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STAT F300 Statistics, Dr. Short
Worksheet 19, Due March 8, 4:15pm

1. Show that if X and Y are random variables, then

$$\text{var}(X+Y) = \text{var}(X) + \text{var}(Y) + 2\text{cov}(X,Y)$$

$$\begin{aligned}
\text{var}(X+Y) &= \text{cov}(X+Y, X+Y) \\
\text{cov}(X+Y, X+Y) &= \mathbb{E}((X+Y)^2) - \mathbb{E}(X+Y)^2 \\
&= (\mathbb{E}(X^2) + \mathbb{E}(Y^2) + \mathbb{E}(XY)) - (\mathbb{E}(X) + \mathbb{E}(Y))^2 \\
&= \mathbb{E}(X^2) + \mathbb{E}(Y^2) + 2\mathbb{E}(XY) - \mathbb{E}(X)^2 - \mathbb{E}(Y)^2 - 2\mathbb{E}(X)\mathbb{E}(Y) \\
&= \text{var}(X) + \text{var}(Y) + 2(\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)) \\
&= \text{var}(X) + \text{var}(Y) + 2\text{cov}(XY)
\end{aligned}$$

2. Let X_1 and X_2 be independent random variables with the following pmf:

x	1	3	5	
$p(x)$	0.5	0.1	0.4	1

(a) Find the pmf of $\overline{X} = \frac{1}{2}(X_1 + X_2)$.

X_1	X_2	$p(X_1, X_2)$	$\overline{X} = \frac{1}{2}(X_1+X_2)$
1	1	$0.5(0.5) = 0.25$	1.0
1	3	$0.5(0.1) = 0.05$	2.0
1	5	$0.5(0.4) = 0.20$	3.0
3	1	$0.1(0.5) = 0.05$	2.0
3	3	$0.1(0.1) = 0.01$	3.0
3	5	$0.1(0.4) = 0.04$	4.0
5	1	$0.4(0.5) = 0.20$	3.0
5	3	$0.4(0.1) = 0.04$	4.0
5	5	$0.4(0.4) = 0.16$	5.0

Let $Y = \overline{X}$. The pmf of $\overline{X}$ is:

y	$p(y)$
1.0	0.25
2.0	0.10
3.0	0.41
4.0	0.08
5.0	0.16

(b) Use your answer to part (a) to find $\mathbb{E}(\overline{X})$, $\mathbb{E}(\overline{X}^2)$, and $\text{var}(\overline{X})$.

$$\begin{aligned}
\mathbb{E}(\overline{X}) &= \sum_Y y p(y) \\
&= 1.0(0.25) + 2.0(0.10) + \cdots + 5.0(0.16) \\
&= \boxed{2.8} \\
\mathbb{E}(\overline{X}^2) &= 1.0^2(0.25) + 2.0^2(0.10) + \cdots + 5.0^2(0.16) \\
&= \boxed{9.62} \\
\text{var}(\overline{X}) &= \mathbb{E}(\overline{X}^2) - \mathbb{E}(\overline{X})^2 \\
&= 9.62 - (2.8)^2 = 9.62 - 7.84 = \boxed{1.78}
\end{aligned}$$

(c) Find $\mathbb{E}(X_1)$ and $\text{var}(X_1)$.

$$\begin{aligned}
\mathbb{E}(X_1) &= \sum_{X_1} xp(x) \\
&= 1(0.5) + 3(0.1) + 5(0.4) = \boxed{2.8} \\
\text{var}(X_1) &= \mathbb{E}(X_1^2) - \mathbb{E}(X_1)^2 \\
&= (0.5 + 0.9 + 10) - 2.8^2 = 11.4 - 7.84 = \boxed{3.56}
\end{aligned}$$

(d) Compare the values of $\mathbb{E}(X_1)$ and $\mathbb{E}(\overline{X})$.

The expected value of X_1 and $\overline{X}$ are the same.

(e) Compare the values of $\text{var}(X_1)$ and $\text{var}(\overline{X})$

The variance of X_1 and $\overline{X}$ are in the ratio of 2:1 such that:

$$\frac{\text{var}(X_1)}{\text{var}(\overline{X})} = \frac{3.56}{1.78} = \frac{2}{1}$$