

TODO: Vector calculus,  $\nabla$ ,  $\nabla \cdot$ ,  $\nabla \times$  Spherical coordinate vector operators

## Analysis 2

Books:

- Understanding Analysis: <https://link.springer.com/book/10.1007/978-1-4939-2712-8>  
Good for reviewing and developing intuition of Analysis 1
- Analysis II Amann Escher: <https://link.springer.com/book/10.1007/3-7643-7402-0> <https://link.springer.com/book/10.1007/978-3-7643-7478-5> too rigorous / abstract
- Zorich looks great

## Linear Differential Equations

Differential equations are functions like any other - they can also be represented as linear combination of some basis, for example the infinite-dimensional Fourier basis which can represent any (long-term periodic or bounded) function.

### Differential Operator

Basic differential operator -  $\frac{d^n}{dx^n}$  - A linear mapping between a function and its n'th (partial) derivative within some vector space, which can be represented as a matrix if finite.

The diagram illustrates the differential operator  $\frac{d}{dx}$  as a linear mapping between basis functions and their derivatives. On the right, a list of basis functions is shown:  $b_0(x) = 1$ ,  $b_1(x) = x$ ,  $b_2(x) = x^2$ ,  $b_3(x) = x^3$ , and so on. On the left, the derivatives of these functions are calculated:  $\frac{d}{dx}b_0(x) = \frac{d}{dx}(1) = 0$ ,  $\frac{d}{dx}b_1(x) = \frac{d}{dx}(x) = 1$ , and  $\frac{d}{dx}b_2(x) = \frac{d}{dx}(x^2) = 2x$ . These derivatives are represented as column vectors:  $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ , and  $\begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ . A matrix representation of the operator  $\frac{d}{dx}$  is shown in the center, with rows corresponding to the derivatives of the basis functions. The matrix is: 
$$\frac{d}{dx} = \begin{bmatrix} 0 & 1 & 0 & ? & \dots \\ 0 & 0 & 2 & ? & \dots \\ 0 & 0 & 0 & ? & \dots \\ 0 & 0 & 0 & ? & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$
 Arrows indicate the mapping from the basis functions to the matrix and then to the derivative vectors.

A linear differential operator is a linear combination of basic differential operators, in other words a mapping from a function to a differential equation, for example:

$$D(f) = \ddot{f} + 2\beta\dot{f} + \omega_n^2 f$$

Which is the differential operator for damped simple harmonic motion.

Differential operators (TODO: check) never have full rank, because the range is restricted to a subset of the domain linear space. **Carathéodory's existence theorem states that the kernel of an order-n differential operator has dimensions n** (and of course infinite solutions are spanned by n basis elements):

$$Dx(t) = 0$$

$$x(t) \in \text{Kernel}(D)$$

TODO: Exact conditions for Carathéodory's Theorem to hold true

*Particular Solution* - Single element that satisfies a differential equation

*Homogeneous Differential Equation* - Can be defined by a linear operator with constant scalars

TODO: Inhomogeneous equations and how to solve them (see solutions to driven-damped oscillator)