Day 3 - 1/19/2024

Dot Product

Suppose we have the vectors $\vec{a} = \left\langle a_x, a_y, a_z \right\rangle$ and $\vec{b} = \left\langle b_x, b_y, b_z \right\rangle$.

Define $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$. Dot Products produce scalar values.

Example:

$$\vec{a} = \langle 1, 2, 0 \rangle \quad \vec{b} = \langle 7, 5, 7 \rangle$$

$$\vec{a} \cdot \vec{b} = 1(7) + 2(5) + 0(7) = 7 + 10 + 0 = \underline{17}$$

Properties:

- Commutivity/Symmetry: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- Linearity: $\vec{a}, \vec{b}, \vec{b}$ - Vectors, r, s - Scalars

$$\left(\vec{a} + \vec{b}\right) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$$

$$\vec{a} \cdot \left(\vec{b} + \vec{c}\right) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

  Possible if $r = 0$

$$(s\vec{a}) \cdot \vec{b} = s\left(\vec{a} \cdot \vec{b}\right) = \vec{a}\left(s\vec{b}\right)$$

$$\left(s\vec{a} + r\vec{b}\right) \cdot \vec{c} = s(\vec{a} \cdot \vec{c}) + r\left(\vec{b} \cdot \vec{c}\right)$$

- Length: $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$

$$\vec{a} = \langle 1, 4, 5 \rangle; \vec{b} = \langle 3, 0, 2 \rangle, \vec{c} = \langle 1, 7, 7 \rangle$$

$$\left(\vec{a} \cdot \vec{b}\right)\vec{c} = (3 + 0 + 10)\vec{c} = 13\vec{c} = \langle 13, 91, 91 \rangle$$

Angles between Vectors

Suppose we have two vectors $\vec{a}$ and $\vec{u}$ and we want to find the angle between the two.

1. Find $\vec{v} - \vec{u}$.
2. Use the *Law of cosines.*

$$\|v - u\|^2 = \|u\|^2 + \|v\|^2 - 2(\|u\| \cdot \|v\|) \cos \alpha$$

$$(v - u) \cdot (v - u) = v \cdot (v - u) - u(v - u)$$

$$v \cdot v - v \cdot u - u \cdot v + u \cdot u$$

$$\text{since } u \cdot v = v \cdot u : \|v\|^2 - 2u \cdot v + \|u\|^2$$

$$\|u\|^2 + \|v\|^2 - 2 \ \|u\| \|v\| \cos \alpha = \|v\|^2 - 2(u \cdot v) + \|u\|^2$$

$$u \cdot v = \|u\| \|v\| \cos \alpha$$

$$\cos \alpha = \frac{u \cdot v}{\|u\| \|v\|}$$

$$\alpha = \arccos\left(\frac{u \cdot v}{\|u\| \|v\|}\right)$$

Properties:

$$u, v \text{ - Vectors;}$$

$$u \cdot v > 0 \Rightarrow \alpha < 90^{\circ}$$

$$u \cdot v = 0 \Rightarrow \alpha = 90^{\circ} \left(\text{given } u \text{ and } v \text{ are not } \vec{0}\right); u \perp v$$

$$\text{if } \vec{a} \parallel \vec{b} \Rightarrow \vec{a} = k\vec{b} \text{ where } k \in \mathbb{R}$$

$$a \cdot b = kb \cdot b = k \ \|b\|^2$$

$$\|a\| \cdot \|b\| = |k| \ \|b\|^2$$

Orthogonal Projections

Define a vector $\vec{d} \perp \vec{a}$ where $\vec{d}$ is the project of $\vec{b}$ onto $\vec{a}$ and $c \cdot \vec{d} = \vec{a}$.

$$\vec{b} = c \cdot \vec{a} + \vec{d}$$

$$\vec{a} \cdot \vec{b} = c\vec{a} \cdot \vec{a} + \vec{d} \cdot \vec{a}$$

$$\vec{a} \cdot \vec{b} = c \; \|a\|^2$$

$$\mathrm{proj}_{\vec{a}}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{\|a\|^2}$$