

Analysis 1

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- *Analysis 1 ITET, F Ziltener* - https://metaphor.ethz.ch/x/2024/hs/401-0231-10L/Ziltener_Notizen_Analysis_1_ITET_RW.pdf
- *Analysis für Informatik, M Struwe* - <https://people.math.ethz.ch/~struwe/Skripten/InfAnalysis-bbm-8-11-2010.pdf>

Logik

Aussage - Eine Aussage, die entweder wahr oder falsch ist

Luegner Paradox - Das ist keine Aussage: "Dieser Satz ist falsch"

Menge (Set) - eine ungeordnete Zusammenfassung verschiedener Objekte zu einem Ganzen

\wedge - and

\vee - or

\vee (XOR) - either ... or ...

Materiale Aequivalenz (\Leftrightarrow)

Logische Aequivalenz (\equiv) $A \Leftrightarrow B \equiv (A \Rightarrow B) \wedge (B \Rightarrow A)$ - Sie haben die gleichen

Wahrheitstabellen

$A \Leftrightarrow B$ - A genau dann wenn B

$A \Rightarrow B$ - Wenn A, dann B

$\neg B \Rightarrow \neg A$ - Kontraposition

$A \Rightarrow B \equiv \neg B \Rightarrow \neg A$

Zum Beispiel:

Es hat geregnet \Rightarrow die Strasse ist nass

Kontraposition: Die Strasse ist nicht nass \Rightarrow Es hat nicht geregnet

Das ist genauso wahr aufgrund der Physik.

Wahr: $0 < 0 \Rightarrow 1 + 1 = 2$

Falsch: $0 < 0 \Leftrightarrow 1 + 1 = 2$

Distributive:

$$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$$

$$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$$

Proofs

Beweis - eine Herleitung einer Aussage aus den Axiomen

Satz - eine Bewiesene Aussage

Lemma (oder Hilfssatz) - ein Satz, der dazu dient, einen anderen Satz zu beweisen

q.e.d. (■) - end of proof

Beweiss formalisieren - Express a proof formally in terms of symbols and Lemmas, can be checked by a computer.

Divide et impera - divide and conquer *Zermelo + Fraenkel Axioms* - Foundational axioms of all proofs

Beweis Methode

Modus ponens - Wird (meistens mehrmals) verwendet, um etwas zu beweisen:

A := Es hat geregnet (Premise)

Wenn es geregnet hat, dann ist die Strasse nass (Regel: $A \Rightarrow B$)

B := Die Strasse ist nass (Konklusion)

Kontraposition - Prove the Kontraposition, which subsequently proves the original statement (they are logically equivalent)

Beweisen, dass $\sqrt{2} < \sqrt{3}$:

$$A := \sqrt{2} \geq \sqrt{3} \equiv \neg \sqrt{2} < \sqrt{3}$$

Monotonie des Quadrierens:

$$x, y \geq 0$$

$$\text{Wenn } x \leq y, \text{ dann ist } x^2 \leq y^2$$

Laut der Monotonie des Quadrierens, $B := 2 \geq 3$ ist wahr

$$A \Rightarrow B \equiv \neg B \Rightarrow \neg A \equiv 2 < 3 \Rightarrow \sqrt{2} < \sqrt{3} \blacksquare$$

Widerspruch beweisen

Um A zu beweisen, nehmen wir an, dass A falsch ist.

Widerspruch finden - das beweist die Aussage A

Zum Beispiel:

Beweis des Satzes $\sqrt{2} < \sqrt{3}$

Nehmen wir an, dass $\sqrt{2} \geq \sqrt{3}$ wahr ist

Lemma (Monotonie des Quadrierens): $\sqrt{2} \geq \sqrt{3} \Rightarrow 2 \geq 3$

Widerspruch: $2 \geq 3$ ist falsch, deshalb ist $\sqrt{2} \geq \sqrt{3}$ auch falsch.

$$\neg(\sqrt{2} \geq \sqrt{3}) \equiv \sqrt{2} < \sqrt{3} \blacksquare$$

It is more rigorous to prove / rewrite something through Contraposition, because we start with a false statement in contradiction.

Vollständige Induktion

$n \in N_0$, $P(n)$ ist eine Aussage

$P(0)$ ist wahr

Wenn $\forall k \in N_0$ gilt $P(k) \Rightarrow P(k+1)$

Dann ist $\forall n \in N_0$, $P(n) \equiv$ wahr

Zum Beispiel:

$$\text{Satz: } \forall n \in N_0, P(n) := \sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$P(0) = \frac{0(1)}{2} = 0$$

$$\text{Sei } P(k) = \frac{k(k+1)}{2}$$

$$\text{Zu zeigen } P(k+1) = \frac{(k+1)((k+1)+1)}{2}$$

$$\begin{aligned} P(k+1) &= P(k) + k + 1 = \frac{k(k+1)}{2} + k + 1 \\ &= 2k^2 + 3k + 1 = \frac{k^2 + \frac{3}{2}k + \frac{1}{2}}{2} \\ &= \frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2} \end{aligned}$$

Vollständige Induktion gibt, dass $\forall n \in N_0$, $P(n)$ wahr ist. \blacksquare

Mengenlehre

Eine ungeordnete Zusammenfassung von Elementen.

\emptyset - Leere Menge, hat keine Elemente

$\{\emptyset\}$ hat genau ein Element

Aussageform $\{x \mid P(x)\}$ or $\{x; P(x)\}$ - die Menge aller x , fuer die $P(x)$ gilt

Example: $\{x \mid x \in \mathbb{N}_0, x \text{ ist gerade}\}$

Russelsche Antonomie - $\{x \mid x \in X, x \notin x\}$ ist ein Paradox

Loesung: Es muss immer so definiert werden $\{x \in X \mid P(x)\}$, wo X eine andere Menge ist.

$A \cap B = \{x \mid x \in A \wedge x \in B\}$ - Intersection

$A \cup B = \{x \mid x \in A \vee x \in B\}$ - Union

$A \setminus B = \{x \in A \mid x \notin B\}$ - Without

$A \subseteq B$ - Jedes Element von A liegt in B (between two sets, unlike $x \in A$ which describes a single element x being inside the set A)

$A \subset B$ - Jedes Element von A liegt in B und A enthaelt weniger Elemente als B

$A \subseteq X, A^c = X \setminus A$, wo X die Grundmenge ist, die jeder Element die wir betrachten enthaelt.

Distributive:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$(1, 2, 3)$ - *Tuple* - Ordered set

Kartesische Product / Potenz - $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$

Example:

$$X := \{0, 1\}, Y := \{\alpha, \beta\}$$

$$X \times Y := \{(0, \alpha), (0, \beta), (1, \alpha), (1, \beta)\}$$

$$|X \times Y| = |X| \times |Y|$$

\mathbb{R}^n := n-dimensionalen Koordinatenraum

$$\mathbb{R}^2 = X \times Y$$

$$\mathbb{R}^3 = X \times Y \times Z$$

Interval Notation

$$[a, b] - a \leq x \leq b$$

$$(a, b) - a < x < b$$

Open bounds cannot be the maximum / minimum of a set, as they are not contained in the set (and $0.\dot{9} \equiv 1$ etc.).

Let $A \subseteq \mathbb{R}$

Supremum

$$\sup A = \begin{cases} \text{Smallest upper bound} & \text{if } A \text{ has an upper bound} \\ \infty & \text{if } A \text{ doesn't have an upper bound} \\ -\infty & \text{if } A = \emptyset \end{cases}$$

Infimum - Largest lower bound

$$\inf A = \begin{cases} \text{Largest lower bound} & \text{if } A \text{ has a lower bound} \\ -\infty & \text{if } A \text{ doesn't have a lower bound} \\ \infty & \text{if } A = \emptyset \end{cases}$$

Infinity cannot be a Supre/Infimum, because $\infty \notin \mathbb{R}$

De Morgan's Laws

Also apply to boolean logic, where $A, B := 1, 0$

$$(A \cap B)^c = A^c \cup B^c$$

$$(A \cup B)^c = A^c \cap B^c$$

Quantoren

They cannot simply be swapped! See the largest natural number problem in script.

\exists - Existenzquantor - Es gibt

\forall - Allquantor - Fuer alle

$\exists!$ - Es gibt genau ein element

$$\neg(\forall x \in X \mid P(x)) = \exists x \in X \mid \neg P(x)$$

$$\neg(\exists x \in X \mid P(x)) = \forall x \in X \mid \neg P(x)$$

Goethe Prinzip - When a variable is renamed correctly, a statement is still logically equivalent

Funktionen

Eine Funktion ist ein Tripel $f = (X, Y, G)$, wobei X und Y Mengen sind und $G \subseteq X \times Y$, sodass $\forall x \in X \exists y \in Y$, sodass $(x, y) \in G$

Domain - Set of possible inputs for a function

Codomain (Range) - Set of possible outputs of a function

Example:

Both are Quadratic funktions but are not equal:

$$X := Y := \mathbb{R}, G = \{(x, x^2) \mid x \in \mathbb{R}^2\}$$

$$X := \mathbb{R}, Y :=]0, \infty[, G = \{(x, x^2) \mid x \in \mathbb{R}^2\}$$

$$X \rightarrow X, \text{id}(x) := x - \text{Identitaets Funktion}$$

Bild und Urbild - Muss nicht bijektiv sein

$$\text{im}(X) := f(X) - \text{Bild von } f$$

$$f : X \rightarrow Y, f^{-1}(Y) := \{x \in X \mid f(x) \in Y\} - \text{Urbild von } y \text{ unter } f$$

Surjektiv - $\forall y \in Y \exists x \in X : f(x) = y$ - Es gibt fuer jeder Ausgang einige dazugehoerige Eingange

Injektiv - $\forall x, x' \in X : x \neq x' \Rightarrow f(x) \neq f(x')$ - Es gibt genau eine Ausgang fuer jeder Eingang in dem Definitionsbereich

Bijektiv - Es ist Surjektiv und Injektiv, weshalb es eine Inverse hat

Umkehrfunktion

Sei $f : X \rightarrow Y$ eine Bijektive funktion, $f^{<-1>} := Y \rightarrow X$ - *Umkehr Funktion*

The inverse can ONLY be defined when the function is Bijektiv, unlike the Urbild. When $X = Y = \mathbb{R}$ it is the reflection of the original function over the line $y = x$. It is sometimes notated as f^{-1} when the context is clear.

Do not forget to consider the given domain / range when considering if a function is bijektiv!

Zum Beispiel:

$$f : \mathbb{R} \rightarrow \mathbb{R}, f(x) := x^2$$

$$\text{im}(f) = f(\mathbb{R}) = [0, \infty]$$

$$f^{-1}([-\infty, 4]) = [-2, 2]$$

The inverse can be only be defined if f is Bijektiv:

$$f : [0, \infty] \rightarrow [0, \infty], f(x) := x^2$$
$$f^{<-1>} = \sqrt{x}$$

$g \circ f := g(f(x))$ - Only possible if the $\text{codom}(f) = \text{dom}(g)$

Zahlen und Vektoren

$$\mathbb{N}_0 := \{0, 1, 2, \dots\}$$

$$\mathbb{N} := \{1, 2, 3, \dots\}$$

$$\mathbb{Z} := \{\dots, -1, 0, 1, \dots\}$$

$$\mathbb{Q} := \left\{ \frac{m}{n} \mid m \in \mathbb{Z} \wedge n \in \mathbb{N} \right\}$$

$$\mathbb{N}_0 \subseteq \mathbb{Z} \subseteq \mathbb{Q}$$

There are infinite gaps in the number line of rational numbers. These can be filled with $\mathbb{R} \setminus \mathbb{Q}$ - Irrational numbers, for example $\sqrt{2}, \pi, e$. For example: $\nexists s \in \mathbb{Q} \mid s^2 = 2$.

Reellen Zahlen

Dedekind Cut

A Dedekind cut is a way of representing the real numbers using the rational numbers by cutting the number line into two sections around a “gap” represented by an irrational number. Let $x \subset \mathbb{Q}$ (x contains less elements than \mathbb{Q}), the following properties describe the cut:

$$x \neq \emptyset$$
$$\forall r \in x \forall s \in \mathbb{Q} : s > r \Rightarrow s \in x$$
$$\forall r \in x \exists s_0 \in x : s_0 < r$$

This definition can of course include $\mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$ and therefore the entire \mathbb{R} set.

The elementary number operations (addition, subtraction, multiplication, inequalities etc.) can be defined in terms of Dedekind cuts, precisely defining our understanding of arithmetic. \mathbb{R} (und deshalb auch \mathbb{Q}) ist eine sogenannte “total geordneter K rper”.

Dedekind Completeness - Every nonempty subset of \mathbb{R} with an upper / lower limit has a smallest / largest upper / lower limit.

This proves that the irrational numbers are not complete: $\{r \in \mathbb{Q} \mid r^2 < 2\}$ has no smallest upper limit.

b-adischer Bruch

This is the formal name of the place value system which is defined for all bases ≥ 2 . The values of the digits before the radix point are nb , and $\frac{1}{nb}$ after the radix.

Youngsche Ungleichung

$$x, y, c \in \mathbb{R}$$
$$c > 0$$
$$2|xy| \leq cx^2 + \frac{y^2}{c}$$

Cardinality (M chtigkeit)

Two sets have the same cardinality if they have the same size and therefore a bijective mapping between them exists (see Cantor’s Diagonalmethod).

$$|\mathbb{N}_0| = |\mathbb{Z}| = |\mathbb{Q}| \neq |\mathbb{R}|$$

Complex Numbers

The Real numbers contain no solution for $x^2 = -1$, which is why the imaginary number $i = \sqrt{-1}$ was introduced, first considered by Cardano. They can be used to solve real world problems throughout electrical engineering, particularly for oscillations because powers of i^n have a repetitive nature.

Complex addition is identical to real addition $+_{\mathbb{R}^2}$ between the real and imaginary parts.

Complex multiplication is defined as:

$$\begin{aligned}\cdot_{\mathbb{C}} : \mathbb{R}^2 \times \mathbb{R}^2 &\rightarrow \mathbb{R}^2, \begin{pmatrix} r \\ m \end{pmatrix} \cdot_{\mathbb{C}} \begin{pmatrix} r' \\ m' \end{pmatrix} := \begin{pmatrix} rr' - mm' \\ rm' + r'm \end{pmatrix} \\ (r + mi)(r' + m'i) &= rr' + rm'i + mr'i + mm'i^2 \\ &= rr' - mm' + (rm' + r'm)i\end{aligned}$$

Therefore the complex body is defined as a tuple with the operations:

$$\begin{aligned}\mathbb{C} &:= (\mathbb{R}^2, +_{\mathbb{R}^2}, \cdot_{\mathbb{C}}) \\ i \in \mathbb{C}, i &:= \begin{pmatrix} 0 \\ 1 \end{pmatrix}\end{aligned}$$

It is not a complete body as it doesn't contain any definitions for inequalities, like \mathbb{R} .

The following injective, non surjective function maps real numbers to complex numbers:

$$\mathbb{R} \rightarrow \mathbb{C} : x \in \mathbb{R}, \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathbb{C}$$

There exists a root for -1 in the complex body:

$$\begin{aligned}i^2 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot_{\mathbb{C}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\ \sqrt{-1} &= \pm i\end{aligned}$$

The complex conjugate is defined as follows:

$$\begin{aligned}z &:= a + bi \\ \Re(z) &= a \\ \Im(z) &= b \\ \bar{z} &= a - bi\end{aligned}$$

The euclidian norm is defined as:

$$|z| = \sqrt{|z_1|^2 + |z_2|^2 + \dots + |z_n|^2}$$

General identities:

$$\begin{aligned}z\bar{z} &= |z|^2 \\ \overline{z + z'} &= \bar{z} + \bar{z'} \\ \overline{zz'} &= \bar{z} \cdot \bar{z'}\end{aligned}$$

The function cis can be used to handle complex numbers in polar form:

$$\text{cis}(\theta) = \cos(\theta) + i \sin(\theta)$$

$$\text{cis}(\theta)\text{cis}(\varphi) = \text{cis}(\theta + \varphi)$$

$$\text{cis}\left(k\frac{\pi}{2}\right) = i^k \forall k \in \mathbb{Z}$$

$$z = |z| \text{cis}(\varphi) = |z|e^{i\varphi}$$

$$zz' = |z||z'| \text{cis}(\varphi + \varphi')$$

$$\bar{z} = |z|\text{cis}(-\varphi)$$

Polar form can also be written in terms of Euler's equation, which was derived from the Taylor Series' of e^x and the trigonometric functions:

$$z = |z|e^{i\theta}$$

De Moivre's Theorem:

$$z^k = |z|^k \text{cis}(k\varphi)$$

k'th Roots

For a complex number z , the k 'th roots w are straightforward to determine:

$$w^k = z$$

$$w_j = |z|^{\frac{1}{k}} \text{cis}\left(\frac{\varphi + 2j\pi}{k}\right), j := 0, 1, \dots, k-1$$

Any of these roots to the power of k is equal to z , as well the product of all of them together. If $j \geq k$ the angle completes a full circle and the same roots are found.

The roots of $z = 1$ are called roots of unity, these will be important later in Fourier transforms:

$$w^k = 1$$

$$\zeta_k(j) = e^{\frac{2j\pi i}{k}}, j := 0, 1, \dots, k-1$$

Fundamental Theroem of Algebra - Every non-constant single variable polynomial contains at least 1 complex root.

Sequences and Series

Sequence - A function that maps a natural index $n \in \mathbb{N}_0 \rightarrow \mathbb{C}$

Series - Sequence of partial sums of the terms in a sequence

Taylor Series - A series of derivatives of a function at a point, that converges towards the value of the function at that same point, more on this later..

Geometric Sequence - $n \in \mathbb{N}_0, a_n \rightarrow z^n$ - Converges towards 0 when $|z| < 1$

Harmonic Sequence - $n \in \mathbb{N}_0, a_n \rightarrow \frac{1}{n}$ - Converges towards 0

Archimedes' Axiom

$$\forall x \in \mathbb{R} \exists n \in \mathbb{N}_0, x \leq n$$

Triangle Inequality

$$|x + y| \leq |x| + |y|$$

$$|x - y| \geq |x| - |y|$$

Valid in all dimensions - for example in 2D, the hypotenuse always shorter than the two cathetes.

Bernoulli Inequality

$$(1 + a)^n \geq 1 + na$$

Important: Check which conditions apply when $a, n \in$ different sets

Convergence

A sequence converges towards $A \Leftrightarrow \exists A \in \mathbb{C} \forall \varepsilon \in (0, \infty) \exists n_0 \in \mathbb{N}_0 \forall n \in \mathbb{N}_0 : n \geq n_0, |a_n - A| < \varepsilon$
 $a_n \rightarrow A$ (converges towards A)

We can also express this as a limit:

$$\lim_{n \rightarrow \infty} a_n = A$$

Note: The index n can never actually be set as ∞ , as infinity is not a natural number.

Divergence can be proved by proving the conjugate of the definition of convergence:

$$\text{A sequence diverges} \Leftrightarrow \forall A \in \mathbb{C} \exists \varepsilon \in (0, \infty) \forall n_0 \in \mathbb{N}_0 \exists n \in \mathbb{N}_0 : n \geq n_0, |a_n - A| > \varepsilon$$

The same definitions extend to \mathbb{R}^d by replacing $|a_n - A|$ as the Euclidian norm in that dimension.

Furthermore, a sequence in \mathbb{R}^d converges towards A when each of its components a_n^i converge towards A^i .

Convergence Criteria

Monotone increasing - $a_0 \leq a_1 \leq a_2 \leq a_3 \dots$

Monotonic Criteria - Every monotone increasing / decreasing sequence with an upper / lower limit converges at that limit (the supremum / infimum of the set of members).

If a sequence is defined as the sum, product, quotient or inequality of two convergent sequences, the resulting sequence also converges towards the sum, product, etc. of the contained limits.

Euler's Number

An irrational number defined as:

$$e := \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

This converging sequence was discovered by Bernoulli whilst calculating the effect of frequency of payments on compound interest: [https://en.wikipedia.org/wiki/E_\(mathematical_constant\)#Compound_interest](https://en.wikipedia.org/wiki/E_(mathematical_constant)#Compound_interest)

The exponential growth function e^x also displays the unique property that its derivative at any point is e^x .

Extended Real Numbers

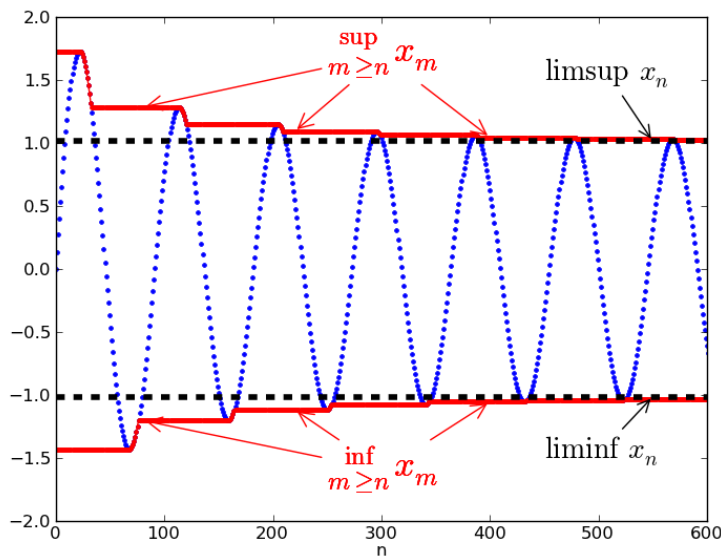
For practical reasons, we can assume the extended real numbers to be defined as:

$$[-\infty, \infty] := \mathbb{R} \cup \{-\infty, \infty\}$$

A sequence diverges towards infinity when:

$$\forall C \in \mathbb{C} \exists n_0 \in \mathbb{N}_0 \forall n \in \mathbb{N}_0 \mid n > n_0 \Rightarrow a_n > C$$

Limes Superior / Inferior



$$\limsup_{n \rightarrow \infty} a_n := \lim_{n \rightarrow \infty} \sup_{i \in \mathbb{N}_0, i \geq n} a_i$$

$$\liminf_{n \rightarrow \infty} a_n := \lim_{n \rightarrow \infty} \inf_{i \in \mathbb{N}_0, i \geq n} a_i$$

If a sequence converges, then:

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sup a_n = \liminf_{n \rightarrow \infty} a_n$$

Cauchy Sequence

Convergence can also be proved without any clue about which value A the sequence converges to by the converging distance between subsequent members. If a sequence satisfies this criteria, it is known as a Cauchy sequence:

$$\forall \varepsilon \in (0, \infty) \exists n \in \mathbb{N}_0 \forall m, n \in \mathbb{N}_0 \mid m, n \geq n_0 \Rightarrow |a_m - a_n| < \varepsilon \Leftrightarrow a_n \text{ converges}$$

Convergence Criteria for Series

The Geometric series can be written as:

$$n \in \mathbb{N}_0, a_n \rightarrow \sum_{k=0}^n z^k = \frac{1 - z^{n+1}}{1 - z}$$

When $z \leq 1$, it converges towards:

$$a_\infty = \sum_{k=0}^{\infty} z^k = \lim_{n \rightarrow \infty} \sum_{k=0}^n z^k = \frac{1}{1 - z}$$

Since series are essentially just a sequence of partial sums, the cauchy convergence criteria can be represented as:

$$\sup_{n \geq m} \left| \sum_{k=m}^n a_k \right| \rightarrow 0 \quad (m \rightarrow \infty)$$

- If a series converges, then the underlying sequence must converge to 0
- On the other hand, a sequence converging to 0 does not imply that the series converges, for example the harmonic series: $\sum \frac{1}{k}$ continues to grow infinitely (albeit extremely slowly)

An alternative convergence criteria is:

$$\lim_{k \rightarrow \infty} \sup \left| \frac{a_{k+1}}{a_k} \right| < 1$$

$$\text{Diverges: } \lim_{k \rightarrow \infty} \inf \left| \frac{a_{k+1}}{a_k} \right| > 1$$

Exponential Taylor Series

TODO