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Final Review

Part I

1. Evaluate $\int_1^\infty \frac{1}{1+x^2}dx$

$$\int_1^\infty \frac{1}{1+x^2}dx = \lim_{a\to\infty}\int_1^a \frac{1}{1+x^2}dx$$
$$\lim_{a\to\infty}\int_1^a \frac{1}{1+x^2}dx = \lim_{a\to\infty}\Big[\arctan(x)\Big]_1^a = \lim_{a\to\infty}\arctan(a)-\arctan(1)$$
$$= \frac{\pi}{2}-\frac{\pi}{4} = \boxed{\frac{\pi}{4}}$$

2. Suppose:

   $x = 4-\ln(t)$

   $y = 1+\ln(7t)$

   $1 \le t \le e$

   Compute the arc length.

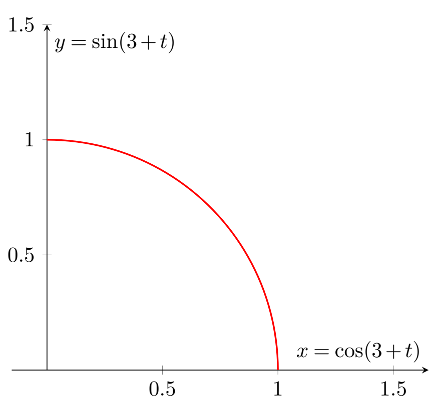
$$\text{Arc length: } \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt$$
$$\frac{dx}{dt} = -\frac{1}{t}$$
$$\frac{dy}{dt} = \frac{7}{7t} = \frac{dt}{t}$$
$$\int_1^e \sqrt{\left(-\frac{1}{t}\right)^2+\left(\frac{1}{t}\right)^2}dt = \int_1^e \sqrt{\frac{1}{t^2}+\frac{1}{t^2}}dt = \int_1^e \sqrt{\frac{2}{t^2}}dt = \sqrt{2}\int_1^e \frac{1}{t}dt$$
$$= \sqrt{2}\Big[\ln(t)\Big]_1^e = \sqrt{2}\Big[\ln(e)-\ln(1)\Big] = \boxed{\sqrt{2}}$$

3. Suppose:

   $x = \cos(3+t)$

   $y = \sin(3+t)$

   What is the area of the region bounded by the graph and the positive x-axis and the positive y-axis?



To find where the curve strikes the axes:

$$x = \cos(3+t) = 0; \quad 3+t = \arccos(0)$$
$$3+t = \frac{\pi}{2}, \quad t = \frac{\pi}{2}-3 \text{ when } x = 0$$
$$y = \sin(3+t) = 0; \quad 3+t = \arcsin(0)$$
$$3+t = 0, \quad t = -3 \text{ when } y = 0$$
$$A = \int_{-3}^{\frac{\pi}{2}-3} \sin(3+t)(-\sin(3+t))dt$$
$$= -\int_{-3}^{\frac{\pi}{2}-3} \sin^2(3+t)dt$$

$$-\int_{-3}^{\frac{\pi}{2}-3}\sin^2(3+t)dt = -\int_{-3}^{\frac{\pi}{2}-3}\frac{1-\cos(6+2t)}{2}dt$$

$$= -\left[\frac{1}{2}t - \frac{\sin(6+2t)}{4}\right]_{-3}^{\frac{\pi}{2}-3} = -\left[\left(\frac{\pi-12}{4} - \frac{\sin(6+\pi-3)}{4}\right) - \left(\frac{-3}{2} - \frac{\sin(6-6)}{2}\right)\right]$$

$$= -\left(\frac{\pi-6}{4} - \frac{\sin(3+\pi)}{4}\right) \approx \boxed{0.679}$$

4. Use the root test to tell if the series converges: $\sum\sqrt{\frac{1+n^2}{1+3^n}}$

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{1+n^2}{1+3^n}\right|} = \lim_{n\to\infty}\frac{\sqrt[n]{1+n^2}}{\sqrt[n]{1+3^n}}$$

$$\text{Note that: } \underbrace{\lim_{n\to\infty}\sqrt[n]{n^2}}_{\sqrt[n]{n^2}=\sqrt[n]{n}^2\to 1} \leq \lim_{n\to\infty}\sqrt[n]{1+n^2} \leq \underbrace{\lim_{n\to\infty}\sqrt[n]{2n^2}}_{\sqrt[n]{2n^2}=\sqrt[n]{2}\cdot\sqrt[n]{n}^2\to 1}$$

$$1 \leq \lim_{n\to\infty}\sqrt[n]{1+n^2} \leq 1, \quad \therefore \lim_{n\to\infty}\sqrt[n]{1+n^2} = 1$$

$$\text{Note that: } \underbrace{\lim_{n\to\infty}\sqrt[n]{3^n}}_{\sqrt[n]{3^n}=3} \leq \lim_{n\to\infty}\sqrt[n]{1+3^n} \leq \underbrace{\lim_{n\to\infty}\sqrt[n]{2\cdot 3^n}}_{\sqrt[n]{2\cdot 3^n}=\sqrt[n]{2}\cdot\sqrt[n]{3}^n\to 1\cdot 3=3}$$

$$3 \leq \lim_{n\to\infty}\sqrt[n]{1+3^n} \leq 3, \quad \therefore \lim_{n\to\infty}\sqrt[n]{1+3^n} = 3$$

$$\therefore \lim_{n\to\infty}\frac{\sqrt[n]{1+n^2}}{\sqrt[n]{1+3^n}} = \frac{1}{3}, \quad \frac{1}{3} < 1, \quad \boxed{\text{Converges}}$$

Part II