1 Motivation

Mirror symmetry is an enormous area of research. Here we provide motivation from just one perspective, which is Fano classification. Let X be a smooth n-dimensional algebraic variety over $\mathbb{C}$. Suppose we also have a line bundle $L \to X$, with some global sections $s_1, s_2, ..., s_m \in \Gamma(X, L)$. Then we can try and write down a map

$$\begin{aligned} \iota : X &\to \mathbb{P}^{m-1} \\ x &\to [s_1(x) : s_2(x) : ... : s_m(x)]. \end{aligned}$$

This is well defined as long as there is no $x \in X$ where all the sections vanish; $s_i(x) = 0$ for all $i = 1, ..., m$.

Definition 1. A line bundle L over an algebraic variety X is called *very ample* if there exist some global sections $s_1, ..., s_m$ of L for which the map ι defined above is an embedding of X into $\mathbb{P}^{m-1}$.

If there exists a natural number k such that $L^{\otimes k}$ is very ample, then we say L is *ample*.

For example, the line bundles $\mathcal{O}(n) \to \mathbb{P}^{m-1}$ (not to be confused with orthogonal groups!) are very ample for all $n \geq 1$.

Definition 2. The variety X is *Fano* if $-K_X := \bigwedge^n TX$ is *ample.*

If X is Fano, then it is projective, since $\bigwedge^n TX$ is very ample, meaning it has some sections which define an embedding ι of X into projective space. Some examples of Fano varieties include $\mathbb{P}^n$, any degree d projective curve in $\mathbb{P}^n$ with $d < n + 1$, and Grassmannians. Naturally then we can ask *Why study Fano varieties?*

- Fano varieties are often the ambient spaces in algebraic geometry. For example, Calabi-Yaus can be cut-out from Fano varieties.
- Fano varieties are special in that there are only finitely many of them in any given dimension.

Theorem 1 (Kollár-Miyaoka-Mori)**.** *Up to* deformation*, there are finitely many Fano varieties in each dimension.*

Here X_1 and X_2 are considered equivalent up to deformation if there exists a flat family $\mathcal{X} \to B$ over an irreducible base B such that X_1 and X_2 are fibers over some points $b_1, b_2 \in B$.

This raises the big question: Can we classify the Fano varieties? The current progress is:

- In dimension 1, there is just one Fano; $\mathbb{P}^1$.

- In dimension 2, there are 10, called the del Pezzo surfaces.
- In dimension 3, there are 105, which were classified throughout the 70s and 90s.
- All higher dimensions are yet to be classified.

In this course, we are also concerned with *mirror symmetries* for Fano varieties. A conjectured mirror symmetry is between n-dimensional Fano varieties and Laurent polynomials in n-dimensions up to an equivalence called *mutation*. Loosely, we can say

Definition 3. A variety X is *mirror* to a polynomial f, if you can determine *enumerative info* about X from f.

By enumerative info for X, we mean things like Gromov-Witten invariants, quantum cohomology and quantum periods. The mirror symmetry conjecture is that these can be computed in terms of correponding quantities of f. For example:

Definition 4. Let $f \in \mathbb{C}[x_1^{\pm 1}, ..., x_n^{\pm 1}]$ be a Laurent polynomial. The *classical period* of f is the quantity

$$\begin{aligned}\pi_f(t) &= \int\limits_{(S^1)^n} \frac{1}{1-tf} \frac{dx_1}{x_1} ... \frac{dx_n}{x_n} \\ &= \sum_{k=0}^{\infty} \frac{c_1(f^k)}{k!} t^k.\end{aligned}$$

where $c_1(f^k)$ means the coefficient of the constant term of f^k.

The idea therefore, is that mirror symmetry can help us compute hard things in geometry by using easier polynomial data. Then what is the status of this conjecture?

- Established in dimensions 1 and 2 by checking all cases.
- In dimension 3, all Fanos X have a mirror polynomial f, but the classification of other maximally mutable polynomials f is unknown.
- In all dimensions, the symmetry is established for *toric varieties*.

Toric varieties are Fano varieties which have the form $V /\!/ T$ where V is a (complex) vector space and $T = (\mathbb{C}^*)^k$, which is called the *algebraic torus*. The double slash // indicates a *geometric invariant theory* quotient, which will be discussed in the first part of the course. Recently, there is a lot of work on extending mirror symmetry to GIT quotients $V /\!/ G$ more generally, for G a reductive algebraic group.

As we will see, toric varieties are very nice to work with. This is because they are extremely computable. Essentially, there is a dictionary between the geometry of a toric variety X and the combintorics of a polytope P corresponding to X. The basic question then, is can a similar correspondence be generalised to other Fano varieties? What should play the role of the polytope? Mirror symmetry answers this question: X corresponds to f, which has a *Newton polytope* with some additional coefficient data.

Exercises:

1. Show that a degree d hypersurface in $\mathbb{P}^n$ is Fano for $d < n + 1$.
2. Find a closed formula for the classical period of $f(x, y) = x + y + \frac{1}{xy}$, and find a differential equation that it satisfies.

2 Quotients in Algebraic Geometry

Definition 5. An *algebraic* group is a group which is also an algebraic variety. An *action* of an algebraic group G on a variety X is a morphism

$$\begin{aligned} G \times X &\to X \\ (g, x) &\to g \cdot x \end{aligned}$$

such that for all $g, g' \in G$ and $x \in X$, we have $(gg') \cdot x = g \cdot (g' \cdot x)$ and $e \cdot x = x$.

For example: $\mathbb{C}^*$, $GL(n)$ and $SL(n)$.

Definition 6. Given an action of G on X and some $x \in X$, the *orbit* of x is

$$G \cdot x = \{g \cdot x, \mid g \in G\}. \tag{1}$$

The *stabiliser* of x is

$$G_x = \{g \in G \text{s.t.} g \cdot x = x\}. \tag{2}$$

Note that G_x is a closed subgroup of G.

Example: Let $T = \mathbb{C}^*$ and $V = \mathbb{C}^2$. Define an action of T on V by $\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda z_2)$ for all $\lambda \in T$, $(z_1, z_2) \in V$. Then the orbit of $(z_1, z_2) \neq (0, 0)$ is the line through (z_1, z_2), except the origin. The stabilizer is $\{1\}$. If $(z_1, z_2) = (0, 0)$ then the orbit is $\{(0, 0)\}$ and the stabilizer is all of T. Notice that $(0, 0)$ is in the closure of $G \cdot z$ for all $z \in \mathbb{C}^2$.

Proposition 1. *For any G, X and $x \in X$,*

- *The orbit $G \cdot x$ is locally closed and a smooth subvariety of X.*
- *Each of its irreducible components has dimension $\dim(G) - \dim(G_x)$.*
- *The closure of $G \cdot x$ is a union of $G \cdot x$ and orbits of strictly smaller dimension.*

The last point implies that minimal dimension orbits must be closed, and $\overline{G \cdot x}$ always contains a closed orbit.

Definition 7. The action of a group G is called *closed* if every orbit of G is closed.

Definition 8. A *linear algebraic group* is a closed subgroup of $GL(n)$.

Goal: If we have an action $G \circlearrowright X$, we want to build some quotient X/G in an algebraio-geometric way. As a naive attempt we can just take the quotient as topological spaces. Consider the action of $\mathbb{C}^*$ on $\mathbb{C}^2$ from before. If we endow the set $\mathbb{C}^2/\mathbb{C}^*$ with the quotient topology, then since $[(0, 0)]$ is in every open neighbourhood of every other point (as we can always take a sequence of $\lambda_i \in \mathbb{C}^*$ approaching zero), this quotient is not even Hausdorff.

To solve this, we essentially want to delete the origin, and obtain $\left(\mathbb{C}^2 - \{(0, 0)\}\right)/\mathbb{C}^* = \mathbb{P}^1$. The putative quotient Y we want to define must have the following properties:

- There exists a surjection $p : X \to Y$ which is G-invariant.
- Y is separated.
- Y satisfies the following universal property: if $f : X \to Z$ is G invariant, then it factors uniquely through p. That is:

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[dr, "f"'] & Y \arrow[d, dashed] \\ & Z \end{tikzcd}$$

- For all U open, $\mathcal{O}_Y(U) \cong \mathcal{O}_X(p^{-1}(U))^G$, where the superscript denotes G-invariant functions.
- If $Z \subset X$ is closed and G-invariant, then $p(Z)$ is closed. If Z_1, Z_2 are disjoint and closed then $p(Z_1)$ and $p(Z_2)$ are disjoint.

If p satisfies all these properties, we say it is a *good quotient*. Moving forward, we will talk about affine and projective GIT quotients, symplectic reduction, and comparison between these two methods of constructing good quotients.

We can also consider a geometric quotient, which is a good quotient whose points are orbits of $G \circlearrowright X$.

Remark: The properties of being good or geometric are local on the base, meaning that $p : X \to Y$ is good or geometric if and only if there exists an open cover of Y with the restrictions of p being good or geometric.

Lemma 1. *If $p : X \to Y$ is good, then it is categorical.*

Proof. Suppose $g : X \to Z$ is another H invariant morphism and $p : X \to Y$ is good. Then we want to define $h : Y \to Z$ such that $p \circ h = g$. Consider $g(p^{-1}(y))$ for some $y \in Y$, which we claim is a singleton set. Suppose for contradiction that there are $z_1 \neq z_2 \in g(p^{-1}(y))$. Then $g^{-1}(z_1) \cap g^{-1}(z_2) = \emptyset$, and these are closed, G-invariant sets because g is continuous and G-invariant. Hence, by the hypothesis that p is good we have:

$$p(g^{-1}(z_1)) \cap p(g^{-1}(z_2)) = \emptyset. \tag{3}$$

However, we must also have that $y \in p(g^{-1}(z_i)), i = 1, 2$ because $z_i \in g(p^{-1}(y))$; hence we have a contradiction and must have that $g(p^{-1}(y))$ is a singleton.

Therefore, we can define a map $h : Y \to Z$ by $y \to g(p^{-1}(y))$ and it is well-defined and clearly $p \circ h = g$. It remains to show that this is a morphism of schemes (namely we need it to be locally induced by ring morphisms $\mathcal{O}_X \to \mathcal{O}_Y$). Let $\{U_i\}$ be a finite open affine cover of Z. Let $W_i = X - g^{-1}(U_i) = g(U_i)^c$. Then U_i being open implies that $g^{-1}(U_i)$ is open and hence W_i is closed. Similarly, g being G-invariant implies W_i is also. Finally, since U_i is a cover, $\bigcap W_i = \emptyset$. Thus by goodness of p we have that $p(W_i)$ are all closed and

$$\bigcap p(W_i) = \emptyset. \tag{4}$$

Define $V_i = Y - p(W_i) = p(W_i)^c$ Then the V_i are an open cover of Y by equation 4. Note further that $p^{-1}(V_i) \subset g^{-1}(U_i)$. Thus we have a sequence of maps

$$\mathcal{O}_Z(U_i) \xrightarrow{\mathcal{O}}_X (g^{-1}(U_i))^G \xrightarrow{\mathrm{res}|_{p^{-1}(V_i)}} \mathcal{O}_X(p^{-1}(V_i))^G \cong_{p \text{ good}} \mathcal{O}_Y(V_i). \tag{5}$$

The G-invariance on the second ring comes from the invariance of g. The last isomorphism is one of the hypothesis conditions of p being good. Thus since U_i and V_i are affine, this defines a local morphism $h_i : V_i \to U_i$, and it suffices to verify that $h|_{V_i} = h$. $\square$

Proposition 2. *Let $p : X \to Y$ be a good quotient. Then*

1. $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset \iff p(x_1) = p(x_2)$.
2. *For alL $y \in Y$, there exists a unique closed orbit in $p^{-1}(y)$.*

Proof. 1) Suppose $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset$. Since p is continuous and constant on orbits, it is constant on orbit closures and hence $p(\overline{G \cdot x_1}) = p(\overline{G \cdot x_2})$ and in particular $p(x_1) = p(x_2)$. On the other hand if $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} = \emptyset$, then since p is good, the image of each orbit is disjoint.

2) Suppose there exist two closed orbits in $p^{-1}(y)$. If they are not equal, then again since p is continuous and constant on orbits they must be disjoint and hence their image is disjoint by goodness. However they must each contain y so this is a contradiction. (Existence was not given in class, maybe it is obvious?) $\square$

Now, let us construct good quotients for affine varieties.

2.1 Affine GIT Quotient

Suppose we have the action of a group G on an affine variety X. Then $X = \mathrm{Spec}(\mathcal{O}_X)$ by definition, and we want our quotient to be good; so we want it to have functions $\mathcal{O}_X^G$. Thus the idea is to let our quotient be $\mathrm{Spec}(\mathcal{O}_X^G)$. However $\mathcal{O}_X^G$ is not finitely generated in general, so we will restrict ourselves to *reductive* groups.

Definition 9. Let G be a linear algebraic group.

- We say G is *reductive* if every smooth connected unipotent normal subgroup of G is trivial.

- We say G is *linearly reductive* if, for every representation $G \to GL(V)$ and every non-zero fixed point $v \in V$, there exists a homogeneous G-invariant degree-1 polynomial f on V such that $f(V) \neq 0$.
- We say G is *geometrically reductive* if, for every representation $G \to GL(V)$ and every non-zero fixed point $v \in V$, there exists a homogeneous G-invariant polynomial f on V such that $f(V) \neq 0$.

Theorem 2. *The three properties above are equivalent over $\mathbb{C}$.*

For example, $GL(n,\mathbb{C})$, $SL(n,\mathbb{C})$ and $PGL(n,\mathbb{C})$ are reductive.

Theorem 3 (Nagata's Theorem)**.** *Let G be geometrically reductive acting on a finitely generated $\mathbb{C}$-algebra R. Then R^G is finitely generated.*

Lemma 2. *Let G be a geometrically reductive group acting on an affine variety X. Let Z_1 and Z_2 be two closed, G-invariant disjoint subsets of X. Then there exists a G-invariant function $\psi \in \mathcal{O}_X^G$ such that $\psi(Z_1) = 1$ and $\psi(Z_2) = 0$.*

Proof. Firstly

$$\langle 1 \rangle = I(\emptyset) = I(Z_1 \cap Z_2) = I(Z_1) + I(Z_2), \tag{6}$$

therefore $1 = f_1 + f_2$ for some f_1, f_2 with $f_i(Z_j) = \delta_{ij}$. Claim: (c.f. Hoskins) The subspace spanned by $\{g \cdot f, \mid g \in G\} \subset \mathcal{O}_X$ is G-invariant and finite dimensional. Therefore we can pick a basis $h_1, .., h_n$, and because G acts on all of the h_i, we get an induced action of G on $\mathbb{C}^n$ such that the map

$$\begin{aligned} \phi : X &\to \mathbb{C}^n \\ x &\to (h_i(x)) \end{aligned}$$

is G-equivariant, meaning $\phi(g \cdot x) = g\phi(x)$. Note then that $\phi(Z_1) = 0$ and $\Phi(Z_2) \neq 0$, and define $v = \Phi(Z_2) \in \mathbb{C}^n$.

Since ϕ is G-equivariant, v is fixed by the action of G. Then by the hypothesis of geometric reductivity, there exists some G-invariant homogenous f_0 such that $f_0(v) \neq 0$ and $f_0(0) = 0$. Finally, let

$$\psi = \frac{1}{f_0(v)} f_0 \circ \phi. \tag{7}$$

□

Definition 10. The *affine GIT quotient* of an affine variety X under reductive group G, denoted $X /\!/ G$ is $\mathrm{Spec}(\mathcal{O}_X^G)$.

Theorem 4. *Let X be an affine variety and G a reductive group acting on X. Then $p : X \to Y = Spec(\mathcal{O}_X^G)$ is a good quotient.*

Proof. First we show p is G-invariant. Suppose for contradiction there exist $x \in X$, $g \in G$ such that $p(x) \neq p(g \cdot x)$. Since Y is affine, there exists an $x \in \mathcal{O}_Y$ such that $f(x) \neq f(g \cdot x)$. However $\mathcal{O}_Y = \mathcal{O}_X^G$ by definition, so f must be G invariant, giving a contradiction.

Next we show p is surjective. Let $y \in Y$ and let $\langle f_1, ..., f_n \rangle$ be the ideal defining y. Let $\mathfrak{m}$ be the maximal ideal containing $\langle f_1, ..., f_n \rangle$. The point corresponding to $\mathfrak{m}$ in X maps to y under p.

Now let $U \subset Y$ be open. We want to show $\mathcal{O}_Y(U) \cong \mathcal{O}_X(p^{-1}(U))^G$; it suffices to show this for $U = D_f^Y$ for any $f \in \mathcal{O}_Y$.

$$\begin{aligned}
\mathcal{O}_Y(D_f^Y) &= (\mathcal{O}_Y)_f \\
&= [\mathcal{O}_X(X)^G]_F \\
&= [\mathcal{O}_X(X)_f]^G \\
&= [\mathcal{O}_X(D_f^X)]^G \\
&= \mathcal{O}_X(p^{-1}(D_f^Y))^G
\end{aligned}$$

Let Z_1, Z_2 be G-invariant closed disjoint subsets. By the lemma, there exists $\psi \in \mathcal{O}_X^G$ with $\psi(Z_1) = 0$ and $\psi(Z_2) = 1$. Then $\overline{p(Z_1)} \cap \overline{p(Z_2)} = \emptyset$, because there is a G-invariant function which separates them. This turns out to be equivalent to the topological condition for goodness, that $p(Z_1) \cap p(Z_2) = \emptyset$. To prove this, it suffices to prove that if Z is closed and G-invariant then $p(Z)$ is closed.

Suppose Z is closed and G-invariant. For contradiction, suppose there exists $g \in \overline{p(Z)} - p(Z)$. Then Z and $p^{-1}(y)$ are both closed and G-invariant, so

$$\overline{p(Z)} \cap \overline{p(p^{-1}(y)} = \emptyset. \tag{8}$$

however y must be in this intersection, giving a contradiction. □

Proposition 3. *If the action of G is closed then $X /\!/ G$ is a geometric quotient.*

This GIT construction separates orbits as much as possible while still being good.

Recall the example of $\mathbb{C}^* \circlearrowright \mathbb{C}^2$ by scaling. We saw that $(0, 0)$ is in the closure of every orbit. Hence $\mathbb{C}^2/\mathbb{C}^*$ is just one point. This is the same as saying that all the $\mathbb{C}^*$ invariants in $\mathbb{C}[x_1, x_2]$ are just the constants. Projective GIT will allow us to loosen the definition of G-invariance and get that $\mathbb{C}^2 /\!/ \mathbb{C}^* = \P^1$. Recall, if f is homogeneous of degree k then $f(\lambda x, \lambda y) = \lambda^k f(x, y)$, so f is projectively invariant.