

# Netzwerke und Schaltungen 1

## Contents

Das Coulomb'sche Gesetz .....	2
Das elektrostatische Feld .....	2
Das elektrostatische Potential .....	2
Elektrische Fluss (Flux) .....	3
Gauss'sche Gesetz .....	4
Line Charge .....	4
Infinite Charged Plane .....	4
Leitenden Koerper (Macroscopic Level) .....	5
Dielectrics .....	5
Dipoles .....	6
Permittivity .....	6
Boundary Conditions .....	7
Capacitance .....	7
Capacitor Networks .....	8
Real-world capacitors .....	8
Energy stored by capacitor .....	9
Current .....	9
Current Density .....	9
Resistance .....	9
Specific Resistivity ( $\rho_R$ ) & Conductivity ( $\kappa$ ) .....	10
Ohm's Law .....	10

## Surface / Volume Integration nudge factors:

Cylindrical coordinate system:  $r$

Spherical coordinate system:  $r^2 \sin(\vartheta)$  (where  $\vartheta$  is the angle from the z axis)

## Das Coulomb'sche Gesetz

$$\vec{F}_2 = \frac{Q_1 Q_2 \vec{e}_{12}}{4\pi\epsilon |\vec{r}_{12}|^2}$$

$\epsilon$  is often replaced with  $\epsilon_0$  as calculations are assumed to take place in a vacuum.

## Das elektrostatische Feld

- Distance is from centre of point charge
- A positive test charge of 1C is used to plot electric fields - this means a positive charge has arrows away from it
- Unit vector is always  $\frac{\vec{r}_{12}}{|\vec{r}_{12}|} = \vec{e}_{12}$
- Electric field lines cannot be closed loops with a constant direction - this would lead to perpetual motion (see electric potential closed integral)

*Feld* - die eigenschaften eines Raums, die eine Wirkung auf andere Ladungstoffen haben

*Homogene Feld* - Field with the same strength and direction in all points. In reality it exists only when zooming in on a small area. Antonym: *inhomogene / ortsabhaengige Feld*.

$$\vec{E} = \frac{\vec{F}}{Q_2}$$

Um die Kraft in einer gewisse Punkt im Raum zu finden, man muss die Ladung mit dem Feld an diesem Punkt multiplizieren.

Any point in a field always has a unique direction. The electric field at the equidistant point between two like charges is 0 as they cancel each other out.

*Quantitative* - Numbers based

*Qualitative* - Interpretation based

A cluster of like charges behaves like a point charge with the sum of their charges on the macroscopic scale. Opposite charge clusters (ex. Dipole) lead to no field around them on a large scale.

## Das elektrostatische Potential

Electrostatic potential energy of a particle as it moves from  $P_0 \rightarrow P_1$ :

$$\vec{F} = \vec{E}Q$$
$$W_e = - \int_{P_0}^{P_1} \vec{F} \cdot d\vec{s} = -Q \int_{P_0}^{P_1} \vec{E} \cdot d\vec{s}$$

The work done by the electric field as a positive test charge moves towards another positive is negative, as external energy is needed in order to overcome the repulsive force. The potential energy of the test charge increases due to the incoming energy, so  $W_e$  is made positive.

$\oint$  - Closed integral, when a line begins and ends at the same point in space.

Provided the speed as the charge moves through the field tends towards 0 (to minimize the arising electromagnetic field):

$$\oint_C \vec{E} \cdot d\vec{s} = 0$$

Electrostatic potential is the electrostatic potential energy of a particle at a point in an electric field.

$$\varphi = \frac{W_e}{Q}$$

A reference potential (ground) must always be defined, often the Earth's surface / an infinitely far away point is taken as  $\varphi_e = 0$ . In a circuit, the negative terminal is often used.

Taking  $r_1$  as an infinitely far away point with potential 0, the electrostatic potential in the space surrounding a point charge  $Q$  as a scalar is:<sup>1</sup>

$$\begin{aligned}\varphi(P_0) &= 0 \\ \varphi(r) &= - \int_{P_0}^r \vec{E} \cdot d\vec{s}\end{aligned}$$

Electrostatic potential is positive in a positive electric field.

The change in electrostatic potential does not depend on the path taken through the field, only the start and end point.

$$W_e = -Q \int_{P_0}^{P_1} \vec{E} \cdot d\vec{s} = Q [\varphi_{e(P_1)} - \varphi_{e(P_0)}]$$

*Voltage (U)* - Difference between two potentials with the same reference potential.

$$U_{12} = \varphi(P_1) - \varphi(P_2) = \int_{P_1}^{P_2} \vec{E} \cdot d\vec{s}$$

## Elektrische Fluss (Flux)

*Electric Displacement Field (aka Flux Density)*: An imaginary field that describes the effect of a charge on space, independent of if it's through a vacuum, dielectric or conductor.

$$\vec{D} = \epsilon \vec{E}$$

$$\text{Point charge} = \vec{e}_r \frac{Q}{4\pi r^2}$$

*Elektrische Fluss (Ψ)* - Total flux density flowing through a surface. Considering a charge  $Q$  inside a sphere with radius  $r$ :

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<sup>1</sup>Derivation in Elektrotechnik, Albach 1.8.1

$r = \text{constant}$

$$\Psi_D := \oint \vec{D} \cdot d\vec{A}$$

Nudge factor needed for spherical coordinate system

$$= \int_0^{2\pi} \int_0^\pi r^2 \sin(\vartheta) \varepsilon_0 \overrightarrow{E(r, \vartheta, \varphi)} \cdot d\vartheta d\varphi$$

$$= \frac{\varepsilon_0 Q r^2}{4\pi \varepsilon_0 r^2} \int_0^{2\pi} \int_0^\pi \sin(\vartheta) \vec{e}_r \cdot d\vartheta d\varphi$$

$\vec{e}_r \cdot d\vartheta d\varphi$  is 1, as they are always parallel.

$$= \frac{Q}{4\pi} \int_0^{2\pi} (-\cos(\pi)) - (-\cos(0)) d\varphi = \frac{Q}{4\pi} \int_0^{2\pi} 2 d\varphi$$

$$= \frac{Q}{4\pi} ((4\pi) - (0)) = \frac{4\pi Q}{4\pi} = Q$$

## Gauss'sche Gesetz

The above derived relationship is known as Gauss's law:

$$\Psi_E := \oint \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0}$$

$$\Psi_D := \oint \vec{D} \cdot d\vec{A} = Q$$

The total electric flux density through an arbitrary closed surface (electric flux) is equal to the charge enclosed inside, regardless of the charges position / area of the surface.

This law is one of Maxwell's equations and can be used in reverse with infinitely small symmetric Gaussian surfaces to calculate the electric field around certain charge distributions, for example, the electric field at the surface of any point on a charged plane.<sup>2</sup>

## Line Charge

The electric field at distance  $r$  from a line charge with uniform charge distribution  $\lambda$  can be calculated using either the integral of contributions to the field by small sections of the line, or far simpler using a cylinder and Gauss's law:

$$\overrightarrow{E(r)} = \frac{\lambda}{2\pi \varepsilon r}$$

## Infinite Charged Plane

As derived using Gauss's law, the electric field around an infinitely large charged plane with surface charge density  $\sigma$  is:

$$\vec{E} = \frac{\sigma}{2\varepsilon}$$

This is the same regardless of distance from the plane, as when moving further away, more area contributes to the superposed field.

The direction of the field is perpendicular to the surface of the plane.

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<sup>2</sup>See *Electricity and Magnetism* | Purcell for some great derivations of electric fields by various charge distributions.

When two charged planes with equal and opposite charge densities are parallel to one another (capacitor), the field behind each one is cancelled due to their superposing opposite fields. The electric field between the two planes is twice as strong:

$$\vec{E} = \frac{\sigma}{\varepsilon}$$

If they have an equal charge distribution, the electric field between plates of a capacitor is homogenous. This results in the potential difference between them:

$$U = Ed = \frac{\sigma d}{\varepsilon}$$

If an isolated, conductive plate is brought in between them, the total distance between the plates throughout which a test charge is under the influence of an electric field decreases by the thickness of the conductive plate. This results in a lower potential difference between the plates.

### Leitenden Koerper (Macroscopic Level)

The electric lines are always perpendicular to the surface of a conductor, as any tangential component of the field redistributes charges to prevent this.

The electric field inside a conductor is always 0, as the free charges repel each other and arrange themselves on the surface, of which the superposed electric fields at any point in the conductor are 0. The same applies if the conductor is brought into an external electric field.

The electrostatic potential inside a conductor is constant and equal to the surface, otherwise work would have to be done to move to the surface, which is not the case. Another way of thinking about it, is that energy was required to move a test charge into the conductor from infinitely far away.

The negative charge on the surface of a conductor arises from a **surplus of electrons**. Positive charge arises due to lack of electrons - a **surplus of holes**. Protons do not move throughout the conductor.

*Influenzierten Ladungen* - The separation of charges influenced by an electric field.

*Faraday Cage* - A hollow, conductive, closed volume. In an external electric field, the free charges are influenced so that the electric field inside the cage (the walls of the cage and the hollow inside it) is cancelled out. This phenomenon is very useful for electromagnetic shielding. A conductive hollow volume with gaps also reduces the inner field vastly, although it is not perfect.

### Dielectrics

*Dielectric* - An electric insulator that can be polarised by an electric field. Regular “insulators” tend to have a low degree of polarisation.

Although dielectrics do not contain free charges, they still experience a small degree of polarisation at the atomic level, as the cloud of electrons surrounding the nucleus is distorted in the direction of the applied electric field.

When a dielectric is placed between two plates, there is still a (macroscopic) electric field throughout the dielectric (albeit lower than through a perfect insulator, as the dipoles generate fields in the opposite direction). This results in a lower potential difference between the plates, although still higher than if it were a conductor.

## Dipoles

*Dipole* - A phenomenon when electric charges / magnetic poles are separated. Examples of electric dipoles are ions under the influence of an external electric field or certain molecules, such as  $H_2O$  which are permanent dipoles due to their asymmetric structure.

*Electric Dipole Moment* - for a dipole with charge  $Q$  on the positive end and vector  $\vec{d}$  from the main negative point to the main positive point, the electric dipole moment is:

$$\vec{p} = Q\vec{d}$$

It can be thought of as a measure of a systems polarity.

When a material containing dipoles is in an external electric field, the dipoles align in the same direction, resulting in an overall polarization of the material (essentially a larger dipole).

Temperature has a large effect on this, as the random movements of particles disrupt the perfect alignment encouraged by the electric field.

The overall polarization of a dielectric with volume  $V$  is:

$$\vec{P} = \frac{\sum_{i=1}^n \vec{p}_n}{V}$$

A polarized conductor / dielectric with a net 0 charge in a non-homogeneous field, such as near a point charge will usually have a resultant force, for example a dipole near a point charge (regardless of polarity) will experience a stronger attractive force and the dipole will have a resultant force towards the point charge.

## Permittivity

*Absolute Permittivity / Dielectric Constant* - A measure of the electric polarizability of a material, ie. the capacity of it to oppose / how well it stores energy from an external electric field through polarisation. Higher for conductors / dielectrics than insulators.

$$\epsilon = \epsilon_r \epsilon_0$$

*Permittivity of free space* -  $\epsilon_0 = 8.85418782 \times 10^{-12}$

*Relative Permittivity* -  $\epsilon_r$  (sometimes  $\kappa$ ) - The ratio of a material's permittivity  $\epsilon$  to the permittivity of free space  $\epsilon_0$ . The permittivity of air is very close to that of a vacuum  $\epsilon_0$ , hence its relative permittivity is nearly 1.

$$\epsilon_r = \frac{\epsilon}{\epsilon_0}$$

The higher the permittivity of a medium, the more its electric field is reduced due to polarisation. Electric field strength is inversely linearly proportional to the permittivity, as seen in Coulomb's law (but not for all materials, for example ferroelectric crystals).

**LTD: The following thinking is not completely correct, I am still wrapping my head around what exactly the displacement field and permittivity is. Revisit later after reading Purcell**

Taking place in a medium with permittivity  $\epsilon$ :

The difference in the displacement field before and after polarisation:

$$\vec{P} = \epsilon \vec{E} - \epsilon_0 \vec{E}$$

Therefore the resulting electric displacement field (field only due to charges, polarization) is equal to the permittivity of the space  $\times$  (electric field after polarization + the change in electric field due to polarization):

$$\begin{aligned}\vec{D} &= \varepsilon_0 \overrightarrow{E_{\text{if it happened in vacuum}}} = \varepsilon_0 (\vec{E} + \vec{P}) = \varepsilon (\vec{E} - (\varepsilon \vec{E} - \varepsilon_0 \vec{E})) \\ &= \varepsilon (\vec{E}(1 - \varepsilon) + \varepsilon_0 \vec{E}) \\ &= \varepsilon(1 - \varepsilon) \vec{E} + \varepsilon \varepsilon_0 \vec{E}\end{aligned}$$

Where  $E$  is the net electric field after polarisation

**END OF IDEA**

## Boundary Conditions

The electric field strength changes as it passes through the border of two different dielectric materials. However, it can be broken down into the normal and tangential components in order to calculate the resulting field:

As an external electric field passes through the boundary of dielectrics with absolute permittivity  $\varepsilon_1$  and  $\varepsilon_2$ :

$$\begin{aligned}\varepsilon_1 E_{n1} &= \varepsilon_2 E_{n2} \\ E_{t1} &= E_{t2}\end{aligned}$$

The normal component is derived using Gauss's law and an infinitely thin cylinder at the boundary. LTD: Revisit tangential proof after learning Faraday's law <https://ximera.osu.edu/electromagnetics/electromagnetics/electrostatics/digInElectricFieldBoundaryConditions>

Supposing the external field enters at angle  $\alpha_1$  to the normal and leaves with angle  $\alpha_2$ , the fields at the boundary can be expressed as:

$$\begin{aligned}\tan \alpha_1 &= \frac{E_{t1}}{E_{n1}} \\ \tan \alpha_2 &= \frac{E_{t2}}{E_{n2}} \\ \frac{\tan \alpha_1}{\tan \alpha_2} &= \frac{\frac{E_{t1}}{E_{n1}}}{\frac{E_{t2}}{E_{n2}}} = \frac{E_{n2}}{E_{n1}} = \frac{\varepsilon_1}{\varepsilon_2}\end{aligned}$$

## Capacitance

Considering the equation for the potential difference between two equally and oppositely charged plates of a capacitor, each with charge  $Q$  and area  $A$ :

$$\begin{aligned}U &= Ed = \frac{\sigma d}{\varepsilon} = \frac{Qd}{\varepsilon A} \\ Q &= CU, \text{ where } C = \frac{\varepsilon A}{d} \\ C &= \frac{Q}{U}\end{aligned}$$

$Q$  is directly proportional to  $U$  regardless of the geometry of the capacitor. The constant of proportionality  $C$  is called the capacitance of such an arrangement and has the unit **F** Farad, ie. the ability of a body to store electrical charge. Formally defined:

$$C = \frac{\epsilon \oint_A \vec{E} \cdot d\vec{A}}{\int_S \vec{E} \cdot d\vec{s}}$$

LTD: Compute capacitance of arbitrary systems in physics simulator

### Spherical Capacitor

Considering two spherical shells, the inner with radius  $a$  and outer with radius  $b$ , each with an equal and opposite charge  $Q$ . The capacitance of such a body is:

$$\begin{aligned} C &= \frac{Q}{U} = \frac{Q}{\int_a^b \vec{E} dr} \\ &= \frac{Q 4\pi\epsilon}{Q \int_a^b \frac{1}{r^2} dr} = \frac{4\pi\epsilon}{\frac{b-a}{ab}} \\ &= \frac{4\pi\epsilon ab}{b-a} \end{aligned}$$

### Capacitor Networks

Large capacitor networks can be broken down into a single capacitor using the following rules:

	Parallel	Series
Total Capacitance	$\sum_{i=1}^n C_n$	$\left(\sum_{i=1}^n C_n^{-1}\right)^{-1}$
Derivation	<p>All capacitors in parallel have the same voltage across them. Therefore an equivalent single capacitor would have capacitance:</p> $\frac{Q_{\text{total}}}{V} = \sum_{i=1}^n C_i$	<p>The total voltage dropped across the series of capacitors is equal to the voltage between the first and last terminal. Due to the conservation of charge, no new charge can be brought in between the inner capacitors:</p> $U_{\text{total}} = \sum_{i=1}^n U_i = \sum_{i=1}^n \frac{Q}{C_i}$ $\therefore \frac{1}{C_{\text{total}}} = \sum_{i=1}^n \frac{1}{C_i}$

### Real-world capacitors

A capacitor consisting of two plates would need to be extremely large to be of any practical use. In real life, various forms of layered capacitors are used.<sup>3</sup>

In general, their capacitance can be calculated by identifying the number of individual plate capacitors they form between layers and summing their capacitance (they are essentially connected in parallel). The capacitance around the edges of layers is negligible.

*Variable capacitor* - A layered capacitor, whose overlapping area can be adjusted mechanically. Capacitance can be expressed in terms of an angle, depending on its construction.

*Film capacitor* - This is essentially a plate capacitor wrapped into a cylinder. Its capacitance can be calculated in the same fashion as a plate capacitor, but **multiplied by 2**, as both sides of the conductor contribute to the capacitance.

<sup>3</sup>See chapter 1.19 of Elektrotechnik, Albach



## Energy stored by capacitor

The energy stored by a capacitance  $C$  with voltage  $U$  across it's terminals is:

$$W_e = \frac{1}{2}CU^2$$

This is derived by integrating a differential equation of the change in energy as a function of the change in charge whilst the capacitor is connected to a constant voltage source (and is therefore independent of the geometry of the capacitor):

$$\begin{aligned}dW_e &= Udq = \frac{1}{C}q dq \\W_e &= \frac{1}{C} \int_0^Q q dq = \frac{1}{2} \frac{Q^2}{C} \\&= \frac{1}{2}CU^2\end{aligned}$$

The stored energy can also be expressed in terms of fields (with energy density  $Jm^{-3}$ ) through the following equation (LTD: Revisit after reading Purcell):

$$w_e = \frac{1}{2} \vec{E} \cdot \vec{D}$$

## Current

The net rate of flow of positive charge through a surface (**A** mperes). In conventional current, holes flow from higher to lower potential, although the electrons actually flow from low to high potential.

$$I(t) = \frac{dQ}{dt}$$

*Convection current* - Current caused by the transport of charges.

## Current Density

The current that is flowing per unit area at a point in space. It can be expressed in terms of volume charge density  $\rho$  and velocity of the charges  $\vec{v}$ :

$$\vec{J} = \rho \vec{v}$$

It is always positive regardless of the polarity of the charge density, as in that case the charge will be moving in the opposite direction anyway.

The total current flowing through an area can be calculated with the help of a surface integral:

$$I = \iint_A \vec{J} \cdot d\vec{A}$$

## Resistance

As electrons in a conductor are accelerated in the same direction by an external electric field, they bump into stationary nuclei in the lattice, causing them to decelerate and scatter. The resulting average velocity of electrons (hence the negative sign) at a point is called the **drift velocity** and is directly proportional to the electric field. The constant of proportionality  $\mu$  is called **electron mobility (Beweglichkeit)**:

$$\vec{v}_e = -\mu_e \vec{E}$$

### Specific Resistivity ( $\rho_R$ ) & Conductivity ( $\kappa$ )

These are two fundamental properties of a material which are reciprocals of each other. They describe how well a material resists / conducts electricity:

$$\rho_R = \frac{1}{\kappa}$$

	Resistivity	Conductivity
Symbol	$\rho_R$	$\kappa$
Units	$\Omega m$ Ohm meters	$S m^{-1}$ Siemens per meter

For a metal with  $n$  electrons per unit volume,  $\rho = -en$ . The current density can be expressed in terms of the specific conductivity  $\kappa$  as:

$$\vec{J} = -ne\vec{v} = ne\mu_e\vec{E} = \kappa\vec{E}$$

Resistivity is heavily dependant on temperature and pressure. A higher temperature results in nuclei oscillating around their equilibrium faster and thus increasing resistivity. On the other hand, the resistivity of semiconductors generally decreases with increasing temperature, as the oscillations free more electrons from the individual atoms. A factor  $\alpha$  is often used to calculate the resistivity with temperature taken into account:

$$\rho_{R(T_{old}, T_{new})} = \rho_{R \text{ T old}}(1 + \alpha\Delta T)$$

where  $\Delta T = T_{new} - T_{old}$

This is of course only accurate within a specific range of temperatures. A better approximation can sometimes be calculated with a 2nd factor  $\beta$ :

$$\rho_{R(T_{old}, T_{new})} = \rho_{R \text{ T old}}(1 + \alpha\Delta T + \beta(\Delta T)^2)$$

Alloys usually have worse conductivity than pure metals due to their irregular atomic structure.

### Ohm's Law

In differential form:

$$\vec{J} = \kappa\vec{E}$$

Considering a cylinder with length  $l$ , homogenous conductivity  $\kappa$  and cross sectional area  $A$  laying orthogonal to a homogenous electric field caused by electrodes with constant potentials  $\varphi_1$  and  $\varphi_2$  which continuously provides charges / holes. By integrating the fields in this equation we can derive the most fundamental equation in electronics.

The current through any cross-section in the cylinder is:

$$I = \iint_A \vec{J} \cdot d\vec{A} = J_x A$$
$$J_x = \frac{I}{A}$$

Thus we can express the electric field through the cylinder as:

$$E_x = \frac{1}{\kappa} J_x = \frac{I}{\kappa A}$$

The potential difference between the ends of the cylinder is:

$$U_{12} = \varphi_1 - \varphi_2 = \int_0^l \vec{E} d\vec{s} = E_x l = \frac{Il}{\kappa A}$$

Resistance can be expressed in terms of resistivity (and thus  $\kappa$ ) as:

$$R = \frac{\rho_R l}{A} = \frac{l}{\kappa A}$$

And thus:

$$U_{12} = IR \blacksquare$$

In summary:

$$R = \frac{U}{I} = \frac{\int_s \vec{E} d\vec{s}}{\kappa \oint \vec{E} d\vec{A}}$$

*Aside:* Since the resistance is directly proportional to the resistivity of the material, the temperature change formulas can be applied in exactly the same way.

Sometimes the conductance of a component is expressed as:

$$G = \frac{1}{R}$$