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Final Review

Part I

1. Evaluate $\int_1^\infty \frac{1}{1+x^2}dx$

$$\int_1^\infty \frac{1}{1+x^2}dx = \lim_{a\to\infty}\int_1^a \frac{1}{1+x^2}dx$$
$$\lim_{a\to\infty}\int_1^a \frac{1}{1+x^2}dx = \lim_{a\to\infty}\Big[\arctan(x)\Big]_1^a = \lim_{a\to\infty}\arctan(a)-\arctan(1)$$
$$= \frac{\pi}{2}-\frac{\pi}{4} = \boxed{\frac{\pi}{4}}$$

2. Suppose:

$x = 4-\ln(t)$

$y = 1+\ln(7t)$

$1 \le t \le e$

Compute the arc length.

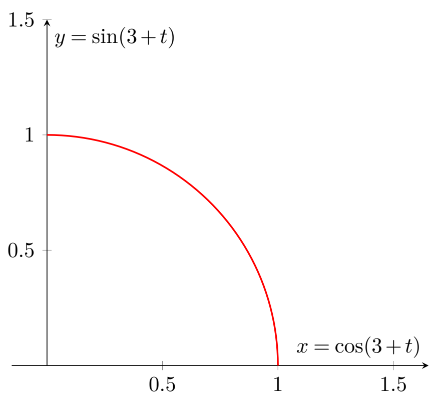
$$\text{Arc length: } \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$
$$\frac{dx}{dt} = -\frac{1}{t}$$
$$\frac{dy}{dt} = \frac{7}{7t} = \frac{1}{t}$$
$$\int_1^e \sqrt{\left(-\frac{1}{t}\right)^2+\left(\frac{1}{t}\right)^2}\,dt = \int_1^e \sqrt{\frac{1}{t^2}+\frac{1}{t^2}}\,dt = \int_1^e \sqrt{\frac{2}{t^2}}\,dt = \sqrt{2}\int_1^e \frac{1}{t}dt$$
$$= \sqrt{2}\Big[\ln(t)\Big]_1^e = \sqrt{2}\Big[\ln(e)-\ln(1)\Big] = \boxed{\sqrt{2}}$$

3. Suppose:

$x = \cos(3+t)$

$y = \sin(3+t)$

What is the area of the region bounded by the graph and the positive x-axis and the positive y-axis?



To find where the curve strikes the axes:
$$x = \cos(3+t) = 0; \quad 3+t = \arccos(0)$$
$$3+t = \frac{\pi}{2}, \quad t = \frac{\pi}{2}-3 \text{ when } x = 0$$
$$y = \sin(3+t) = 0; \quad 3+t = \arcsin(0)$$
$$3+t = 0, \quad t = -3 \text{ when } y = 0$$
$$A = \int_{-3}^{\frac{\pi}{2}-3} \sin(3+t)(-\sin(3+t))dt$$
$$= -\int_{-3}^{\frac{\pi}{2}-3} \sin^2(3+t)dt$$

$$-\int_{-3}^{\frac{\pi}{2}-3} \sin^2(3+t)dt = -\int_{-3}^{\frac{\pi}{2}-3} \frac{1-\cos(6+2t)}{2}dt$$
$$= -\left[\frac{1}{2}t - \frac{\sin(6+2t)}{4}\right]_{-3}^{\frac{\pi}{2}-3} = -\left[\left(\frac{\pi-12}{4} - \frac{\sin(6+\pi-3)}{4}\right) - \left(\frac{-3}{2} - \frac{\sin(6-6)}{2}\right)\right]$$
$$= -\left(\frac{\pi-6}{4} - \frac{\sin(3+\pi)}{4}\right) \approx \boxed{0.679}$$

4. Use the root test to tell if the series converges: $\sum\sqrt{\frac{1+n^2}{1+3^n}}$

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{1+n^2}{1+3^n}\right|} = \lim_{n\to\infty} \frac{\sqrt[n]{1+n^2}}{\sqrt[n]{1+3^n}}$$

$$\text{Note that: } \underbrace{\lim_{n\to\infty} \sqrt[n]{n^2}}_{\sqrt[n]{n^2}=\sqrt[n]{n}^2\to 1} \leq \lim_{n\to\infty} \sqrt[n]{1+n^2} \leq \underbrace{\lim_{n\to\infty} \sqrt[n]{2n^2}}_{\sqrt[n]{2n^2}=\sqrt[n]{2}\cdot\sqrt[n]{n}^2\to 1}$$
$$1 \leq \lim_{n\to\infty} \sqrt[n]{1+n^2} \leq 1, \quad \therefore \lim_{n\to\infty} \sqrt[n]{1+n^2} = 1$$

$$\text{Note that: } \underbrace{\lim_{n\to\infty} \sqrt[n]{3^n}}_{\sqrt[n]{3^n}=3} \leq \lim_{n\to\infty} \sqrt[n]{1+3^n} \leq \underbrace{\lim_{n\to\infty} \sqrt[n]{2\cdot 3^n}}_{\sqrt[n]{2\cdot 3^n}=\sqrt[n]{2}\cdot\sqrt[n]{3}^n\to 1\cdot 3=3}$$
$$3 \leq \lim_{n\to\infty} \sqrt[n]{1+3^n} \leq 3, \quad \therefore \lim_{n\to\infty} \sqrt[n]{1+3^n} = 3$$

$$\therefore \lim_{n\to\infty} \frac{\sqrt[n]{1+n^2}}{\sqrt[n]{1+3^n}} = \frac{1}{3}, \quad \frac{1}{3} < 1, \quad \boxed{\text{Converges}}$$

5. Express the following as a closed-form expression: $\sum\limits_{k=0}^{\infty}(-36)^k\frac{x^{10k}}{(2k)!}$.

$$\text{Note that: } \cos(x) = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}$$
$$\sum_{k=0}^{\infty}(-36)^k\frac{x^{10k}}{(2k)!} = \sum_{k=0}^{\infty}(-1)^k\frac{(6x^5)^{2k}}{(2k)!}$$
$$\therefore \sum_{k=0}^{\infty}(-36)^k\frac{x^{10k}}{(2k)!} = \boxed{\cos(6x^5)}$$

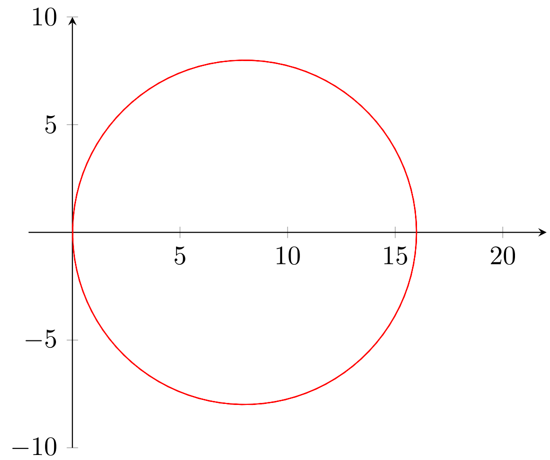
6. Find the Maclaurin series of e^{x-5}

$$\text{Note that: } e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!} \text{ and } e^{x-5} = \frac{e^x}{e^5}$$
$$\therefore \quad e^{x-5} = \boxed{\sum_{k=0}^{\infty}\frac{x^k}{e^5 k!}}$$

7. Graph $r = 16\cos(\theta)$.

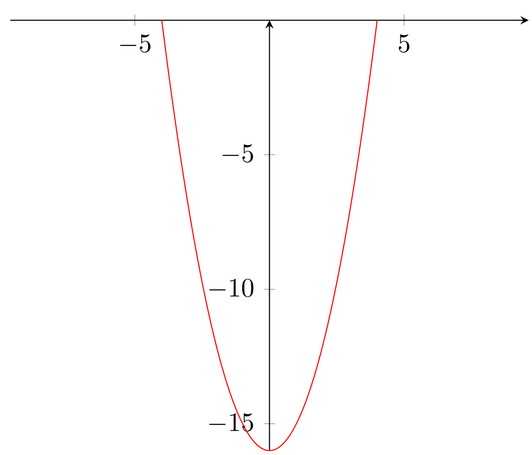
I get to cheat this one with technology, but note that:

- For $r = \cos(\theta)$, its path circular.
- The max distance the graph goes from $y = 0$ is 16 $(16 \cdot \cos(n \cdot \pi))$.
- It has a radius of 8.
- Its center is on $x = 8$.



8. A region is bounded by the x-axis and the line $y = x^2 - 16$. A solid object sits on this region. Cross sections perpendicular to the y-axis are squares. What is the volume of this object?

The intersection looks like this:



Note that the values here will be inverted: $y = x^2 - 16 \rightarrow y = 16 - x^2$ to get positive results.

The area to integrate is: $0 = 16 - x^2, x = \pm 4, [-4, 4]$.

$$\text{Note that for } V \text{ volume: } V = \int_a^b A dx$$

$$\text{Where } A = (16 - x^2)^2 = x^4 - 32x^2 + 256$$

$$\int_{-4}^{4} x^4 - 32x^2 + 256 dx = 2\left[\frac{x^5}{5} - \frac{32x^3}{3} + 256x\right]_0^4 = 2\left[\frac{x^5}{5} - \frac{32x^3}{3} + 256x\right]$$

$$= 2\left[\frac{(4)^5}{5} - \frac{32(4)^3}{3} + 256(4)\right] = \boxed{1092.2\overline{66}}$$

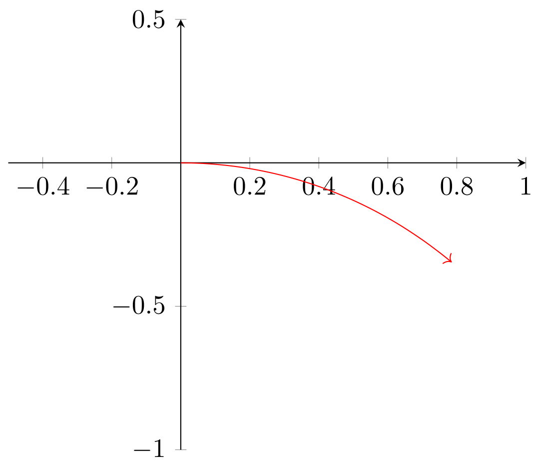
9. A force of two pounds stretches a spring one inch. How much work is required to pull the spring an entire foot? Express your answer in foot-pounds.

$$\text{Note that: } W = \int_a^b F dx$$

$$\text{And that: } F = 2$$

$$\int_0^{12} 2dx = \Big[2x\Big]_0^{12} = \boxed{24 \text{ foot-pounds}}$$

10. Consider the graph of $y = \ln(\cos(x))$ where $0 \leq x \leq \frac{\pi}{4}$. Compute the arc length.



$$\text{Note that arc length is: } \int_a^b \sqrt{1 + |f'(x)|^2} dx$$

$$f'(x) = -\frac{\sin(x)}{\cos(x)} = -\tan(x)$$

$$\int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2(x)} dx = \int_0^{\frac{\pi}{4}} \sqrt{\sec^2(x)} dx = \int_0^{\frac{\pi}{4}} \sec(x) dx$$

$$= \Big[\ln|\sec(x) + \tan(x)|\Big]_0^{\frac{\pi}{4}} = \Big[\ln|\sec(\pi/4) + \tan(\pi/4)| - \ln|\sec(0) + \tan(0)|\Big]$$

$$= \Big[\ln|\sqrt{(}2) + 1| - \ln|1|\Big] = \ln\left(\sqrt{2} + 1\right) \approx \boxed{0.881}$$

Part II

1. Find the second degree Taylor polynomial of $\sqrt{x}$ expanded about four. Use this to approximate $\sqrt{4.1}$.

$$\text{Taylor series: } \sum_{k=0}^{\infty} \frac{f^k(a)(x-a)^k}{k!}$$

$$\text{Second degree of } \sqrt{x} \text{ at 4: } \boxed{2+\frac{x-4}{8}+\frac{(x-4)^2}{48}}$$

$$\text{Approximation at } \sqrt{4.1}\text{: } 2+\frac{0.1}{8}+\frac{0.1^2}{48} \approx \boxed{2.0127}$$

2. Consider the expression $r^2 = \sin(2\theta)$. Convert this from polar to cartesian coordinates. Also, convert $x^2+y^2 = 2x$ from cartesian to polar coordinates.

$$r^2 = \sin(2\theta) \Rightarrow x^2+y^2 = \sin(2\theta)$$

$$x^2+y^2 = 2\sin(\theta)\cos(\theta) \Rightarrow \boxed{x^2+y^2 = 2xy}$$

$$x^2+y^2 = 2x \Rightarrow r^2 = 2r\cos(\theta) \Rightarrow \boxed{r = 2\cos(\theta)}$$

3. Evaluate $\int_0^\pi \sin(x)\cos^5(x)dx$

$$\int_0^\pi \sin(x)\cos^5(x)dx = \int_0^\pi \sin(x)(1-\sin^2(x))\cos^3(x)dx = \int_0^\pi \sin(x)-\sin^3(x)\cos^3(x)dx$$

$$\int_0^\pi \sin(x) - \int_0^\pi \sin^3(x)\cos^3(x)dx$$

$$\int_0^\pi \sin(x) = 0$$

$$-\int_0^\pi \sin^3(x)\cos^3(x)dx : \text{ Let } u = \cos(x) \quad du = -\sin(x)dx$$

$$\int_1^{-1} u^3 dx = -\int_{-1}^{1} u^3 dx = -\left[\frac{u^4}{4}\right]_{-1}^{1} = -\left[\frac{1}{4}-\frac{1}{4}\right] = \boxed{0}$$

4. Find the Maclaurin series expansion of $x\sin(3x^2)$.

$$\text{Note that: } \sin(x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\therefore\ x\sin(3x^2) = x\sum_{k=0}^{\infty}(-1)^k\frac{(3x^2)^{2k+1}}{(2k+1)!} = \boxed{\sum_{k=0}^{\infty} 9(-27)^k \frac{x^{4k+5}}{(2k+1)!}}$$

5. Find the area enclosed by the graph $r = 2 + \cos(\theta)$.

$$A = \frac{1}{2}\int_0^{2\pi} f(\theta)^2 dx$$

$$\frac{1}{2}\int_0^{2\pi} (2+\cos(\theta))^2 d\theta = \frac{1}{2}\int_0^{2\pi} \left[\cos^2(\theta) + 4\cos(\theta) + 4\right] d\theta = \frac{1}{2}\int_0^{2\pi} \left[\frac{1+\cos(2\theta)}{2} + 4\cos(\theta) + 4\right] d\theta$$

$$= \frac{1}{4}\int_0^{2\pi} [\cos(2\theta) + 8\cos(\theta) + 9]\, d\theta = \left[9\theta - \frac{\sin(2\theta)}{2} - 8\sin(\theta)\right]_0^{2\pi} = 9(2\pi) = \boxed{18\pi}$$

6. Determine if the following converges or diverges:

(a) $\sum \frac{k^5+1}{k^6+1}$

Using limit comparison test:

Note that: $\sum \frac{1}{k}$ Diverges

$$\lim_{k\to\infty} \frac{\frac{k^5+1}{k^6+1}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k^6+k}{k^6+1} \overset{(H)}{=} \lim_{k\to\infty} \frac{k^5+1}{k^5} \overset{(H)}{=} \lim_{k\to\infty} \frac{k^4}{k^4} = \lim_{k\to\infty} 1 = 1$$

$$1 \neq 0, \quad \therefore \sum \frac{k^5+1}{k^6+1} \boxed{\text{Diverges}}$$

(b) $\sum \frac{k!}{(2k)!}$

Using ratio test:

$$\lim_{k\to\infty} \left| \frac{\frac{(k+1)!}{(2k+1)!}}{\frac{k!}{(2k)!}} \right| = \lim_{k\to\infty} \frac{(k+1)!}{(k)!}\frac{(2k)!}{(2k+1)!} = \lim_{k\to\infty} \frac{k+1}{2k+1} \overset{(H)}{=} \lim_{k\to\infty} \frac{1}{2} = \frac{1}{2}$$

$$\frac{1}{2} < 1, \quad \therefore \sum \frac{k!}{(2k)!} \boxed{\text{Converges}}$$