1 – Unitary Gates (notes) – by Kelvin Ma

Unitary Gates

Qubits

The quantum state of a single qubit can be written as:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Where α and β are both complex numbers. The probability of the qubit being measured as $|0\rangle$ is $|\alpha|^2$ and the probability of the qubit being measured as $|1\rangle$ is $|\beta|^2$. Due the probabilistic nature of quantum mechanics and the axioms of probability, we know that $|\alpha|^2 + |\beta|^2 = 1$. In vector form, we can write the following:

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

We also know that the global phase of undetectable, i.e. $|\psi = e^{i\delta}|\psi\rangle$. Therefore, we only need two real numbers to describe a single qubit in quantum state.

Bloch Sphere Representation

Note that: $|\alpha|^2 + |\beta|^2 = 1$. If we let $\alpha = \sin(\theta)$ and $\beta = \sin(\theta)$, this equation still holds true.

Then a very convenient way to write out first equation is:

$$|\psi\rangle = \cos(\theta)\,|0\rangle + \sin(\theta)\,e^{i\phi}|1\rangle$$

Where (as with standard spherical form), $0 \le \phi \le 2\pi$ and $0 \le \theta \le \frac{\pi}{2}$. Then it is easy to see that to describe a quantum state existing in $\boldsymbol{C^2}$, on the complex plane, we need a sphere existing in $\boldsymbol{R^3}$ existing on the real plane.

This notation is called the Bloch Sphere Representation of a qubit.

Using Qiskit

In **Qiskit**, the range of the two angles in the Bloch Sphere Representation (BSR) change slightly, thus our equation must also change slightly. If we let θ be in the range of: $0 \le \theta \le \pi$, then we can write our equation as

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|1\rangle$$

Unitary Gates

In Qiskit, the general single-qubit unitary gate is given by:

$$u3(\theta,\phi,\lambda) = U(\theta,\phi,\lambda) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -e^{i\lambda}\sin\left(\frac{\theta}{2}\right) \\ e^{i\theta}\sin\left(\frac{\theta}{2}\right) & e^{i\lambda+i}\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

Where $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$ and $0 \le \lambda \le 2\pi$

U3 gate

We can write `u3` to call this function in Qiskit as follows:

$$u3(\theta, \phi, \lambda) = U(\theta, \phi, \lambda)$$

```
qc = QuantumCircuit(q)
qc.u3(pi/2,pi/2,pi/2,q)
qc.draw(output='mpl')
```

This allows us to create superpositions.

U2 gate

The U2 gate is another gate. The U2 gate applies the U3 gate when $\theta = \frac{\pi}{2}$, i.e. $u2(\phi, \lambda) = u3(\frac{\pi}{2}, \phi, \lambda)$.

The matrix form of this gate is:

$$u2(\phi, \lambda) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -e^{i\lambda} \\ e^{i\phi} & e^{i(\phi+\lambda)} \end{pmatrix}$$

The U2 gate allows us to quickly and easily create superpositions.

Important behavior of the U2 gate:

$$u2(0, \pi) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -e^{i\pi} \\ e^{i(0)} & e^{i(0+\pi)} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -e^{i\pi} \\ e^{0} & e^{i\pi} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

The last matrix should look familiar because this is the quantum **Hadamard gate (H-gate).**

U1 gate

The U1 gate is when both $\theta = 0$ and $\phi = 0$. i.e. $u1(\lambda) = u3(0,0,\lambda)$

The matrix form of this gate is:

$$u1(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\lambda} \end{pmatrix}$$

This allows us to apply a quantum phase easily.