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STAT F300 Statistics, Dr. Short
Homework 5, Due Friday, February 17, 23:59

1. Let X be a discrete random variable having the cdf, $F(X)$, as defined below.

$$F(X) = \begin{cases} 0 & \text{if } x \leq 0 \\ 0.2 & \text{if } 0 \leq x < 2 \\ 0.5 & \text{if } 2 \leq x < 4 \\ 0.9 & \text{if } 4 \leq x < 7 \\ 1 & \text{if } 7 \leq x \end{cases}$$

(a) Find the pmf of X.

$$p(X) = \begin{cases} 0.2 & \text{if } x = 0 \\ 0.3 & \text{if } x = 2 \\ 0.4 & \text{if } x = 4 \\ 0.1 & \text{if } x = 7 \\ 0 & \text{otherwise} \end{cases}$$

(b) Find $P(X < 3)$ and $P(X \leq 3)$

$$\begin{aligned} p(X < 3) = P(X \leq 3) &= F(3) \\ &= \boxed{0.5} \end{aligned}$$

(c) Find $P(X < 4)$ and $P(X \leq 4)$.

$$\begin{aligned} p(X < 4) &= \lim_{X \to 4} F(X) \\ &= \boxed{0.5} \\ p(X \leq 4) &= F(4) \\ &= \boxed{0.9} \end{aligned}$$

(d) Find $P(2 \leq X < 7)$

$$\begin{aligned} p(2 \leq X < 7) &= \lim_{X \to 7} F(X) - F(2) \\ &= 0.9 - 0.5 = \boxed{0.4} \end{aligned}$$

2. We say that X is a discrete uniform random variable from 1 to N if its pmf is:

$$p(x) = 1/N, \quad x = 1,2,3,\dots,N$$

(a) Show that $\mathbb{E}(X) = (N+1)/2$.

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{x=1}^{N} xp(x) \text{ where } p(x) = \frac{1}{N} \\
&= \frac{1}{N}\sum_{x=1} Nx \\
&= \frac{1}{N}(1+2+3+\cdots+N) \\
&= \frac{1}{N}\frac{N(N+1)}{2} = \frac{N+1}{2} \\
\therefore \quad \mathbb{E}(X) &= (N+1)/2
\end{aligned}$$

(b) Show that $\text{var}(X) = (N+1)(N-1)/12$.

$$\begin{aligned}
var(x) &= \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \\
\mathbb{E}(X^2) &= \sum_{x=1}^{N} x^2 p(x) \text{ where } p(x) = \frac{1}{N} \\
&= \frac{1}{N}\sum_{x=1} Nx^2 \\
&= \frac{1}{N}(1^2+2^2+3^2+\cdots+N^2) \\
&= \frac{1}{N}\frac{N(N+1)(2N+1)}{6} = \frac{(N+1)(2N+1)}{6} \\
\therefore \quad \mathbb{E}(X^2) &= (N+1)/2 \\
var(x) &= \frac{(N+1)(2N+1)}{6} - \left(\frac{N+1}{2}\right)^2 \\
&= \frac{(N+1)(2N+1)}{6} - \frac{(N+1)^2}{4} \\
&= \frac{2(N+1)(2N+1) - 3(N+1)^2}{12} \\
&= \frac{(4N^2+6N+2) - (3N^2+6N+3)}{12} \\
&= \frac{N^2-1}{12} = \frac{(N+1)(N-1)}{12} \therefore \qquad var(x) = \frac{(N+1)(N-1)}{12}
\end{aligned}$$

(c) Let $Y = X + 3$. Find the pmf of Y. Find $\mathbb{E}(Y)$ and $var(Y)$.

$$\begin{aligned}
Y &= X + 3 \\
p(Y = y) &= p(X + 3 = y) \\
p(Y = y - 3) &= 1/N, \quad \text{for} \quad y = 4, 5, \cdots, N + 3 \\
\mathbb{E}(Y) &= \mathbb{E}(X + 3) = \mathbb{E}(X) + \mathbb{E}(3) \\
&= \frac{N+1}{2} + 3 = \frac{N+7}{2} \\
var(Y) &= var(X + 3) = var(X) \text{ since constants don't affect variance}
\end{aligned}$$

3. The number of male mates that the females of a certain type of wasp have has a Poisson distribution with $\lambda = 3.1$.

(a) Find the probability that a female will have at most 4 mates.

$$\begin{aligned}
Pr(X = 0) &= e^{(-3.1)} * (3.1^0)/0! = 0.044 \\
Pr(X = 1) &= e^{(-3.1)} * (3.1^1)/1! = 0.136 \\
Pr(X = 2) &= e^{(-3.1)} * (3.1^2)/2! = 0.211 \\
Pr(X = 3) &= e^{(-3.1)} * (3.1^3)/3! = 0.206 \\
Pr(X = 4) &= e^{(-3.1)} * (3.1^4)/4! = 0.160 \\
Pr(X \leq 4) &= 0.044 + 0.136 + 0.211 + 0.206 + 0.160 = \boxed{0.757}
\end{aligned}$$

(b) Find the probability that a female will have at least 2 mates.

$$\begin{aligned}
Pr(X \leq 2) = 1 - Pr(2 \leq X) = 1 - (0.044 + 0.136) = 1 - 0.180 \\
= \boxed{0.820}
\end{aligned}$$

(c) Find the probability that a female will have no mates.

$$Pr(X = 0) = \boxed{0.045} \leftarrow \text{ Precomputed in part (a)}$$

(d) Find the probability that the number of mates a female has lies with 2 standard deviations of the mean.

$$\text{Mean} = \lambda = 3.1$$
$$\text{sd} = \sqrt{\lambda} = \sqrt{3.1} = 1.7607$$