PMATH 965 A2
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Question 1
Let $K = U(n)$ acting on $X = M(n,k)$ by left multiplication. For any $A, B \in \Gamma(TM(n,k))$, define a symplectic form on X by

$$\omega(A,B) = \frac{1}{2i}\mathrm{Tr}(A^\dagger B - B^\dagger A).$$

Then I claim that $\mu : X \to \mathfrak{u}(n)^*$, defined by

$$\mu(x) = \left(\alpha \to \left(\frac{1}{2i}\mathrm{Tr}\left(x^\dagger \alpha^\dagger x\right)\right)\right)$$

is a moment map for the $U(n)$ action on (X, ω). First we check it is equivariant. Suppose $U \in U(n)$ and $\alpha \in \mathfrak{u}(n)$.

$$\begin{aligned}\mu(U \cdot x)(\alpha) &= \frac{1}{2i}\mathrm{Tr}(x^\dagger U^\dagger \alpha^\dagger U x)\\ &= \frac{1}{2i}\mathrm{Tr}\left((U^\dagger \alpha U)x\right)\\ &= \mu(x)(U^\dagger \alpha U)\end{aligned}$$

Now since U is unitary, $U^\dagger = U^{-1}$, so this becomes

$$\begin{aligned}\mu(U \cdot x)(\alpha) &= \mu(x)(U^{-1}\alpha U)\\ &= \mu(x)(U \cdot \alpha).\end{aligned}$$

Hence μ is $U(n)$-equivariant. Next we check that

$$d\left(\mu(x)(A)\right) = \omega_x(X_\alpha, A). \tag{1}$$

First, we can find a more explicit formula for X_α since $U(n)$ is a matrix Lie group:

$$X_\alpha(x) := \frac{d}{dt}|_{t=0} e^{t\alpha} \cdot x = \alpha x$$

Then the right side of equation (1) is

$$\begin{aligned}\omega_x(X_\alpha, A) &= \frac{1}{2i}\mathrm{Tr}(A^\dagger X_\alpha - X_\alpha^\dagger A)\\ &= \frac{1}{2i}\mathrm{Tr}(A^\dagger \alpha x - x^\dagger \alpha^\dagger A).\end{aligned}$$

For the left side we first compute

$$\mu(x)(\alpha) = \frac{1}{2i}\mathrm{Tr}(x^\dagger \alpha x)$$

To take d of this, we consider a path $x(t) = x + At$ in $U(n)$, for an arbitrary $A \in \Gamma(TX)$. Then

$$\begin{aligned}(d\mu(x)(\alpha))(A) &= \frac{1}{2i}\frac{d}{dt}|_{t=0}\mathrm{Tr}(x(t)^\dagger \alpha x(t))\\ &= \frac{1}{2i}\mathrm{Tr}(A^\dagger \alpha x + x^\dagger \alpha A)\\ &= \frac{1}{2i}\mathrm{Tr}(A^\dagger \alpha x - x^\dagger \alpha^\dagger A).\end{aligned}$$

Where in the last step we used that since $\alpha \in \mathfrak{u}(n)$, $\alpha^\dagger = -\alpha$. Thus the left and right hand sides are equal, and μ is indeed a moment map.

Now, we can form the symplectic quotient. Fix some $\eta \neq 0$ in $\mathfrak{u}^*$. We identify $\mathfrak{u}^*$ with $\mathfrak{u}$ by the map

$$B \to (A \to \mathrm{Tr}(B^\dagger A)). \tag{2}$$

Composing this with our moment map, we can view $\mu : X \to \mathfrak{u}(n)$ given by

$$\mu(x) = \frac{1}{2i} x x^{\dagger}.$$

Then let η be a regular point of μ in $\mathfrak{u}(n)$, say (up to redefining μ), $\eta = \frac{1}{2i} I_n$, the identity matrix. The symplectic quotient is defined to be

$$X /\!/ K := \mu^{-1}(\eta)/U(n).$$

To compute this, I claim first that every matrix in $\mu^{-1}(\eta)$ has rank n. If x has rank less than n, then $xx^{\dagger}$ has rank less than n and hence cannot be I_n. Furthermore, if $xx^{\dagger} = I_m$ then x must be "unitary" i.e. its n linearly independent columns form an orthonormal basis of $\mathbb{C}^n$. These n columns define an n-dimensional subspace of $\mathbb{C}^k$. Two such matrices x and y yield the same n-dimensional subspace exactly when there is a change-of-basis matrix between the columns. This matrix will be unitary since the sets of columns are orthonormal. Hence we finally arrive at:

$$X /\!/ K = \{n - \text{dimensional subspaces of } \mathbb{C}^k\} = \mathrm{Gr}(n, k).$$

Question 2

Let $K = \mathbb{C}^*$ and let it act on $\mathbb{C}^{m+1}$ via weight matrix $A = [a_1, ..., a_m, -d]$, with $\gcd(a_1, ..., a_m) = 1$, $a_i | d$ and $a_i, d > 0$ integers.

a) Let $\omega = 1$. To compute the semi-stable locus, we will use the proposition which tells us

$$V_\omega^{ss} = \bigcup_{I \in \mathcal{A}_\omega} (\mathbb{C}^*)^I \times \mathbb{C}^{\overline{I}}. \tag{3}$$

We need to find $\mathcal{A}_\omega$. Since all the a_i are positive integers, 1 is in any rational cone containing any of the a_i, but not the cone over just $-d$. Hence $\mathcal{A}_\omega = \{I \subset [m+1] \mid I \neq \{m+1\}\}$. This tells us that the semi-stable locus is

$$V_\omega^{ss} = \{z = (z_1, ..., z_{m+1}) \in \mathbb{C}^{m+1} \mid \exists i \neq m+1 \text{ s.t. } z_i \neq 0\} = (\mathbb{C}^m - \{0\}) \times \mathbb{C}. \tag{4}$$

Now we can take the $\mathbb{C}^*$ quotient. We will adopt the viewpoint that we have a trivial line bundle over $(\mathbb{C}^m - \{0\})$ which we are quotienting. The factor of $(\mathbb{C}^m - \{0\})/\mathbb{C}^*$ with weights $[a_1, ..., a_m]$ gives exactly the definition of $\mathbb{P}(a_1, ..., a_m)$. The "sections" of the trivial line bundle that descend are those families of points $(z_1, ..., z_m, l) \in V_\omega^{ss}$ for which

$$t^{-d} l(t \cdot z) = l(z). \tag{5}$$

These are precisely the degree-d homogeneous functions with respect to the weighted grading on $\mathbb{C}[x_1, ..., x_m]$. Hence we see that our quotient is the total space of $\mathcal{O}(d)$ over $\mathbb{P}(a_1, ..., a_m)$.

b) Now let $\omega = -1$. Again we find the anti-cones; this time an anticone will contain ω if and only if $-d$ is one of the generating rays. This is because the rest of the weights $a_1, ..., a_m$ are positive and hence any cones they define consist only of positive numbers. The semi-stable locus is therefore

$$V_\omega^{ss} = \{(z_1, ..., z_m) \in \mathbb{C}^{m+1} \mid z_{m+1} \neq 0\} = \mathbb{C}^m \times \mathbb{C}^*. \tag{6}$$

Next we try to take the $\mathbb{C}^*$ quotient. We will adopt the viewpoint that we have a trivial $\mathbb{C}^m$ bundle over $(\mathbb{C}^*)$ that we are quotienting. The quotient of $\mathbb{C}^*$ by $\mathbb{C}^*$ with weight d gives one point, because

$$\sqrt[d]{\frac{w}{z}}^{d} z = w$$

so $z \sim w$ for all $z, w \in \mathbb{C}^*$. The sections of the bundle that descend are those semi-stable points for which

$$(t^{a_1} s_1(t \cdot z_{m+1}), ..., t^{a_m} s_m(t \cdot z_{m+1})) = (s_1(z_{m+1}), ..., s_m(z_{m+1})).$$

I.e. we have $t^{a_i} s_i(t \cdot z_{m+1}) = s_i(z_{m+1})$ is a degree a_i homogeneous polynomial, so our quotient is $\bigoplus_{i=1}^m \mathcal{O}(a_i)$ over a single point. There is only one homogeneous polynomial of degree a_i in one variable (z^{a_i}), so our quotient is isomorphic to the vector space $\mathbb{C}^m$.

Question 3
a) Let v be a vertex of P. Then v is the intersection of n facets of P, say $v = \bigcap_{i\in I} F_i$. These facets have inward facing vectors $\{\rho_i\}_{i\in I}$. Thus, we can associate an n-dimensional cone σ to v: the cone over $\{\rho_i\}_{i\in I}$.

Let $U_\sigma = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$. A point $\mathfrak{p}$ in U_σ defines a map $\mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}$ by quotienting. By thinking of $\sigma^\vee \cap M \subset \mathbb{C}[\sigma^\vee \cap M]$ we can also consider $\mathfrak{p}$ to be a map $\sigma^\vee \cap M \to \mathbb{C}$. Similarly, an element of the torus $T = \mathrm{Spec}(\mathbb{C}[M])$ is a map $M \to \mathbb{C}^*$. In this view, the torus action T on U_σ is given by the map $t \cdot \mathfrak{p}$ defined to be

$$t \cdot \mathfrak{p}(m) = t(m)\mathfrak{p}(m).$$

Therefore a fixed point is an ideal $\mathfrak{p}$ in U_σ for which $\mathfrak{p}(m) = t(m)\mathfrak{p}(m)$ for all $m \in \sigma^\vee \cap M$. Define a map $f : \sigma^\vee \cap M \to \mathbb{C}$ by

$$f(m) = \begin{cases} 0, & m \neq 0 \\ 1, & m = 0. \end{cases}$$

For $m \neq 0$ we have $f(m) = 0 = t(m)f(m)$, and for $m = 0$ we have $f(m) = 1 = t(m)f(m)$. The latter is because $t(0)$ must be 1 as t is a homomorphism. Finally, we must show that this map f indeed defines a point $\mathfrak{p} \in U_\sigma$; i.e. that f is a homomorphism. Suppose $m_1, m_2 \in \sigma^\vee \cap M$. We want

$$f(m_1 + m_2) = f(m_1)f(m_2)$$

If $m_1 + m_2 \neq 0$ then this holds. Suppose $m_1 + m_2 = 0$. Then either $m_1 = m_2 = 0$, in which case the equation holds again, or $m_1 = -m_2$. If $m_1 = -m_2$ non-zero then $\sigma^\vee$ is not strongly convex. This contradicts the fact that σ is dimension n. Hence f is a homomorphism and it defines a fixed point in U_σ. Since f sends all the non-constant maps in $\mathbb{C}[\sigma^\vee \cap M]$ to 0, $\mathfrak{p}$ is defined by the maximal ideal $\mathfrak{p} = \langle \rho_1, ..., \rho_n \rangle$.

b) Let $v = \bigcap_{i\in I} F_i$ with F_i defined by inward facing normal ρ_i. Let $u_1, ..., u_n$ be the inward facing edges of v. If $u_j \in F_i$, then $\langle \rho_i, u_j \rangle = 0$. This implies that $v_j \in F_i$, as

$$\langle v_j, \rho_i \rangle = \langle v, \rho_i \rangle + \epsilon \langle u_j, \rho_i \rangle = \lambda_i.$$

A facet of a dimension n Delzant polytope has $n-1$ edges, so for each F_i there are $n-1$ of the vertices v_i in F_i. Re-label the edges u_i so that $u_j \in F_i$ for $i \neq j$. Furthermore, since P is Delzant, the set $\{\rho_i\}$ is a $\mathbb{Z}$-basis, and up to an lattice isomorphism we can assume ρ_i are orthonormal.

Lemma 1. *Let G be the hyperplane in $N_\mathbb{R} \cong \mathbb{R}^n$ through $\{v_i\}_{i=1}^n$. Then there exists a λ_G such that*

$$G = \{\phi \in N_\mathbb{R} \mid \langle \phi, \rho_1 + ... + \rho_n \rangle = \lambda_G\}.$$

Proof. Let $\lambda_G = \sum_{i=1}^n \langle v, \rho_i \rangle + 1$. Then for each v_i,

$$\langle v_i, \rho_1 + ... + \rho_n \rangle = \sum_{j=1}^n \langle v, \rho_j \rangle + \epsilon \langle u_i, \rho_j \rangle = \lambda_G + \epsilon \langle u_i, \rho_i \rangle.$$

Furthermore, since the ρ_i are an orthonormal basis, and $\langle u_i, \rho_j \rangle = 0$ for $i \neq j$, we must have that $u_i = c\rho_i$, and we can scale u_i to take $c = 1$. Hence we get

$$\langle v_i, \rho_1 + ... + \rho_n \rangle = \lambda_G$$

Hence $v_i \in G$, proving it is the hyperplane through $\{v_i\}_{i=1}^n$. □

Let $\rho_0 = \rho_1 + ... + \rho_n$. This lemma tells us that the dual cone $\sigma_i^\vee$ associated to v_i is the cone over $\{\rho_0, ..., \rho_{i-1}, \rho_{i+1}, ..., \rho_n\}$. The normal fan Σ_ϵ associated to P_ϵ has rays $\{\rho_0, \rho_1 + ... + \rho_n\}$ (plus all the rays associated to the rest of the polytope away from v).

Proposition 1. *Let σ be the open cone over an orthonormal $\mathbb{Z}$ basis $\{\rho_1, ..., \rho_n\}$ of $N_\mathbb{R}$. Let $\rho_0 = \rho_1 + ... + \rho_n$ and let σ_i be the cone over $\{\rho_0, ...\rho_{i-1}, \rho_{i+1}, ..., \rho_n\}$. Let Σ_ϵ be the fan generated by $\{\sigma_i\}_{i=1}^n$. Then the toric variety X_ϵ associated to this fan is the blow up of $\mathbb{C}^n$ at the origin.*

Proof. The hypothesis that $\{\rho_1, ..., \rho_n\}$ is an orthnormal $\mathbb{Z}$ basis implies that $X := \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M]) \cong \mathrm{Spec}(\mathbb{C}[x_1, ..., x_n]) = \mathbb{C}^m$. Let U_i be the open affine set $\mathrm{Spec}(\mathbb{C}[\sigma_i^\vee \cap M])$. Then

$$\mathbb{C}[\sigma_i^\vee \cap M] = \mathbb{C}[x_1 x_2 ... x_n, x_1, ..., x_{i-1}, x_{i+1}, ..., x_n] \cong \mathbb{C}\left[y_i, \frac{x_1}{x_i}, ..., \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, ..., \frac{x_n}{x_i}\right].$$

The isomorphism is by first dividing all co-ordinates by x_i and then sending $x_1 x_2 ... x_{i-1} x_{i+1} ... x_n \to y$. Hence the sets U_i are the standard affine cover for $\mathrm{Bl}_0(\mathbb{C}^n)$.

It remains to check that the gluing maps are those for $\mathrm{Bl}_0(\mathbb{C}^n)$. I am running low on time to complete the assignment, so I will check for $n = 2$. In this case, we have two open affines U_1, U_2 with co-ordinate rings

$$\mathbb{C}[\sigma_1^\vee \cap M] = \mathbb{C}[y_1, x_2/x_1], \qquad \mathbb{C}[\sigma_2^\vee \cap M] = \mathbb{C}[y_2, x_1/x_2].$$

They intersect along the face τ which is the open cone over $\{\rho_0\}$, hence the dual cone is spanned by ρ_1, ρ_2 and we have

$$U_\tau = \mathrm{Spec}\left(\mathbb{C}[x_1, x_2]\right)$$

Therefore we have an injective morphism $U_1 \to U_\tau$ by sending $y_1 \to x_1$ and $(x_2/x_1) \to x_2$. Similarly, we have a morphism $U_2 \to U_\tau$ by sending $y_2 \to x_1$ and $(x_1/x_2) \to x_2$. Composing these, we have transition map $U_\tau \to U_\tau$ by $y_1 \to y_2$ and $(x_1/x_2) \to (x_2/x_1)$. Finally, remembering that $y_i := (x_1 x_2)/(x_i)$ we have

$$y_1 \to (x_1 x_2)/(x_2) = (x_1/x_1)(x_1 x_2)/(x_2) = x_1 (x_1 x_2)/(x_1)(x_2)^{-1} = x_1 y_2 (x_2)^{-1}$$

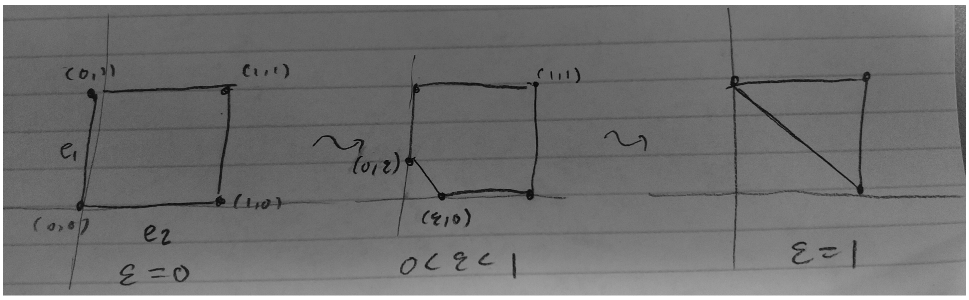
Finally, we arrive at $y_1 x_2 = x_1 y_2$, the gluing map for $\mathrm{Bl}_0(\mathbb{C}^2)$. $\square$

Now consider P_ϵ. The cones associated to all vertices except v are unchanged (for sufficiently small ϵ!), so $X_\epsilon \cong X$ away from the open affine set U_σ associated to v. Thus it suffices to show that the fan Σ_ϵ generated by the cones σ_i associated to each vertex v_i is the fan of the blow up over U_σ. Furthermore, we can identify $U_\sigma \cong \mathbb{C}^n$ in general, as $\rho_i, ..., \rho_j$ are orthnormal. Proposition 1 then tells us that Σ_ϵ is the blow-up of U_σ at the origin. Hence it remains to show that the fixed point $\mathfrak{p} \in U_\sigma$ is sent to the origin when we choose $\rho_i, ..., \rho_j$ to be orthnormal. Recall that $\mathfrak{p}$ is the maximal ideal $\mathfrak{p} = \langle \rho_1, ..., \rho_n \rangle$. Thus when we identify

$$\mathbb{C}[x_1, ..., x_n] \cong \mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[\chi^{\rho_1}, ..., \chi^{\rho_n}]$$

we identify $\langle \rho_1, ..., \rho_n \rangle$ with $\langle x_1, ..., x_n \rangle$ which is the origin (by the Nullstellensatz).

c) Consider the following polytope:



Let v denote the vertex $(0, 0)$ in the bottom left. Let σ denote the open affine associated to v. When $\epsilon = 0$, $U_\sigma = \mathrm{Spec}(\mathbb{C}[x, y]) = \mathbb{C}^2$. For $0 < \epsilon < 1$, as we analysed above we get that X_ϵ will have a blow up at the origin. Finally at $\epsilon = 1$ we obtain the polytope corresponding to the toric variety $\mathbb{P}^2$. How can we interpret this?

For each value of ϵ, we have an inclusion map $\iota : P_\epsilon \to P_0$. This inclusion induces a toric morphism $X_\epsilon \to X_0$. Thus we can image X_ϵ as a $[0, 1]$-family of schemes over X_0. For small ϵ, the X_ϵ are all isomorphic, and so the ϵ data is lost. In fact for $\epsilon' > \epsilon$ we have a morphism $X_{\epsilon'} \to X_\epsilon$, so I imagine that as ϵ increases we are blowing up ""more"", in some vague sense.

Question 4
Suppose X_1 and X_2 are two smooth toric varieties, given by the GIT data $\mathbb{C}^{m_1}, (\mathbb{C}^*)^{k_1}, (A_1, \omega_1)$ and $\mathbb{C}^{m_2}, (\mathbb{C}^*)^{k_2}, (A_2, \omega_2)$ respectively. Let P_i, Σ_i denote their respective polytopes and fans. Further let $\{\rho_i^j\}$ denote the set of rays defining the cones in Σ_i. Then I claim that we can let $n = m_1 + m_2$, $k = k_1 + k_2$ and $A : \mathbb{Z}^n \to \mathbb{Z}^k$ be given by $A_1 \oplus A_2$, $\omega = (\omega_1, \omega_2)$ be GIT data which will yield $X_1 \times X_2$.

A defines an exact sequence (top row), taking Gale dual and $\otimes\mathbb{R}$ yields the bottom row.

$$\begin{tikzcd}
M \arrow[r, hook] & \mathbb{Z}^n \arrow[r, two heads, "A_1\oplus A_2"] \arrow[d, Rightarrow] & \mathbb{Z}^k \\
\mathbb{R}^k \arrow[r, hook] & \mathbb{R}^n \arrow[r, two heads, "(B_1\oplus B_2)^T"'] & M_{\mathbb{R}}^{\vee}
\end{tikzcd}$$

Letting ρ_i be the image of the standard basis vector $e_i \in \mathbb{R}^n$, and $m_i = n_i - k_i$, we have that

$$\rho_i = \begin{cases} (\rho_i^1, \vec{0}_{m_2}) & i \in \{1, ..., m_1\} \\ (\vec{0}_{m_1}, \rho_{i-m_1}^2), & i \in \{m_1 + 1, ..., m\} \end{cases}$$

To build the fan corresponding to this combined GIT data, we need to find the set of anticones $\mathcal{A}_\omega$. For any $I \subset [m] = [m_1 + m_2]$, let $I_1 = I \cap \{1, ..., m_1\}$ and $I_2 = I \cap \{m_1 + 1, ..., m\}$. Then I claim that the set of anticones is

$$\mathcal{A}_\omega = \{I_1 \cup I_2, \mid I_1 \in \mathcal{A}_\omega^1, I_2 \in \mathcal{A}_\omega^2\}.$$

This is because we can write $M_{\mathbb{R}}^{\vee}$ as $(M_1)_{\mathbb{R}}^{\vee} \oplus (M_2)_{\mathbb{R}}^{\vee}$ with $M_i = (\mathbb{Z}^{n_i - k_i})$, our stability condition also splits along this decomposition, and we have that the rays ρ_i in the secondary fan also split into the rays of X_1 and X_2's secondary fans. Finally, note that for $I = I_1 \cup I_2$ in $\mathcal{A}_\omega$ we have

$$\sigma_I = \left\{ \sum_{i \in I} a_i \rho_i \mid a_i > 0 \in \mathbb{Q} \right\} = \left\{ \sum_{i \in I_1} a_i \rho_i + \sum_{i \in I_2} b_i \rho_i \mid a_i, b_i > 0 \in \mathbb{Q} \right\} = \sigma_{I_1} \oplus \sigma_{I_2}.$$

Therefore we have that

$$\Sigma_{X_1 \times X_2} = \Sigma_1 \oplus \Sigma_2 := \{\sigma \oplus \tau \mid \sigma \in \Sigma_1, \tau \in \Sigma_2\}.$$

For the polytope, the polytope P_i is defined by vertices v_i^j. These vertices correspond to highest-dimensional fans in Σ_i. The highest dimensional fans of $\Sigma_{X_1 \times X_2}$ are sums of the highest dimensional fans of Σ_1 and those of Σ_2. If $\sigma \in \Sigma_1$ has rays $\{\rho_i\}$ and $\tau \in \Sigma_2$ has rays $\{\gamma_i\}$, then the rays in $\sigma \oplus \tau$ are $\{\rho_i + \gamma_j\}$. Let v,w be vertices of P_1 and P_2 corresponding to σ and τ, and let $v \oplus w = (v, w)$ in $M_{\mathbb{R}}^{\vee}$. Then the inward facing vectors defining $v \oplus w$ are exactly the rays $\{\rho_i + \gamma_j\}$ defining $\sigma \oplus \tau$. Thus, $P_{X_1 \times X_2}$ is the convex hull of the points $\{v \oplus w \mid v \text{ a vertex of } P_1, w \text{ a vertex of } P_2\}$.