Day 4 - 1/22/2024

Determinants

Suppose we have the vectors $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{u} = \langle u_1, u_2 \rangle$.

$$\binom{u}{v} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = u_1 v_2 - v_1 u_2$$

$$\vec{v} = \langle a, 0 \rangle; u = \langle 0, b \rangle$$

$$\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

Suppose $\vec{u} \parallel \vec{v} \Leftrightarrow \vec{u} = k\vec{v}$ for some $k \in \mathbb{R}$, then

$$\binom{\vec{u}}{\vec{v}} = \begin{vmatrix} kv_1 & kv_2 \\ v_1 & v_2 \end{vmatrix} = kv_1 v_2 - v_1 k v_2 = 0$$

$$\binom{u}{v + w} = \binom{u}{v} + \binom{u}{w}$$

$$\begin{aligned} \begin{vmatrix} u_1 & u_2 \\ v_1 + w_1 & v_2 + w_2 \end{vmatrix} &= u_1(v_2 + w_2) - u_2(v_1 + w_1) \\ &= (u_1 v_2 - v_1 u_2) + (u_1 w_2 - w_1 u_2) \\ &= \binom{u}{v} + \binom{u}{w} \end{aligned}$$

$$\binom{ku}{v} = k\binom{u}{v}$$

This is terrible.

Determinants in RR^3

$$a, b, c; \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Example:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & 1 & -1 \\ 2 & 0 & 3 \end{vmatrix} = 1 \begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} - 2 \begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix} + 0 \begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix}$$

$$= 3 \cdot 0 - 2 \cdot 11 + 0 = 3 - 22 = \underline{-19}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = - \begin{pmatrix} a \\ c \\ d \end{pmatrix}$$

Vector Cross Product

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

$$\left(\vec{a} \times \vec{b}\right) \cdot a = (a_2 b_3) a_1 + (a_3 b_1 - a_1 b_3) a_2 + (a_1 b_2 - a_2 b_1) a_3 = 0$$

$$a \perp b$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Properties:

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$k\left(\vec{a}\vec{b}\right) = k\vec{a} \times \vec{b} = \vec{a} \times \left(k\vec{b}\right)$$

$$\vec{a} \times 0 = 0$$

$$\vec{a} \times \vec{a} = 0$$

Scalar Triple Product:

$$u \cdot (v \times w) = (u \times v) \cdot w$$

$$\vec{i} \times \vec{j} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0\vec{i} + 0\vec{j} + 1\vec{k}$$

$$\vec{i} \times \vec{j} = \vec{k} \quad \vec{j} \times \vec{i} = -\vec{k}$$

$$\vec{j} \times \vec{k} = \vec{i} \quad \vec{k} \times \vec{j} = -\vec{i}$$

$$\vec{k} \times \vec{i} = \vec{j} \quad \vec{i} \times \vec{k} = -\vec{j}$$

Length of Cross Products

$$\|a \times b\|$$

$$\vec{a} = k\vec{i}\vec{b} = \|b\| \cos\theta\vec{i} + \|b\| \sin\theta\vec{j}$$

$$a \times b = a \times \left(\|b\| \cos\theta\vec{i} + \|b\| \sin\theta\vec{j}\right)$$

$$\|a\|\vec{i}\left(\|b\| \cos\theta\vec{i} + \|b\| \sin(\theta)\left(\vec{i} \times \vec{k}\right)\right)$$

$$\|a\| \ \|b\| \cos\theta(i \times i) + \|b\| \ \|a\| \sin\theta k = \|a\| \ \|b\| \sin(\theta)k$$

Find the area of the triangle: $P(1, 2, 0), Q(0, 1, 0), R(0, 0, 0)$

$$A = \frac{a \cdot b \cdot \sin\theta}{2}$$

$$A = \frac{\|RQ\| \cdot \|RP\| \cdot \sin(\angle PRQ)}{2}$$

$$= \frac{\|\overrightarrow{RQ} \times \overrightarrow{RP}\|}{2}$$

$$\begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{vmatrix} = i \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} - j \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} + k \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= i(0 - 0) - j(0 - 0) + k(0 - 1) = -k$$

$$\frac{\| - k \|}{2} = \underline{\frac{1}{2}}$$