HHL Algorithm

Quantum Algorithm for solving a linear system of equations.

About HHL

- one of the fundamental algorithms expected to provide a speed-up over classical counter parts.

  - Time complexity : $O(\log(N)\kappa^2)$
    where: N - number of variables
    κ - condition number
    ↳ sensitivity of the system to small changes or errors in the input

  classical complexity = $O(N\kappa)$
  ↳ or $O(N\sqrt{\kappa})$ for positive semi-definite matrices.

How the algorithm works

The problem:

Given a Hermitian matrix (→ required condition.), A and a unit vector $\vec{b}$, find x, such that:

$$A\vec{x} = \vec{b}$$

1. Represent $\vec{b}$ as a quantum state:

$$|b\rangle = \sum_{i=1}^{N} b_i |i\rangle.$$

2. Decompose $|b\rangle$ into the eigenbasis of A and find the corresponding eigenvalues λ_j (with quantum phase estimation). → QPE.

   Hamiltonian simulation techniques are used to apply the unitary operator (e^{iAt}) to $|b\rangle$ for superposition of different times t -

   The system after decomposition and application is:

$$\sum_{j=1}^{N} \beta_j |u_j\rangle |\lambda_j\rangle \rightarrow$$ where u_j is the eigenvector basis of A and $|b\rangle = \sum_{j=1}^{N} \beta_j |u_j\rangle$.

$$\therefore \sum_{j=1}^{N} |b\rangle |\lambda_j\rangle.$$

3. Perform linear mapping by taking $|\lambda_j\rangle$ to $C\lambda_j^{-1}|\lambda_j\rangle$, where C is the normalization constant.

   NOTE: The linear mapping is not unitary, and has a chance to fail. To solve this problem, it must be repeated.

4. After the sucess of linear mapping → we can uncompute the $|\lambda_j\rangle$ register, and we can find x.

$$\sum_{j=1}^{N} \beta_i \lambda_j^{-1} |u_j\rangle = A^{-1}|b\rangle = |x\rangle.$$

This is the general idea of the algorithm. Qiskit does something similar —

HHL Algorithm with Qiskit:

1. First, we must express $\vec{b}$ as a quantum state $|b\rangle$ on a quantum register
2. Decompose $\vec{b}$ into a superposition of eigenvectors of A by using the Quantum Phase Estimation algorithm (QPE).

   - A is invertible (because it is Hermitian).
3. The inversion of the Eigenvector base of A is achieved by rotating an ancillary qubit by an angle (θ)

   $$\theta = \sin^{-1}\left(\frac{c}{\lambda_i}\right) = \arcsin\left(\frac{c}{\lambda_i}\right)$$

   around the y-axis.

   where λ_i are now the eigenvalues of A.

   $$\therefore A^{-1}|b\rangle = |x\rangle$$

4. Next, we need to uncompute the register storing the eigenvalues using the inverse QPE

   we measure the ancillary qubit where the measurement of 1 indicates that the matrix inversion was successful.

   The inverse QPE leaves the system proportional to the solution vector $|x\rangle$. In many cases, one is not interested in the single vector elements of $|x\rangle$ but only on certain properties.

   These properties are accessible by applying an operator M to the state $|x\rangle$. where M is a <u>problem-specific</u> operator.

NOTES: A must be hermitian of dimension 2^n.

↳ This is a requirement, all other matrices are <u>not</u> <u>supported</u> as of 08/05/2019.