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MATH F252, Dr. J. Gimbel, University of Alaska Fairbanks
Final Review

Part I

1. Evaluate $\int_1^\infty \frac{1}{1+x^2}dx$

$$\int_1^\infty \frac{1}{1+x^2}dx = \lim_{a\to\infty}\int_1^a \frac{1}{1+x^2}dx$$
$$\lim_{a\to\infty}\int_1^a \frac{1}{1+x^2}dx = \lim_{a\to\infty}\Big[\arctan(x)\Big]_1^a = \lim_{a\to\infty}\arctan(a)-\arctan(1)$$
$$= \frac{\pi}{2}-\frac{\pi}{4} = \boxed{\frac{\pi}{4}}$$

2. Suppose:

$x = 4-\ln(t)$

$y = 1+\ln(7t)$

$1 \le t \le e$

Compute the arc length.

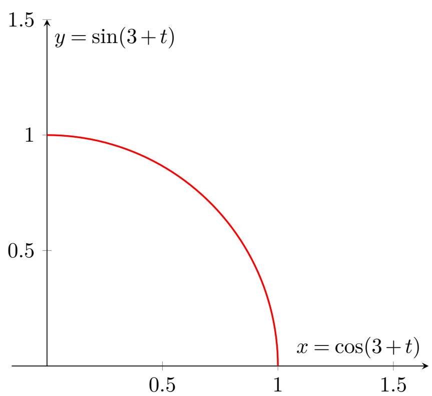
$$\text{Arc length: } \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$
$$\frac{dx}{dt} = -\frac{1}{t}$$
$$\frac{dy}{dt} = \frac{7}{7t} = \frac{1}{t}$$
$$\int_1^e \sqrt{\left(-\frac{1}{t}\right)^2+\left(\frac{1}{t}\right)^2}\,dt = \int_1^e \sqrt{\frac{1}{t^2}+\frac{1}{t^2}}\,dt = \int_1^e \sqrt{\frac{2}{t^2}}\,dt = \sqrt{2}\int_1^e \frac{1}{t}dt$$
$$= \sqrt{2}\Big[\ln(t)\Big]_1^e = \sqrt{2}\Big[\ln(e)-\ln(1)\Big] = \boxed{\sqrt{2}}$$

3. Suppose:

$x = \cos(3+t)$

$y = \sin(3+t)$

What is the area of the region bounded by the graph and the positive x-axis and the positive y-axis?



To find where the curve strikes the axes:
$$x = \cos(3+t) = 0; \quad 3+t = \arccos(0)$$
$$3+t = \frac{\pi}{2}, \quad t = \frac{\pi}{2}-3 \text{ when } x = 0$$
$$y = \sin(3+t) = 0; \quad 3+t = \arcsin(0)$$
$$3+t = 0, \quad t = -3 \text{ when } y = 0$$
$$A = \int_{-3}^{\frac{\pi}{2}-3} \sin(3+t)(-\sin(3+t))dt$$
$$= -\int_{-3}^{\frac{\pi}{2}-3} \sin^2(3+t)dt$$

$$-\int_{-3}^{\frac{\pi}{2}-3} \sin^2(3+t)dt = -\int_{-3}^{\frac{\pi}{2}-3} \frac{1-\cos(6+2t)}{2}dt$$
$$= -\left[\frac{1}{2}t - \frac{\sin(6+2t)}{4}\right]_{-3}^{\frac{\pi}{2}-3} = -\left[\left(\frac{\pi-12}{4} - \frac{\sin(6+\pi-3)}{4}\right) - \left(\frac{-3}{2} - \frac{\sin(6-6)}{2}\right)\right]$$
$$= -\left(\frac{\pi-6}{4} - \frac{\sin(3+\pi)}{4}\right) \approx \boxed{0.679}$$

4. Use the root test to tell if the series converges: $\sum \sqrt{\frac{1+n^2}{1+3^n}}$

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{1+n^2}{1+3^n}\right|} = \lim_{n\to\infty} \frac{\sqrt[n]{1+n^2}}{\sqrt[n]{1+3^n}}$$

$$\text{Note that: } \underbrace{\lim_{n\to\infty} \sqrt[n]{n^2}}_{\sqrt[n]{n^2} = \sqrt[n]{n}^2 \to 1} \le \lim_{n\to\infty} \sqrt[n]{1+n^2} \le \underbrace{\lim_{n\to\infty} \sqrt[n]{2n^2}}_{\sqrt[n]{2n^2} = \sqrt[n]{2}\cdot \sqrt[n]{n}^2 \to 1}$$
$$1 \le \lim_{n\to\infty} \sqrt[n]{1+n^2} \le 1, \quad \therefore \lim_{n\to\infty} \sqrt[n]{1+n^2} = 1$$

$$\text{Note that: } \underbrace{\lim_{n\to\infty} \sqrt[n]{3^n}}_{\sqrt[n]{3^n}=3} \le \lim_{n\to\infty} \sqrt[n]{1+3^n} \le \underbrace{\lim_{n\to\infty} \sqrt[n]{2\cdot 3^n}}_{\sqrt[n]{2\cdot 3^n} = \sqrt[n]{2}\cdot \sqrt[n]{3^n} \to 1\cdot 3 = 3}$$
$$3 \le \lim_{n\to\infty} \sqrt[n]{1+3^n} \le 3, \quad \therefore \lim_{n\to\infty} \sqrt[n]{1+3^n} = 3$$

$$\therefore \lim_{n\to\infty} \frac{\sqrt[n]{1+n^2}}{\sqrt[n]{1+3^n}} = \frac{1}{3}, \quad \frac{1}{3} < 1, \quad \boxed{\text{Converges}}$$

5. Express the following as a closed-form expression: $\sum_{k=0}^{\infty} (-36)^k \frac{x^{10k}}{(2k)!}$.

$$\text{Note that: } \cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$
$$\sum_{k=0}^{\infty} (-36)^k \frac{x^{10k}}{(2k)!} = \sum_{k=0}^{\infty} (-1)^k \frac{(6x^5)^{2k}}{(2k)!}$$
$$\therefore \sum_{k=0}^{\infty} (-36)^k \frac{x^{10k}}{(2k)!} = \boxed{\cos(6x^5)}$$

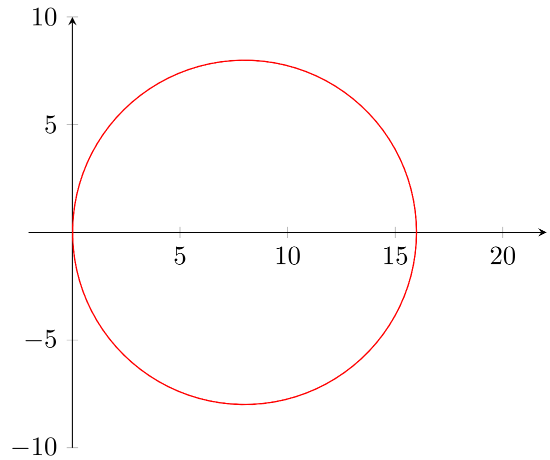
6. Find the Maclaurin series of e^{x-5}

$$\text{Note that: } e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \text{ and } e^{x-5} = \frac{e^x}{e^5}$$
$$\therefore \quad e^{x-5} = \boxed{\sum_{k=0}^{\infty} \frac{x^k}{e^5 k!}}$$

7. Graph $r = 16\cos(\theta)$.

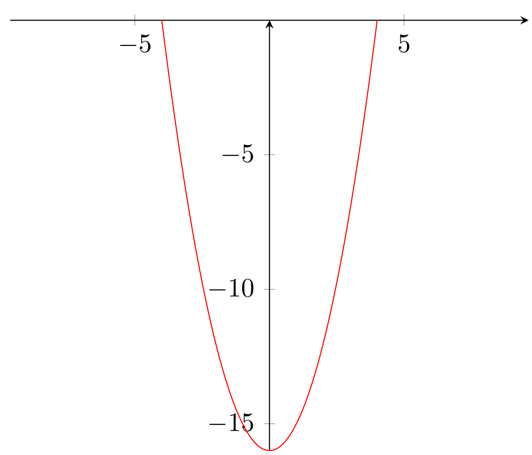
I get to cheat this one with technology, but note that:

- For $r = \cos(\theta)$, its path circular.
- The max distance the graph goes from $y = 0$ is 16 $(16 \cdot \cos(n \cdot \pi))$.
- It has a radius of 8.
- Its center is on $x = 8$.



8. A region is bounded by the x-axis and the line $y = x^2 - 16$. A solid object sits on this region. Cross sections perpendicular to the y-axis are squares. What is the volume of this object?

The intersection looks like this:



Note that the values here will be inverted: $y = x^2 - 16 \to y = 16 - x^2$ to get positive results.

The area to integrate is: $0 = 16 - x^2$, $x = \pm 4, [-4, 4]$.

$$\text{Note that for } V \text{ volume: } V = \int_a^b A dx$$

$$\text{Where } A = 16 - x^2$$

$$\int_{-4}^{4} 16 - x^2 dx = 2\left[16x - \frac{x^3}{3}\right]_0^4 = 2\left[16(4) - \frac{(4)^3}{3}\right] = 2\left[64 - \frac{64}{3}\right] = 2 \cdot \frac{128}{3} = \boxed{\frac{256}{3}}$$

Part II