Analysis 1

Contents

- *Analysis 1 ITET, F Ziltener* - https://metaphor.ethz.ch/x/2024/hs/401-0231-10L/Ziltener_Notizen_Analysis_1_ITET_RW.pdf
- *Analysis für Informatik, M Struwe* - https://people.math.ethz.ch/~struwe/Skripten/InfAnalysis-bbm-8-11-2010.pdf

Logik

Aussage - Eine Aeusserung, die entweder wahr oder falsch ist
Luegner Paradox - Das ist keine Aussage: “Dieser Satz ist falsch”

Menge (Set) - eine ungeordnete Zusammenfassung verschiedener Objekte zu einem Ganzen

$\wedge$ - and
$\vee$ - or
$\veebar$ (XOR) - either … or …
Materiale Aequivalenz ($\Leftrightarrow$)
Logische Aequivalenz ($\equiv$) $A \Leftrightarrow B \equiv (A \Rightarrow B) \wedge (B \Rightarrow A)$ - Sie haben die gleichen Wahrheitstabellen

$A \Leftrightarrow B$ - A genau dann wenn B
$A \Rightarrow B$ - Wenn A, dann B
$\neg B \Rightarrow \neg A$ - Kontraposition
$A \Rightarrow B \equiv \neg B \Rightarrow \neg A$

Zum Beispiel:
Es hat geregnet $\Rightarrow$ die Strasse ist nas
Kontraposition: Die Strasse ist nicht nass $\Rightarrow$ Es hat nicht geregnet
Das ist genauso wahr aufgrund der Physik.

Wahr: $0 < 0 \Rightarrow 1 + 1 = 2$
Falsch: $0 < 0 \Leftrightarrow 1 + 1 = 2$

Distributive:

$$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$$
$$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$$

Proofs

Beweis - eine Herleitung einer Aussage aus den Axiomen
Satz - eine Bewiesene Aussage
Lemma (oder Hilfssatz) - ein Satz, der dazu dient, einen anderen Satz zu beweisen

q.e.d. (■) - end of proof

Beweiss formalisieren - Express a proof formally in terms of symbols and Limmas, can be checked by a computer.
Divide et impera - divide and conquer *Zermelo + Fraenkel Axioms* - Foundational axioms of all proofs

Beweis Methode

Modus ponens - Wird (meistens mehrmals) verwendet, um etwas zu beweisen:
$A :=$ Es hat geregnet (Premise)
Wenn es geregnet hat, dann ist die Strasse nass (Regel: $A \Rightarrow B$)
$B :=$ Die Strasse ist nass (Konklusion)

Kontraposition - Prove the Kontraposition, which subsequently proves the original statement (they are logically equivalent)
Beweisen, dass $\sqrt{2} < \sqrt{3}$:

$$A := \sqrt{2} \geq \sqrt{3} \equiv \neg\sqrt{2} < \sqrt{3}$$

Monotonie des Quadrierens:

$$x, y \geq 0$$

$$\text{Wenn } x \leq y, \text{dann ist } x^2 \leq y^2$$

$$\text{Laut der Monotonie des Quadrierens, } B := 2 \geq 3 \text{ ist wahr}$$

$$A \Rightarrow B \equiv \neg B \Rightarrow \neg A \equiv 2 < 3 \Rightarrow \sqrt{2} < \sqrt{3} \blacksquare$$

Widerspruch beweis
Um A zu beweisen, nehmen wir an, dass A falsch ist.
Widerspruch finden - das beweist die Aussage A

Zum Beispiel:
Beweis des Satzes $\sqrt{2} < \sqrt{3}$
Nehmen wir an, dass $\sqrt{2} \geq \sqrt{3}$ wahr ist
Lemma (Monotonie des Quadrierens): $\sqrt{2} \geq \sqrt{3} \Rightarrow 2 \geq 3$
Widerspruch: $2 \geq 3$ ist falsch, deshalb ist $\sqrt{2} \geq \sqrt{3}$ auch falsch.
$\neg\left(\sqrt{2} \geq \sqrt{3}\right) \equiv \sqrt{2} < \sqrt{3}$ $\blacksquare$

It is more rigorous to prove / rewrite something through Contraposition, because we start with a false statement in contradiction.

Vollstaendige Induktion
$n \in N_0, P(n)$ ist eine Aussage
$P(0)$ ist wahr
Wenn $\forall k \in N_0$ gilt $P(k) \Rightarrow P(k+1)$
Dann ist $\forall n \in N_0, P(n) \equiv$ wahr
Zum Beispiel:

$$\text{Satz: } \forall n \in N_0, P(n) := \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$P(0) = \frac{0(1)}{2} = 0$$

$$\text{Sei } P(k) = \frac{k(k+1)}{2}$$

$$\text{Zu zeigen } P(k+1) = \frac{(k+1)((k+1)+1)}{2}$$

$$P(k+1) = P(k) + k + 1 = \frac{k(k+1)}{2} + k + 1$$

$$= 2k^2 + 3k + 1 = \frac{k^2 + \frac{3}{2}k + \frac{1}{2}}{2}$$

$$= \frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2}$$

Vollstaendige Induktion gibt, dass $\forall n \in N_0, P(n)$ wahr ist. $\blacksquare$

Mengenlehre
Eine ungeordnete Zusammenfassung von Elemente.

$\emptyset$ - Leere Menge, hat keine Elemente
$\{\emptyset\}$ hat genau ein Element

Aussageform $\{x \mid P(x)\}$ or $\{x; P(x)\}$ - die Menge aller x, fuer die $P(x)$ gilt
Example: $\{x \mid x \in \mathbb{N}_0, x \text{ ist gerade}\}$

Russelsche Antonomie - $\{x \mid x \in X, x \notin x\}$ ist ein Paradox
Loesung: Es muss immer so definiert werden $\{x \in X \mid P(x)\}$, wo X eine andere Menge ist.

$A \cap B - \{x \mid x \in A \wedge x \in B\}$ - Intersection
$A \cup B - \{x \mid x \in A \vee x \in B\}$ - Union
$A \setminus B - \{x \in A \mid x \notin B\}$ - Without
$A \subseteq B$ - Jedes Element von A liegt in B (between two sets, unlike $x \in A$ which describes a single element x being inside the set A)
$A \subset B$ - Jedes Element von A liegt in B und A enthaelt weniger Elemente als B
$A \subseteq X, A^{\mathsf{C}} = X \setminus A$, wo X die Grundmenge ist, die jeder Element die wir betrachten enthaelt.

Distributive:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$(1, 2, 3)$ - *Tuple* - Ordered set

Kartesische Product / Potenz - $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$
Example:

$$X := \{0, 1\}, Y := \{\alpha, \beta\}$$
$$X \times Y := \{(0, \alpha), (0, \beta), (1, \alpha), (1, \beta)\}$$
$$|X \times Y| = |X| \times |Y|$$

$\mathbb{R}^n$:= n-dimensionalen Koordinatenraum
$\mathbb{R}^2 = X \times Y$
$\mathbb{R}^3 = X \times Y \times Z$

Interval Notation

$$[a, b] - a \leq x \leq b$$
$$(a, b) - a < x < b$$

Open bounds cannot be the maximum / minimum of a set, as they are not contained in the set (and $0.\dot{9} \equiv 1$ etc.).

Let $A \subseteq \mathbb{R}$
Supremum

$$\sup A = \begin{cases} \text{Smallest upper bound} & \text{if A has an upper bound} \\ \infty & \text{if A doesn't have an upper bound} \\ -\infty & \text{if } A = \emptyset \end{cases}$$

Infimum - Largest lower bound

$$\inf A = \begin{cases} \text{Largest lower bound} & \text{if A has a lower bound} \\ -\infty & \text{if A doesn't have a lower bound} \\ \infty & \text{if } A = \emptyset \end{cases}$$

Infinity cannot be a Supre/Infimum, becuase $\infty \notin \mathbb{R}$

De Morgan's Laws

Also apply to boolean logic, where $A, B := 1, 0$

$$(A \cap B)^{\complement} = A^{\complement} \cup B^{\complement}$$
$$(A \cup B)^{\complement} = A^{\complement} \cap B^{\complement}$$

Quantoren

They cannot simply be swapped! See the largest natural number problem in script.

$\exists$ - Existenzquantor - Es gibt
$\forall$ - Allquantor - Fuer alle
$\exists!$ - Es gibt genau ein element

$\neg(\forall x \in X |\ P(x)) = \exists x \in X \mid \neg P(x)$
$\neg(\exists x \in X |\ P(x)) = \forall x \in X \mid \neg P(x)$

Goethe Prinzip - When a variable is renamed correctly, a statement is still logically equivalent

Funktionen

Eine Funktion ist ein Tripel $f = (X, Y, G)$, wobei X und Y Mengen sind und $G \subseteq X \times Y$, sodass $\forall x \in X \exists y \in Y$, sodass $(x, y) \in G$

Domain - Set of possible inputs for a function
Codomain (Range) - Set of possible outputs of a function

Example:
Both are Quadratic funktions but are not equal:
$X := Y := \mathbb{R}, G = \{(x, x^2) \mid x \in \mathbb{R}^2\}$
$X := \mathbb{R}, Y :=]0, \infty[, G = \{(x, x^2) \mid x \in \mathbb{R}^2\}$

$X \to X, \mathrm{id}(x) := x$ - *Identitaets Funktion*

Bild und Urbild - Muss nicht bijektiv sein
$\mathrm{im}(X) := f(X)$ - *Bild von f*
$f : X \to \alpha, f^{-1}(Y) := \{x \in X \mid f(x) \in Y\}$ - *Urbild von y unter f*

Surjektiv - $\forall y \in Y \exists x \in X : f(x) = y$ - Es gibt fuer jeder Ausgang einige dazugehoerige Eingange
Injektiv - $\forall x, x' \in X : x \neq x' \Rightarrow f(x) \neq f(x')$ - Es gibt genau eine Ausgang fuer jeder Eingang in dem Definitionsbereich
Bijektiv - Es ist Surjektiv und Injektiv, weshalb es eine Inverse hat

Umkehrfunktion

Sei $f : X \to Y$ eine Bijektive funktion, $f^{<-1>} := Y \to X$ - *Umkehr Funktion*
The inverse can ONLY be defined when the function is Bijektiv, unlike the Urbild. When $X = Y = \mathbb{R}$ it is the reflection of the original function over the line $y = x$. It is sometimes notated as f^{-1} when the context is clear.

Do not forget to consider the given domain / range when considering if a function is bijektiv!

Zum Beispiel:

$$f : \mathbb{R} \to \mathbb{R}, f(x) := x^2$$
$$\mathrm{im}(f) = f(\mathbb{R}) = [0, \infty]$$
$$f^{-1}([-\infty, 4]) = [-2, 2]$$

The inverse can be only be defined if f is Bijektiv:

$$f : [0, \infty] \to [0, \infty], f(X) := x^2$$
$$f^{<-1>} = \sqrt{X}$$

$g \circ f := g(f(x))$ - Only possible if the $\operatorname{codom}(f) = \operatorname{dom}(g)$

Zahlen und Vektoren

$\mathbb{N}_0 := \{0, 1, 2, ...\}$
$\mathbb{N} := \{1, 2, 3, ...\}$
$\mathbb{Z} := \{..., -1, 0, 1, ...\}$
$\mathbb{Q} := \left\{\frac{m}{n} \mid m \in Z \land n \in N\right\}$
$\mathbb{N}_0 \subseteq \mathbb{Z} \subseteq \mathbb{Q}$

There are infinite gaps in the number line of rational numbers. These can be filled with $\mathbb{R} \setminus \mathbb{Q}$ - Irrational numbers, for example $\sqrt{2}, \pi, e$. For example: $\nexists s \in \mathbb{Q} \mid s^2 = 2$.

Reelen Zahlen

Dedekind Cut

A Dedekind cut is a way of representing the real numbers using the rational numbers by cutting the number line into two sections around a "gap" represented by an irrational number. Let $x \subset \mathbb{Q}$ (x contains less elements than $\mathbb{Q}$), the following properties describe the cut:

$$x \notin \emptyset$$
$$\forall r \in x \forall s \in \mathbb{Q} : s > r \Rightarrow s \in x$$
$$\forall r \in x \exists s_0 \in x : s_0 < r$$

This definition can of course include $\mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$ and therefore the entire $\mathbb{R}$ set.

The elementary number operations (addition, subtraction, multiplication, inequalities etc.) can be defined in terms of Dedekind cuts, precisely defining our understanding of arithmetic. $\mathbb{R}$ (und deshalb auch $\mathbb{Q}$) ist eine sogennante "total geordneter Koerper".

Dedekind Completeness - Every nonempty subset of $\mathbb{R}$ with an upper / lower limit has a smallest / largest upper / lower limit.
This proves that the irrational numbers are not complete: $\{r \in Q \mid r^2 < 2\}$ has no smallest upper limit.

b-adischer Bruch

This is the formal name of the place value system which is defined for all bases ≥ 2. The values of the digits before the radix point are nb, and $\frac{1}{nb}$ after the radix.

Youngsche Ungleichung

$$x, y, c \in \mathbb{R}$$
$$c > 0$$
$$2|xy| \leq cx^2 + \frac{y^2}{c}$$

Cardinality (Mächtigkeit)

Two sets have the same cardinality if they have the same size and therefore a bijective mapping between them exists (see Cantor's Diagonalmethod).

$$|\mathbb{N}_0| = |\mathbb{Z}| = |\mathbb{Q}| \neq |\mathbb{R}|$$

Complex Numbers

The Real numbers contain no solution for $x^2 = -1$, which is why the imaginary number $i = \sqrt{-1}$ was introduced, first considered by Cardano. They can be used to solve real world problems throughout electrical engineering, particularly for oscillations because powers of i^n have a repetitive nature.

Complex addition is identical to real addition $+_{\mathbb{R}^2}$ between the real and imaginary parts.

Complex multiplication is defined as:

$$\cdot_{\mathbb{C}} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2, \begin{pmatrix} r \\ m \end{pmatrix} \cdot_{\mathbb{C}} \begin{pmatrix} r' \\ m' \end{pmatrix} := \begin{pmatrix} rr' - mm' \\ rm' + r'm \end{pmatrix}$$

$$\begin{aligned} (r + mi)(r' + m'i) &= rr' + rm'i + mr'i + mm'i^2 \\ &= rr' - mm' + (rm' + r'm)i \end{aligned}$$

Therefore the complex body is defined as a tuple with the operations:

$$\mathbb{C} := (\mathbb{R}^2, +_{\mathbb{R}^2}, \cdot_{\mathbb{C}})$$

$$i \in \mathbb{C}, i := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

It is not a complete body as it doesn't contain any definitions for inequalities, like $\mathbb{R}$.

The following injective, non surjective function maps real numbers to complex numbers:

$$\mathbb{R} \to \mathbb{C} : x \in \mathbb{R}, \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathbb{C}$$

There exists a root for −1 in the complex body:

$$i^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot_{\mathbb{C}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\sqrt{-1} = \pm i$$

The complex conjugate is defined as follows:

$$z := a + bi$$

$$\Re(z) = a$$

$$\Im(z) = b$$

$$\overline{z} = a - bi$$

The euclidian norm is defined as:

$$|z| = \sqrt{|z_1|^2 + |z_2|^2 + ... + |z_n|^2}$$

General identities:

$$z\overline{z} = |z|^2$$

$$\overline{z + z'} = \overline{z} + \overline{z'}$$

$$\overline{zz'} = \overline{z} \cdot \overline{z'}$$

The function cis can be used to handle complex numbers in polar form:

$$\text{cis}(\theta) = \cos(\theta) + i\sin(\theta)$$
$$\text{cis}(\theta)\text{cis}(\varphi) = \text{cis}(\theta + \varphi)$$
$$\text{cis}\left(k\frac{\pi}{2}\right) = i^k \forall k \in \mathbb{Z}$$
$$z = |z|\ \text{cis}(\varphi) = |z|e^{i\varphi}$$
$$zz' = |z||z'|\ \text{cis}(\varphi + \varphi')$$
$$\overline{z} = |z|\text{cis}(-\varphi)$$

Polar form can also be written in terms of Euler's equation, which was derived from the Taylor Series' of e^x and the trigonometric functions:

$$z = |z|e^{i\theta}$$

De Moivre's Theorem:

$$z^k = |z|^k\ \text{cis}(k\varphi)$$

k'th Roots

For a complex number z, the k'th roots w are straightforward to determine:

$$w^k = z$$
$$w_j = |z|^{\frac{1}{k}}\ \text{cis}\left(\frac{\varphi + 2j\pi}{k}\right), j := 0, 1, ..., k - 1$$

Any of these roots to the power of k is equal to z, as well the product of all of them together. If $j \geq k$ the angle completes a full circle and the same roots are found.

The roots of $z = 1$ are called roots of unity, these will be important later in Fourier transforms:

$$w^k = 1$$
$$\zeta_k(j) = e^{\frac{2j\pi i}{k}}, j := 0, 1, ..., k - 1$$

Fundamental Theroem of Algebra - Every non-constant single variable polynomial contains at least 1 complex root.

Sequences and Series

Sequence - A function that maps a natural index $n \in \mathbb{N}_0 \to \mathbb{C}$
Series - Sequence of partial sums of the terms in a sequence

Taylor Series - A series of derivatives of a function at a point, that converges towards the value of the function at that same point, more on this later...

Geometric Sequence - $n \in \mathbb{N}_0, a_n \to z^n$ - Converges towards 0 when $z < 1$
Geometric Series - $n \in \mathbb{N}_0, a_n \to \sum_{k=0}^{n} z^k$

Harmonic Sequence - $n \in \mathbb{N}_0, a_n \to \frac{1}{n}$ - Converges towards 0

Archimedes' Axiom

$$\forall x \in \mathbb{R} \exists n \in \mathbb{N}_0, x \leq n$$

Triangle Inequality

$$|x + y| \leq |x| + |y|$$
$$|x - y| \geq |x| - |y|$$

Convergence

A sequence converges towards $A \Leftrightarrow \exists A \in \mathbb{C} \forall \varepsilon \in (0, \infty) \exists n_0 \in \mathbb{N}_0 \forall n \in \mathbb{N}_0 : n \geq n_0, |a_n - A| \leq \varepsilon$

$$a_n \to A \text{ (converges towards A)}$$

We can also express this as a limit:

$$\lim_{n \to \infty} a_n = A$$

Note: The index n cannot be set as ∞, as infinity is not a natural number.

Divergence can be proved by proving the conjugate of the definition of convergence:

$$\text{A sequence diverges} \Leftrightarrow \forall A \in \mathbb{C} \exists \varepsilon \in (0, \infty) \forall n_0 \in \mathbb{N}_0 \exists n \in \mathbb{N}_0 : n \geq n_0, |a_n - A| > \varepsilon$$

Convergence Criteria

Monotone Increasing - $a_0 \leq a_1 \leq a_2 \leq a_3 ...$

The Geometric series can be written as:

$$a_n = \frac{1 - x^{n+1}}{1 - x}$$

It converges towards:

$$a_\infty = \frac{1}{1 - x}$$

If a sequence is defined as the sum, product, quotient or inequality of two convergent sequences, the resulting sequence also converges towards the sum, product, etc. of the contained limits.

Euler's Number

An irrational number defined as:

$$e := \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

This converging sequence was discovered by Bernoulli whilst calculating the effect of frequency of payments on compound interest: https://en.wikipedia.org/wiki/E_(mathematical_constant)#Compound_interest

The exponential growth function e^x also displays the unique property that its derivative at any point is e^x.