

Part I

1. Evaluate $\int_1^\infty \frac{1}{1+x^2} dx$

$$\begin{aligned}\int_1^\infty \frac{1}{1+x^2} dx &= \lim_{a \rightarrow \infty} \int_1^a \frac{1}{1+x^2} dx \\ \lim_{a \rightarrow \infty} \int_1^a \frac{1}{1+x^2} dx &= \lim_{a \rightarrow \infty} \left[\arctan(x) \right]_1^a = \lim_{a \rightarrow \infty} \arctan(a) - \arctan(1) \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \boxed{\frac{\pi}{4}}\end{aligned}$$

2. Suppose:

$$x = 4 - \ln(t)$$

$$y = 1 + \ln(7t)$$

$$1 \leq t \leq e$$

Compute the arc length.

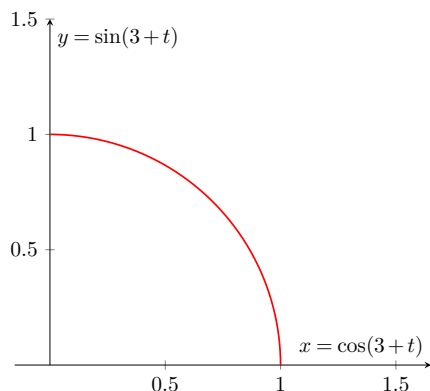
$$\begin{aligned}\text{Arc length: } \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ \frac{dx}{dt} = -\frac{1}{t} \\ \frac{dy}{dt} = \frac{7}{7t} = \frac{1}{t} \\ \int_1^e \sqrt{\left(-\frac{1}{t}\right)^2 + \left(\frac{1}{t}\right)^2} dt = \int_1^e \sqrt{\frac{1}{t^2} + \frac{1}{t^2}} dt = \int_1^e \sqrt{\frac{2}{t^2}} dt = \sqrt{2} \int_1^e \frac{1}{t} dt \\ = \sqrt{2} \left[\ln(t) \right]_1^e = \sqrt{2} \left[\ln(e) - \ln(1) \right] = \boxed{\sqrt{2}}\end{aligned}$$

3. Suppose:

$$x = \cos(3+t)$$

$$y = \sin(3+t)$$

What is the area of the region bounded by the graph and the positive x-axis and the positive y-axis?



To find where the curve strikes the axes:

$$x = \cos(3+t) = 0; \quad 3+t = \arccos(0)$$

$$3+t = \frac{\pi}{2}, \quad t = \frac{\pi}{2} - 3 \text{ when } x = 0$$

$$y = \sin(3+t) = 0; \quad 3+t = \arcsin(0)$$

$$3+t = 0, \quad t = -3 \text{ when } y = 0$$

$$\begin{aligned}A &= \int_{-3}^{\frac{\pi}{2}-3} \sin(3+t)(-\sin(3+t)) dt \\ &= - \int_{-3}^{\frac{\pi}{2}-3} \sin^2(3+t) dt\end{aligned}$$

$$\begin{aligned}
& - \int_{-3}^{\frac{\pi}{2}-3} \sin^2(3+t) dt = - \int_{-3}^{\frac{\pi}{2}-3} \frac{1 - \cos(6+2t)}{2} dt \\
& = - \left[\frac{1}{2}t - \frac{\sin(6+2t)}{4} \right]_{-3}^{\frac{\pi}{2}-3} = - \left[\left(\frac{\pi-12}{4} - \frac{\sin(6+\pi-3)}{4} \right) - \left(\frac{-3}{2} - \frac{\sin(6-6)}{2} \right) \right] \\
& = - \left(\frac{\pi-6}{4} - \frac{\sin(3+\pi)}{4} \right) \approx \boxed{0.679}
\end{aligned}$$

4. Use the root test to tell if the series converges: $\sum \sqrt[n]{\frac{1+n^2}{1+3^n}}$

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{1+n^2}{1+3^n} \right|} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{1+n^2}}{\sqrt[n]{1+3^n}}$$

$$\begin{aligned}
& \text{Note that: } \underbrace{\lim_{n \rightarrow \infty} \sqrt[n]{n^2}}_{\sqrt[n]{n^2} = \sqrt[n^2]{n^2} \rightarrow 1} \leq \lim_{n \rightarrow \infty} \sqrt[n]{1+n^2} \leq \underbrace{\lim_{n \rightarrow \infty} \sqrt[n]{2n^2}}_{\sqrt[n]{2n^2} = \sqrt[n]{2} \cdot \sqrt[n]{n^2} \rightarrow 1} \\
& 1 \leq \lim_{n \rightarrow \infty} \sqrt[n]{1+n^2} \leq 1, \quad \therefore \lim_{n \rightarrow \infty} \sqrt[n]{1+n^2} = 1
\end{aligned}$$

$$\begin{aligned}
& \text{Note that: } \underbrace{\lim_{n \rightarrow \infty} \sqrt[n]{3^n}}_{\sqrt[n]{3^n} = 3} \leq \lim_{n \rightarrow \infty} \sqrt[n]{1+3^n} \leq \underbrace{\lim_{n \rightarrow \infty} \sqrt[n]{2 \cdot 3^n}}_{\sqrt[n]{2 \cdot 3^n} = \sqrt[n]{2} \cdot \sqrt[n]{3^n} \rightarrow 1 \cdot 3 = 3} \\
& 3 \leq \lim_{n \rightarrow \infty} \sqrt[n]{1+3^n} \leq 3, \quad \therefore \lim_{n \rightarrow \infty} \sqrt[n]{1+3^n} = 3
\end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{\sqrt[n]{1+n^2}}{\sqrt[n]{1+3^n}} = \frac{1}{3}, \quad \frac{1}{3} < 1, \quad \boxed{\text{Converges}}$$

5. Express the following as a closed-form expression: $\sum_{k=0}^{\infty} (-36)^k \frac{x^{10k}}{(2k)!}$

$$\begin{aligned}
& \text{Note that: } \cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \\
& \sum_{k=0}^{\infty} (-36)^k \frac{x^{10k}}{(2k)!} = \sum_{k=0}^{\infty} (-1)^k \frac{(6x^5)^{2k}}{(2k)!} \\
& \therefore \sum_{k=0}^{\infty} (-36)^k \frac{x^{10k}}{(2k)!} = \boxed{\cos(6x^5)}
\end{aligned}$$

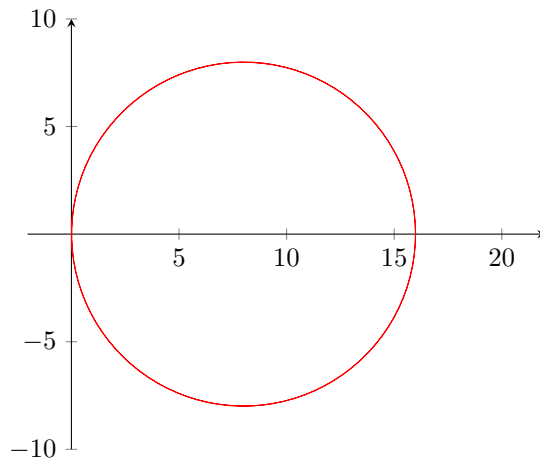
6. Find the Maclaurin series of e^{x-5}

$$\begin{aligned}
& \text{Note that: } e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \text{ and } e^{x-5} = \frac{e^x}{e^5} \\
& \therefore e^{x-5} = \boxed{\sum_{k=0}^{\infty} \frac{x^k}{e^5 k!}}
\end{aligned}$$

7. Graph $r = 16\cos(\theta)$.

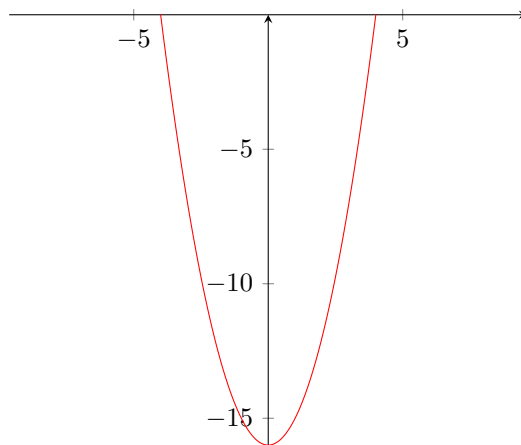
I get to cheat this one with technology, but note that:

- For $r = \cos(\theta)$, its path circular.
- The max distance the graph goes from $y = 0$ is 16 ($16 \cdot \cos(n \cdot \pi)$).
- It has a radius of 8.
- Its center is on $x = 8$.



8. A region is bounded by the x-axis and the line $y = x^2 - 16$. A solid object sits on this region. Cross sections perpendicular to the y-axis are squares. What is the volume of this object?

The intersection looks like this:



Note that the values here will be inverted: $y = x^2 - 16 \rightarrow y = 16 - x^2$ to get positive results.

The area to integrate is: $0 = 16 - x^2, x = \pm 4, [-4, 4]$.

Note that for V volume: $V = \int_a^b A dx$

Where $A = (16 - x^2)^2 = x^4 - 32x^2 + 256$

$$\begin{aligned} \int_{-4}^4 x^4 - 32x^2 + 256 dx &= 2 \left[\frac{x^5}{5} - \frac{32x^3}{3} + 256x \right]_0^4 = 2 \left[\frac{x^5}{5} - \frac{32x^3}{3} + 256x \right] \\ &= 2 \left[\frac{(4)^5}{5} - \frac{32(4)^3}{3} + 256(4) \right] = \boxed{1092.2\overline{66}} \end{aligned}$$

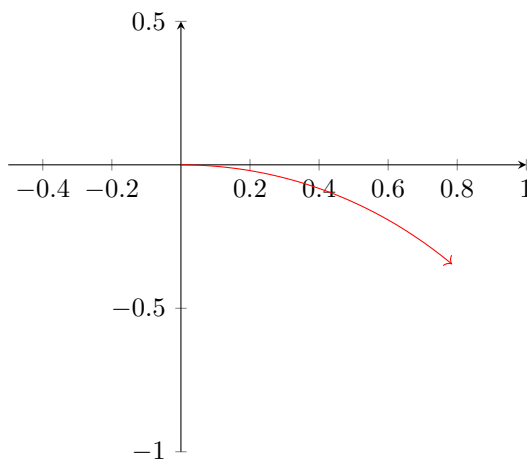
9. A force of two pounds stretches a spring one inch. How much work is required to pull the spring an entire foot? Express your answer in foot-pounds.

Note that: $W = \int_a^b F dx$

And that: $F = 2$

$$\int_0^{12} 2 dx = \left[2x \right]_0^{12} = \boxed{24 \text{ foot-pounds}}$$

10. Consider the graph of $y = \ln(\cos(x))$ where $0 \leq x \leq \frac{\pi}{4}$. Compute the arc length.



Note that arc length is: $\int_a^b \sqrt{1 + |f'(x)|^2} dx$

$$f'(x) = -\frac{\sin(x)}{\cos(x)} = -\tan(x)$$

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2(x)} dx &= \int_0^{\frac{\pi}{4}} \sqrt{\sec^2(x)} dx = \int_0^{\frac{\pi}{4}} \sec(x) dx \\ &= \left[\ln |\sec(x) + \tan(x)| \right]_0^{\frac{\pi}{4}} = \left[\ln |\sec(\pi/4) + \tan(\pi/4)| - \ln |\sec(0) + \tan(0)| \right] \\ &= \left[\ln |\sqrt{2} + 1| - \ln |1| \right] = \ln(\sqrt{2} + 1) \approx \boxed{0.881} \end{aligned}$$

5. Find the area enclosed by the graph $r = 2 + \cos(\theta)$.

$$\begin{aligned}
 A &= \frac{1}{2} \int_0^{2\pi} f(\theta)^2 d\theta \\
 \frac{1}{2} \int_0^{2\pi} (2 + \cos(\theta))^2 d\theta &= \frac{1}{2} \int_0^{2\pi} [\cos^2(\theta) + 4\cos(\theta) + 4] d\theta = \frac{1}{2} \int_0^{2\pi} \left[\frac{1 + \cos(2\theta)}{2} + 4\cos(\theta) + 4 \right] d\theta \\
 &= \frac{1}{4} \int_0^{2\pi} [\cos(2\theta) + 8\cos(\theta) + 9] d\theta = \left[9\theta - \frac{\sin(2\theta)}{2} - 8\sin(\theta) \right]_0^{2\pi} = 9(2\pi) = \boxed{18\pi}
 \end{aligned}$$

6. Determine if the following converges or diverges:

(a) $\sum \frac{k^5 + 1}{k^6 + 1}$

Using ratio test:

$$\lim_{k \rightarrow \infty} \sqrt[n]{\left| \frac{k^5 + 1}{k^6 + 1} \right|}$$

(b)