

# PMATH 965: Mirror Symmetry for Toric Varieties

Content by Elana Kalashnikov  
Typeset by Kaleb Ruscitti  
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## 1 Motivation

Mirror symmetry is an enormous area of research. Here we provide motivation from just one perspective, which is Fano classification. Let  $X$  be a smooth  $n$ -dimensional algebraic variety over  $\mathbb{C}$ . Suppose we also have a line bundle  $L \rightarrow X$ , with some global sections  $s_1, s_2, \dots, s_m \in \Gamma(X, L)$ . Then we can try and write down a map

$$\begin{aligned}\iota : X &\rightarrow \mathbb{P}^{m-1} \\ x &\rightarrow [s_1(x) : s_2(x) : \dots : s_m(x)].\end{aligned}$$

This is well defined as long as there is no  $x \in X$  where all the sections vanish;  $s_i(x) = 0$  for all  $i = 1, \dots, m$ .

**Definition 1.** A line bundle  $L$  over an algebraic variety  $X$  is called *very ample* if there exist some global sections  $s_1, \dots, s_m$  of  $L$  for which the map  $\iota$  defined above is an embedding of  $X$  into  $\mathbb{P}^{m-1}$ .

If there exists a natural number  $k$  such that  $L^{\otimes k}$  is very ample, then we say  $L$  is *ample*.

For example, the line bundles  $\mathcal{O}(n) \rightarrow \mathbb{P}^{m-1}$  (not to be confused with orthogonal groups!) are very ample for all  $n \geq 1$ .

**Definition 2.** The variety  $X$  is *Fano* if  $-K_X := \bigwedge^n TX$  is *ample*.

If  $X$  is Fano, then it is projective, since  $\bigwedge^n TX$  is very ample, meaning it has some sections which define an embedding  $\iota$  of  $X$  into projective space. Some examples of Fano varieties include  $\mathbb{P}^n$ , any degree  $d$  projective curve in  $\mathbb{P}^n$  with  $d < n + 1$ , and Grassmannians. Naturally then we can ask *Why study Fano varieties?*

- Fano varieties are often the ambient spaces in algebraic geometry. For example, Calabi-Yaus can be cut-out from Fano varieties.
- Fano varieties are special in that there are only finitely many of them in any given dimension.

**Theorem 1** (Kollár-Miyaoka-Mori). *Up to deformation, there are finitely many Fano varieties in each dimension.*

Here  $X_1$  and  $X_2$  are considered equivalent up to deformation if there exists a flat family  $\mathcal{X} \rightarrow B$  over an irreducible base  $B$  such that  $X_1$  and  $X_2$  are fibers over some points  $b_1, b_2 \in B$ .

This raises the big question: Can we classify the Fano varieties? The current progress is:

- In dimension 1, there is just one Fano;  $\mathbb{P}^1$ .

- In dimension 2, there are 10, called the del Pezzo surfaces.
- In dimension 3, there are 105, which were classified throughout the 70s and 90s.
- All higher dimensions are yet to be classified.

In this course, we are also concerned with *mirror symmetries* for Fano varieties. A conjectured mirror symmetry is between  $n$ -dimensional Fano varieties and Laurent polynomials in  $n$ -dimensions up to an equivalence called *mutation*. Loosely, we can say

**Definition 3.** A variety  $X$  is *mirror* to a polynomial  $f$ , if you can determine *enumerative info* about  $X$  from  $f$ .

By enumerative info for  $X$ , we mean things like Gromov-Witten invariants, quantum cohomology and quantum periods. The mirror symmetry conjecture is that these can be computed in terms of corresponding quantities of  $f$ . For example:

**Definition 4.** Let  $f \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$  be a Laurent polynomial. The *classical period* of  $f$  is the quantity

$$\begin{aligned}\pi_f(t) &= \int_{(S^1)^n} \frac{1}{1 - tf} \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n} \\ &= \sum_{k=0}^{\infty} \frac{c_1(f^k)}{k!} t^k.\end{aligned}$$

where  $c_1(f^k)$  means the coefficient of the constant term of  $f^k$ .

The idea therefore, is that mirror symmetry can help us compute hard things in geometry by using easier polynomial data. Then what is the status of this conjecture?

- Established in dimensions 1 and 2 by checking all cases.
- In dimension 3, all Fanos  $X$  have a mirror polynomial  $f$ , but the classification of other maximally mutable polynomials  $f$  is unknown.
- In all dimensions, the symmetry is established for *toric varieties*.

Toric varieties are Fano varieties which have the form  $V // T$  where  $V$  is a (complex) vector space and  $T = (\mathbb{C}^*)^k$ , which is called the *algebraic torus*. The double slash  $//$  indicates a *geometric invariant theory* quotient, which will be discussed in the first part of the course. Recently, there is a lot of work on extending mirror symmetry to GIT quotients  $V // G$  more generally, for  $G$  a reductive algebraic group.

As we will see, toric varieties are very nice to work with. This is because they are extremely computable. Essentially, there is a dictionary between the geometry of a toric variety  $X$  and the combinatorics of a polytope  $P$  corresponding to  $X$ . The basic question then, is can a similar correspondence be generalised to other Fano varieties? What should play the role of the polytope? Mirror symmetry answers this question:  $X$  corresponds to  $f$ , which has a *Newton polytope* with some additional coefficient data.

Exercises:

1. Show that a degree  $d$  hypersurface in  $\mathbb{P}^n$  is Fano for  $d < n + 1$ .
2. Find a closed formula for the classical period of  $f(x, y) = x + y + \frac{1}{xy}$ , and find a differential equation that it satisfies.

## 2 Quotients in Algebraic Geometry

**Definition 5.** An *algebraic group* is a group which is also an algebraic variety. An *action* of an algebraic group  $G$  on a variety  $X$  is a morphism

$$\begin{aligned} G \times X &\rightarrow X \\ (g, x) &\rightarrow g \cdot x \end{aligned}$$

such that for all  $g, g' \in G$  and  $x \in X$ , we have  $(gg') \cdot x = g \cdot (g' \cdot x)$  and  $e \cdot x = x$ .

For example:  $\mathbb{C}^*$ ,  $GL(n)$  and  $SL(n)$ .

**Definition 6.** Given an action of  $G$  on  $X$  and some  $x \in X$ , the *orbit* of  $x$  is

$$G \cdot x = \{g \cdot x, \mid g \in G\}. \quad (1)$$

The *stabiliser* of  $x$  is

$$G_x = \{g \in G \text{ s.t. } g \cdot x = x\}. \quad (2)$$

Note that  $G_x$  is a closed subgroup of  $G$ .

Example: Let  $T = \mathbb{C}^*$  and  $V = \mathbb{C}^2$ . Define an action of  $T$  on  $V$  by  $\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda z_2)$  for all  $\lambda \in T$ ,  $(z_1, z_2) \in V$ . Then the orbit of  $(z_1, z_2) \neq (0, 0)$  is the line through  $(z_1, z_2)$ , except the origin. The stabilizer is  $\{1\}$ . If  $(z_1, z_2) = (0, 0)$  then the orbit is  $\{(0, 0)\}$  and the stabilizer is all of  $T$ . Notice that  $(0, 0)$  is in the closure of  $G \cdot z$  for all  $z \in \mathbb{C}^2$ .

**Proposition 1.** For any  $G$ ,  $X$  and  $x \in X$ ,

- The orbit  $G \cdot x$  is locally closed and a smooth subvariety of  $X$ .
- Each of its irreducible components has dimension  $\dim(G) - \dim(G_x)$ .
- The closure of  $G \cdot x$  is a union of  $G \cdot x$  and orbits of strictly smaller dimension.

The last point implies that minimal dimension orbits must be closed, and  $\overline{G \cdot x}$  always contains a closed orbit.

**Definition 7.** The action of a group  $G$  is called *closed* if every orbit of  $G$  is closed.

**Definition 8.** A *linear algebraic group* is a closed subgroup of  $GL(n)$ .

Goal: If we have an action  $G \curvearrowright X$ , we want to build some quotient  $X/G$  in an algebraic-geometric way. As a naive attempt we can just take the quotient as topological spaces. Consider the action of  $\mathbb{C}^*$  on  $\mathbb{C}^2$  from before. If we endow the set  $\mathbb{C}^2/\mathbb{C}^*$  with the quotient topology, then since  $[(0, 0)]$  is in every open neighbourhood of every other point (as we can always take a sequence of  $\lambda_i \in \mathbb{C}^*$  approaching zero), this quotient is not even Hausdorff.

To solve this, we essentially want to delete the origin, and obtain  $(\mathbb{C}^2 - \{(0, 0)\})/\mathbb{C}^* = \mathbb{P}^1$ . The putative quotient  $Y$  we want to define must have the following properties:

- There exists a surjection  $p : X \rightarrow Y$  which is  $G$ -invariant.
- $Y$  is separated.

- $Y$  satisfies the following universal property: if  $f : X \rightarrow Z$  is  $G$  invariant, then it factors uniquely through  $p$ . That is:

$$\begin{array}{ccc} X & \xrightarrow{p} & Y \\ & \searrow f & \downarrow \\ & & Z \end{array}$$

- For all  $U$  open,  $\mathcal{O}_Y(U) \cong \mathcal{O}_X(p^{-1}(U))^G$ , where the superscript denotes  $G$ -invariant functions.
- If  $Z \subset X$  is closed and  $G$ -invariant, then  $p(Z)$  is closed. If  $Z_1, Z_2$  are disjoint and closed then  $p(Z_1)$  and  $p(Z_2)$  are disjoint.

If  $p$  satisfies all these properties, we say it is a *good quotient*. Moving forward, we will talk about affine and projective GIT quotients, symplectic reduction, and comparison between these two methods of constructing good quotients.

We can also consider a geometric quotient, which is a good quotient whose points are orbits of  $G \curvearrowright X$ .

Remark: The properties of being good or geometric are local on the base, meaning that  $p : X \rightarrow Y$  is good or geometric if and only if there exists an open cover of  $Y$  with the restrictions of  $p$  being good or geometric.

**Lemma 1.** *If  $p : X \rightarrow Y$  is good, then it is categorical.*

*Proof.* Suppose  $g : X \rightarrow Z$  is another  $H$  invariant morphism and  $p : X \rightarrow Y$  is good. Then we want to define  $h : Y \rightarrow Z$  such that  $p \circ h = g$ . Consider  $g(p^{-1}(y))$  for some  $y \in Y$ , which we claim is a singleton set. Suppose for contradiction that there are  $z_1 \neq z_2 \in g(p^{-1}(y))$ . Then  $g^{-1}(z_1) \cap g^{-1}(z_2) = \emptyset$ , and these are closed,  $G$ -invariant sets because  $g$  is continuous and  $G$ -invariant. Hence, by the hypothesis that  $p$  is good we have:

$$p(g^{-1}(z_1)) \cap p(g^{-1}(z_2)) = \emptyset. \quad (3)$$

However, we must also have that  $y \in p(g^{-1}(z_i)), i = 1, 2$  because  $z_i \in g(p^{-1}(y))$ ; hence we have a contradiction and must have that  $g(p^{-1}(y))$  is a singleton.

Therefore, we can define a map  $h : Y \rightarrow Z$  by  $y \mapsto g(p^{-1}(y))$  and it is well-defined and clearly  $p \circ h = g$ . It remains to show that this is a morphism of schemes (namely we need it to be locally induced by ring morphisms  $\mathcal{O}_X \rightarrow \mathcal{O}_Y$ ). Let  $\{U_i\}$  be a finite open affine cover of  $Z$ . Let  $W_i = X - g^{-1}(U_i) = g(U_i)^c$ . Then  $U_i$  being open implies that  $g^{-1}(U_i)$  is open and hence  $W_i$  is closed. Similarly,  $g$  being  $G$ -invariant implies  $W_i$  is also. Finally, since  $U_i$  is a cover,  $\bigcap W_i = \emptyset$ . Thus by goodness of  $p$  we have that  $p(W_i)$  are all closed and

$$\bigcap p(W_i) = \emptyset. \quad (4)$$

Define  $V_i = Y - p(W_i) = p(W_i)^c$ . Then the  $V_i$  are an open cover of  $Y$  by equation 4. Note further that  $p^{-1}(V_i) \subset g^{-1}(U_i)$ . Thus we have a sequence of maps

$$\mathcal{O}_Z(U_i) \xrightarrow{\mathcal{O}} \mathcal{O}_X(g^{-1}(U_i))^G \xrightarrow{\text{res}|_{p^{-1}(V_i)}} \mathcal{O}_X(p^{-1}(V_i))^G \cong_{p \text{ good}} \mathcal{O}_Y(V_i). \quad (5)$$

The  $G$ -invariance on the second ring comes from the invariance of  $g$ . The last isomorphism is one of the hypothesis conditions of  $p$  being good. Thus since  $U_i$  and  $V_i$  are affine, this defines a local morphism  $h_i : V_i \rightarrow U_i$ , and it suffices to verify that  $h|_{V_i} = h$ .  $\square$

**Proposition 2.** *Let  $p : X \rightarrow Y$  be a good quotient. Then*

1.  $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset \iff p(x_1) = p(x_2)$ .
2. For all  $y \in Y$ , there exists a unique closed orbit in  $p^{-1}(y)$ .

*Proof.* 1) Suppose  $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset$ . Since  $p$  is continuous and constant on orbits, it is constant on orbit closures and hence  $p(\overline{G \cdot x_1}) = p(\overline{G \cdot x_2})$  and in particular  $p(x_1) = p(x_2)$ . On the other hand if  $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} = \emptyset$ , then since  $p$  is good, the image of each orbit is disjoint.

2) Suppose there exist two closed orbits in  $p^{-1}(y)$ . If they are not equal, then again since  $p$  is continuous and constant on orbits they must be disjoint and hence their image is disjoint by goodness. However they must each contain  $y$  so this is a contradiction. (Existence was not given in class, maybe it is obvious?)  $\square$

Now, let us construct good quotients for affine varieties.

## 2.1 Affine GIT Quotient

Suppose we have the action of a group  $G$  on an affine variety  $X$ . Then  $X = \text{Spec}(\mathcal{O}_X)$  by definition, and we want our quotient to be good; so we want it to have functions  $\mathcal{O}_X^G$ . Thus the idea is to let our quotient be  $\text{Spec}(\mathcal{O}_X^G)$ . However  $\mathcal{O}_X^G$  is not finitely generated in general, so we will restrict ourselves to *reductive* groups.

**Definition 9.** Let  $G$  be a linear algebraic group.

- We say  $G$  is *reductive* if every smooth connected unipotent normal subgroup of  $G$  is trivial.
- We say  $G$  is *linearly reductive* if, for every representation  $G \rightarrow GL(V)$  and every non-zero fixed point  $v \in V$ , there exists a homogeneous  $G$ -invariant degree-1 polynomial  $f$  on  $V$  such that  $f(v) \neq 0$ .
- We say  $G$  is *geometrically reductive* if, for every representation  $G \rightarrow GL(V)$  and every non-zero fixed point  $v \in V$ , there exists a homogeneous  $G$ -invariant polynomial  $f$  on  $V$  such that  $f(v) \neq 0$ .

**Theorem 2.** *The three properties above are equivalent over  $\mathbb{C}$ .*

For example,  $GL(n, \mathbb{C})$ ,  $SL(n, \mathbb{C})$  and  $PGL(n, \mathbb{C})$  are reductive.

**Theorem 3** (Nagata's Theorem). *Let  $G$  be geometrically reductive acting on a finitely generated  $\mathbb{C}$ -algebra  $R$ . Then  $R^G$  is finitely generated.*

**Lemma 2.** *Let  $G$  be a geometrically reductive group acting on an affine variety  $X$ . Let  $Z_1$  and  $Z_2$  be two closed,  $G$ -invariant disjoint subsets of  $X$ . Then there exists a  $G$ -invariant function  $\psi \in \mathcal{O}_X^G$  such that  $\psi(Z_1) = 1$  and  $\psi(Z_2) = 0$ .*

*Proof.* Firstly

$$\langle 1 \rangle = I(\emptyset) = I(Z_1 \cap Z_2) = I(Z_1) + I(Z_2), \quad (6)$$

therefore  $1 = f_1 + f_2$  for some  $f_1, f_2$  with  $f_i(Z_j) = \delta_{ij}$ . Claim: (c.f. Hoskins) The subspace spanned by  $\{g \cdot f, \mid g \in G\} \subset \mathcal{O}_X$  is  $G$ -invariant and finite dimensional. Therefore we can pick a basis  $h_1, \dots, h_n$ , and because  $G$  acts on all of the  $h_i$ , we get an induced action of  $G$  on  $\mathbb{C}^n$  such that the map

$$\begin{aligned} \phi : X &\rightarrow \mathbb{C}^n \\ x &\rightarrow (h_i(x)) \end{aligned}$$

is  $G$ -equivariant, meaning  $\phi(g \cdot x) = g\phi(x)$ . Note then that  $\phi(Z_1) = 0$  and  $\phi(Z_2) \neq 0$ , and define  $v = \phi(Z_2) \in \mathbb{C}^n$ .

Since  $\phi$  is  $G$ -equivariant,  $v$  is fixed by the action of  $G$ . Then by the hypothesis of geometric reductivity, there exists some  $G$ -invariant homogenous  $f_0$  such that  $f_0(v) \neq 0$  and  $f_0(0) = 0$ . Finally, let

$$\psi = \frac{1}{f_0(v)} f_0 \circ \phi. \quad (7)$$

□

**Definition 10.** The *affine GIT quotient* of an affine variety  $X$  under reductive group  $G$ , denoted  $X // G$  is  $\text{Spec}(\mathcal{O}_X^G)$ .

**Theorem 4.** Let  $X$  be an affine variety and  $G$  a reductive group acting on  $X$ . Then  $p : X \rightarrow Y = \text{Spec}(\mathcal{O}_X^G)$  is a good quotient.

*Proof.* First we show  $p$  is  $G$ -invariant. Suppose for contradiction there exist  $x \in X$ ,  $g \in G$  such that  $p(x) \neq p(g \cdot x)$ . Since  $Y$  is affine, there exists an  $x \in \mathcal{O}_Y$  such that  $f(x) \neq f(g \cdot x)$ . However  $\mathcal{O}_Y = \mathcal{O}_X^G$  by definition, so  $f$  must be  $G$  invariant, giving a contradiction.

Next we show  $p$  is surjective. Let  $y \in Y$  and let  $\langle f_1, \dots, f_n \rangle$  be the ideal defining  $y$ . Let  $\mathfrak{m}$  be the maximal ideal containing  $\langle f_1, \dots, f_n \rangle$ . The point corresponding to  $\mathfrak{m}$  in  $X$  maps to  $y$  under  $p$ .

Now let  $U \subset Y$  be open. We want to show  $\mathcal{O}_Y(U) \cong \mathcal{O}_X(p^{-1}(U))^G$ ; it suffices to show this for  $U = D_f^Y$  for any  $f \in \mathcal{O}_Y$ .

$$\begin{aligned} \mathcal{O}_Y(D_f^Y) &= (\mathcal{O}_Y)_f \\ &= [\mathcal{O}_X(X)^G]_f \\ &= [\mathcal{O}_X(X)_f]^G \\ &= [\mathcal{O}_X(D_f^X)]^G \\ &= \mathcal{O}_X(p^{-1}(D_f^Y))^G \end{aligned}$$

Let  $Z_1, Z_2$  be  $G$ -invariant closed disjoint subsets. By the lemma, there exists  $\psi \in \mathcal{O}_X^G$  with  $\psi(Z_1) = 0$  and  $\psi(Z_2) = 1$ . Then  $\overline{p(Z_1)} \cap \overline{p(Z_2)} = \emptyset$ , because there is a  $G$ -invariant function which separates them. This turns out to be equivalent to the topological condition for goodness, that  $p(Z_1) \cap p(Z_2) = \emptyset$ . To prove this, it suffices to prove that if  $Z$  is closed and  $G$ -invariant then  $p(Z)$  is closed.

Suppose  $Z$  is closed and  $G$ -invariant. For contradiction, suppose there exists  $g \in \overline{p(Z)} - p(Z)$ . Then  $Z$  and  $p^{-1}(y)$  are both closed and  $G$ -invariant, so

$$\overline{p(Z)} \cap \overline{p(p^{-1}(y))} = \emptyset. \quad (8)$$

however  $y$  must be in this intersection, giving a contradiction.  $\square$

**Proposition 3.** *If the action of  $G$  is closed then  $X // G$  is a geometric quotient.*

This GIT construction separates orbits as much as possible while still being good.

Example: Consider  $\mathbb{C}^* \curvearrowright \mathbb{C}^2$  by  $t(x, y) = (tx, t^{-1}y)$ . Then the affine GIT quotient is given by  $\mathbb{C}^2 // \mathbb{C}^* = \text{Spec}(\mathbb{C}[x, y]^G)$ , and  $\mathbb{C}[x, y]^G = \mathbb{C}[xy] \cong \mathbb{C}[z]$ . Therefore  $\mathbb{C}^2 // \mathbb{C}^* = \mathbb{C}$ . The quotient map is  $(x, y) \rightarrow xy$  and the orbits come in three types:

1. The orbit of the origin is  $G \cdot (0, 0) = (0, 0)$ .
2. The orbits of  $(x, 0)$  and  $(0, y)$ , for  $x, y \neq 0$  are the  $x$  and  $y$  axes in  $\mathbb{C}^2$ .
3. The remaining orbits have the form  $G \cdot (x, y) = \{(z_1, z_2) \mid z_1 z_2 = \lambda\}$  for some  $\lambda$ , which are conics.

The GIT quotient sends the type 1 and 2 orbits to the same point,  $0 \in \mathbb{C}$ , so this is not a geometric quotient.

Example: Consider the additive complex group  $G_a = (\mathbb{C}, +)$ . Let it act on  $\mathbb{C}^4$  by embedding it into  $GL(4, \mathbb{C})$  by the map

$$s \rightarrow \begin{pmatrix} 1 & s & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (9)$$

$G_a$  is not reductive, but in this case the ring of invariants is still finitely generated. Note however that in our proof that the GIT quotient is surjective, we again used that  $G$  is reductive. In this case, we will not have a good quotient. If  $f$  is invariant, it must send

$$\begin{array}{ll} x_1 \rightarrow x_2 & x_2 \rightarrow sx_1 + x_2 \\ x_3 \rightarrow x_3 & x_4 \rightarrow sx_3 + x_4 \end{array}$$

So  $x_1, x_3$  are invariant, and  $x_1 x_4 - x_2 x_3$  is invariant. It turns out these three generate all the invariants, and so  $\mathcal{O}_{\mathbb{C}^4}^{G_a} = \mathbb{C}[x_1, x_3, x_1 x_4 - x_2 x_3]$ . Furthermore,  $\text{Spec}(\mathbb{C}[x_1, x_3, x_1 x_4 - x_2 x_3]) = \mathbb{C}^3$ . The quotient map is

$$(x_1, x_2, x_3, x_4) \rightarrow (x_1, x_3, x_1 x_4 - x_2 x_3), \quad (10)$$

which is not surjective as  $(0, 0, \lambda)$  is not in its image for  $\lambda \neq 0$ .

## 2.2 Projective GIT Quotient

Recall the example of  $\mathbb{C}^* \curvearrowright \mathbb{C}^2$  by scaling. We saw that  $(0,0)$  is in the closure of every orbit. Hence  $\mathbb{C}^2/\mathbb{C}^*$  is just one point. This is the same as saying that all the  $\mathbb{C}^*$  invariants in  $\mathbb{C}[x_1, x_2]$  are just the constants. Projective GIT will allow us to loosen the definition of  $G$ -invariance and get that  $\mathbb{C}^2 // \mathbb{C}^* = \mathbb{P}^1$ . Recall, if  $f$  is homogeneous of degree  $k$  then  $f(\lambda x, \lambda y) = \lambda^k f(x, y)$ , so  $f$  is projectively invariant.

Consider  $G \curvearrowright X$ , with  $X$  a projective variety. We can think of  $X$  being projective in two ways, either there is an embedding  $X \subset \mathbb{P}^k$ , or  $X$  is equipped with an ample line bundle  $L \rightarrow X$ . We will swap between these two pictures as convenient. The idea of projective GIT is to replace Spec with Proj. If  $R$  is the graded ring with  $X = \text{Proj}(R)$ , then we want to define  $X // G$  to be  $\text{Proj}(R^G)$ . To make sense of this in the  $(X, L)$  perspective, we need a  $G$ -action on the sections of the bundle  $L$ .

**Definition 11.** Let  $X$  be an algebraic variety and  $\pi : L \rightarrow X$  a line bundle. Suppose  $G \curvearrowright X$  via  $\sigma : G \times X \rightarrow X$ . Then a  $G$ -linearisation of  $L$  is a lift of  $\sigma$  to  $\bar{\sigma} : G \times L \rightarrow L$  which commutes with  $\sigma$  under the projection  $\pi$ ;  $\sigma(g, \pi(s)) = \pi(\bar{\sigma}(g, s))$  for all  $s \in \Gamma(X, L)$ , and such that the 0 section is invariant.

Remark: A linearisation defines a linear map between fibres of  $L$ ,  $\bar{\sigma} : L_x \rightarrow L_{g \cdot x}$ .

Example: Let  $X = \mathbb{C}^n$  and  $L = \mathbb{C} \times \mathbb{C}^n$  be the trivial bundle. Then a linearisation of  $L$  is a character in  $\chi(G)$ . If we fix  $\theta \in \chi(G)$ , then the linearisation of  $L$  is

$$g \cdot (a, v) = (\theta(g)a, g \cdot v). \quad (11)$$

This defines an action on the sections of  $L$ ; for  $U \subset X$  open and  $s \in \Gamma(U, L)$ , let  $(g \cdot s)(x) = \theta(g)s(g^{-1}x)$ .

In the other perspective, when  $X \subset \mathbb{P}^k$  explicitly, then a linearisation is a way to think of  $G \curvearrowright X$  via an embedding  $G \hookrightarrow GL(k+1, \mathbb{C}) \curvearrowright \mathbb{P}^k$ . In particular, if  $L$  is very ample, then  $X \hookrightarrow \mathbb{P}(\Gamma(X, L)^\vee) = \mathbb{P}^k$ . Then these two notions of linearisation agree. If  $X = \text{Proj}(R)$ , then a linearisation is an action  $G \curvearrowright R$  which preserves the grading.

In any case, we can now define projective GIT.

**Definition 12.** The *projective GIT quotient* of  $(X, L)$  by  $G$ , with respect to a given linearisation, is

$$X // G = \text{Proj} \left( \bigoplus_{r \geq 0} \Gamma(X, L^r)^G \right) \quad (12)$$

with the quotient map induced by the injection  $R^G \hookrightarrow R$ .

Example: We construct  $\mathbb{P}^n$  as a GIT quotient of  $X = \mathbb{C}^{n+1}$  by  $\mathbb{C}^*$  under scaling. A linearisation is given by a character of  $\mathbb{C}^*$ .

$$\begin{aligned} \chi(\mathbb{C}^*) &\cong \mathbb{Z} \\ (\lambda \rightarrow \lambda^a) &\leftrightarrow a \end{aligned}$$

Let  $a \in \mathbb{Z}$  be a character, then  $\mathbb{C}^*$  acts on the trivial line bundle  $L$  over  $\mathbb{C}^{n+1}$  by  $\lambda \cdot s(x) = \lambda^a s(x)$ . We have that  $\Gamma(\mathbb{C}^{n+1}, L^k) = \mathbb{C}[x_0, \dots, x_n]$ . If we want an element  $f$  to be  $\mathbb{C}^*$  invariant, we need

$$t \cdot f(x_0, \dots, x_n) = t^a f(t^{-1}x_0, \dots, t^{-1}x_n) = f(x_0, \dots, x_n). \quad (13)$$



If  $a = 1$  then equation 13 exactly means that  $f$  is a degree- $k$  homogenous polynomial. Then

$$X // G = \text{Proj}(\bigoplus_{k \geq 0} \text{degree } k \text{ homogenous polynomials}) = \mathbb{P}^n.$$

If  $a = 0$ , then equation 13 is only solved by constants. In this case,  $X // G$  has only one point and we recover the affine GIT quotient.

If  $a < 0$  then equation 13 has no solutions and the quotient is the empty set. Finally, the case with  $a > 1 \in \mathbb{N}$  is left as an exercise.

We can also think of  $\mathbb{C}^{n+1}$  as  $\text{Proj}(\mathbb{C}[x_0, \dots, x_n, y])$ , with the grading that lets  $x_i$  have degree 0 and  $y$  have degree 1. Then  $\mathbb{C}^* \curvearrowright \mathbb{C}^{n+1}$  by  $\lambda \cdot (x_0, \dots, x_n, y) = (\lambda x_0, \dots, \lambda x_n, \lambda^{-a} y)$  for  $a \in \chi(\mathbb{C}^*)$ . The quotient in each case works out exactly the same as above.

Let us try to get an intuitive sense for  $X // G$ . Suppose that  $L$  is very ample. Suppose further that some sections  $s_0, \dots, s_n$  generate the  $G$ -invariant sections in all degrees. Then the Proj construction is essentially doing

$$\begin{aligned} X &\rightarrow \mathbb{P}^n \\ x &\rightarrow [s_0(x) : \dots : s_n(x)]. \end{aligned}$$

This is defined where not all of the  $s_i(x)$  vanish; the image is  $X // G$ , which contains all the points  $x$  that have some non-vanishing  $G$ -invariant section.