Prove that the Legendre Polynomials P_{0-3} are a basis for $\mathcal{P}_4(\mathbb{R}_3[x])$:

To show that they are a basis, we need to show that:

1. They span (and as a basis are also a subset of) $\mathcal{P}_4$
2. They are linearly independent

Firstly we express them in terms of the Monomes p_n:

$$
\begin{aligned}
P_0 &= 1 = p_0 \\
P_1 &= x = p_1 \\
P_2 &= \frac{1}{2}(3x^2 - 1) = \frac{3}{2}p_2 - \frac{1}{2}p_0 \\
P_3 &= \frac{1}{2}(5x^3 - 3x) = \frac{5}{2}p_3 - \frac{3}{2}p_1
\end{aligned}
$$

They span $\mathcal{P}_4$:

$$
\begin{gathered}
\boldsymbol{x} \in \mathbb{R}^4 \\
\text{Let } q = x_0P_0 + x_1P_1 + x_2P_2 + x_3P_3 \\
= x_0p_0 + x_1p_1 + x_2\left(\frac{3}{2}p_2 - \frac{1}{2}p_0\right) + x_3\left(\frac{5}{2}p_3 - \frac{3}{2}p_1\right) \\
= \left(x_0 - \frac{1}{2}x_2\right)p_0 + \left(x_1 - \frac{3}{2}x_3\right)p_1 + \left(\frac{3}{2}x_2\right)p_2 + \left(\frac{5}{2}x_3\right)p_3
\end{gathered}
$$

Therefore q can be expressed as a linear combination of Monomes p_{0-3} and $\therefore q \in \mathcal{P}_4$.

$\mathcal{P}_4$ can be shown to be a subset of $\operatorname{Span}\{P_0, P_1, P_2, P_3\}$ in the same manner.

They are linearly independent:

If the 0 vector is the only solution for $\boldsymbol{x}$, they are linearly independent.

$$
\begin{gathered}
\boldsymbol{x} \in \mathbb{R}^4 \\
x_0P_0 + x_1P_1 + x_2P_2 + x_3P_3 = 0 \\
\left(x_0 - \frac{1}{2}x_2\right)p_0 + \left(x_1 - \frac{3}{2}x_3\right)p_1 + \left(\frac{3}{2}x_2\right)p_2 + \left(\frac{5}{2}x_3\right)p_3 = 0
\end{gathered}
$$

The Monomes have already been proven to be linearly independent in Dr Gradinaru's script, hence:

$$
\begin{aligned}
x_0 - \frac{1}{2}x_2 &= 0 \\
x_1 - \frac{3}{2}x_3 &= 0 \\
\frac{3}{2}x_2 &= 0 \\
\frac{5}{2}x_3 &= 0
\end{aligned}
$$

As a Lineare Gleichungs System:

$$\begin{pmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -\frac{3}{2} \\ 0 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & \frac{5}{2} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

This matrix has full rank and therefore only has the trivial solution (0) for $\boldsymbol{x}$.

Therefore these Legendre Polynomials are linearly independent and their Bild is equal to $\mathcal{P}_4$, meaning they are a basis for the space $\mathbb{R}_3[x]$ ■.