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MATH F252, Dr. J. Gimbel, University of Alaska Fairbanks
Final Review

Part I

1. Evaluate $\int_1^\infty \frac{1}{1+x^2}dx$

$$\int_1^\infty \frac{1}{1+x^2}dx = \lim_{a\to\infty}\int_1^a \frac{1}{1+x^2}dx$$
$$\lim_{a\to\infty}\int_1^a \frac{1}{1+x^2}dx = \lim_{a\to\infty}\Big[\arctan(x)\Big]_1^a = \lim_{a\to\infty}\arctan(a) - \arctan(1)$$
$$= \frac{\pi}{2} - \frac{\pi}{4} = \boxed{\frac{\pi}{4}}$$

2. Suppose:

$x = 4 - \ln(t)$

$y = 1 + \ln(7t)$

$1 \le t \le e$

Compute the arc length.

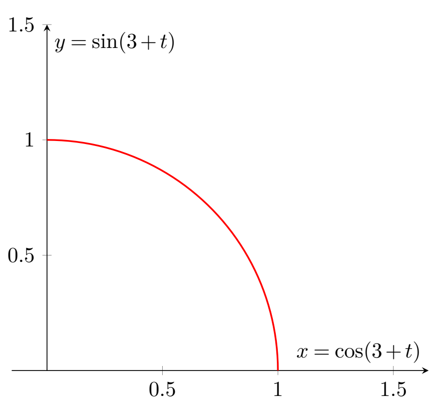
$$\text{Arc length: } \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$
$$\frac{dx}{dt} = -\frac{1}{t}$$
$$\frac{dy}{dt} = \frac{7}{7t} = \frac{1}{t}$$
$$\int_1^e \sqrt{\left(-\frac{1}{t}\right)^2 + \left(\frac{1}{t}\right)^2}\,dt = \int_1^e \sqrt{\frac{1}{t^2}+\frac{1}{t^2}}\,dt = \int_1^e \sqrt{\frac{2}{t^2}}\,dt = \sqrt{2}\int_1^e \frac{1}{t}dt$$
$$= \sqrt{2}\Big[\ln(t)\Big]_1^e = \sqrt{2}\Big[\ln(e) - \ln(1)\Big] = \boxed{\sqrt{2}}$$

3. Suppose:

$x = \cos(3+t)$

$y = \sin(3+t)$

What is the area of the region bounded by the graph and the positive x-axis and the positive y-axis?



To find where the curve strikes the axes:

$$x = \cos(3+t) = 0; \quad 3+t = \arccos(0)$$
$$3+t = \frac{\pi}{2}, \quad t = \frac{\pi}{2} - 3 \text{ when } x = 0$$
$$y = \sin(3+t) = 0; \quad 3+t = \arcsin(0)$$
$$3+t = 0, \quad t = -3 \text{ when } y = 0$$
$$A = \int_{-3}^{\frac{\pi}{2}-3} \sin(3+t)(-\sin(3+t))dt$$
$$= -\int_{-3}^{\frac{\pi}{2}-3} \sin^2(3+t)dt$$

$$-\int_{-3}^{\frac{\pi}{2}-3}\sin^2(3+t)dt = -\int_{-3}^{\frac{\pi}{2}-3}\frac{1-\cos(6+2t)}{2}dt$$

$$= -\left[\frac{1}{2}t - \frac{\sin(6+2t)}{4}\right]_{-3}^{\frac{\pi}{2}-3} = -\left[\left(\frac{\pi-12}{4} - \frac{\sin(6+\pi-3)}{4}\right) - \left(\frac{-3}{2} - \frac{\sin(6-6)}{2}\right)\right]$$

$$= -\left(\frac{\pi-6}{4} - \frac{\sin(3+\pi)}{4}\right) \approx \boxed{0.679}$$

4. Use the root test to tell if the series converges: $\sum\sqrt{\frac{1+n^2}{1+3^n}}$

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{1+n^2}{1+3^n}\right|} = \lim_{n\to\infty}\frac{\sqrt[n]{1+n^2}}{\sqrt[n]{1+3^n}}$$

$$\text{Note that: } \underbrace{\lim_{n\to\infty}\sqrt[n]{n^2}}_{\sqrt[n]{n^2}=\sqrt[n]{n}^2\to 1} \le \lim_{n\to\infty}\sqrt[n]{1+n^2} \le \underbrace{\lim_{n\to\infty}\sqrt[n]{2n^2}}_{\sqrt[n]{2n^2}=\sqrt[n]{2}\cdot\sqrt[n]{n}^2\to 1}$$

$$1 \le \lim_{n\to\infty}\sqrt[n]{1+n^2} \le 1, \quad \therefore \lim_{n\to\infty}\sqrt[n]{1+n^2} = 1$$

$$\text{Note that: } \underbrace{\lim_{n\to\infty}\sqrt[n]{3^n}}_{\sqrt[n]{3^n}=3} \le \lim_{n\to\infty}\sqrt[n]{1+3^n} \le \underbrace{\lim_{n\to\infty}\sqrt[n]{2\cdot 3^n}}_{\sqrt[n]{2\cdot 3^n}=\sqrt[n]{2}\cdot\sqrt[n]{3^n}\to 1\cdot 3=3}$$

$$3 \le \lim_{n\to\infty}\sqrt[n]{1+3^n} \le 3, \quad \therefore \lim_{n\to\infty}\sqrt[n]{1+3^n} = 3$$

$$\therefore \lim_{n\to\infty}\frac{\sqrt[n]{1+n^2}}{\sqrt[n]{1+3^n}} = \frac{1}{3}, \quad \frac{1}{3} < 1, \quad \boxed{\text{Converges}}$$

5. Express the following as a closed-form expression: $\sum_{k=0}^{\infty}(-36)^k\frac{x^{10k}}{(2k)!}$.

$$\text{Note that: } \cos(x) = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}$$

$$\sum_{k=0}^{\infty}(-36)^k\frac{x^{10k}}{(2k)!} = \sum_{k=0}^{\infty}(-1)^k\frac{(6x^5)^{2k}}{(2k)!}$$

$$\therefore \sum_{k=0}^{\infty}(-36)^k\frac{x^{10k}}{(2k)!} = \boxed{\cos(6x^5)}$$

6. Find the Maclaurin series of e^{x-5}

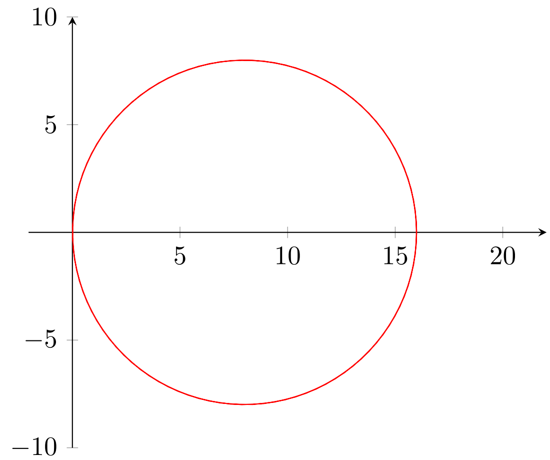
$$\text{Note that: } e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!} \text{ and } e^{x-5} = \frac{e^x}{e^5}$$

$$\therefore \quad e^{x-5} = \boxed{\sum_{k=0}^{\infty}\frac{x^k}{e^5 k!}}$$

7. Graph $r = 16\cos(\theta)$.

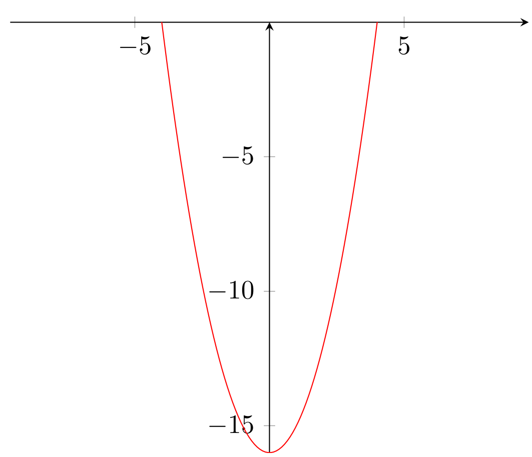
I get to cheat this one with technology, but note that:

- For $r = \cos(\theta)$, its path circular.
- The max distance the graph goes from $y = 0$ is 16 $(16 \cdot \cos(n \cdot \pi))$.
- It has a radius of 8.
- Its center is on $x = 8$.



8. A region is bounded by the x-axis and the line $y = x^2 - 16$. A solid object sits on this region. Cross sections perpendicular to the y-axis are squares. What is the volume of this object?

The intersection looks like this:



Note that the values here will be inverted: $y = x^2 - 16 \rightarrow y = 16 - x^2$ to get positive results.

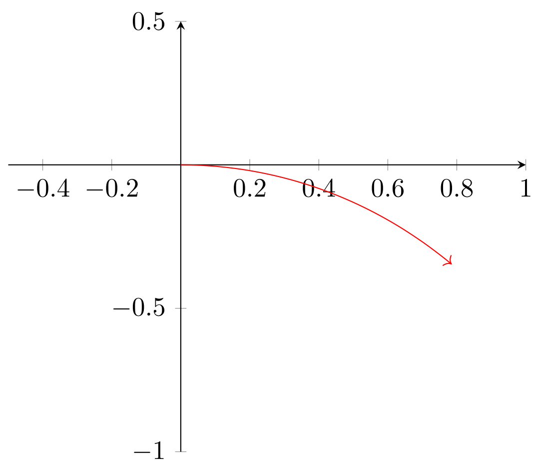
The area to integrate is: $0 = 16 - x^2, x = \pm 4, [-4, 4]$.

$$\text{Note that for } V \text{ volume: } V = \int_a^b A dx$$
$$\text{Where } A = (16 - x^2)^2 = x^4 - 32x^2 + 256$$
$$\int_{-4}^{4} x^4 - 32x^2 + 256dx = 2\left[\frac{x^5}{5} - \frac{32x^3}{3} + 256x\right]_0^4 = 2\left[\frac{x^5}{5} - \frac{32x^3}{3} + 256x\right]$$
$$= 2\left[\frac{(4)^5}{5} - \frac{32(4)^3}{3} + 256(4)\right] = \boxed{1092.2\overline{66}}$$

9. A force of two pounds stretches a spring one inch. How much work is required to pull the spring an entire foot? Express your answer in foot-pounds.

$$\text{Note that: } W = \int_a^b F dx$$
$$\text{And that: } F = 2$$
$$\int_0^{12} 2dx = \Big[2x\Big]_0^{12} = \boxed{24 \text{ foot-pounds}}$$

10. Consider the graph of $y = \ln(\cos(x))$ where $0 \le x \le \dfrac{\pi}{4}$. Compute the arc length.



$$\text{Note that arc length is: } \int_a^b \sqrt{1 + |f'(x)|^2} dx$$
$$f'(x) = -\frac{\sin(x)}{\cos(x)} = -\tan(x)$$
$$\int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2(x)} dx = \int_0^{\frac{\pi}{4}} \sqrt{\sec^2(x)} dx = \int_0^{\frac{\pi}{4}} \sec(x) dx$$
$$= \Big[\ln|\sec(x) + \tan(x)|\Big]_0^{\frac{\pi}{4}} = \Big[\ln|\sec(\pi/4) + \tan(\pi/4)| - \ln|\sec(0) + \tan(0)|\Big]$$
$$= \Big[\ln|\sqrt{(}2) + 1| - \ln|1|\Big] = \ln\left(\sqrt{2} + 1\right) \approx \boxed{0.881}$$

Part II

1. Find the second degree Taylor polynomial of $\sqrt{x}$ expanded about four. Use this to approximate $\sqrt{4.1}$.

$$\text{Taylor series: } \sum_{k=0}^{\infty} \frac{f^k(a)(x-a)^k}{k!}$$

$$\text{Second degree of } \sqrt{x} \text{ at 4: } \boxed{2+\frac{x-4}{8}+\frac{(x-4)^2}{48}}$$

$$\text{Approximation at } \sqrt{4.1}\text{: } 2+\frac{0.1}{8}+\frac{0.1^2}{48} \approx \boxed{2.0127}$$

2. Consider the expression $r^2 = \sin(2\theta)$. Convert this from polar to cartesian coordinates. Also, convert $x^2+y^2 = 2x$ from cartesian to polar coordinates.

$$r^2 = \sin(2\theta) \Rightarrow x^2+y^2 = \sin(2\theta)$$

$$x^2+y^2 = 2\sin(\theta)\cos(\theta) \Rightarrow \boxed{x^2+y^2 = 2xy}$$

$$x^2+y^2 = 2x \Rightarrow r^2 = 2r\cos(\theta) \Rightarrow \boxed{r = 2\cos(\theta)}$$

3. Evaluate $\int\limits_0^\pi \sin(x)\cos^5(x)dx$

$$\int_0^\pi \sin(x)\cos^5(x)dx = \int_0^\pi \sin(x)(1-\sin^2(x))\cos^3(x)dx = \int_0^\pi \sin(x) - \sin^3(x)\cos^3(x)dx$$

$$\int_0^\pi \sin(x) - \int_0^\pi \sin^3(x)\cos^3(x)dx$$

$$\int_0^\pi \sin(x) = 0$$

$$-\int_0^\pi \sin^3(x)\cos^3(x)dx: \text{ Let } u = \cos(x) \quad du = -\sin(x)dx$$

$$\int_1^{-1} u^3dx = -\int_{-1}^1 u^3dx = -\left[\frac{u^4}{4}\right]_{-1}^1 = -\left[\frac{1}{4}-\frac{1}{4}\right] = \boxed{0}$$