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MATH F252, Dr. J. Gimbel
Homework Set III, Due Friday, January 27th, 2023

1. Evaluate $\int\limits_1^e \frac{(1+\ln(x))^2}{x}\mathrm{d}x$

$$\text{let} \quad u = \ln x + 1 \quad \mathrm{d}u = \frac{1}{x}\mathrm{d}x$$
$$u_{lower} = \ln(1) + 1 = 1 \quad u_{upper} = \ln(e) + 1 = 2$$
$$\rightarrow \int_1^2 u^2 \mathrm{d}u = \frac{2}{3}u^3\bigg|_1^2 = \frac{2(2^3)-2}{3} = \boxed{\frac{14}{3}}$$

2. Evaluate $\int \frac{\mathrm{d}x}{\sqrt{25-x^2}}$

$$\text{Using} \quad a^2 - u^2 \implies u = a\sin\theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$
$$\Rightarrow x = 5\sin\theta \quad \mathrm{d}x = 5\cos\theta\mathrm{d}\theta \quad \theta = \sin^{-1}\left(\frac{x}{5}\right)$$
$$\int \frac{5\cos\theta\mathrm{d}\theta}{\sqrt{25-25\sin^2\theta}} = \int \frac{\cancel{5}\cos\theta\mathrm{d}\theta}{\cancel{5}\sqrt{1-\sin^2\theta}} = \int \frac{\cos\theta\mathrm{d}\theta}{\sqrt{\cos^2\theta}}$$
$$= \int \frac{\cos\theta\mathrm{d}\theta}{\cos\theta} = \int \mathrm{d}\theta = \theta + C = \boxed{\sin^{-1}\left(\frac{x}{5}\right) + C}$$

3. Evaluate $\int \frac{\mathrm{d}x}{\sqrt{25+4x^2}}$

$$\text{Using} \quad a^2 + u^2 \implies u = a\tan\theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$
$$\Rightarrow 2x = 5\tan\theta \quad \mathrm{d}x = \frac{5}{2}\sec^2\theta\mathrm{d}\theta \quad \theta = \tan^{-1}\left(\frac{2x}{5}\right)$$
$$\int \frac{\frac{5}{2}\sec^2\theta\mathrm{d}\theta}{\sqrt{25+25\tan^2\theta}} = \frac{5}{2}\int \frac{\sec^2\theta\mathrm{d}\theta}{5\sqrt{1+\tan^2\theta}} = \frac{1}{2}\int \frac{\sec^2\theta\mathrm{d}\theta}{\sqrt{\sec^2\theta}}$$
$$= \frac{1}{2}\int \sec\theta\mathrm{d}\theta = \frac{1}{2}(\ln|\sec\theta + \tan\theta|) + C$$
$$= \frac{1}{2}\left[\ln\left|\sec\left(\tan^{-1}\left(\frac{2x}{5}\right)\right) + \tan\left(\tan^{-1}\left(\frac{2x}{5}\right)\right)\right|\right] + C = \boxed{\frac{1}{2}\left[\ln\left|\sec\left(\tan^{-1}\left(\frac{2x}{5}\right)\right) + \frac{2x}{5}\right|\right] + C}$$

4. Evaluate $\int \frac{e^x}{1+e^{2x}}$

$$\text{Using} \quad a^2 + u^2 \implies u = a\tan\theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$
$$\Rightarrow e^x = \tan\theta \quad e^x\mathrm{d}x = \sec^2\theta\mathrm{d}\theta \quad \theta = \tan^{-1}(e^x)$$
$$\int \frac{\sec^2\theta\mathrm{d}\theta}{1+\tan^2\theta} = \int \frac{\sec^2\theta\mathrm{d}\theta}{\sec^2\theta} = \int \theta\mathrm{d}\theta = \theta + C = \boxed{\tan^{-1}(e^x) + C}$$

4. Find the area between the curves: $y = x^2 + 1$ and $y = x + 1$

$$\begin{aligned} x + 1 &= x^2 + 1 \\ x - x^2 &= 0 \\ x(1-x) &= 0 \\ x &= (0, 1) \end{aligned}$$

$$\int_0^1 (x - x^2)\mathrm{d}x$$
$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\bigg|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{3-1}{6} = \boxed{\frac{1}{6}}$$