Technische Mechanik

Contents

Kinematiks - How a model is currently at motion without concern for the causes (forces)
Statics - Which conditions (forces & moments) are needed to keep a system at rest
Dynamics - Which conditions are needed to create movement in a system in a certain way

Vector Identities:
$\vec{a} \times (b + c) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
$a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b)$ (Pacman Identity)

Trig Identities:
$\cos(\frac{\pi}{2} - \theta) = \sin(\theta)$
$\sin(\frac{\pi}{2} - \theta) = \cos(\theta)$
$\sin(-\theta) = -\sin(\theta)$
$\cos(-\theta) = \cos(\theta)$
$\tan(-\theta) = -\tan(\theta)$

Calculus Rules:
$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
$\frac{d(uv)}{dx} = u\frac{dv}{dx}v\frac{du}{dx}$

Notation:

κ - Set of all points in a body

Time derivative - $\dot{x} = \frac{dy}{dt}$

Coordinate Systems

Orthogonale Koordinatensystemen:

$$\text{Cartesian: } e_x \times e_y = e_z$$
$$\text{Cylindrical: } e_\rho \times e_\varphi = e_z$$

Position Vectors:

$$\text{Cartesian: } \boldsymbol{r} = x e_x + y e_y + z e_z$$
$$\text{Cylindrical: } \boldsymbol{r} = \rho e_\rho + z e_z$$

There is no separate e_φ component in a cylindrical position vector, as it's already accounted for by the changing e_ρ unit vector. The following relations are useful for calcutions in the cylindrical co-ordinate system:

$$\text{Wegen des Einheitskreises: } e_\rho = \cos(\varphi) e_x + \sin(\varphi) e_y$$
$$\text{Intuitiv: } e_\varphi = \frac{d(e_\rho)}{d\varphi} = -\sin(\varphi) e_x + \cos(\varphi) e_y$$
$$\frac{d(e_\varphi)}{d\varphi} = -\cos(\varphi) e_x - \sin(\varphi) e_y = -e_\rho \text{ (Centripetal acceleration!)}$$
$$\frac{d(e_\rho)}{d\varphi} = e_\varphi$$

The time derivatives can be found by deriving the cartesian formulae with respect to time and doing some substitution:

$$\frac{d(e_\varphi)}{dt} = -\dot{\varphi} e_\rho$$
$$\frac{d(e_\rho)}{dt} = \dot{\varphi} e_\varphi$$

Thus the velocity formula in the cylindrical co-ordinate system:

$$\begin{aligned}\vec{v} &= \frac{d(\vec{r})}{dt} \\ &= \dot{\rho} e_\rho + \rho \dot{\varphi} e_\varphi + \dot{z} e_z\end{aligned}$$

Rigid bodies

A body in which deformation is negligible. There are no ideal rigid bodies in real life.

Let P, Q be points in a rigid body

$$\forall P, Q \in \mathbb{R}^3, |\boldsymbol{r_Q} - \boldsymbol{r_P}| = \text{Constant}$$

Satz der Projizierten Geschwindigkeiten

The velocities of any two points in a rigid body projected on the vector between them is always the same. This means the body is not getting shorter or longer (deforming).
Useful for determining the velocities of points on rigid bodies with relation to each other.

$$\overrightarrow{v_Q} \cdot e = \overrightarrow{v_P} \cdot e$$
$$\text{wo } e = \frac{r_Q - r_P}{|r_Q - r_P|}$$

$\overrightarrow{v_A} \cdot e = v'_A$ = Velocity of A projected onto the vector alongside a rigid body.

Translation - for all points $P, \overrightarrow{v_P}$ is equal

Movement across a plane

- All velocities are parallel to a certain plane
- All points along a normal to the plane have the same velocity
- It is either a translation or a rotation at any point in time

Rotation

If at least two points in a rigid body do not have the same velocity, it is currently rotating. The momentary, static center of rotation is the intersection of lines perpendicular to the velocities of two points. All points rotate around the center with the **same angular velocity** ω.

Considering a point with vector $\overrightarrow{r_P}$ from the center of rotation, rotating with angular velocity $\omega = \frac{d\Theta}{dt}$. Its velocity vector can be determined as:

$$\overrightarrow{v_P} = (\omega e_z) \times \overrightarrow{r_P}$$

The unit vector e_z is simply needed so the resulting direction is anticlockwise (for a positive ω) and perpendicular to $\overrightarrow{r_P}$.

Polbahn - The path traced by the series of momentary centers of rotation of a rigid body.

Movement in space

In 3D space, simultaneous translation & rotation is possible due to the extra dimension.

Rotation Axis - The body rotates around an entire axis instead of a single point. The velocity of all points on this axis are the same and equal to the object's overall translational velocity. $\vec{\omega}$ is defined as the unit vector in the direction of the axis times the angular velocity: $\vec{\omega} = \omega \vec{e_r}$

LTD: Parametric equation of points along the rotational axis

Starrkörperformel

The following extremely useful formula can be used to link the unique angular velocity vector to the velocity of any two points in a rotating body:[1]

$$\overrightarrow{v_P} = \overrightarrow{v_B} + \vec{\omega} \times \overrightarrow{r_{BP}}$$

This essentially shows that every point in a rigid body rotates around every other point in the body with the same angular velocity.

The following properties of movement in space are constant and called "Invariants":

1. $I_1 = \vec{\omega} \forall P \in \kappa$ - The angular velocity is the same regardless of the reference point
2. $I_2 = \vec{\omega} \cdot v_P \forall P \in \kappa$ - The component of the velocity of a point in the direction of the rotation axis is the same for all points in the body. This is the translation velocity of the body.

Therefore the momentary movement of any point in the body can be described with just two values called the **Kinemate**: $\{\overrightarrow{v_B}, \vec{\omega}\}$

[1]Derivation available in the 5th Powerpoint of Dr. P Tiso

LTD: Test these in a simulation :)

Schraubung - The combination of a rotation with a translation in the direction of the rotation axis

Types of movement in space:
1. Translation: $\vec{\omega} = 0$
2. Rotation: $\vec{\omega} \neq 0 \wedge I_2 = 0$
3. Schraubung: $I_2 \neq 0$

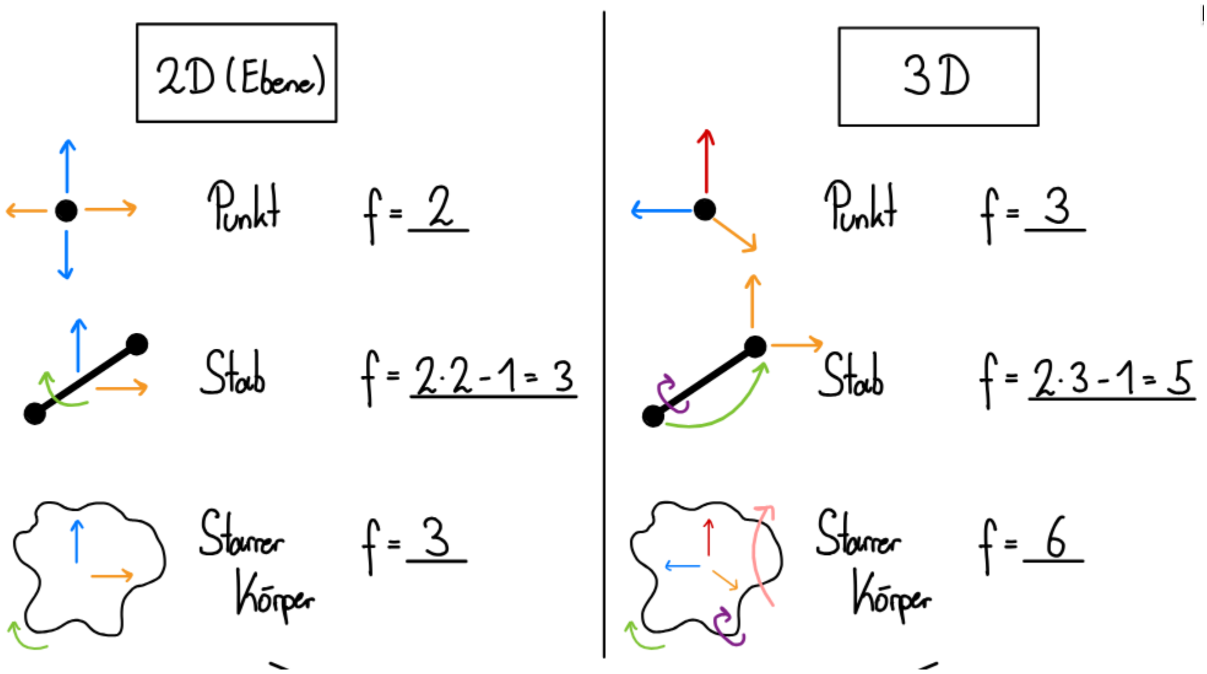
LTD: Understand Rechteck Beispiel in script

Degrees of freedom

The minimum number of coordinates (in an arbitrary optimal coordinate system for this specific system) to clearly determine the state of a system. This could for example be the location of a slider and the angle at which a rod is attached to it (much more concise than for example the cartesian coordinates of the slider and the other end of the rod).

Considering a system with several bodies. For a sum of degrees of freedom of n, and b restricted degrees of freedom due to connections, the resulting degrees of freedom of the whole system is:

$$f = n - b$$



IMPORTANT: The joint at a roller / pivot must be accounted for too! (using the n-Gelenk formula)

b von	2D	b von	2D
Auflager (beidseitig)	1	Gelenk (2 SKs verbunden)	2
Gelenk (Festlager)	2	Rollen ohne gleiten	2
Einspannung	3	Gelenk (n SK verbunden)	(n-1)·2

TODO: b-value for slider? 1

Forces

Force - An influence that can cause an object's velocity to change. It is a vector quantity applied at an attack point. The line through the attack point in the direction of the force is called the line of action.

In reality, there are 4 fundamental forces (electromagnetic, gravitational, weak and strong nuclear) but in many practical applications we consider integral values such as contact forces and friction.

Inner vs outer forces

Every inner force in a system exists in a pair with its corresponding reaction force. Forces without a corresponding reaction are so called *external forces*.

$$\sum \text{Inner Forces} = 0$$

Force groups

The set of forces acting on a body is known as the force group.

Two force groups are statically equivalent when:

$$\mathcal{P}_{\text{tot}}(G_1) = \mathcal{P}_{\text{tot}}(G_2)$$

Static equivalence in a rigid body

Due to the total power of a rigid body formula (see power), static equivalence in a rigid body requires:

$$R_1 = R_2$$
$$(M_B)_1 = (M_B)_2$$

Furthermore, two forces are equivalent if they have the same magnitude and line of action.

Forces with lines of action going through the same point have only a resultant force - no moment:

$$M_P = 0, R \neq 0$$

Moments

A moment is a concept for describing the capacity of a force to rotate an object around an arbitrary center of rotation with units Nm.

The moment of a force around the center of rotation O in vector form is:

$$M_O = \overrightarrow{r_{OP}} \times \vec{F}$$

The resulting moment lies along the axis of rotation and describes the angular direction of the rotation caused by the moment.

Alternatively, it can be expressed as a scalar with the perpendicular distance from the line of action of F to O: d:

$$M_O = dF$$

Transformation of moments

The moment of a force can be transformed with respect to a different point using the following formula:

$$M_A = M_B + r_{AB} \times R$$

Torque

A torque (also known as couple) is a pair of moments with respect to the same point with equal magnitude F and opposite direction. Considering the perpendicular distance between their lines of action d, these result in:

$$R = 0$$
$$M_P = dF$$

Whenever the resultant force is 0, the moment around all points in the body is the same.

Dynamic

The dynamic of a force group with respect to a point O describes the entire set of forces on the body:

$$\{R, M_O\}$$

Where R is the resultant force and M_O is the resultant moment around O: $\sum \overrightarrow{r_{OP_i}} \times \vec{F_i}$

The following invariants apply to the dynamic:

- $I_1 = R \forall P \in \kappa$
- $I_1 = R \cdot M_O \forall P \in \kappa$ - the component of the resultant in the direction of the moment with respect to the same point is the same for all points

Power

The rate of transfer of energy.

Due to the work done by a force $\int_c \vec{F} d\vec{s}$, the power exerted by a force at a point in time can be expressed as:

$$\mathcal{P} = \vec{F} \cdot \vec{v}$$

- *Accelerating force* ($\frac{\pi}{2} < \alpha \leq \pi$) - A force with a positive component in the direction of the velocity is contributing kinetic energy to the object and increasing the power
- *Braking Force* ($0 < \alpha < \frac{\pi}{2}$) - Reduces the kinetic energy of the object
- A force perpendicular to the velocity of an object does not contribute to its power until the object begins moving with a component in the direction of the perpendicular force.

Total power of a rigid body

The total power of an object is the sum of powers for each force acting on the body:

$$\mathcal{P}_{\text{tot}} = \sum_{i=1}^{n} \vec{F}_i \cdot \vec{v}_i$$

When the kinematic $\{v_B, \omega\}$ and dynamic $\{R, M_B\}$ with respect to a point B are known, we can calculate the total power thanks to the rigid body formula and the "pacman" identity:

$$\mathcal{P}_{\text{tot}} = R \cdot v_B + M_B \cdot \omega$$

Parallel Forces

When all forces acting on a body point point in the same direction, they can be written as:

$$\vec{F}_i = F_i \vec{e}$$

Where $\vec{e}$ is the unit vector of their common direction.

The dynamic with reference to a point O is:

$$R = \sum \vec{F}_i = \vec{e} \sum F_i$$
$$M = \sum \vec{r}_i \times \vec{F}_i = \sum \vec{r}_i F_i \times \vec{e}$$

The sum $\sum \vec{r}_i F_i$ is called the dipole moment (sometimes written as N) of a set of parallel forces. It is independent of the point O, as long as it's consistent for each force included.

Center of Forces

This is the point on which a pivot can be placed and no resultant moment would act on the body. In other words, an equal and opposite resultant force can act on this point resulting in a net 0 dynamic.

It is unique to a dipole moment, the direction of $\vec{e}$ is irrelevant (although gravity always acts in the same direction, so in practice this fact isn't too important).

The position vector of the center of forces from the point O can be calculated using a dipole moment with **the same point O** as its origin:

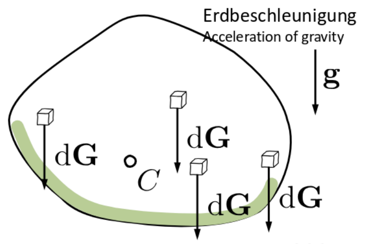
$$r_{OC} = \frac{\sum \vec{r}_i F_i}{\sum F_i}$$

For a dipole moment with the specific direction of the forces $\vec{e}$, an entire line can serve as a center of forces:

$$\overrightarrow{r_{OC}}(\lambda) = \overrightarrow{r_{OC}} + \lambda \vec{e}$$

Center of Mass

This is the average location of all the weight of an object. It can be calculated using a volume integral over the density of the body:



Infinitesimale Schwerkraft: $d\mathbf{G} = \mathbf{g}dm$

Infinitesimal weight force:

Schwerkraft: $\mathbf{G} = m\mathbf{g}$

Weight force:

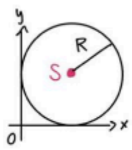
$$\mathbf{r}_{OC} = \frac{\sum F_i \mathbf{r}_i}{\sum F_i} \rightarrow \mathbf{r}_{OC} = \frac{\iiint g\mathbf{r}dm}{\iiint g dm} = \frac{g\iiint \mathbf{r}dm}{g\iiint dm} = \frac{g}{mg}\iiint \mathbf{r}dm = \frac{1}{m}\iiint \mathbf{r}dm$$

$$\boxed{\mathbf{r}_{OC} = \frac{1}{m}\iiint \mathbf{r}dm \qquad \begin{aligned} x_{OC} &= \frac{1}{m}\iiint x dm \\ y_{OC} &= \frac{1}{m}\iiint y dm \\ z_{OC} &= \frac{1}{m}\iiint z dm \end{aligned}}$$

ETH zürich

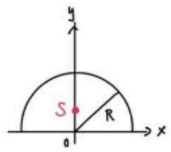
Where dm is effectively the density at the point of each differential.

Kreis:



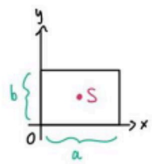
$$\vec{r}_{OS} = \begin{pmatrix} R \\ R \end{pmatrix} \qquad A = \pi r^2$$

Halbkreis:



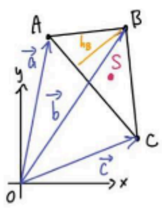
$$\vec{r}_{OS} = \begin{pmatrix} 0 \\ \frac{4R}{3\pi} \end{pmatrix} \qquad A = \frac{1}{2}\pi r^2$$

Rechteck:



$$\vec{r}_{OS} = \begin{pmatrix} a/2 \\ b/2 \end{pmatrix} \qquad A = a \cdot b$$

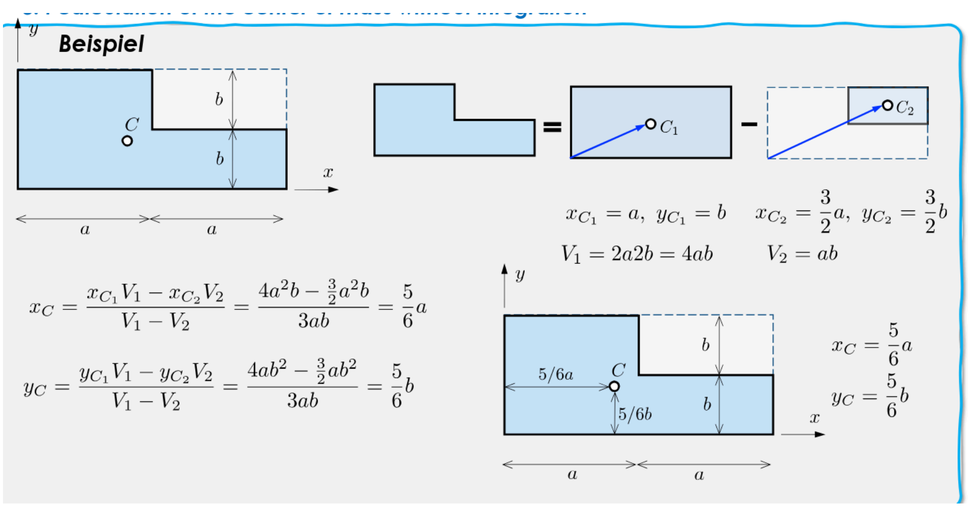
Dreieck:



$$\vec{r}_{OS} = \frac{\vec{a} + \vec{b} + \vec{c}}{3} \qquad A = \frac{1}{2} AC h_B$$

Integration is a linear transformation - a center of mass can be calculated as the sum of separate integrals (which of course can also be negative):

$$\overrightarrow{r_{OC}} = \frac{\sum \oiiint \vec{r} dm}{\sum m_i}$$



LTD: General formula

$$\overrightarrow{r_C} = \frac{\sum m_i \overrightarrow{r_{Ci}}}{\sum m_i}$$

Rest

A system is at rest when all of its velocities are 0.

- Instantaneous rest - $\vec{v_p} = 0 \forall p \in \kappa \mid t = t_0$
- State of rest - $\vec{v_p} = 0 \forall p \in \kappa, \forall t$

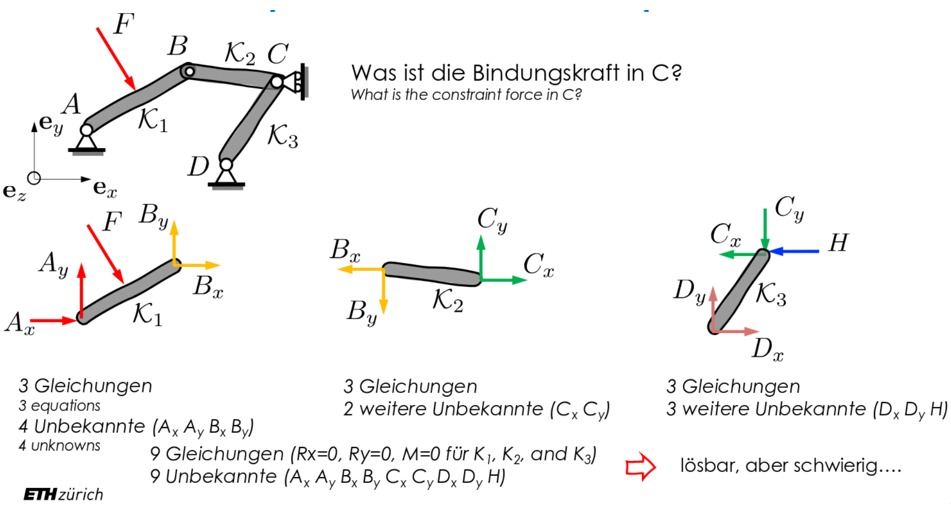
Fundamental Theorem of Statics

This also means that the resultant force and moment around any point (the moment is always the same if $R = 0$) is 0:

$$\vec{R} = 0$$
$$\vec{M} = 0$$

Constraints exert equal and opposite forces to prevent an object moving through them.

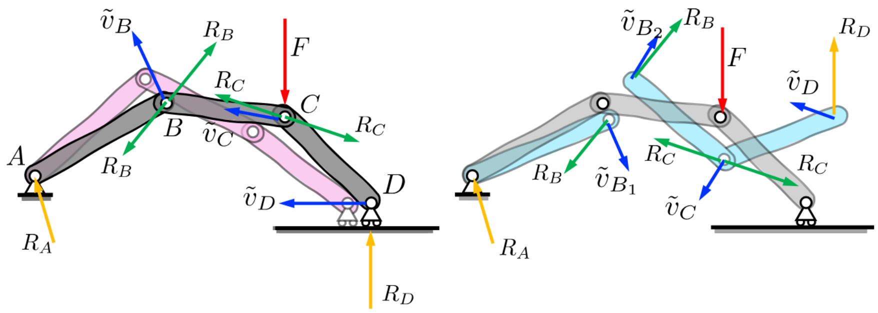
Forces in a system at rest with multipled bodies can be solved by including forces at constraints:



These systems of equations can be solved through Gaussian elimination.

Virtual Motion

A system can be modelled as in virtual motion when virtual velocities are modelled at each point, such that the total power of constraint forces is 0:



Zulässige virtuelle Bewegunszustände:
Leistung der Bindungskräfte = 0
Admissible state of motion:
The power of the contraint forces is 0

Unzulässige virtuelle Bewegunszustände:
Leistung der Bindungskräfte≠0
Non admissible state of motion:
The power of the contraint forces≠0

- For example, velocity at the slider is perpendicular to its constraint force, and equal and opposite constraint forces at joints cancel each other's accelerating and braking power out.
- There can be no virtual velocity at a pivot, as it exerts constraint forces in both the x and y component - they are orthogonal and the power of this force would never be 0.
- There can however exist a rotational velocity around a pivot.

These virtual velocities, denoted as $\tilde{v}$ can be found using techniques in the kinematics of rigid bodies.

Theorem of Virtual Power

A system is at rest when the virtual total power is 0 for every virtual state of motion:

$$\mathcal{P}_{\text{tot}} = \sum_{i=1}^{n} \vec{F}_i \cdot \tilde{v}_i = 0$$

This is useful to calculate a few external / constraint forces, by strategically allowing virtual motion which involves that force. Other forces with and equal and opposite reaction can be ignored as the resultant at that point is 0.

If many forces in a system are needed, then a full analysis using the theorem of statics is more appropriate.

Framework

The constraints in a framework can be calculated by removing one of the rods, whose compression (or tension, this becomes apparent if a negative value is calculated) is acting as a constraint force.

LTD: Force cut

Power of Torque

It is useful to solve statics problems with pulleys using the virtual power of a torque (the cable around a pulley has the same tension throughout):

$$\mathcal{P} = \vec{\omega} \cdot \overrightarrow{M}$$

As usual in a torque, the origin of the moment is irrelevant.

LTD: Do not assume all tensions are equal in pulleys exam!

TODO: Add useful trig identities, special triangles

Static Determinacy
If the number of constraint forces and moments is equal to the number of equilibrium conditions. TODO: What does this imply? A static can always be determined? Clean up

This essentially means that the system of equations doesn't have full rank and has infinite / no solutions.

$f > 0$ - the system is statically undetermined, movement means the forces can take on infinite values $f = 0$ - statically determined $f < 0$ - constraint forces cannot be determined, overconstrained intuition? is it because either one of the constraints can be high enough to stop movement (we can't say for sure which one), multiple constraints per possible movement?

Kinematic Certainty

Stability

Surface integral of friction forces on contact area. TODO: Maths

System is stable if the average point on which the normal force acts is within the contact surface

ie weight is acting through or before the tipping point

This is tipping point from A levels :D