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Math 307 - Discrete Mathematics
Assignment 3

Homework Problems

Section 1.7

Problem 4

Consider the following functions from $\mathbb{N}$ into $\mathbb{N}$: $1_\mathbb{N}(n) = n$, $f(n) = 3n$, $g(n) = n + (-1)^n$, $h(n) = \min[n, 100]$, $k(n) = \max[0, n - 5]$.

a) Which of these functions are one-to-one?

The functions $1_\mathbb{N}$, f, and g.

In the case of $1_\mathbb{N}$, the function simply maps any n to itself, and thus cannot map it to another number.

In the case of f, the logic is the same as $1_\mathbb{N}$, just that the values being mapped to are tripled.

In the case of g, we get an odd pattern such that $\{g(0), g(1), g(2), g(3), g(4), g(5), ..\} = \{1, 0, 3, 2, 5, 4, ..\}$ We can see a pattern here that we swap every pair of numbers such that $\{0, 1, 2, 3, 4, 5\} = \{1, 0, 3, 2, 5, 4\}$. With this mapping, g is therefore one-to-one.

b) Which of these functions map $\mathbb{N}$ onto $\mathbb{N}$?

The functions $1_\mathbb{N}$, g, and k map $\mathbb{N}$ onto $\mathbb{N}$.

The case for $1_\mathbb{N}$ is trivial as explained in my answer for `4a`, $1_\mathbb{N}$ maps any given n to itself, and thus is onto.

In the case of g, since g merely swaps neighboring pairs, it also is onto as it maps every possible output with an input n.

In the case of k, this function would provide an issue if $\mathbb{N}$ did not include 0. However, the max function in this case maps all inputs $n \leq 5$ to 0, and all inputs $n > 5$ to $n - 5$. This still maps them to every possible value in $\mathbb{N}$, making it onto.

Problem 5cde

Here are two "shift functions" mapping $\mathbb{N}$ into $\mathbb{N}$: $f(n) = n + 1$ and $g(n) = \max[0, n - 1]$ for $n \in \mathbb{N}$.

c) Show that f is one-to-one but does not map $\mathbb{N}$ onto $\mathbb{N}$.

f is one-to-one because f maps all n to $n + 1$, thus never mapping any two n to the same output.

f is not onto as all $f(n) \geq 1$. $0 \in \mathbb{N}$ and not the range of f, thus making it not onto.

d) Show that g maps $\mathbb{N}$ onto $\mathbb{N}$ but is not one-to-one.

g maps $\mathbb{N}$ onto $\mathbb{N}$ because the range of g is $\{0, 1, 2, 3, 4, ..\}$. We can see that by computing g:

$$\begin{aligned}
g(0) &= \max[0, 0 - 1] = \max[0, -1] = 0 \\
g(1) &= \max[0, 1 - 1] = \max[0, 0] = 0 \\
g(2) &= \max[0, 2 - 1] = \max[0, 1] = 1 \\
g(3) &= \max[0, 3 - 1] = \max[0, 2] = 2 \\
&\vdots
\end{aligned}$$

g merely maps the first two elements of $\mathbb{N}$ to 0 and then to $n - 1$. This also makes it not one-to-one as two input elements, (0 and 1) both map to 0.

e) Show that $g \circ f(n) = n$ for all n, but that $f \circ g(n) = n$ does not hold for all n.

$$g \circ f(n) = g(n + 1)$$
$$= \max[0, n + 1 - 1] = \max[0, n];$$
$$\max[0, n] = n \text{ as } \min(\mathbb{N}) = 0.$$

$$f \circ g(n) = g(\max[0, n - 1])$$
$$= \max[0, n - 1] + 1;$$
$$g(f(0)) = \max[0, 0 - 1] + 1 = 1.0 \neq 1$$
$$\text{therefore } f \circ g(n) \neq n$$

Problem 6bc

Let $\Sigma = \{a, b, c\}$ and let Σ^* be the set of all words w using letters from Σ; see Example 2(b). Define $L(w) = \text{length}(w)$ for all $w \in \Sigma^*$.

b) Is L a one-to-one function? Explain.

No. L maps $\Sigma^* \to \mathbb{N}$ by the length of any word definable with Σ. We can see though that the words "a" and "b" both have length 1, or $L(\text{"a"}) = 1, L(\text{"b"}) = 1$, which makes this function not one-to-one.

c) The function L maps Σ^* into $\mathbb{N}$. Does L map Σ^* onto $\mathbb{N}$? Explain.

Yes, L does map Σ^* onto $\mathbb{N}$, as any given word in Σ^* can be of any possible length ≥ 0. Any word can have any number of letters, but also $\varepsilon \in \Sigma^*$, satisfying the needed $L(n) = 0$.

Problem 11

Here are some functions from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$: $\text{SUM}(m, n) = m + n$, $\text{PROD}(m, n) = m * n$, $\text{MAX}(m, n) = \max[m, n]$, $\text{MIN}(m, n) = \min[m, n]$; here $*$ denotes multiplication of integers.

a) Which of these functions map $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$?

All functions here map $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$.

$$\text{For SUM} : \text{SUM}(0, n) = n$$
$$\text{For PROD} : \text{PROD}(1, n) = n$$
$$\text{For MIN} : \text{MIN}(n, n) = n$$
$$\text{For MAX} : \text{MAX}(0, n) = n$$

b) Show that none of these functions are one-to-one.

$$\text{For SUM} : \text{SUM}(0, 1) = 1 \text{ and } \text{SUM}(1, 0) = 1$$
$$\text{For PROD} : \text{PROD}(2, 3) = 6 \text{ and } \text{PROD}(3, 2) = 6$$
$$\text{For MIN} : \text{MIN}(0, 1) = 0 \text{ and } \text{MIN}(1, 0) = 0$$
$$\text{For MAX} : \text{MAX}(0, 1) = 1 \text{ and } \text{MAX}(1, 0) = 1$$

c) For each of these functions F, how big is the set $F^{\leftarrow}(4)$?

For SUM: Find all pairs where $m + n = 4$:

$$|\{(0, 4), (1, 3), (2, 2), (3, 1), (4, 0)\}| = 5$$

For PROD: Find all pairs where $m \times n = 4$

$$|\{(1, 4), (2, 2), (4, 1)\}| = 3$$

For MIN: Find all pairs where $\min[m, n] = 4$

$$|\{(4, 4), (4, 5), (4, 6), .., (5, 4), (6, 4), ..\}| = \infty$$

For MAX: Find all pairs where $\max[m, n] = 4$

$$|\{(0, 4), (1, 4), (2, 4), (3, 4), (4, 4), (4, 0), (4, 1), (4, 2), (4, 3)\}| = 9$$