+1}$$
$$+ \beta\frac{1-\theta\beta}{2}G_{t+1}\pi_{t+1} + \beta\frac{\epsilon-1}{2(1-\theta)}\pi_{t+1}^2 + \beta\frac{\epsilon}{2}\pi_{t+1}^2 \ , \tag{A.24}$$

where $\widehat{x}_{1t} \equiv mc_t - \sigma\widehat{c}_t + \widehat{y}_t$ and $\widehat{x}_{2t} \equiv \widehat{y}_t - \sigma\widehat{c}_t$, and:

$$G_t = \sum_{\tau=t}^{\infty}(\theta\beta)^{\tau-t}(x_{1,t,\tau} + x_{2,t,\tau}) \ ,$$

where $\widehat{x}_{1,t,\tau} \equiv \widehat{x}_{1\tau} + \epsilon \sum_{s=t+1}^{\tau} \pi_s$ and $\widehat{x}_{1,t,\tau} \equiv \widehat{x}_{1\tau} + (\epsilon - 1) \sum_{s=t+1}^{\tau} \pi_s$. Defining $Y_t \equiv \pi_t + \frac{\epsilon-1}{2(1-\theta}\pi_t^2 + \frac{1-\theta\beta}{2}G_t\pi_t + \frac{\epsilon}{2}\pi_t^2$, equation (A.24) can be rewritten as:

$$Y_t = \kappa\left[\widehat{x}_{1t} - \widehat{x}_{2t} + \frac{1}{2}(\widehat{x}_{1t}^2 - \widehat{x}_{2t}^2)\right] + \beta\frac{\epsilon}{2}\pi_t^2 + \beta Y_{t+1}$$

Hence:

$$Y_0 = \kappa\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\widehat{x}_{1t} - \widehat{x}_{2t} + \frac{1}{2}(\widehat{x}_{1t}^2 - \widehat{x}_{2t}^2)\right\}\right] + \frac{\epsilon}{2}\sum_{t=0}^{\infty}\beta^t\pi_t^2 \tag{A.25}$$

The difference between the Calvo terms reduces to the marginal costs $\widehat{mc}_t$, which, using households labor-supply condition and the production technology, can be expressed as

$$\begin{aligned}\widehat{x}_{1t} - \widehat{x}_{2t} = \widehat{mc}_t =& \widehat{w}_t - a_t + \gamma\widehat{y}_t \\ =& \varphi\widehat{n}_t + \sigma\widehat{c}_t - a_t + \gamma\widehat{y}_t \\ =& \varphi\left(\widehat{\Delta}_t + \widehat{y}_t - a_t + \gamma\widehat{y}_t\right) + \sigma\widehat{c}_t - a_t + \gamma\widehat{y}_t\end{aligned}$$

Hence:

$$\widehat{x}_{1t} - \widehat{x}_{2t} = \varphi\widehat{\Delta}_t + \left(\varphi + \gamma(1+\varphi) + \sigma\right)\widehat{y}_t - (1+\varphi)a_t - \widehat{x}_{1t} - \widehat{x}_{2t} = \varphi\widehat{\Delta}_t + \zeta y_t - (1+\varphi)a_t$$

Ignoring higher-order terms and terms independent of policy, we can then rewrite Y_0 as :

$$\begin{aligned}Y_0 \approx \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t y_t\right] + \frac{1}{2\zeta}\mathbb{E}_0\Bigg[\sum_{t=0}^{\infty}\beta^t\Bigg\{&\frac{\epsilon(1+\varphi)}{\kappa}\pi_t^2 + \left[\left(1+\varphi+\gamma(1+\varphi)\right)^2 - (1-\sigma)^2\right]\widehat{y}_t^2 \\ &- 2(1+\varphi)(1+\varphi+\gamma(1+\varphi))\widehat{y}_t a_t\Bigg\}\Bigg]\end{aligned}$$

Since V_0 is given we then have:

$$\begin{aligned}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t y_t\right] \approx -\frac{1}{2\zeta}\mathbb{E}_0\Bigg[\sum_{t=0}^{\infty}\beta^t\Bigg\{&\frac{\epsilon(1+\varphi)}{\kappa}\pi_t^2 + \left[(1+\varphi+\gamma(1+\varphi))^2 - (1-\sigma)^2\right]\widehat{y}_t^2 \\ &- 2(1+\varphi)(1+\varphi+\gamma(1+\varphi))\widehat{y}_t a_t\Bigg\}\Bigg] + t.i.p.\end{aligned}$$

Rewriting in terms of the output gap $\widehat{x}_t^e$ we get:

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\widehat{x}_t\right] \approx X_1 + X_2 + X_3 + t.i.p.$$

where

$$X_1 = -\frac{1}{2\zeta}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\frac{\epsilon(1+\varphi)}{\kappa}\pi_t^2\right\}\right]$$

$$X_2 = -\frac{1}{2\zeta}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\left[(1+\varphi+\gamma(1+\varphi))^2-(1-\sigma)^2\right](\widehat{x}_t^e)^2\right\}\right]$$

$$X_3 = -\frac{1}{\zeta}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\left[(1+\varphi+\gamma(1+\varphi))^2-(1-\sigma)^2\right]\widehat{x}_t^e\widehat{y}_t^e-(1+\varphi)(1+\varphi+\gamma(1+\varphi))\widehat{x}_t^e a_t\right\}\right]$$

Using $\widehat{y}_t^e = \frac{1+\varphi}{\zeta+\frac{\gamma}{1+\gamma}}a_t$ and $\zeta = \sigma+\varphi+\gamma(1+\varphi)$ into X_3 to get:

$$X_3 = -\frac{1+\varphi}{\zeta\left(\zeta+\frac{\gamma}{1+\gamma}\right)}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\zeta\left(1-\sigma\right)-\zeta\frac{\gamma}{1+\gamma}-\frac{\gamma}{1+\gamma}\left(1-\sigma\right)\right\}\right]\widehat{x}_t^e a_t$$