Analysis 1

Contents

Literature:
- Alessio Figalli's 2023 Lecture Notes
- Analysis 1 by H. Amann & J. Escher

What I would like to take notes of:
1. State the theorem / definition, expand with some intuition / memory aids
2. Write the proof by myself if deemed useful for an exam
3. Name some examples only if very helpful

These notes should serve as condensed revision material - only the minimal, important facts to remember before solving problems

1. The Real Numbers

Definition - Set:
An **unordered** collection of **distinct** ($\{x, x\} \equiv \{x\}$) elements such that:
1. It is defined by the elements it contains
2. It is not an element of itself, this prevents Russell's Paradox: $\{x \mid x \notin x\}$
3. Its elements can be filtered by a series of statements which hold true, for example the set of even integers:

$$\{n \in \mathbb{Z} \mid \exists m \in \mathbb{Z} : n = 2m\}$$

   Where $\mid$ and $:$ both mean "such that".
4. The empty set $\emptyset$ contains no elements

1.1. Axioms of The Real Numbers

An axiomatic approach to defining the set of real numbers $\mathbb{R}$.

Definition - Group:
A **non-empty** set G endowed with an operation $\star$ which satisfies the following criteria $\forall a, b, c \in G$:
1. *Associativity* - $a \star (b \star c) = (a \star b) \star c$
2. $\exists$ *Neutral Element* n - $a \star n = n \star a = a$ - Examples:
   - $a + 0 = 0 + a = a$
   - $a \cdot 1 = 1 \cdot a = a$
3. $\forall a \exists$ *Inverse Element* i - $a \star i = i \star a = n$ - Examples:
   - $a + (-a) = (-a) + a = 0$
   - $a \neq 0 \Rightarrow a \cdot a^{-1} = a^{-1} \cdot a = 1$
4. If $a \star b = b \star a$ it is a **commutative group**, although this is not required.

Properties:
- The *Neutral Element* is unique. Proof:

$$\begin{aligned} \text{Let } n, n' \in G \text{ be neutral elements} \\ n \star n' = n = n' \end{aligned}$$

- There is unique *Inverse Element* for all elements. Proof:

$$\begin{aligned} \text{Let } i, i' \in G \text{ be inverse elements for } a \\ i \star (a \star i') = (i \star a) \star i' \\ i \star n = n \star i' \\ i = i' \end{aligned}$$

Examples:
- The non-zero rational numbers $\mathbb{Q} := \left\{\frac{p}{q} \mid p, q \in \mathbb{Z} : p, q \neq 0\right\}$ with operation $\cdot$ is a group, where $n = \frac{1}{1}$ and $i\left(\frac{p}{q}\right) = \frac{q}{p}$
- The natural numbers $\mathbb{N}$ with operation $+$ is **not** a group, as there are no negative inverse elements

Definition - Ring:
A non-empty set R with operations $+$ and $\cdot$.
1. Addition is **always commutative** with $n = 0, i = -a$
2. Multiplication is not necessarily commutative, for example a matrix ring

3. If multiplication is commutative, it is a **commutative ring** and has neutral element $n = 1$
4. It is **not necessarily** a group for multiplication as 0 may be included and has no inverse element $0 \cdot i \neq 1$

Definition - Field:
A commutative ring K (Körper) where $\forall a \in K \mid k \neq 0$ the inverse element for multiplication exists.
1. Addition: $n = 0, i = -a, -(-a) = a$
2. Multiplication: $n = 1, i = a^{-1}, \left(a^{-1}\right)^{-1} = a \mid a \neq 0$

Examples:
- $\mathbb{Z}$ is a ring but not a field as there is no multiplicative inverse element for all non-zero elements
- The complete set of rational numbers $\mathbb{Q} := \left\{\frac{p}{q} \mid p, q \in \mathbb{Z} : q \neq 0\right\}$ is a commutative ring and a field.
- $0 \cdot a = 0$. Proof:

$$0 = 0 \cdot a - 0 \cdot a = (0 - 0) \cdot a = 0 \cdot a$$

Definition - Cartesian Product:
For the sets X, Y, the Cartesian product is the set of tuples (ordered lists): $X \times Y := \{(x, y) \mid x \in X, y \in Y\}$
- The number of elements in the set is:

$$|X \times Y| = |X| \cdot |Y|$$

Example:

$$X := \{0, 1\}, Y := \{\alpha, \beta\}$$
$$X \times Y := \{(0, \alpha), (0, \beta), (1, \alpha), (1, \beta)\}$$

Definition - Subset:
A set whose elements are entirely contained in a parent set with the following notation:
- $P \subseteq Q$ - P is a subset of Q and they may be equal
- $P \subsetneq Q$ - P is a **proper** subset of Q; Q has at least 1 additional element
- $P \not\subset Q$ - There is at least one element in P that is not in Q
- The same applies in reverse using sup(er)set notation $\supseteq$
- The symbols $\subset$ and $\supset$ are ambiguous in meaning
- Two sets can be shown to be equal if $P \subseteq Q \land Q \subseteq P$ holds true

Definition - Relationship:
A relationship on X is the subset $\mathfrak{R} := \{(a, b) \in X \times X \mid a \sim b\}$ where $\sim$ is an operator for expressing conditions called a relation and may have the following properties if the corresponding condition holds true $\forall x, y, z \in X$:
- Reflexive - $x \sim x$ - Example: $\leq$
- Transitive - $x \sim y \land y \sim z \Rightarrow x \sim z$ - Example: $<$
- Symmetric - $x \sim y \Rightarrow y \sim x$ - Example: $=$
- Anti-Symmetric - $x \sim y \land y \sim x \Rightarrow x = y$ - Example: $\leq$ - Although such relations are often reflexive, this is not a requirement, consider $\mathfrak{R} = \emptyset, X = \{1\}$
- A relation is called **equivalence relation** if it is reflexive, transitive and symmetric

- A relation is called **order relation** if it is reflexive, transitive and anti-symmetric

TODO: As part of set operations
Definition - Complement:
$A \subseteq X, A^{\complement} = X \setminus A$ The elements of a set excluding those that appear in a set.

2. Functions of one Real Variable