

## Part I

1. Evaluate  $\int_1^\infty \frac{1}{1+x^2} dx$

$$\begin{aligned}\int_1^\infty \frac{1}{1+x^2} dx &= \lim_{a \rightarrow \infty} \int_1^a \frac{1}{1+x^2} dx \\ \lim_{a \rightarrow \infty} \int_1^a \frac{1}{1+x^2} dx &= \lim_{a \rightarrow \infty} \left[ \arctan(x) \right]_1^a = \lim_{a \rightarrow \infty} \arctan(a) - \arctan(1) \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \boxed{\frac{\pi}{4}}\end{aligned}$$

2. Suppose:

$$x = 4 - \ln(t)$$

$$y = 1 + \ln(7t)$$

$$1 \leq t \leq e$$

Compute the arc length.

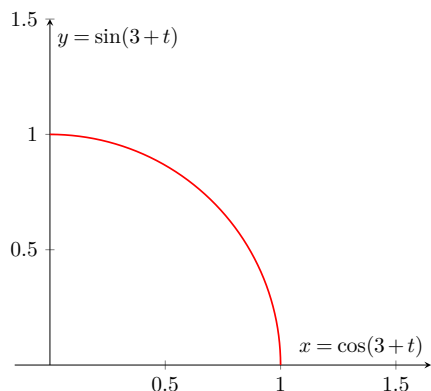
$$\begin{aligned}\text{Arc length: } \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ \frac{dx}{dt} = -\frac{1}{t} \\ \frac{dy}{dt} = \frac{7}{7t} = \frac{1}{t} \\ \int_1^e \sqrt{\left(-\frac{1}{t}\right)^2 + \left(\frac{1}{t}\right)^2} dt = \int_1^e \sqrt{\frac{1}{t^2} + \frac{1}{t^2}} dt = \int_1^e \sqrt{\frac{2}{t^2}} dt = \sqrt{2} \int_1^e \frac{1}{t} dt \\ = \sqrt{2} \left[ \ln(t) \right]_1^e = \sqrt{2} \left[ \ln(e) - \ln(1) \right] = \boxed{\sqrt{2}}\end{aligned}$$

3. Suppose:

$$x = \cos(3+t)$$

$$y = \sin(3+t)$$

What is the area of the region bounded by the graph and the positive x-axis and the positive y-axis?



To find where the curve strikes the axes:

$$x = \cos(3+t) = 0; \quad 3+t = \arccos(0)$$

$$3+t = \frac{\pi}{2}, \quad t = \frac{\pi}{2} - 3 \text{ when } x = 0$$

$$y = \sin(3+t) = 0; \quad 3+t = \arcsin(0)$$

$$3+t = 0, \quad t = -3 \text{ when } y = 0$$

$$\begin{aligned}A &= \int_{-3}^{\frac{\pi}{2}-3} \sin(3+t)(-\sin(3+t)) dt \\ &= - \int_{-3}^{\frac{\pi}{2}-3} \sin^2(3+t) dt\end{aligned}$$

$$\begin{aligned}
& - \int_{-3}^{\frac{\pi}{2}-3} \sin^2(3+t) dt = - \int_{-3}^{\frac{\pi}{2}-3} \frac{1 - \cos(6+2t)}{2} dt \\
& = - \left[ \frac{1}{2}t - \frac{\sin(6+2t)}{4} \right]_{-3}^{\frac{\pi}{2}-3} = - \left[ \left( \frac{\pi-12}{4} - \frac{\sin(6+\pi-3)}{4} \right) - \left( \frac{-3}{2} - \frac{\sin(6-6)}{2} \right) \right] \\
& = - \left( \frac{\pi-6}{4} - \frac{\sin(3+\pi)}{4} \right) \approx \boxed{0.679}
\end{aligned}$$

4. Use the root test to tell if the series converges:  $\sum \sqrt[n]{\frac{1+n^2}{1+3^n}}$

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{1+n^2}{1+3^n} \right|} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{1+n^2}}{\sqrt[n]{1+3^n}}$$

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Note that:  $\underbrace{\lim_{n \rightarrow \infty} \sqrt[n]{n^2}}_{\sqrt[n]{n^2} = \sqrt[n^2]{n^2} \rightarrow 1} \leq \lim_{n \rightarrow \infty} \sqrt[n]{1+n^2} \leq \underbrace{\lim_{n \rightarrow \infty} \sqrt[n]{2n^2}}_{\sqrt[n]{2n^2} = \sqrt[n]{2} \cdot \sqrt[n]{n^2} \rightarrow 1}$

$$1 \leq \lim_{n \rightarrow \infty} \sqrt[n]{1+n^2} \leq 1, \quad \therefore \lim_{n \rightarrow \infty} \sqrt[n]{1+n^2} = 1$$


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Note that:  $\underbrace{\lim_{n \rightarrow \infty} \sqrt[n]{3^n}}_{\sqrt[n]{3^n} = 3} \leq \lim_{n \rightarrow \infty} \sqrt[n]{1+3^n} \leq \underbrace{\lim_{n \rightarrow \infty} \sqrt[n]{2 \cdot 3^n}}_{\sqrt[n]{2 \cdot 3^n} = \sqrt[n]{2} \cdot \sqrt[n]{3^n} \rightarrow 1 \cdot 3 = 3}$

$$3 \leq \lim_{n \rightarrow \infty} \sqrt[n]{1+3^n} \leq 3, \quad \therefore \lim_{n \rightarrow \infty} \sqrt[n]{1+3^n} = 3$$


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$$\therefore \lim_{n \rightarrow \infty} \frac{\sqrt[n]{1+n^2}}{\sqrt[n]{1+3^n}} = \frac{1}{3}, \quad \frac{1}{3} < 1, \quad \boxed{\text{Converges}}$$

5. Express the following as a closed-form expression:  $\sum_{k=0}^{\infty} (-36)^k \frac{x^{10k}}{(2k)!}$

$$\begin{aligned}
& \text{Note that: } \cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \\
& \sum_{k=0}^{\infty} (-36)^k \frac{x^{10k}}{(2k)!} = \sum_{k=0}^{\infty} (-1)^k \frac{(6x^5)^{2k}}{(2k)!} \\
& \therefore \sum_{k=0}^{\infty} (-36)^k \frac{x^{10k}}{(2k)!} = \boxed{\cos(6x^5)}
\end{aligned}$$

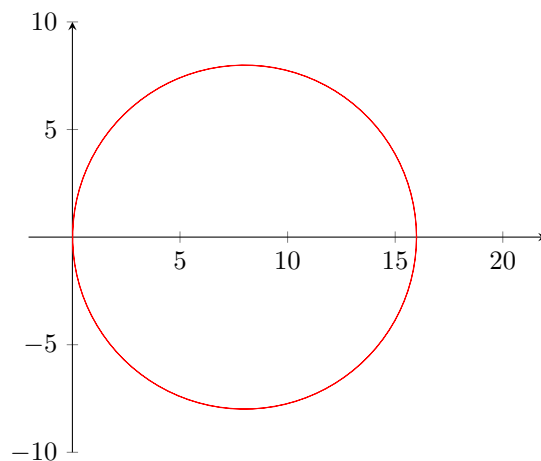
6. Find the Maclaurin series of  $e^{x-5}$

$$\begin{aligned}
& \text{Note that: } e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \text{ and } e^{x-5} = \frac{e^x}{e^5} \\
& \therefore e^{x-5} = \boxed{\sum_{k=0}^{\infty} \frac{x^k}{e^5 k!}}
\end{aligned}$$

7. Graph  $r = 16\cos(\theta)$ .

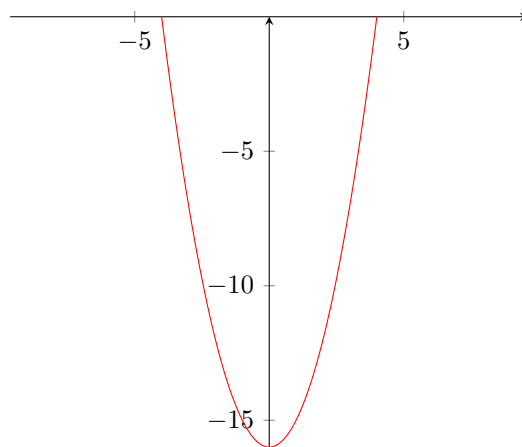
I get to cheat this one with technology, but note that:

- For  $r = \cos(\theta)$ , its path circular.
- The max distance the graph goes from  $y = 0$  is 16 ( $16 \cdot \cos(n \cdot \pi)$ ).
- It has a radius of 8.
- Its center is on  $x = 8$ .



8. A region is bounded by the x-axis and the line  $y = x^2 - 16$ . A solid object sits on this region. Cross sections perpendicular to the y-axis are squares. What is the volume of this object?

The intersection looks like this:



Note that the values here will be inverted:  $y = x^2 - 16 \rightarrow y = 16 - x^2$  to get positive results.

The area to integrate is:  $0 = 16 - x^2, x = \pm 4, [-4, 4]$ .

Note that for  $V$  volume:  $V = \int_a^b A dx$

Where  $A = 16 - x^2$

$$\int_{-4}^4 16 - x^2 dx = 2 \left[ 16x - \frac{x^3}{3} \right]_0^4 = 2 \left[ 16(4) - \frac{(4)^3}{3} \right] = 2 \left[ 64 - \frac{64}{3} \right] = 2 \cdot \frac{128}{3} = \boxed{\frac{256}{3}}$$

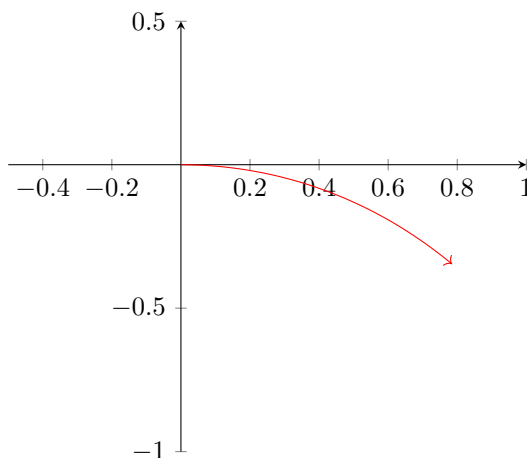
9. A force of two pounds stretches a spring one inch. How much work is required to pull the spring an entire foot? Express your answer in foot-pounds.

$$\text{Note that: } W = \int_a^b F dx$$

$$\text{And that: } F = 2$$

$$\int_0^{12} 2 dx = \left[ 2x \right]_0^{12} = \boxed{24 \text{ foot-pounds}}$$

10. Consider the graph of  $y = \ln(\cos(x))$  where  $0 \leq x \leq \frac{\pi}{4}$ . Compute the arc length.



$$\text{Note that arc length is: } \int_a^b \sqrt{1 + |f'(x)|^2} dx$$

$$f'(x) = -\frac{\sin(x)}{\cos(x)} = -\tan(x)$$

$$\begin{aligned} \int_0^{\pi/4} \sqrt{1 + \tan^2(x)} dx &= \int_0^{\pi/4} \sqrt{\sec^2(x)} dx = \int_0^{\pi/4} \sec(x) dx \\ &= \left[ \ln |\sec(x) + \tan(x)| \right]_0^{\pi/4} = \left[ \ln |\sec(\pi/4) + \tan(\pi/4)| - \ln |\sec(0) + \tan(0)| \right] \\ &= \left[ \ln |\sqrt{2} + 1| - \ln |1| \right] = \ln(\sqrt{2} + 1) \approx \boxed{0.881} \end{aligned}$$

## Part II

1. Find the second degree Taylor polynomial of  $\sqrt{x}$  expanded about four. Use this to approximate  $\sqrt{4.1}$ .

$$\text{Taylor series: } \sum_{k=0}^{\infty} \frac{f^k(a)(x-a)^k}{k!}$$

$$\text{Second degree of } \sqrt{x} \text{ at } 4: \boxed{1 + \frac{x-4}{8} + \frac{(x-4)^2}{48}}$$

$$\text{Approximation at } \sqrt{4.1}: 1 + \frac{0.1}{8} + \frac{0.1^2}{48}$$