

Analysis 1

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Literature:

- Alessio Figalli's 2023 Lecture Notes
- Analysis 1 by H. Amann & J. Escher

What I would like to take notes of:

1. State the theorem / definition, expand with some intuition / memory aids
2. Write the proof by myself if deemed useful for an exam
3. Name some examples only if very helpful

These notes should serve as condensed revision material - only the minimal, important facts to remember before solving problems

Note on order: Many scripts used for teaching introduce concepts as they become relevant. My goal is to build a revision reference, not a learning resource, therefore over time the order will be rearranged to group relevant definitions and theorems together.

Proofs heavily involve decomposition; to progress, smaller Lemmas need to be brought in along the way and proven (or taken as true since someone else proved them). However first of all, you need to understand and remember the axioms (rules of the game). Intuition is helpful but doesn't prove anything unless it can be formulated as a series of statements a computer can verify

Mathematics - Abstracting enough to focus on the matter

Contradiction is a useful tool for linking statements about $>$ and \geq . When in doubt, assume by contradiction

Conjecture - A conclusion formed on the basis of incomplete information Prove uniqueness through trichotomy, existence by completeness axiom. The convergence of a sequence can be simplified by considering long-term values of n , for example $n > |x|$ which often simplifies the n 'th term (as in the case of Bernoulli's sequence for e) Finding two convergent boundaries around a sequence can be used to find its limit (squeeze theorem)

TODO: Read Einsiedler einführung

1. Fundamentals

1.1.1. Definition - Set

An **unordered** collection of **distinct** ($\{x, x\} \equiv \{x\}$) elements such that:

1. It is defined by the elements it contains
2. It is not an element of itself, this prevents Russell's Paradox: $\{x \mid x \notin x\}$
3. Its elements can be filtered by a series of statements which hold true, for example the set of even integers:

$$\{n \in \mathbb{Z} \mid \exists m \in \mathbb{Z} : n = 2m\}$$

Where \mid and $:$ both mean "such that".

4. The empty set \emptyset contains no elements

1.1.2. Definition - Cartesian Product

For the sets X, Y , the Cartesian product is the set of tuples (ordered lists): $X \times Y := \{(x, y) \mid x \in X, y \in Y\}$

- The number of elements in the set is:

$$|X \times Y| = |X| \cdot |Y|$$

Example:

$$\begin{aligned} X &:= \{0, 1\}, Y := \{\alpha, \beta\} \\ X \times Y &:= \{(0, \alpha), (0, \beta), (1, \alpha), (1, \beta)\} \end{aligned}$$

1.1.3. Definition - Subset

A set whose elements are entirely contained in a parent set with the following notation:

- $P \subseteq Q$ - P is a subset of Q and they may be equal
- $P \subsetneq Q$ - P is a **proper** subset of Q ; Q has at least 1 additional element
- $P \not\subseteq Q$ - There is at least one element in P that is not in Q
- The same applies in reverse using sup(er)set notation \supseteq
- The symbols \subset and \supset are ambiguous in meaning
- Two sets can be shown to be equal if $P \subseteq Q \wedge Q \subseteq P$ holds true

1.1.4. Definition - Power Set

The power set of a set X is the set of all subsets:

$$\mathcal{P}(X) := \{\text{Set } Q \mid Q \subseteq X\}$$

Example:

$$\begin{aligned} X &= \{0, 1, 2\} \\ \mathcal{P}(X) &= \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\} \end{aligned}$$

1.1.5. Definition - Interval Notation

Interval notation allows us to succinctly express common sets of real numbers between limits $a, b \in \mathbb{R}$:

- Closed interval

$$[a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

- Open interval

$$(a, b) := \{x \in \mathbb{R} \mid a < x < b\}$$

- Half-open interval

$$[a, b) := \{x \in \mathbb{R} \mid a \leq x < b\}$$

- Unbounded interval

$$[a, \infty) := \{x \in \mathbb{R} \mid a \leq x\}$$

- For a non-empty interval, the **length** is defined as $b - a$
- Sometimes inverted square brackets are used to specify an open bound, ex. $[a, b[$
- The intersection of a finite number of intervals is also an interval, such that the lower bound is the smallest lower bound and vice versa for the upper bound
- Sets aren't doors, they don't need to be either open or closed.

1.1.6. Definition - Set Operations

This allows us to construct common sets from component sets P, Q :

- Intersection:

$$P \cap Q := \{x \in P \mid x \in Q\}$$

- Union:

$$P \cup Q := \{x \in P \vee x \in Q\}$$

- Relative Complement:

$$P \setminus Q := \{x \in P \mid x \notin Q\}$$

- Complement of a Subset:

$$R \subseteq X$$

$$R^c := \{x \in X \mid x \notin R\}$$

- Symmetric Difference:

$$P \triangleright Q := (P \cup Q) \setminus (P \cap Q)$$

- Addition:

$$P + Q := \{p + q \mid p \in P \wedge q \in Q\}$$

- Multiplication:

$$P \cdot Q := \{p \cdot q \mid p \in P \wedge q \in Q\}$$

- They are distributive:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- Let \mathbb{A} be a set of sets, we can also define:

$$\bigcap_{A \in \mathbb{A}} A := \{x \mid \exists A \in \mathbb{A} \mid x \in A\}$$

$$\bigcup_{A \in \mathbb{A}} A := \{x \mid \forall A \in \mathbb{A} \mid x \in A\}$$

1.1.7. Theorem - De Morgan's Laws

This states:

$$(A \cap B \cap \dots)^c = A^c \cup B^c \cup \dots^c$$

$$(A \cup B \cup \dots)^c = A^c \cap B^c \cap \dots^c$$

It is commonly applied to Boolean logic, where $A, B \subseteq \{0, 1\}$:

$$\overline{A \wedge B \wedge \dots} \equiv \overline{A} \vee \overline{B} \vee \dots$$

1.1.8. Definition - Maxima and Minima

The maximum of a set is the smallest upper bound, which is **contained** in the set:

$$X \subseteq \mathbb{R}$$

$$\max(X) := m \in X \mid \forall x \in X x \leq m$$

The **minimum** is defined analogously:

$$\min(X) := m \in X \mid \forall x \in X x \geq m$$

- An open bound has no maximum or minimum defined as there is always some number slightly larger / smaller than a number we can express inside it. An open bound itself is not in the set
- The maximum / minimum is unique. Proof: Let m_1, m_2 be 2 maxima of the set. It follows $m_1 \leq m_2$ and $m_2 \leq m_1$, therefore $m_1 = m_2$ (trichotomy)

1.1.9. Definition - Supremum and Infimum

Let $B = \{b \in \mathbb{R} \mid \forall x \in X x \leq (\geq) b\}$ be the set of upper (lower) bounds for the set X . The supremum (infimum) is defined as the smallest (largest) such bound:

$$\sup(X) := \min(B)$$

$$\inf(X) := \max(B)$$

- Due to the $\leq (\geq)$ the supremum infimum may be the same as the maximum / minimum for a set with closed bounds
- An alternative characterization states there is no smaller bound, anything smaller is not a bound of X :

$$\forall x \in X \mid x \leq \sup(X) \exists x' \in X : x < x'$$

- Formally, the supremum / infimum does **not** exist for an unbounded or empty set, as there are no strictly real bounds ($\infty \notin \mathbb{R}$). However by convention the compactified reals are used and $\pm\infty$ is permitted.
- For all bounded, non-empty sets X , the supremum / infimum exists.

Proof:

The set of bounds $B = \{b \in \mathbb{R} \mid \forall x \in X x \leq (\geq) b\} \neq \emptyset$

We need to show that $\exists \sup(X) \in \mathbb{R} \mid \forall b \in B, \sup(X) \leq b$

Lemma: Completeness Axiom $\forall x \in X \forall b \in B, x \leq b \Rightarrow \exists c \in \mathbb{R} \mid x \leq c \leq b \forall x \in X \forall b \in B$

This c is an upper bound **and** minimum of B , therefore it is the supremum ■

Let X, Y be non-empty sets with an upper bound:

- $\sup(X \cup Y) = \max(\sup(X), \sup(Y))$
- $\sup(X \cap Y) = \min(\sup(X), \sup(Y)) \mid (X \cap Y) \neq \emptyset$
- $\sup(X + Y) = \sup(X) + \sup(Y)$
- $\sup(X \cdot Y) = \sup(X) \cdot \sup(Y) \mid \forall x \in X \forall y \in Y, y \geq 0$ (two “large” negative elements can make a larger supremum)

TODO: Review proof 2.59

1.1.10. Definition - Identity Function

This function simply outputs its input and is needed to define the inverse of a function:

$$\begin{aligned}\text{id} : X &\rightarrow X \\ \text{id}(x) &:= x\end{aligned}$$

1.1.11. Definition - Characteristic Function

$X \subseteq Y$, the characteristic / indicator function $\chi_X : Y \rightarrow \{0, 1\}$ indicates whether an element is part of a set:

$$\chi_X(x) := \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{otherwise} \end{cases}$$

1.1.12. Definition - Restriction Function

A new function can be defined with a smaller domain, which is useful for drawing conclusions from its properties over that domain:

$$\begin{aligned}f : X &\rightarrow Y, A \subseteq X \\ f|_A : A &\rightarrow Y\end{aligned}$$

2. Topology

2.1.1. Definition - Ball / Disk

A topological ball with radius r and center $x_0 \in \mathbb{R}^d$ in dimension \mathbb{R}^d is defined as the set of points:

$$\begin{aligned}B_r^d(x_0) &= \{x \in \mathbb{R}^d \mid |x - x_0| < r\} - \text{Open ball} \\ \overline{B}_r^d(x_0) &= \{x \in \mathbb{R}^d \mid |x - x_0| \leq r\} - \text{Closed ball} \\ S_r^{d-1}(x_0) &= \{x \in \mathbb{R}^d \mid |x - x_0| = r\} - \text{Sphere (boundary of ball)}\end{aligned}$$

Where $|x - x_0|$ is the length of the vector from $x_0 \rightarrow x$ ie the radius. This can also be defined using complex numbers and the complex absolute function. It follows:

$$\begin{aligned}B_0(x_0) &= \emptyset \\ \overline{B}_0(x_0) &= \{x_0\} \\ B_\infty^d(x_0) &= \overline{B}_\infty^d(x_0) = \mathbb{R}^d\end{aligned}$$

- Man muss immer am B_r^d bleiben!

The sphere has dimensions $d - 1$ because its points only form a subspace in the dimension below the ball which it is enclosing:

- S_r^1 - Is the line of points around a circle ie 1 dimensional
- S_r^2 - Every point in the surface of a 3D ball can be reached with linear combinations of two basis vectors (such that they stay within the subspace).

2.1.2. Definition - Inner Point

A point $x \in S^n$ is inner $\Leftrightarrow \exists r \in (0, \infty) \mid B_r^n(x) \subseteq S$ - there is an open ball with a radius > 0 around x such that it is entirely a subset of / equal to S .

2.1.3. Definition - Interior

- The interior of a set is the set of all its inner points:

$$\text{Int } S := \{\text{inner points of } S\}$$

- $\text{Int } S \subset S$ is always true.
- Alternatively, the interior can be defined as the union of open balls:

$$\text{Int } S := \bigcup_{B_r^n(x) \mid B \subseteq S} B$$

2.1.4. Definition - Open Set

A set which is equal to its interior: $S = \text{Int } S$. In other words, it is defined with $>$ or $<$ relations.

- It has no maximum / minimum, only an infimum / supremum.
- Every point of an open ball is an inner point, hence making the ball “open”.

$$\text{Int } \overline{B_r^d}(x_0) = B_r^d(x_0)$$

$$x \in \mathbb{R}, \{x\} \text{ is not open}$$

- The union of arbitrarily many open sets is open (the outer boundaries will remain open no matter what)
- The intersection of finitely many open sets is also open

2.1.5. Definition - Closed Set

The definition is built upon that of an open set: Let $A \subseteq \mathbb{R}^n$:

$$A \text{ is closed} \Leftrightarrow (\mathbb{R}^n \setminus A) \text{ is open}$$

For example, $[a, b] \mid a < b$ can instead be expressed as $(-\infty, a) \cup (b, \infty)$, which is open.

- $\{x\}$ is closed
- \emptyset, \mathbb{R}^n are both open and closed, since $\emptyset^c = \mathbb{R}^n$ and $\mathbb{R}^c = \emptyset$
- $[a, b) \subsetneq \mathbb{R}$ is neither open or closed

2.1.6. Definition - Closure

The closure of \overline{S} is the smallest possible closed set which entirely includes the set S , this can be formed using the intersection of all possible closed balls with different radii and centers, as long as they entirely contain S :

$$\overline{S} := \text{clos}(S) := \bigcap_{\text{All } \overline{B_r^n}(c) \mid S \subseteq \overline{B}} \overline{B}$$

For example:

$$S := (0, 1]$$

$$\overline{S} = [0, 1]$$

- $S \subseteq \overline{S}$ - The closure of a set contains the set itself
- A topological set can only be called **closed** if it is equal to its closure

2.1.7. Definition - Boundary

The boundary of a set ∂S is:

$$\partial S := \overline{S} \setminus \text{Int } S$$

Characterized more fundamentally:

$$\partial S := \{x \in \mathbb{R}^n \mid (\forall r \in (0, \infty) \mid (B_r^n(x) \cap S) \neq \emptyset \neq B_r^n(x) \setminus S)\}$$

The boundary of a set S is the set of points such that:

- A ball with increasing radius (starting just above 0) always continues to overlap with some elements of S ($(B_r^n(x) \cap S) \neq \emptyset$), ie the ball must be actually in or right next to S
- The points themselves are part of $B_r^n(x) \setminus S$, which is never equal to the empty set, ie the point itself is never in S

By definition, a topological sphere is the boundary of a ball:

$$\partial B_r^n(x) = \partial \overline{B_r^n}(x) = S_r^{n-1}(x)$$

Furthermore, a boundary is a closed set:

$$(\mathbb{R}^n \setminus \partial S = (\text{Int } S \cup \mathbb{R}^n \setminus \overline{S})) \text{ which is open} \Rightarrow \partial S \text{ is closed}$$

2.1.8. Definition - Bounded

A set which is a subset of a closed set (other than \mathbb{R}^n). In other words, the set of bounds $B = \{b \in \mathbb{R}^n \mid \forall x \in X x \leq (\geq) b\} \neq \emptyset$.

2.1.9. Definition - Compact

A set which is closed **and** bounded

- A closed ball is by definition compact.
- \mathbb{R}^n is not compact, because it is an infinitely large (albeit open & closed) set.

2.1.10. Definition - Neighborhood (Umgebung)

A subset $U \subseteq X$ is considered a **neighborhood** of a point x_0 relative to a set X if:

$$\exists O \mid x_0 \in O \wedge O \subseteq U \subseteq X$$

Where O is a non-empty open set.

- For example, there are many possible neighborhoods around a point in the middle of a non-empty set.
- Points on the boundary of X have no neighborhood U as no non-empty open set contains only points which remain in X

3. Axioms of The Real Numbers

An axiomatic approach to defining the set of real numbers \mathbb{R} .

3.1.1. Definition - Group

A **non-empty** set G endowed with an operation \star which satisfies the following criteria $\forall a, b, c \in G$:

1. *Associativity* - $a \star (b \star c) = (a \star b) \star c$
2. \exists *Neutral Element* n - $a \star n = n \star a = a$ - Examples:
 - $a + 0 = 0 + a = a$
 - $a \cdot 1 = 1 \cdot a = a$
3. $\forall a \exists$ *Inverse Element* i - $a \star i = i \star a = n$ - Examples:
 - $a + (-a) = (-a) + a = 0$
 - $a \neq 0 \Rightarrow a \cdot a^{-1} = a^{-1} \cdot a = 1$
4. If $a \star b = b \star a$ it is a **commutative group**, although this is not required.

Properties:

- The *Neutral Element* is unique. Proof:

Let $n, n' \in G$ be neutral elements

$$n \star n' = n = n'$$

- There is unique *Inverse Element* for all elements. Proof:

Let $i, i' \in G$ be inverse elements for a

$$i \star (a \star i') = (i \star a) \star i'$$

$$i \star n = n \star i'$$

$$i = i'$$

Examples:

- The non-zero rational numbers $\mathbb{Q} := \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z} : p, q \neq 0 \right\}$ with operation \cdot is a group, where $n = \frac{1}{1}$ and $i\left(\frac{p}{q}\right) = \frac{q}{p}$
- The natural numbers \mathbb{N} with operation $+$ is **not** a group, as there are no negative inverse elements

3.1.2. Definition - Ring

A non-empty set R with operations $+$ and \cdot .

1. Addition is **always commutative** with $n = 0, i = -a$
2. Multiplication is not necessarily commutative, for example a matrix ring
3. If multiplication is commutative, it is a **commutative ring** and has neutral element $n = 1$
4. It is **not necessarily** a group for multiplication as 0 may be included and has no inverse element $0 \cdot i \neq 1$

3.1.3. Definition - Field (Körper)

A commutative ring K (Körper) where $\forall a \in K \mid a \neq 0$ the inverse element for multiplication exists.

1. Addition: $n = 0, i = -a, -(-a) = a$
2. Multiplication: $n = 1, i = a^{-1}, (a^{-1})^{-1} = a \mid a \neq 0$

Examples:

- \mathbb{Z} is a ring but not a field as there is no multiplicative inverse element for all non-zero elements
- The complete set of rational numbers $\mathbb{Q} := \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z} : q \neq 0 \right\}$ is a commutative ring and a field.
- $0 \cdot a = 0$. Proof:

$$0 = 0 \cdot a - 0 \cdot a = (0 - 0) \cdot a = 0 \cdot a$$

3.1.4. Definition - Relationship

A relationship on X is the subset $\mathfrak{R} := \{(a, b) \in X \times X \mid a \sim b\}$ where \sim is an operator for expressing conditions called a relation and may have the following properties if the corresponding condition holds true $\forall x, y, z \in X$:

- Reflexive - $x \sim x$ - Example: \leq
- Transitive - $x \sim y \wedge y \sim z \Rightarrow x \sim z$ - Example: $<$
- Symmetric - $x \sim y \Rightarrow y \sim x$ - Example: $=, \neq$
- Anti-Symmetric - $x \sim y \wedge y \sim x \Rightarrow x = y$ - Example: \leq - Although such relations are often reflexive too, this is not a requirement, consider $<$, which is anti-symmetric (no such x, y exist) but not reflexive.
- A relation is called **equivalence relation** if it is reflexive, transitive and symmetric. For example $=$ is an equivalence relation, \leq is not (not symmetric).
- A relation is called **order relation** if it is reflexive, transitive and anti-symmetric. For example \leq

3.1.5. Definition - Ordered Field

This extends the definition of a field K with the relation \leq , which is denoted as (K, \leq) , under which all elements $x, y, z \in K$ satisfy the following:

1. Linearity of the order:

$$x \leq y \vee y \leq x$$

2. Compatibility of order and addition:

$$x \leq y \Rightarrow x + z \leq y + z$$

3. Compatibility of order and multiplication:

$$0 \leq x \wedge 0 \leq y \Rightarrow 0 \leq x \cdot y$$

This can be combined with the Inverse Element of addition (which exists in all fields) to make statements about multiplication of negative numbers.

Axioms 2 and 3 **also** apply to the relation $<$, which significantly simplifies proofs. Proof: <https://math.stackexchange.com/a/3271338>

These conditions allow us to define conventions such as:

- Positive $:= x > 0$
- Non-negative $:= x \geq 0$
- $(x \leq y = z) \equiv (x \leq y \wedge y = z)$
- An example of an ordered field is the set of rational numbers \mathbb{Q} .
- An example of a non-ordered field is the set of complex numbers \mathbb{C}

The conditions of an ordered field lead to many properties we take as given. Here are some interesting proofs:

- $(x < y \wedge y \leq z) \Rightarrow x < z$ - Proof:

$$x < y \Rightarrow x \leq y. \leq \text{ is a transitive relation, hence } x \leq z$$

We must now show that $x < z$.

Assume by contradiction that $\neg(x < z) \equiv x \geq z$ holds true

$$\text{Due to } x \leq z \wedge x \geq z, x = z$$

Recalling $x < y$ this implies $z < y$ which contradicts $y \leq z$

$$\therefore x < z \blacksquare$$

- If $x \neq 0$, $x^2 > 0$ holds true. Proof:

As $x \neq 0$ there are 2 cases:

- $x > 0$
- $x < 0$

The ring is only guaranteed to be valid for the relation \leq , so we will prove $x^2 \geq 0$ first.

If $x > 0$, $x \geq 0$ also holds true and also $x^2 \geq 0$ per condition 3.

If $x < 0$, $x \leq 0$ also holds true. Applying condition 2, $(x - x \leq 0 - x) \equiv (0 \leq -x)$. Applying condition 3, $-x \cdot -x = x^2 \geq 0$.

Lastly, we must show that $x^2 \geq 0 \Rightarrow x^2 > 0$. Assume by contradiction that $\exists x \neq 0 : x^2 \leq 0 \Rightarrow x^2 < 0$. This contradicts $x^2 \geq 0$, which we have proven for all $x \neq 0$ in the field. Hence $x^2 > 0$ must also be true ■

- $0 < 1$. Proof:

Lemma: $0 \neq 1$ (Neutral Elements of addition and multiplication are not the same)

Lemma: If $x \neq 0$, $x^2 > 0$ holds true. Therefore $1^2 = 1 > 0$ ■

Based on the fact that $0 < 1$ and the compatibility + inverse element of addition, it is clear that the integers $\mathbb{Z} := \dots, < -1 < 0 < 1 < \dots$ are a subset of any ordered field. Furthermore, the rational numbers are defined from the set of integers, which are also a subset of all ordered fields K :

$$\mathbb{Z} \subsetneq \mathbb{Q} \subseteq K$$

3.1.6. Definition - Absolute Function

A function $|x| : K \rightarrow K_+$ defined on every **ordered field** such that:

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

- $(|x| \leq y) \equiv (-y \leq x \leq y)$
- $|xy| \equiv |x||y|$

3.1.7. Definition - Sign Function

A function $\text{sgn}(x) : K \rightarrow \{-1, 0, 1\}$ defined on every **ordered field** such that:

$$\text{sgn}(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

- Every $x \in K$ can be expressed as $x = \text{sgn}(x) \cdot |x|$

3.1.8. Theorem - Triangle Inequality

It holds $\forall x, y \in K$ elements of an ordered field that:

$$|x + y| \leq |x| + |y|$$

The name stems from considering a triangle spanned by two vectors. It is clear that the length of their vector sum is \leq the sum of both side lengths. Proof (on an ordered field):

Lemma: $|x| \Rightarrow -|x| \leq x \leq |x|$

Therefore we can state the following:

$$\begin{aligned} -|x| &\leq x \leq |x| \\ -|y| &\leq y \leq |y| \end{aligned}$$

Lemma: $x \leq y \Rightarrow x + z \leq y + z$:

$$\begin{aligned} -|x| + -|y| &\leq x + -|y| \\ -|y| + x &\leq y + x \end{aligned}$$

Lemma: $x \leq y \wedge y \leq z \Rightarrow x \leq z$

$$\therefore -(|x| + |y|) \leq x + y$$

Applying the same procedure to $x \leq |x|$ and $y \leq |y|$ we get:

$$-(|x| + |y|) \leq x + y \leq |x| + |y|$$

Lemma: $(|x| \leq y) \equiv (-y \leq x \leq y)$

$$\therefore |x + y| \leq |x| + |y| \blacksquare$$

An alternative, the **inverse triangle inequality** can also be useful:

$$|x - y| \geq ||x| - |y||$$

3.1.9. Definition - Completeness Axiom

The ordered field of rational numbers is still unsuitable as we need to “fill in the gaps”. The completeness axiom is an alternative but equivalent approach to Dedekind cuts (which define the cuts first and then operations in terms of cuts) which characterizes a **complete ordered field** if the completeness axiom holds true:

1. Let $X, Y \subseteq K \mid X, Y \neq \emptyset : \forall x \in X \forall y \in Y$ the inequality $x < y$ holds true. If there exists $c \in K \mid x < c < y \forall x, y$ for all such subsets X and Y , the ordered field is complete.
- The field of real numbers \mathbb{R} is a completely ordered extension of \mathbb{Q}
- The reason subsets are checked instead of individual elements x, y is because subsets can be defined in terms of inequalities. For example, consider checking the existence of $\sqrt{2}$ in \mathbb{Q} . The set of rational numbers is **dense**, therefore no matter which lower bound x we choose, there is **always** a rational number closer to $\sqrt{2}$ and therefore the check $x \leq c \leq y$ holds true (although $\sqrt{2}$ is not a member of \mathbb{Q}). On the other hand if we choose the subset $X = \{x \in \mathbb{Q} \mid x < \sqrt{2}\}$, this contains every possible rational $< \sqrt{2}$ and checks **completeness** rather than **density**. Of course, both approaches would involve checking infinitely many elements but luckily we can arrive at such an inequality from the axioms of an ordered set.

3.1.10. Definition - Compactification

The reals can be extended to be compact (closed and bounded) with $-\infty, \infty$ for certain purposes, such as defining the supremum / infimum of an unbounded / empty set:

$$\begin{aligned}\overline{\mathbb{R}} &= \mathbb{R} \cup \{-\infty, \infty\} \\ \forall x \in \mathbb{R}, -\infty &< x < \infty\end{aligned}$$

Certain conventions are defined, however these are ambiguous and should be used sparingly:

$$\begin{aligned}\infty + x &= \infty \\ -\infty + x &= -\infty \\ x \cdot \infty &= \infty \mid x > 0 \\ \sup(\emptyset) &= -\infty \\ \inf(\emptyset) &= \infty\end{aligned}$$

3.1.11. Definition - Archimedean Principle

For every $x \in \mathbb{R}$ there exists **exactly one** $n \in \mathbb{Z} \mid n \leq x < n + 1$. In simpler words, $\forall x \in \mathbb{R} \exists z \in \mathbb{Z} \mid z > x$

3.1.12. Definition - Integer / Fractional Part

The integer part of any $r \in \mathbb{R}$ is given by the floor function $\lfloor \cdot \rfloor : \mathbb{R} \rightarrow \mathbb{Z}$ which returns the lower n which exists due to the Archimedean principle.

The fractional part is given by $r - \lfloor r \rfloor \in [0, 1)$

3.1.13. Corollary - $\frac{1}{n}$ is arbitrarily small

$$\forall \varepsilon \in \mathbb{R} \mid \varepsilon > 0 \exists n \in \mathbb{Z} \mid n \geq 1 \wedge \frac{1}{n} < \varepsilon.$$

Proof:

If $\varepsilon > 1$ this holds true with $n = 1$.

For $\varepsilon \leq 1$, $\frac{1}{\varepsilon} \geq 1$. The Archimedean principle states that there always exists a $n \geq 1 \mid n > \frac{1}{\varepsilon}$, which becomes $\frac{1}{n} < \varepsilon$ ■

3.1.14. Definition - Cardinality

The cardinality of two sets describes their relative “sizes”.

- $X \sim Y$ - We say two sets X and Y have the same cardinality (the same number of elements) if there exists a **bijective** mapping $f : X \rightarrow Y$. Surjectivity guarantees that $|Y| \geq |X|$ and injectivity guarantees $|X| \geq |Y|$, which leads to $|X| = |Y|$ (trichotomy).
- $X < Y$ - X is **smaller than or equal to** Y if there exists an injective mapping $f : X \rightarrow Y$
- $X \overset{\sim}{<} Y \wedge Y \overset{\sim}{<} X \Rightarrow X \sim Y$ - One can find a bijective mapping (Schröder-Bernstein Theorem)

- $|\emptyset| = 0$
- $\exists f : X \rightarrow \{1, 2, \dots, n\}$ is bijective $\Rightarrow |X| = n$, X is finite
- $|\mathbb{N}| = \aleph_0$ - A set which has the same cardinality as \mathbb{N} is called **countable**

3.1.15. Theorem - Cantor's Theorem

The power set $\mathcal{P}(X)$ of any (infinite too) non-empty set X is larger than and not equal to X .

This reveals that $\mathcal{P}(\mathbb{N}) > \mathbb{N} \wedge \mathcal{P}(\mathbb{N}) \neq \mathbb{N}$ which is useful for showing that other sets are larger than or equal to $\mathcal{P}(\mathbb{N})$ (\exists injection) and therefore also uncountable.

Proof:

Although this may seem obvious, when dealing with infinity it is easier to write a formal proof than find logical reasoning behind the intuition.

First we must show that there is an injective mapping $i : X \rightarrow \mathcal{P}(X)$, which indeed exists: $x \in X \rightarrow \{x\}$.

Now we show that there is **no** surjective mapping. Assume by contradiction that such a mapping $s : X \rightarrow \mathcal{P}(X)$ exists.

We will demonstrate its absurdity by defining the subset:

$$B = \{x \in X \mid x \notin s(x)\} \subseteq X$$

For every $x \in X$ there are two cases:

1. $x \in s(x)$, therefore $x \notin B$ and $s(x) \neq B$ because x would need to be a member of B for them to be equal
2. $x \notin s(x)$, therefore $x \in B$ and $s(x) \neq B$ because x would need to be a member of $s(x)$ for them to be equal

We have shown that $\nexists x \in X \mid s(x) = B$ and because $B \in \mathcal{P}(X)$, there exists no surjective mapping $s : X \rightarrow \mathcal{P}(X)$ ■

3.1.16. Theorem - \mathbb{R} is Uncountable

To prove this, we can find an injection $i : \mathcal{P}(\mathbb{N}) \rightarrow \mathbb{R}$, which is given from the decimal expansion of reals TODO: Understand Cantor diagonalization

4. Sequences of Real Numbers

4.1.1. Definition - Sequence

A sequence is a function $a : \mathbb{N} \rightarrow \mathbb{R}$ which is often written as $(a_n)_{n \in \mathbb{N}}$, such that the input n serves as an ordered index for the outputs (hence the natural numbers are used).

- A sequence is called **constant** if $\forall n, m \in \mathbb{N}, a_n = a_m$ and **eventually constant** if $\exists N \in \mathbb{N} \mid \forall n, m \geq N, a_n = a_m$

4.1.2. Definition - Convergence

A sequence is said to **converge towards** A if:

$$\exists A \in \mathbb{R} \forall \varepsilon \in (0, \infty) \exists N \in \mathbb{N} \mid \forall n \in \mathbb{N} : n \geq N, |a_n - A| < \varepsilon$$

$$\lim_{n \rightarrow \infty} a_n = A$$

- **Divergence** can be proved by proving the conjugate:

$$\forall A \in \mathbb{R} \exists \varepsilon \in (0, \infty) \forall N \in \mathbb{N} \exists n \in \mathbb{N} : n \geq N, |a_n - A| > \varepsilon$$

- A convergent sequence has **only one** limit. Proof:

Let $A_1, A_2 \in \mathbb{R}$ be two limits of the sequence a_n .

Due to the convergence criteria:

$$\begin{aligned} \exists N_1, N_2 \in \mathbb{N} \mid \forall n \geq \max(N_1, N_2), |a_n - A_1| < \varepsilon \wedge |a_n - A_2| < \varepsilon, \forall \varepsilon \in (0, \infty) \\ 0 < \varepsilon - |a_n - A_1|, 0 < \varepsilon - |a_n - A_2| \\ \therefore |a_n - A_1| + |a_n - A_2| < 2\varepsilon \end{aligned}$$

Applying the Lemma $|a + b| \leq |a| + |b|$:

$$\begin{aligned} a_n - A_1 - (a_n - A_2) &= A_2 - A_1 \\ 0 \leq |A_2 + (-A_1)| &\leq |a_n - A_1| + |-(a_n - A_2)| < 2\varepsilon \end{aligned}$$

Since $|-x| = |x| \geq 0$, and this is true $\forall \varepsilon > 0$:

$$\begin{aligned} |A_2 - A_1| &= 0 = A_2 - A_1 \\ A_2 &= A_1 \blacksquare \end{aligned}$$

- The sequence $a_n = \frac{1}{n}$ converges to 0, because $\forall \varepsilon > 0, \exists n \in \mathbb{Z} \mid \frac{1}{n} < \varepsilon$, satisfying the criteria of convergence $\left| \frac{1}{n} - 0 \right| < \varepsilon$

4.1.3. Definition - Subsequence

A subsequence of a_n is any sequence obtained by keeping certain elements a_{n_i} indexed by

$$i_{k \in \mathbb{N}} \mid \forall k \in \mathbb{N}, i_{k+1} > i_k$$

- It follows that $i_k \geq k$ (proof by induction invoking the property of natural numbers $x > y \Rightarrow x \geq y + 1$)
- A sequence can have convergent subsequences **without** itself converging, for example $a_n = (-1)^n$ does not converge but the subsequences a_{2n}, a_{2n+1} are constant and convergent

4.1.4. Lemma - Subsequences of a Convergent Sequence are Convergent to the Same Limit

Proof:

Let a_{n_i} (indexed by $i_{k \in \mathbb{N}}$) be a subsequence of a_n , which converges to $A \in \mathbb{R}$, ie $\exists N \in \mathbb{N} \mid \forall n > N, |a_n - A| < \varepsilon$.

$i_k \geq k \Rightarrow j \geq n$ is a term of i_k , which satisfies the convergence condition for the same A , along with all subsequent elements.

4.1.5. Definition - Accumulation Point

A point $A \in \mathbb{R}$ is called an **accumulation point** of a sequence a_n if:

$$\forall \varepsilon > 0 \forall N \in \mathbb{N} \exists n \in \mathbb{N} \mid n \geq N \wedge |a_n - A| < \varepsilon$$

This no longer requires that every $n \geq N$ is close to A , just that such an n can be chosen for every minimum N (which is a similar notion to choosing the terms that make up a converging subsequence). For example, both 1 and -1 are accumulation points of $a_n = (-1)^n$ but not limits. The following corollaries apply:

- A is an accumulation point of a sequence $a_n \Leftrightarrow$ There exists a subsequence of a_n which converges towards A
- $\forall \varepsilon > 0$ there are infinitely many elements of the sequence a_n near an accumulation point ($A - \varepsilon, A + \varepsilon$). This follows from the fact that there is a subsequence that converges to A and all elements of the subsequence after N are both close to A **and** elements of the parent sequence.
- A convergent sequence's limit is its **unique** accumulation point. The Lemma states: "All subsequences of a convergent sequence are convergent to the same limit" and applying the first corollary proves that they all correspond to the same accumulation point.

4.1.6. Definition - Ring of Sequences

Sequences $\in \mathbb{R}$ form a commutative ring together with point wise addition and multiplication and the constant sequences 0_n and 1_n as neutral elements:

$$a_n + b_n = (a + b)_n$$

$$a_n \cdot b_n = (a \cdot b)_n$$

$$\alpha \cdot b_n = (\alpha \cdot b)_n$$

- They do not form a field under pointwise multiplication, as a non-zero sequence may still contain $0 \in \mathbb{R}$ in it, which under real multiplication has no inverse $0 \cdot i \neq 1$.

4.1.7. Theorem - Operations on Limits

Operations on the sequence x_n which converges to X and y_n which converges to Y have the following effects on their limits:

1. $(x_n + y_n)_n \rightarrow X + Y$ Proof:

We can say the following about these sequences:

$$\forall \varepsilon > 0 \exists n \in \mathbb{N} \mid n \geq \max(N_x, N_y) \wedge |x_n - X| < \varepsilon \wedge |y_n - Y| < \varepsilon$$

To show that $(x_n + y_n)_n$ converges to $X + Y$, we need to show $|(x_n + y_n) - (X + Y)| < \varepsilon$ for increasing n . Due to $0 \leq |x|$ and the compatibility of addition in the ordered field \mathbb{R} , we can add these inequalities:

$$|x_n - X| + |y_n - Y| < 2\varepsilon$$

Applying the triangle inequality:

$$|x_n - X + y_n - Y| = |(x_n + y_n) - (X + Y)| < 2\varepsilon \blacksquare$$

Similar proofs for 2, 3 and 4

2. $(x_n \cdot y_n)_n \rightarrow X \cdot Y$
3. $\forall \alpha \in \mathbb{R}, \alpha \cdot x_n \rightarrow \alpha X$
4. $\forall n \in \mathbb{N}, x_n \neq 0 \wedge X \neq 0 \Rightarrow (x_n^{-1})_n \rightarrow X^{-1}$

Furthermore:

1. $X < Y \Rightarrow \exists N \in \mathbb{N} \mid \forall n > N, x_n < y_n$

Proof:

Since both sequences converge, there exists:

$$\exists N \mid \forall \varepsilon > 0, \forall n > N, |x_n - X| < \varepsilon \wedge |y_n - Y| < \varepsilon$$

We want to consider the case of $X < Y$ and look for terms that “surround” this inequality. These inequalities can be rewritten as:

$$-\varepsilon + X < x_n < \varepsilon + X$$

$$-\varepsilon + Y < y_n < \varepsilon + Y$$

$$x_n - \varepsilon < X$$

$$y_n + \varepsilon > Y$$

$$x_n - \varepsilon < X < Y < y_n + \varepsilon$$

$$x_n - \varepsilon < y_n + \varepsilon$$

Since ε can be chosen to tend towards 0, $x_n < y_n \forall n > N \blacksquare$

2. $(\exists N \in \mathbb{N} \mid \forall n > N, x_n < y_n) \Rightarrow X \leq Y$. This can be proved by contradiction using the previous Lemma. We cannot say $X < Y$, for example the two sequences $x_n = -\frac{1}{n}, y_n = \frac{1}{n}$ approach their limit 0 from different sides.

These can be very useful to calculate the limits of complicated expressions, for example:

$$\lim_{n \rightarrow \infty} \frac{7n^4 + 15}{3n^4 + n^3 + n - 1} = \lim_{n \rightarrow \infty} \frac{7 + 15n^{-4}}{3 + n^3n^{-4} + nn^{-4} - 1n^{-4}} = \lim_{n \rightarrow \infty} \frac{7 + 15n^{-4}}{3 + n^{-1} + n^{-3} - n^{-4}}$$

$$x \geq 1, \lim_{n \rightarrow \infty} n^{-x} = 0$$

$$\therefore \frac{7}{3}$$

Care must be taken to not divide by 0 or ∞ when simplifying such limits.

4.1.8. Lemma - Sandwich Lemma

Consider 3 sequences, such that $\exists N \in \mathbb{N} \mid \forall n > N \mid x_n \leq y_n \leq z_n$. If x_n and z_n converge to the same limit, the sequence y_n also converges to the same limit. The proof follows from the previous Lemma $(\exists N \in \mathbb{N} \mid \forall n > N, x_n < y_n) \Rightarrow X \leq Y$ applied for both x_n and z_n and the principle of trichotomy.

4.1.9. Definition - Bounded Sequence

A sequence is bounded if $\exists M \in \mathbb{R} \mid M \geq 0, |x_n| \leq M \forall n \in \mathbb{N}$. This is different from a limit as the sequence may oscillate between negative and positive.

A sequence is unbounded if $\forall M \in \mathbb{R} \mid M \geq 0, \exists n \in \mathbb{N} \mid |x_n| \geq M$

- Every convergent sequence is bounded (but not every bounded sequence is convergent). Proof:
The bound M is $\max(|A|, |x_1|, |x_2|, \dots, |x_{N-1}|)$, where N is finite.
- Bounded sequences have at least 1 accumulation point / a convergent subsequence.

4.1.10. Definition - Monotone Sequence

A sequence x_n is called (strictly) monotonically increasing / decreasing if:

$$(\forall m, n \in \mathbb{N} \mid m > n) \Rightarrow x_m \geq (>)x_n$$

$$(\forall m, n \in \mathbb{N} \mid m > n) \Rightarrow x_m \leq (<)x_n$$

- Consider a monotone sequence. It is bounded \Leftrightarrow it converges, such that:

$$\text{Monotonically increasing: } \lim_{n \rightarrow \infty} x_n = \sup\{x_n \mid n \in \mathbb{N}\}$$

$$\text{Monotonically decreasing: } \lim_{n \rightarrow \infty} x_n = \inf\{x_n \mid n \in \mathbb{N}\}$$

Proof:

I state the following facts:

$$\text{Monotonically increasing: } (\forall m, n \in \mathbb{N} \mid m > n) \Rightarrow x_m \geq x_n$$

$$\text{Bounded: } \exists M \in \mathbb{R} \mid M \geq 0, \forall b \in \mathbb{N}, |x_b| \leq M, -M \leq x_b \leq M$$

$$\text{Supremum: } \min\{b \in \mathbb{R} \mid \forall x \in X, x \leq b\}$$

I aim to show convergence by combining these definitions towards: $\exists A \in \mathbb{R} \forall \varepsilon \in (0, \infty) \exists N \in \mathbb{N} \mid \forall n \in \mathbb{N} : n \geq N, |a_n - A| < \varepsilon$.

The existence of a bound M shows that the bound set is **not** empty and a supremum “on the sequence” exists (it may be smaller than M). Let $A \in \mathbb{R}$ be such a supremum:

$$\begin{aligned}\forall x_n, -A \leq x_n \leq A \\ \therefore x_n - A \leq 0\end{aligned}$$

Since $\varepsilon > 0$, we can rearrange this to:

$$x_n - A < \varepsilon$$

We now wish to show $-\varepsilon < x_n - A$. It is given $x_n \geq x_{n-1}$. Furthermore, x_{n-1} also respects the bound A . Hence:

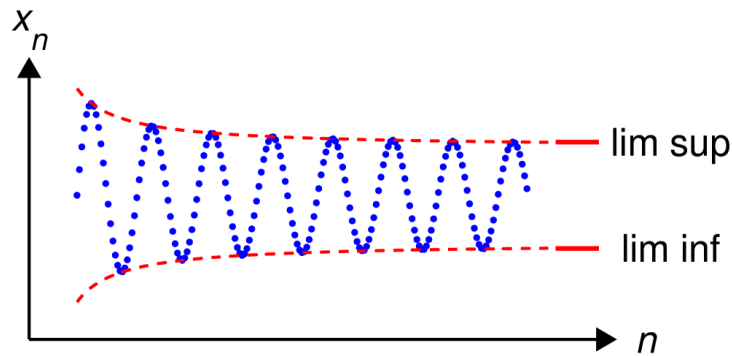
$$-A \leq x_{n-1} \leq x_n$$

TODO: This proof is taking too long :(but I think I am almost there, it is rather important so I certainly need to come back to it later. Hoping to apply the definition of the absolute function, $|x_n - A| < \varepsilon$. I also need to show that there is an N after which this is valid, will try again another time...

4.1.11. Definition - Superior / Inferior Limits

These can be thought of as the steady-state bounds of a sequence. Consider the sequence $s_n = \sup\{x_k \mid k \geq n\}$ based on the sequence x_n . As the starting term to be included n gets larger, the supremum can only stay the same or get smaller (monotonically decreasing) because $m > n$, $\{x_k \mid k \geq m\} \subsetneq \{x_k \mid k \geq n\}$, ie. the starting terms get excluded and $\therefore s_m \leq s_n$. **If the sequence x_n is bounded**, s_n is also bounded as the first (and subsequent) suprema are a real, non-infinite number. Therefore, s_n converges to $\inf\{s_n \mid n \in \mathbb{N}\}$ and vice versa for the inferior limit i_n , such that:

$$\begin{aligned}\forall n \in \mathbb{N}, i_n \leq x_n \leq s_n \\ (\limsup)_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} (\sup\{x_k \mid k \geq n\}) = \inf\{\sup\{x_k \mid k \geq n\} \mid n \in \mathbb{N}\} \\ (\liminf)_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} (\inf\{x_k \mid k \geq n\}) = \sup\{\inf\{x_k \mid k \geq n\} \mid n \in \mathbb{N}\} \\ (\liminf)_{n \rightarrow \infty} x_n \leq (\limsup)_{n \rightarrow \infty} x_n\end{aligned}$$



- The superior and inferior limits of all **bounded** sequences are accumulation points, and therefore have convergent subsequences:

A = Set of accumulation points

$$(\limsup)_{n \rightarrow \infty} x_n = \max(A)$$

$$(\liminf)_{n \rightarrow \infty} x_n = \inf(A)$$

4.1.12. Theorem - Squeeze Theorem

This is known as the **squeeze theorem** as the bounds squeeze towards the limit from either side and is an alternative criteria for convergence:

$$\text{A sequence converges} \Leftrightarrow (\liminf)_{n \rightarrow \infty} x_n = (\limsup)_{n \rightarrow \infty} x_n$$

Proof:

Let $s_n = \sup\{x_k \mid k \geq n\}$, $i_n = \inf\{x_k \mid k \geq n\}$, we know that $\forall n \in \mathbb{N}, i_n \leq x_n \leq s_n$.

It is given that s_n and i_n converge to the same limit, therefore x_n also converges to this limit (sandwich lemma).

To show convergence $\Rightarrow \limsup = \liminf$, as $n \rightarrow \infty$, the minimum index for convergence N will have been reached and:

$$\begin{aligned} \forall n > N \forall \varepsilon \in \mathbb{R} \mid \varepsilon > 0, -\varepsilon < x_n - A < \varepsilon \\ A - \varepsilon < x_n < A + \varepsilon \\ A - \varepsilon < i_n \leq s_n < A + \varepsilon \end{aligned}$$

This can be rearranged to convergence criteria for s_n and i_n , showing that their limits are equal:

$$\begin{aligned} \forall \varepsilon > 0, A - \varepsilon < i_n < A + \varepsilon \Rightarrow |i_n - A| < \varepsilon \\ |s_n - A| < \varepsilon \blacksquare \end{aligned}$$

4.1.13. Definition - Cauchy Sequence

A sequence is a **Cauchy Sequence** if:

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \mid \forall n, m > N, |x_n - x_m| < \varepsilon$$

Important: It is only a Cauchy sequence if $|x_n - x_m| < \varepsilon$ **for all** $n, m > N$. For example, the sequence $1, 1 + \frac{1}{2}, 2, 2 + \frac{1}{3}, 2 + \frac{2}{3}, 3, \dots$ satisfies $|x_n - x_{n-1}| < \varepsilon \forall n > N$, but it is not a Cauchy sequence and is unbounded.

- They are bounded
- A sequence converges \Leftrightarrow it is a Cauchy sequence.

Proof:

Let x_n be a sequence converging to A :

$$\begin{aligned} \forall \varepsilon > 0 \exists N \in \mathbb{N} \mid \forall n > N, |x_n - A| < \varepsilon \\ m > N \Rightarrow |x_m - A| < \varepsilon \\ |x_n - A| + -|x_m - A| < \varepsilon \end{aligned}$$

Applying the triangle inequality:

$$|x_n - x_m| < \varepsilon$$

Since ε is arbitrary, this shows that it is a Cauchy sequence. Now we must show that this implies that it converges, let x_n be a Cauchy sequence. Because it is bounded, there exists a subsequence x_{n_k} which converges to the bound A as $n \rightarrow \infty$. Elements of this subsequence are also elements of x_n and can be included in the Cauchy inequality as $n \rightarrow \infty$:

$$\begin{aligned}
&|x_n - x_{n_k}| < \varepsilon \\
&|x_{n_k} - A| < \varepsilon \\
&|x_n - A| \leq |x_n - x_{n_k}| + |x_{n_k} - A| < 2\varepsilon
\end{aligned}$$

Therefore the entire sequence converges to A ■

4.1.14. Definition - Divergence

We say that a sequence x_n diverges to ∞ ($-\infty$) if:

$$\begin{aligned}
&\forall M \in \mathbb{R} \mid M > (<)0, \exists N \in \mathbb{N} \mid \forall n > N, x_n > (<)M \\
&\lim_{n \rightarrow \infty} x_n = \infty(-\infty)
\end{aligned}$$

The limits ∞ and $-\infty$ are called **improper**.

- An unbounded sequence doesn't necessarily diverge to ∞ , for example $(-1)^n n$ oscillates.
- An unbounded sequence always has a subsequence which diverges to ∞ or $-\infty$.

Proof:

The definition of an unbounded sequence is very similar to that of divergence to ∞ or $-\infty$:

$$\begin{aligned}
&\forall M \in \mathbb{R} \mid M \geq 0, \exists k \in \mathbb{N} \mid |x_k| \geq M \\
&x_k > M \vee x_k < -M
\end{aligned}$$

We now need to show that there are infinitely many such elements with index greater than k .

Assume by contradiction that $\nexists i > k \mid x_i > M \vee x_k < -M$. This violates the criteria for an unbounded sequence, as it implies there are finitely (\mathbb{N} is not dense) many elements $x_k > M$ but this is required for **infinitely** many $M > 0$.

This contradicts the assumption and shows:

$$\forall M \in \mathbb{R} \mid M > (<)0, \exists N \in \mathbb{N} \mid \forall n > N, x_{n_k} > (<)M \blacksquare$$

- The superior / inferior limits of an unbounded sequence are the improper limits ∞ or $-\infty$ depending on if it has an upper or lower bound.

4.1.15. Definition - Complex Sequences

We can study the limits of the real and imaginary parts of a complex number individually. A sequence z_n converges to $A + Bi$ if $\text{Re}(z_n) \rightarrow A, \text{Im}(z_n) \rightarrow B$. Since complex numbers are not ordered, we check divergence using the absolute function $|z_n| \rightarrow \infty$.

5. Complex Numbers

5.1.1. Definition - Complex Numbers

The set of complex numbers \mathbb{C} is defined from the Cartesian coordinates, where the $+$ can be thought of as a substitute for the comma in a tuple, and i is called the **complex unit**:

$$\begin{aligned}
z = a + bi \in \mathbb{C} &:= \mathbb{R}^2 = \{(a, b) \mid a, b \in \mathbb{R}\} \\
a &= \text{Re}(z) \\
b &= \text{Im}(z)
\end{aligned}$$

Complex addition $+\mathbb{C}$ and multiplication $\cdot\mathbb{C}$ are defined such that \mathbb{C} is a field (the operations follow the conditions for a ring excluding division by 0) **and** $i^2 = -1$ holds:

$$(a + bi) +_{\mathbb{C}} (c + di) = (a + b) + (c + d)i$$

$$(a + bi) \cdot_{\mathbb{C}} (c + di) = ac + adi + bci + bdi^2 \\ = (ac - bd) + (ad + bc)i$$

- Addition has Neutral Element $(0, 0) = 0$ and Inverse Element $(-a, -b) = -a - bi$
- Multiplication has Neutral Element $(1, 0) = 1$ and (non-zero) Inverse Element $\left(\frac{a}{a^2+b^2}, -\frac{b}{a^2+b^2}\right)$
- The same notation as in \mathbb{R} is normally used, for example $-(i) = (-0, -1) = -i$ and $i^{-1} = (0, -1) = -i$
- An order relation cannot be defined in a way that satisfies the ordered field axioms. Proof:

Let $0 \leq i$, condition 3. implies $0 \leq i \cdot i = -1$ which is false $\therefore i \leq 0$

Applying condition 2. $i + -i \leq 0 + -i \Rightarrow 0 \leq -i$

Applying condition 3. $0 \leq -i \cdot -i = i^2 = -1$ which is also false and contradicts $i \leq 0$ ■

Nevertheless, they satisfy a generalization of the completeness axiom and we can perform calculus on them.

5.1.2. Definition - Complex Conjugate

The mapping $\bar{\cdot} : \mathbb{C} \rightarrow \mathbb{C}$ of a complex number $z = a + bi \in \mathbb{C}$ is denoted as \bar{z} and defined:

$$\bar{z} := a - bi$$

It has the following properties $\forall z, w \in \mathbb{C}$:

- $z \cdot \bar{z} \in \mathbb{R} \geq 0$
- $\overline{z + w} = \bar{z} + \bar{w}$
- $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$
- $\overline{z \cdot \bar{w}} = \bar{z} \cdot w$
- $\text{Re}(z) = \frac{z + \bar{z}}{2}$
- $\text{Im}(z) = \frac{z - \bar{z}}{2i}$

5.1.3. Definition - Complex Absolute Function

The complex absolute function $|\cdot| : \mathbb{C} \rightarrow \mathbb{R}$ is defined as:

$$|z = a + bi| := \sqrt{z \cdot \bar{z}} = \sqrt{a^2 + b^2}$$

- $|z \cdot w| = \sqrt{(z \cdot w) \cdot \overline{z \cdot w}} = \sqrt{z \cdot \bar{z} \cdot w \cdot \bar{w}} = \sqrt{z \cdot \bar{z}} \cdot \sqrt{w \cdot \bar{w}} = |z| \cdot |w|$
- It has the same notion of length when complex numbers are plotted on an Argand diagram

5.1.4. Theorem - Cauchy-Schwartz Inequality

$\forall z = x_1 + y_1i, w = x_2 + y_2i \in \mathbb{C}$:

$$x_1x_2 + y_1y_2 \leq |z \cdot w|$$

Proof:

Through algebraic rearrangement, we can show that:

$$|z \cdot w| - x_1x_2 + y_1y_2 = (x_1y_2 - x_2y_1)^2$$

Lemma: $x^2 \geq 0$

Therefore $|z \cdot w| - x_1x_2 + y_1y_2 \geq 0$

By applying the compatibility of addition in an ordered field (although \mathbb{C} is not an ordered field, $|x \cdot w|$ can be expressed in terms of the component real numbers), we arrive at:

$$|z \cdot w| \geq x_1x_2 + y_1y_2 \blacksquare$$

5.1.5. Theorem - Complex Triangle Inequality

We can show that the triangle inequality also holds true $\forall z = x_1 + y_1i, w = x_2 + y_2i \in \mathbb{C}$:

$$|z + w| \leq |z| + |w|$$

Proof:

Through algebraic rearrangement, we can show:

$$|z + w|^2 = |z|^2 + |w|^2 + 2(x_1x_2 + y_1y_2)$$

Applying the Cauchy-Schwarz Inequality:

$$|z|^2 + |w|^2 + 2(x_1x_2 + y_1y_2) \leq |z|^2 + |w|^2 + 2|z \cdot w|$$

$$|z + w|^2 \leq (|z| + |w|)^2$$

$$\therefore |z + w| \leq |z| + |w|$$

6. Functions of One Real Variable

6.1.1. Definition - Function

A function $f : X \rightarrow Y$ is a mapping from a domain X (not just the natural numbers like sequences) to range / codomain Y . For now we only discuss single-valued **real** functions: $X, Y \subseteq \mathbb{R}$. It **may** have the following properties:

1. *Injective* - $\forall x, x' \in X : x \neq x' \Rightarrow f(x) \neq f(x')$ - Assigns each element of X a **unique** element in Y
2. *Surjective* - $\forall y \in Y \exists x \in X : f(x) = y$ - Every element in the range is a possible output of the function
3. *Bijjective* - It is both injective and surjective, and therefore an inverse function can be defined
4. Two functions are **equal** $\Leftrightarrow X_1 = X_2 \wedge Y_1 = Y_2 \wedge \forall x \in X, f_1(x) = f_2(x)$

6.1.2. Definition - Image and Preimage (Urbild) of a Function

Consider a function $f : X \rightarrow Y$.

- The Image $f(A)$ of $A \subseteq X$ under f is defined as:

$$f(A) := \{y \in Y \mid \exists x \in A : f(x) = y\}$$

$$f(A) \subseteq Y$$

- The Preimage (Urbild) $f^{-1}(B)$ of $B \subseteq Y$ under f is defined as:

$$f^{-1}(B) := \{x \in X \mid \exists y \in B : f(x) = y\} = \{x \in X \mid f(x) \in B\}$$

$$f^{-1}(B) \subseteq X$$

- A function $f : X \rightarrow Y$ is surjective \Leftrightarrow The set $f(X) = Y$, because the image can only contain domain elements which map to Y by definition.
- For example consider $f : \mathbb{R} \rightarrow \mathbb{R} := x \rightarrow 0$:
 - $f(\mathbb{R}) = \{0\}$ - It is not surjective
 - $f^{-1}(\mathbb{R}) = f^{-1}(\{0\}) = \mathbb{R}$
 - $f^{-1}(\{1\}) = \emptyset$

There is an interesting property of finite sets; consider $f : X \rightarrow Y$, where X and Y are **finite** sets with the same number of elements n :

$$f \text{ is injective} \Leftrightarrow f \text{ is surjective}$$

Proof:

If f is injective, the image $f(X)$ has at least n distinct elements so every distinct $x \in X$ has its own $y \in f(X)$.

Lemma: A function $f : X \rightarrow Y$ is surjective \Leftrightarrow The set $f(X) = Y$

We are given that Y has n elements, and since $f(X) \subseteq Y \Rightarrow f(X) = Y$ showing that it must also be surjective.

We must now show that surjectivity \Rightarrow injectivity. If f is surjective, $f(X) = Y$ (Lemma), therefore $f(X)$ has n elements.

Consider two elements $x_1, x_2 \in X$. Since X is a set, they are distinct $x_1 \neq x_2$.

If $f(x_1) = f(x_2)$ for any two elements, they would “validate” the same member of $f(X)$, leaving out at least one element of Y (deterministic, another input cannot have 2 outputs to make up for it) meaning $f(X)$ would have $n - 1$ elements, which contradicts the lemma about surjectivity, therefore f must also be injective ■

This is **not** necessarily true for infinite sets, for example $f : \mathbb{N} \rightarrow \mathbb{N}, f(x) := x + 1$ is injective but not surjective.

6.1.3. Definition - Square Root

This is the bijective function $\sqrt{\cdot} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ such that $\forall a \in \mathbb{R}_{\geq 0} (\sqrt{a})^2 = a$, whose existence is only possible due to the real numbers being a **complete ordered field**. TODO: Various proofs

Exercise 2.27 Figalli’s script

6.1.4. Definition - Ring of Functions

For a domain X , we can define a **commutative** ring (not a field, there is no inverse element for multiplication) on the set of real valued functions $\mathcal{F}(X) := \{f \mid f : X \rightarrow \mathbb{R}\}$ with the following operations:

$$\begin{aligned} f_1, f_2 &\in \mathcal{F}(X) \\ (f_1 + f_2)(x) &:= f_1(x) + f_2(x) \\ (f_1 \cdot f_2)(x) &:= f_1(x) \cdot f_2(x) \\ \alpha \in \mathbb{R}, (\alpha \cdot f_1)(x) &:= \alpha \cdot f_1(x) \end{aligned}$$

The constant function is defined as $\forall x \in X, f(x) = a$.

- Neutral elements - Addition: $f(x) = 0$, Multiplication: $f(x) = 1$

An order relation \leq is defined:

$$f_1 \leq f_2 \Leftrightarrow \forall x \in X, f_1(x) \leq f_2(x)$$

6.1.5. Definition - Bounded Functions

As with many function definitions, they are very similar to the definitions for sequences. A function $f : X \rightarrow Y$ is bounded if:

$$\exists M \in \mathbb{R} \mid M > 0, \forall x \in X, |f(x)| < M$$

This can be separated into “bounded from above” $f(x) < M$ and below $f(x) < -M$

6.1.6. Definition - Monotone Functions

A function $f : X \rightarrow Y$ is called (strictly) monotonically increasing / decreasing if $\forall m, n \in X$:

$$\begin{aligned} m > n &\Rightarrow f(m) \geq (>)f(n) \\ m > n &\Rightarrow f(m) \leq (<)f(n) \end{aligned}$$

- The rounding function $\lfloor x \rfloor$ is monotonically increasing but not strictly

- A function is constant \Leftrightarrow A function is both monotonically increasing and decreasing. Proof: Trichotomy
- A **strictly** monotone function is always injective. Proof:
Assume by contradiction that it is not injective. $\exists x_1 \neq x_2 \mid f(x_1) = f(x_2)$. However, since $x_1 \neq x_2$, they must be either $>$ or $<$ each other and therefore $f(x_1) \neq f(x_2)$ (monotone) which proves that they are injective by contradiction ■

6.1.7. Definition - $\varepsilon\delta$ Continuity

Intuitively, a function is continuous over an interval if we can draw it without lifting the pencil. A function is continuous at a point $x_0 \in X$ if:

$$\forall \varepsilon > 0 \exists \delta > 0 \mid \forall x \in X, |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

It is continuous over a set X if it is continuous $\forall x_0 \in X$, the definition can be amended to $\forall x_1, x_2 \in X, \exists \delta \mid |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$ (a different δ may be chosen for each pair of points). If there is a jump in the function at the point x , then there exists a small enough ε , so that no matter how close x is to x_0 , the max error ε will never be satisfied.

- The **Dirichlet Function** based on the characteristic function is non-continuous at every point:

$$1_{\mathbb{Q}} : \mathbb{R} \rightarrow \{1, 0\}$$

$$1_{\mathbb{Q}} := \chi_{\mathbb{Q}}$$

This is because there are irrational, real numbers around every rational number so for $0 < \varepsilon < 1 \nexists \delta$ such that points next to each other have an output $< \varepsilon$. This also demonstrates why the $\forall x \in X$ is necessary, otherwise one could simply pick two rational numbers within the δ interval (it is a dense set).

- Constant functions are continuous. Proof:
 $\forall x, x_0 \in X, f(x) - f(x_0) = 0 < \varepsilon$ therefore there always exists such a δ ■
- The function $f(x) = x$ is continuous. Proof:
We need to find a δ such that $\forall x_1, x_2 \in X, |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$.
Because $|f(x_1) - f(x_2)| = |x_1 - x_2|$ and $\delta, \varepsilon > 0$, δ can always be chosen such that the second inequality also holds true $\forall \varepsilon > 0$ ■
- The absolute function $f(x) = |x|$ is continuous. Proof:
Inverse triangle inequality: $|f(x_1) - f(x_2)| = ||x_1| - |x_2|| \leq |x_1 - x_2| < \delta = \varepsilon$ ■
- x^2 is continuous TODO

6.1.8. Lemma - Operations on Continuous Functions

Let $f_1, f_2 : D \rightarrow \mathbb{R}$ be continuous functions at a point $x_0 \in D$. The following functions are also continuous at x_0 :

1. $(f_1 + f_2)(x)$ - Proof:

We are given:

$$\forall \varepsilon > 0 \forall x \in X$$

$$\exists \delta_1 > 0 \mid |x - x_0| < \delta_1 \Rightarrow |f_1(x) - f_1(x_0)| < \varepsilon$$

$$\exists \delta_2 > 0 \mid |x - x_0| < \delta_2 \Rightarrow |f_2(x) - f_2(x_0)| < \varepsilon$$

Setting $\delta = \min\{\delta_1, \delta_2\}$:

$$\exists \delta > 0 \mid |x - x_0| < \delta \Rightarrow 0 \leq |f_1(x) - f_1(x_0)| < \varepsilon \wedge 0 \leq |f_2(x) - f_2(x_0)| < \varepsilon$$

$$|f_1(x) - f_1(x_0)| + |f_2(x) - f_2(x_0)| < 2\varepsilon$$

Since functions form a ring, we want to show $\exists \delta \mid |x - x_0| < \delta \Rightarrow |f_1(x) + f_2(x) - f_1(x_0) + f_2(x_0)|$. Applying the triangle inequality gives:

$$|f_1(x) + f_2(x) + (-(f_1(x_0) + f_2(x_0)))| \leq |f_1(x) - f_1(x_0)| + |f_2(x) - f_2(x_0)| < 2\varepsilon$$

Therefore $(f_1 + f_2)(x)$ is also continuous at x_0 ■

2. $(f_1 \cdot f_2)(x)$ - Proof:

Following the previous definitions, we wish to show that $\exists \delta \mid |x - x_0| < \delta \Rightarrow |f_1(x) \cdot f_2(x) - f_1(x_0) \cdot f_2(x_0)| < \varepsilon$, which we can achieve as follows:

$$\begin{aligned} |f_2(x)| |f_1(x) - f_1(x_0)| &< |f_2(x)| \varepsilon \\ |f_1(x_0)| |f_2(x) - f_2(x_0)| &< |f_1(x_0)| \varepsilon \\ |f_2(x)f_1(x) - f_2(x)f_1(x_0)| + |f_1(x_0)f_2(x) - f_1(x_0)f_2(x_0)| &< \varepsilon(|f_1(x)| + |f_2(x)|) \\ |f_1(x) \cdot f_2(x) - f_1(x_0) \cdot f_2(x_0)| &< \varepsilon(|f_1(x)| + |f_2(x)|) \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this proves the continuity of the product of continuous functions ■

3. $\forall \alpha \in \mathbb{R}, (\alpha \cdot f_2)(x)$ - The proof follows from setting $f_1(x)$ in the previous Lemma to the constant function $f_1(x) = \alpha$, which has been shown to be continuous.

This can be extended to continuity over a common subset if they are both continuous over that set.

6.1.9. Corollary - Polynomials are continuous

All polynomials can be constructed from a linear combination of $f(x) = x$, its powers and constant functions $f(x) = a$, which were both shown to be continuous $\in \mathbb{R}$. Hence polynomials are also continuous for all points in \mathbb{R} .

6.1.10. Definition - Composition of Functions

Functions can be passed as arguments into one another:

$$\begin{aligned} f : X \rightarrow Y, g : Y \rightarrow Z \\ g \circ f : X \rightarrow Z \\ g \circ f := g(f(x)) \end{aligned}$$

- Composition of functions is associative and brackets are irrelevant

6.1.11. Theorem - Composition of Continuous Functions

Let $f : X \rightarrow Y$ which is continuous at x_0 and $g : Y \rightarrow Z$ continuous at $f(x_0)$ such that $X, Y, Z \subseteq \mathbb{R}$. $g \circ f$ is also continuous at x_0 .

Proof:

The following properties apply $\forall \varepsilon > 0$:

$$\begin{aligned} \exists \delta_1 \mid \forall x \in X, |x - x_0| < \delta_1 \Rightarrow |f(x) - f(x_0)| < \varepsilon \\ \exists \delta_2 \mid \forall f(x) \in Y, |f(x) - f(x_0)| < \delta_2 \Rightarrow |g(f(x)) - g(f(x_0))| < \varepsilon \end{aligned}$$

To show continuity of $g \circ f$ at x_0 , I will show $\exists \delta \mid \forall x \in X, |x - x_0| < \delta \Rightarrow |g(f(x)) - g(f(x_0))| < \varepsilon$. By choosing $\delta = \min(\delta_1, \delta_2)$, this is clearly the case as:

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \Rightarrow |g(f(x)) - g(f(x_0))|$$

The surjectivity of $f(x)$ does not matter, as the final inequality only applies if the intermediate $|f(x) - f(x_0)| < \varepsilon$ is true ■

As usual, this extends to the entire domain if both f and g are continuous functions.

6.1.12. Theorem - Sequential Continuity

This is an alternative characterization of a continuous function. The function $f : X \rightarrow \mathbb{R}$ is continuous at $x_0 \in X \Leftrightarrow$ For every sequence $x_n \subseteq X$ converging to x_0 (there always exists at least 1, for example $x_n = x_0$), the sequence $f(x_n)$ converges to $f(x_0)$.

Proof:

The following holds true for all sequences converging to x_0 :

$$\forall \delta > 0 \exists N_1 \mid \forall n > N_1, |x_n - x_0| < \delta$$

If the sequence $f(x_n)$ converges to $f(x_0)$:

$$\forall \varepsilon > 0 \exists N_2 \mid \forall n > N_2, |f(x_n) - f(x_0)| < \varepsilon$$

Choosing $N = \max\{N_1, N_2\}$ both conditions hold true:

$$\forall \varepsilon > 0 \exists N \mid \forall n > N, |x_n - x_0| < \delta \wedge |f(x_n) - f(x_0)| < \varepsilon$$

TODO: Somehow this is equivalent to saying $\forall x \in X$:

$$\forall \varepsilon > 0 \exists \delta \mid \forall x \in X |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \blacksquare$$

6.1.13. Theorem - Intermediate Value Theorem (Zwischenwertsatz)

Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function such that $a \leq b$. $\forall c \in [f(a), f(b)] \exists x \in [a, b] \mid f(x) = c$. In simpler words, a continuous function f takes on every value between $[f(a), f(b)]$ at least once - it is **surjective** over $[f(a), f(b)]$. This is useful to show surjectivity or to prove that an injective, continuous function is **strictly monotone** over an interval (assume it is not and choose a point that violates the monotonicity, there exists two points on either side with the same output due to this theorem).

Proof:

Let c be any value who want to show is in the domain of the continuous function $f : [a, b] \rightarrow X$ such that $f(a) < c < f(b)$ (one can prove for $f(b) < f(a)$ similarly by working with the set $U = \{x \in [a, b] \mid c < f(x)\}$).

Consider the set $L = \{x \in [a, b] \mid f(x) < c\}$. The set is not-empty ($a \in L$) and is bounded, therefore it has a supremum s . We will now show that $s \neq a \wedge s \neq b$.

Due to the continuity of f at a , we can keep $f(x)$ any $\varepsilon > 0$ away (at most) from $f(a)$ such that $|x - a| < \delta$. Choose $\varepsilon < c - f(a)$ (to pull the values away from a towards whichever value outputs c).

The x 's which satisfy this ε must be in L ($\forall x \in [a, b] \wedge f(x) - f(a) < \varepsilon = c - f(a) \Rightarrow f(x) < c$) but not equal to a (because $\varepsilon \neq 0$), therefore L contains at least one element greater than $a \Rightarrow$ the supremum $s > a$. The same can be argued about the continuity at b , any elements close to b are greater than c (due to $c < f(b)$) \Rightarrow not in L , therefore the supremum $s < b$.

We now know that $s \in (a, b)$ and want to show $f(s) = c$. Since f is continuous over $[a, b]$, it is also continuous at s :

$$\forall \varepsilon > 0 \exists \delta > 0 \mid \forall x \in [a, b], |x - s| < \delta \Rightarrow |f(x) - f(s)| < \varepsilon$$

Considering δ for any ε , there must exist some $x_0 \mid -\varepsilon + f(s) < f(x_0) < \varepsilon + f(s)$ which is in the interval $x_0 \in (s - \delta, s]$ (because s is the supremum). Since $f(x_0) < c$, this leads to:

$$\begin{aligned} -\varepsilon + f(s) &< c \\ f(s) &< c + \varepsilon \end{aligned}$$

Furthermore, there exists an $x_1 \in [s, s + \delta) \mid f(x_1) \geq c$ (otherwise it would've been in L), and due to the continuity of f at s :

$$\begin{aligned} -\varepsilon + f(s) &< f(x_1) < \varepsilon + f(s) \\ c &< \varepsilon + f(s) \\ -\varepsilon + c &< f(s) \\ -\varepsilon + c &< f(s) < c + \varepsilon \end{aligned}$$

Choosing $\varepsilon \rightarrow 0$, it is clear that $f(s) = c$ whilst $s \in [a, b]$ ■

Summary:

- Define set $L = \{x \in [a, b] \mid f(x) < c\}$
- Show that its supremum $s \in (a, b)$ due to continuity at both of those points
- Exploit the continuity at s along with points $x_0 \in (s - \delta, s], \in L \Rightarrow f(x_0) < c$ and $x_1 \in [s, s + \delta), \notin L \Rightarrow f(x_1) \geq c$ to show that $f(s) = c$ as $\varepsilon \rightarrow 0$ for any $c \in (f(a), f(b))$

6.1.14. Definition - Inverse Function

Any **bijective** function $f : X \rightarrow Y$ has a corresponding inverse $f^{-1} : Y \rightarrow X$ (not to be confused with the preimage, which is defined for all functions but doesn't take account of every element in Y , not surjective) defined such that:

$$f^{-1} \circ f = f \circ f^{-1} = \text{id}$$

6.1.15. Theorem - Inverse Function Theorem

A function that is strictly monotone and continuous (for example a^x) over an interval $I \subseteq \mathbb{R}$ is bijective and has an inverse function, which is also strictly monotone and continuous.

Proof:

Let $J = f(I)$ and consider the strictly monotone, continuous function $f : I \rightarrow J$. It is surjective by definition and injective due to its strict monotonicity, therefore the inverse $f^{-1} : J \rightarrow I$ exists, which is also strictly monotone:

$$\begin{aligned} \forall x_1, x_2 \in X \\ x_1 < x_2 \Leftrightarrow f(x_1) < (>) f(x_2) \Leftrightarrow f^{-1}(f(x_1)) < f^{-1}(f(x_2)) \end{aligned}$$

The continuity of the inverse can be shown using the sequential continuity criteria but is rather complicated.

6.1.16. Definition - n'th Root Function

This is defined as the inverse of $x^n : [0, \infty) \rightarrow [0, \infty)$ and can be written as either $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$.

6.1.17. Theorem - Compact Domain, Bounded Function

Let $f : [a, b] \rightarrow \mathbb{R}$ be a **continuous** function with a compact (closed and bounded) domain, its range (and by convention f itself) is therefore also bounded.

Proof:

Assume by contradiction that f is not bounded, ie. $\forall N \in \mathbb{R} \mid N > 0, \exists x \in [a, b] \mid |f(x)| > N$.

Because the domain is compact, there exists a convergent subsequence x_n with a limit

$\lim_{n \rightarrow \infty} x_n \in [a, b]$, such that due to the composition of functions $|f(x_n)| \rightarrow |f(\lim_{n \rightarrow \infty} x_n)|$.

TODO: I did not understand the last contradiction step Theorem 3.39 Figalli

- The domain must be compact, not just bounded. For example, the domain of the function $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbb{R}$ is bounded but not closed, therefore the function is unbounded.

6.1.18. Definition - Extrema

The maximum (minimum) of $f : X \rightarrow \mathbb{R}$ is defined as the value $f(m)$ of the point $m \in X$ such that $\forall x \in X, f(x) \leq (\geq) f(m)$. Bounded functions (and therefore any continuous function with a compact domain) have both a maximum and minimum over their domain.

6.1.19. Definition - Uniform Continuity

This is a stricter definition of $\varepsilon\delta$ continuity, where δ satisfies the **the same** ε error interval for all points:

$$f : X \rightarrow Y$$

$$\forall \varepsilon > 0 \exists \delta \mid \forall x_1, x_2 \in X, |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$$

- Generally, uniform means the same ε and δ applies to all points in the domain (see uniform convergence), all points are continuous / converge at the same rate.
- An example of a continuous but not uniformly continuous function is $x^2 : \mathbb{R} \rightarrow [0, \infty)$ and $e^x : \mathbb{R} \rightarrow (0, \infty)$ - they become nearly vertical as x increases, meaning that $\delta \rightarrow 0$ to satisfy the entire domain, which is not allowed as $\delta > 0$. Allowing a different δ for each point (normal continuity) is however allowed. Non-uniform continuity can be proved by choosing $x_2 = x_1 + \frac{\delta}{2}$ and showing that $|f(x_1) - f(x_2)| < \varepsilon$ doesn't hold true $\forall x_1 \in X$, a proof by contradiction.
- A continuous function on a **compact** domain is **uniformly** continuous. Intuition: It is bounded and has a maximum and minimum, therefore we can choose a $\delta > 0$ which satisfies the epsilon interval for all points.

6.1.20. Definition - Bernoulli's Inequality

This states that:

$$\forall a \in \mathbb{R} \mid a \geq -1, n \in \mathbb{N} \mid n \geq 1$$

$$(1 + a)^n \geq 1 + na$$

Proof by induction:

For $n = 1, 1 + a \geq 1 + a$ holds true. Assume it is true for $n = k, n = k + 1$ amounts to:

$$(1 + a)^k \geq 1 + ka$$

$$a \geq -1 \Rightarrow (1 + a) \geq 0$$

$$\therefore (1 + a)^{k+1} = (1 + a)^k \cdot (1 + a) \geq (1 + ka) \cdot (1 + a)$$

$$(1 + a)^{k+1} \geq 1 + a + ka + ka^2 = 1 + a(1 + k) + ka^2$$

$$ka^2 \geq 0, \therefore 1 + a(1 + k) + ka^2 \geq 1 + a(1 + k)$$

$$\therefore (1 + a)^{k+1} \geq 1 + a(1 + k)$$

Therefore it holds true for all $n \in \mathbb{N}$ ■.

6.1.21. Definition - Euler's Number as a Limit

This converging sequence was discovered by Bernoulli whilst calculating the effect of frequency of payments on compound interest:

$$x \in \mathbb{R}$$

$$e^x = \exp(x) := \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n$$

It converges to a positive, real number. Proof:

First we show that it is monotonically increasing. Let $n_0 \geq 1, n_0 > -x$:

$$\begin{aligned}
\forall n \geq n_0, \frac{x}{(n+1)(n+x)} &\leq \frac{x+n}{(n+1)(n+x)} = \frac{1}{n+1} < 1 \\
&n+x > 0 \\
&\therefore \frac{x}{(n+1)(n+x)} \leq 1 \\
&\therefore -\frac{x}{(n+1)(n+x)} \geq -1
\end{aligned}$$

We now want to show that $\frac{x_{n+1}}{x_n} \geq 1$:

$$\frac{\left(1 + \frac{x}{n+1}\right)^{n+1}}{\left(1 + \frac{x}{n}\right)^n} = \left(1 + \frac{x}{n}\right) \left(\frac{1 + \frac{x}{n+1}}{1 + \frac{x}{n}}\right)^{n+1} = \frac{n+x}{n} \left(1 - \frac{x}{(n+1)(n+x)}\right)^{n+1}$$

Applying Bernoulli's inequality to the second term with $a = -\frac{x}{(n+1)(n+x)}$ and $n = n+1$:

$$\begin{aligned}
\frac{n+x}{n} \left(1 - \frac{x}{(n+1)(n+x)}\right)^{n+1} &\geq \frac{n+x}{n} \left(1 + \frac{-x(n+1)}{(n+1)(n+x)}\right) = \frac{n+x}{n} \left(1 - \frac{x}{n+x}\right) \\
&= \frac{n+x}{n} \left(\frac{n}{n+x}\right) = 1 \\
&\therefore \frac{x_{n+1}}{x_n} \geq 1 \forall n \geq n_0 \geq 1
\end{aligned}$$

Showing that the $\left(1 + \frac{x}{n}\right)^n$ is monotonically increasing. It remains to show that it is bounded, consider the case of $x \leq 0$:

$$\forall n \in \mathbb{N}, 1 + \frac{x}{n} \leq 1$$

Showing that 1 is the upper bound. Now we consider the case $x > 0$:

$$\begin{aligned}
\left(\left(1 + \frac{x}{n}\right)\left(1 - \frac{x}{n}\right)\right)^n &= \left(1 - \frac{x^2}{n^2}\right)^n \leq 1 \\
&\therefore \left(1 + \frac{x}{n}\right)^n \leq \frac{1}{\left(1 - \frac{x}{n}\right)^n}
\end{aligned}$$

$\left(1 - \frac{x}{n}\right)^n$ was shown to have upper bound 1 in the case $x \leq 0$, therefore the sequence is also bounded for $x > 0$. Since it is monotonically increasing and bounded, it converges in all cases ■

6.1.22. Definition - Properties of e^x

The exponential function $\exp : \mathbb{R} \rightarrow \mathbb{R}_{>0}$ exhibits many useful and interesting properties:

- $\forall x > -n, \exp(x) \geq \left(1 + \frac{x}{n}\right)^n$ since it is defined as the limit of a monotonically increasing sequence
- $\forall x \in \mathbb{R}, \exp(x) \geq 1 + x$ this is a consequence of the aforementioned Lemma

Since it is defined as the limit of a sequence, we must show that the usual exponential rules also apply before we can treat it as such:

- $\exp(-x) = \exp(x)^{-1}$ Proof:

Lemma: Multiplying converging sequences

$$\exp(x) \exp(-x) = \lim_{n \rightarrow \infty} \left(1 - \frac{x^2}{n^2}\right)^n$$

Eventually $n > |x|$, therefore $-\frac{x^2}{n^2} \geq -1$ and we can apply Bernoulli's inequality:

$$\exp(x) \exp(-x) \geq 1 + -\frac{x^2}{n^2}n = 1 - \frac{x^2}{n}$$

We have shown that $\lim_{n \rightarrow \infty} \left(1 - \frac{x^2}{n^2}\right)^n$ has upper bound 1, therefore:

$$1 - \frac{x^2}{n} \leq \exp(x) \exp(-x) \leq 1$$

$\lim_{n \rightarrow \infty} 1 - \frac{x^2}{n} = 1$ thus $\forall x \in \mathbb{R}, \exp(x) \exp(-x) = 1 = \frac{\exp(x)}{\exp(x)}$ ■

- $\exp(x + y) = \exp(x) \exp(y)$ Proof:

I aim to show that $\frac{\exp(x) \exp(y)}{\exp(x+y)} = 1$

$$\begin{aligned} \exp(x) \exp(y) &= \lim_{n \rightarrow \infty} \left[\left(1 + \frac{x}{n}\right)^n \left(1 + \frac{y}{n}\right)^n = \left(1 + \frac{x+y}{n} + \frac{xy}{n^2}\right)^n \right] \\ \frac{\exp(x) \exp(y)}{\exp(x+y)} &= \frac{\left(1 + \frac{x+y}{n} + \frac{xy}{n^2}\right)^n}{\left(1 + \frac{x+y}{n}\right)^n} = \left(1 + \frac{(xy)}{n^2 \left(1 + \frac{x+y}{n}\right)}\right)^n \end{aligned}$$

As $n \rightarrow \infty$:

$$\begin{aligned} \left(1 + \frac{xy}{2n^2}\right)^n &< \left(1 + \frac{(xy)}{n^2 \left(1 + \frac{x+y}{n}\right)}\right)^n < \left(1 + \frac{2xy}{n^2}\right)^n \\ \lim_{n \rightarrow \infty} \left(1 + \frac{xy}{2n^2}\right)^n &= \lim_{n \rightarrow \infty} \left(1 + \frac{2xy}{n^2}\right)^n = 1 \end{aligned}$$

Due to the squeeze theorem, $\frac{\exp(x) \exp(y)}{\exp(x+y)}$ also converges to 1, proving that multiplication behaves the same as exponentials ■

- It is continuous - This proof requires several lemmas / tricks, dig deeper if part of Kobel-Keller's course.
- It is strictly monotone increasing. Proof:

Consider $x_1, x_2 \in \mathbb{R} \mid x_1 < x_2 \Rightarrow x_2 - x_1 > 0$, we want to show that $\exp(x_1) < \exp(x_2)$.

Applying the Lemma $\exp(x) \geq 1 + x$:

$$\begin{aligned} \exp(x_2 - x_1) &\geq 1 + x_2 - x_1 > 1 \\ \exp(x_1) \exp(x_2 - x_1) &> 1 \exp(x_1) \\ \exp(x_2) &> \exp(x_1) \blacksquare \end{aligned}$$

- It is bijective. Proof:

It is strictly monotonically increasing and thus injective.

Choosing $x_1, x_2 \in \mathbb{R}$ such that $\exp(x_1) < \exp(x_2)$, there always exists some $x_m \in$

$\mathbb{R} \mid \exp(x_m) \in [\exp(x_1), \exp(x_2)]$ due to the intermediate value theorem, proving surjectivity over $\mathbb{R}_{>0}$ ■

- It displays the unique property that its derivative at any point is the same. TODO prove using first principles

$$\frac{de^x}{dx} = \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h}$$

6.1.23. Definition - Natural Logarithm

Due to the inverse function theorem, the inverse of $\exp(x)$ exists and is also strictly monotonic and continuous, denoted as:

$$\ln(x) = \exp(x)^{-1} : \mathbb{R}_{>0} \rightarrow \mathbb{R}$$

The typical log identities apply, by rearranging the exponential identities proven for $\exp(x)$:

- $\ln(1) = 0$
- $\ln(x^{-1}) = -\ln(x)$
- $\ln(ab) = \ln(a) + \ln(b)$
- $\forall x \in \mathbb{R}_{>0}, \ln(x) \leq x - 1$

6.1.24. Theorem - Logarithm Identities

Logarithms are equipped with a variety of useful identities for basic calculations as well as calculus:

- Change of basis, useful with log tables:

$$b^{\log_b x} = x = e^{\ln x} = e^{\ln(b)^{\log_b x}} = e^{\ln(b) \log_{b(x)}}$$

$$\ln x = \ln(b) \log_{b(x)}$$

$$\log_{b(x)} = \frac{\ln x}{\ln b}$$

- $a > 0, x \in \mathbb{R}, a^x = e^{x \ln(a)}$ - Useful for taking the derivative of an arbitrary exponent
- Slide rules were widely used for performing multiplication, division and many other operations before electronic calculators became widespread in the 70s. They contain pairs of scales with some logarithmic base marked, for example $\ln(x)$ such that the x-values were marked at decreasing distances representing the output values. To calculate $1.2 \cdot 2.6$, one could align the start of the first scale at the position where 1.2 is marked on the second, such that distance between the start of the second and 2.6 on the first scale is equal to $\ln(1.2) + \ln(2.6) = \ln(1.2 \cdot 2.6)$ (thanks to $\ln(a) + \ln(b) \equiv \ln(ab)$) which can simply be read off the marking. Division is done in a similar fashion and multiples of 10 can easily be factored out to ensure the result fits in the scale.

6.1.25. Definition - Accumulation Point

Also known as limit points, the output of a function at such points can be approximated based on its neighborhood. Consider a point $x_0 \in \mathbb{R}$, if $\forall \delta > 0, X \cap (x_0 - \delta, x_0 + \delta) \neq \emptyset$ (there exist points immediately next to x_0 which are in the domain of $f : X \rightarrow \mathbb{R}$), such a point is called an **accumulation point** of the domain X and we can define a limit.

6.1.26. Definition - Limit of a Function

This can be used to represent the value of function at a point which is not necessarily in the domain of the function, for example at a gap.

The limit L of f at an **accumulation point** x_0 is formally defined as:

$$\lim_{x \rightarrow x_0} f(x) = L \Leftrightarrow \forall \varepsilon > 0 \exists \delta > 0 \mid \forall x \in (X \cap (x_0 - \delta, x_0 + \delta)), |f(x) - L| < \varepsilon$$

This definition is very similar to continuity at the point x_0 , with the difference that only points in the domain are checked (not necessarily x_0) and L is what the output $f(x_0)$ would be if based on its surrounding points.

- A limit **may not always exist** (for example at a jump in the surrounding points) but if it does, it is unique.
- Endpoints are also accumulation points and the both-sided (normal) limit is defined, based only on the surrounding points from one side.

- If there exists a discontinuity in the function at x_0 but the surrounding points on both sides are “continuous”, we can take the limit of $X \setminus x_0$ which is defined based on the surrounding points and the jump is ignored, this can be used to correct so-called **removable discontinuities** (discontinuities with continuous points immediately next to them) and define the **continuous extension** of $f(x)$:

$$L = \lim_{\substack{x \rightarrow x_0 \\ x \neq x_0}} f(x)$$

$$\tilde{f}(x) : X \rightarrow \mathbb{R} := \begin{cases} f(x) & \text{if } x \neq x_0 \\ L & \text{if } x = x_0 \end{cases}$$

- Linear combinations of functions with limits at x_0 hold:

$$\lim_{x \rightarrow x_0} f(x) = L_1, \lim_{x \rightarrow x_0} g(x) = L_2$$

$$\lim_{x \rightarrow x_0} (f + g)(x) = L_1 + L_2$$

$$\lim_{x \rightarrow x_0} (f \cdot g)(x) = L_1 \cdot L_2$$

$$\lim_{x \rightarrow x_0} (\alpha f)(x) = \alpha L_1$$

- A function is continuous at $x_0 \Leftrightarrow \lim_{x \rightarrow x_0} f(x) = f(x_0)$, this is the same as the continuity definition.

6.1.27. Definition - Indeterminate Form

Although combinations of limits such as $\infty + \infty = \infty$ are allowed, certain combinations cannot be computed through simple addition / multiplication and are named **indeterminate forms**:

$$\infty - \infty, 0 \cdot \infty, \frac{0}{0}, \frac{\infty}{\infty}, 0^0$$

Instead, one can often simplify an expression, for example by dividing polynomials in a fraction by some power of n to reduce terms to $\frac{a}{n^i}$, which can be eliminated as they tend towards 0 as $n \rightarrow \infty$. In more difficult cases, one may be able to apply l'Hôpital's Theorem or rewrite the function as a Taylor expansion.

6.1.28. Theorem - Composition of Limits

Consider a function $f : X \rightarrow Y$ such that the limit $\lim_{x \rightarrow x_0} f(x)$ is defined. If f is composed with a **continuous** function $g : Y \rightarrow Z$, the following holds true:

$$\lim_{x \rightarrow x_0} g \circ f(x) = g\left(\lim_{x \rightarrow x_0} f(x)\right)$$

Proof: Because $g(x)$ is continuous, its limit is equal to its output throughout its domain Y (Lemma), therefore when it is composed with another (potentially non-continuous) function, the output depends purely on the limit of the inner value.

- For example, we can rewrite $x^x = e^{x \ln x}$, breaking it down into limits of well-understood functions.

6.1.29. Definition - Diverging Limit

The limit of a function diverges as $x \rightarrow x_0$ if:

$$\lim_{x \rightarrow x_0} f(x) := (-)\infty \Leftrightarrow$$

$$\forall M \in \mathbb{R} \mid M > 0 \exists \delta > 0 \mid \forall x \in (X \cap (x_0 - \delta, x_0 + \delta)), (-)M < (>)x_0$$

6.1.30. Definition - One-Sided Limit

Limits can be defined with a direction the value is approached from:

$$\left(\lim_{\substack{x \rightarrow x_0 \\ x \geq (\leq) x_0}} f(x) = L \right) := (\forall \varepsilon > 0 \exists \delta > 0 \mid \forall x \in (X \cap [x_0, x_0 + \delta) ((x_0 - \delta, x_0])), |f(x) - L| < \varepsilon$$

A one-sided continuous extension can also be defined by using an open bound $(x_0, x_0 + \delta)$ and is represented using notation such as $\lim_{\substack{x \rightarrow x_0 \\ x > x_0}}$.

- If **both the left and right limits are defined** (not an endpoint), they must be equal for the both-sided limit to exist. For example, the positive indicator function's χ_+ left $\lim_{\substack{x=0 \\ x < 0}} = 0$ and right $\lim_{\substack{x=0 \\ x > 0}} = 1$ limits are different at the same point $x = 0$, therefore the both-sided $\lim_{x \rightarrow 0}$ is not defined. Such a point with different left / right limits is called a **jump point**.
- A function who's left / right limit is equal to $f(x_0)$ is said to be left / right continuous.

6.1.31. Definition - Limits at ∞

It is not meaningful to consider a both-sided limit at $\pm\infty$ therefore the one sided limit is assumed when defining limits $\rightarrow \infty$ if $\forall \delta > 0, X \cap (\delta, \infty) \neq \emptyset$ (∞ is an accumulation point):

$$\lim_{x \rightarrow (-)\infty} f(x) : X \rightarrow \mathbb{R} = L \Leftrightarrow \forall \varepsilon > 0 \exists \delta > 0 \mid \forall x \in (\delta, \infty) ((-\infty, -\delta)) \cap X, |f(x) - L| < \varepsilon$$

- Instead of getting smaller as the error ε reduces, $\delta \rightarrow \infty$ to bring points the considered points closer to ∞ and vice versa for $-\infty$.
- A limit $\rightarrow \infty$ can be transformed into the right-sided limit $\rightarrow 0$ by inputting the reciprocal instead. For example, $\lim_{x \rightarrow \infty} e^{-x} = 0$, we arrive at the same result through $\lim_{x \rightarrow 0^+} e^{-(x)^{-1}}$ because $\lim_{x \rightarrow 0} \frac{1}{x} = \infty$. This is a useful result as many functions have known Taylor expansions at $x = 0$, making them easier to calculate than $x \rightarrow \infty$.

6.1.32. Definition - Big-O Notation

This relates the asymptotic behavior of two functions and is useful for characterizing the computational efficiency of an algorithm by relating it to a simpler, better understood function which behaves similarly as the argument tends to specific values (usually ∞). The function $f : X \rightarrow \mathbb{R}$ is said to be a **Big-O** of $g : X \rightarrow \mathbb{R}$ at a point x_0 if:

$$f(x) \in O(g(x)) \text{ as } x \rightarrow x_0 \Leftrightarrow \exists M > 0, \delta > 0 \mid \forall x \in X \cap (x_0 - \delta, x_0 + \delta) |f(x)| \leq M|g(x)|$$

In the case of $x_0 \rightarrow \infty$:

$$f(x) \in O(g(x)) \text{ as } x \rightarrow \infty \Leftrightarrow \exists M > 0, \delta > 0 \mid \forall x \in X \cap (\delta, \infty) |f(x)| \leq M|g(x)|$$

This means that f is bounded by some multiple of g in the neighborhood of x_0 .

- For example, $x \rightarrow 0, x^2 \in O(x)$ but $x \rightarrow \infty, x^2 \notin O(x)$ because x grows slower than x^2 (real gradient M compared to $2 \cdot \infty$) as $x \rightarrow \infty$ and therefore x cannot bound x^2 over a neighborhood no matter how large M or how small δ is.
- If $f(x)$ is a sum of terms such as a polynomial, only the dominating term plays a role as $x \rightarrow \infty$ and constant coefficients can be ignored, for example $4x^4 + 3x^3 + 2x^2 + x \in O(x^4)$; this can be verified by expanding $|f(x)|$ using the triangle inequality.

- In the context of computer science, $f(x)$ is a function of steps / storage needed to execute an algorithm given an input n . For example the steps needed to loop over an input list with length n once is αn , where α is the number of steps per loop iteration; such an algorithm would be classified as $O(n)$ as $n \rightarrow \infty$ (the worst-case scenario where α dwindles into irrelevance). An algorithm with a flatter Big-O equivalent is said to be more efficient in the long term (for example $O(\log(n))$ is quicker than $O(n^2)$).

TODO: Little-O notation, appears in both scripts

6.1.33. Definition - Sequence of Functions

A set of functions over the same domain X can be indexed by \mathbb{N} as a sequence $(f_n)_{n \in \mathbb{N}}$, the order plays a role like in any normal sequence.

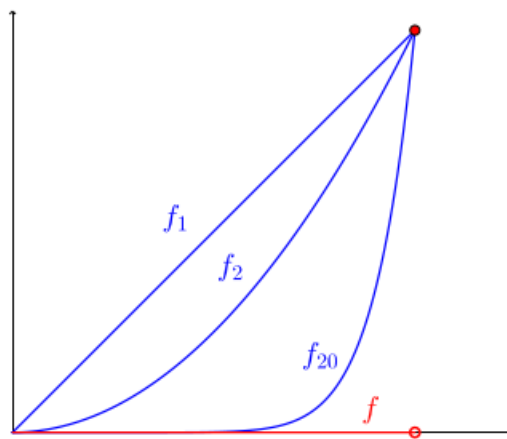
6.1.34. Definition - Point-Wise Convergence

A sequence of functions is said to **converge point-wise** to a function $l : X \rightarrow \mathbb{R}$ if:

$$\forall x \in X, \lim_{n \rightarrow \infty} f_n(x) = l(x)$$

- The so-called point-wise limit $l : X \rightarrow \mathbb{R}$ is unique. Proof: Simple by contradiction.

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} x^n = \begin{cases} 0 & \text{for } x < 1 \\ 1 & \text{for } x = 1. \end{cases}$$



6.1.35. Definition - Uniform Convergence

This is a stronger measure of convergence than point-wise, a sequence of functions f_n converges uniformly to $l(x)$ if:

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \mid \forall n > N \forall x \in X |f_n(x) - l(x)| < \varepsilon$$

This differs by applying the same error ε for all points in the domain at a given index N , meaning that all points need to converge towards l at the same rate.

- Uniform convergence \Rightarrow point-wise convergence. This is proved by acknowledging that $\{n \in \mathbb{N} \mid n > N\} = \mathbb{N} \cap (N, \infty)$, leading to the point-wise definition.

6.1.36. Theorem - Uniform Limit Continuity

The uniform limit $l : X \rightarrow \mathbb{R}$ of a uniformly converging sequence of **continuous functions** is also a continuous function, this can be interpreted as yet another characterization of continuity.

Proof:

The functions f_n are given to be continuous at all points, therefore the following holds true:

$$\forall n \in \mathbb{N} \forall x_0 \in X \forall \varepsilon > 0 \exists \delta > 0 \mid \forall x \in X, |x - x_0| < \delta \Rightarrow |f_n(x) - f_n(x_0)| < \varepsilon$$

This also applies to functions indexed past $n > N$ for any given uniform convergence error E , such that $|f_n(x) - l(x)| < E$. Now we must show that $|f_n(x) - f_n(x_0)| < \varepsilon \Rightarrow |l(x) - l(x_0)| < \varepsilon$.

Both x and x_0 are elements of X , hence $|f_n(x_{(0)}) - l(x_{(0)})| < E$ holds true $\forall E > 0$, which we can combine using the Triangle inequality to achieve our goal:

$$\begin{aligned} |l(x) - l(x_0)| &\leq |l(x) - f_n(x)| + |f_n(x_0) - l(x_0)| + |f_n(x) - f_n(x_0)| < 3E \\ &\Rightarrow |l(x) - l(x_0)| < \varepsilon \blacksquare \end{aligned}$$

- Unlike uniform convergence, point-wise convergence does not guarantee continuity of l .

6.1.37. Theorem - Bounded Uniform Limit

The uniform limit of a sequence of bounded functions is also bounded.

Proof:

As $\varepsilon \rightarrow 0, \forall n > N \forall x \in X, f_n(x) \approx l(x)$, therefore we can extrapolate $|f_n(x)| \leq B$ to $|l(x)| \leq B$, where l is the uniform limit.

- This does not apply to the point-wise limit.

7. Series (Reihen)

7.1.1. Definition - Convergent Series

A series $s_n : \mathbb{N} \rightarrow \mathbb{R} := \sum_{k=0}^n a_k$ is a sequence of partial sums of a sequence $(a_k)_{k \in \mathbb{N}} \in \mathbb{R}$. It may converge to a value $A \in \mathbb{R}$ if the following limit exists:

$$\begin{aligned} \lim_{n \rightarrow \infty} s_n = \sum_{k=0}^{\infty} a_k = A &\Leftrightarrow \\ \forall \varepsilon > 0 \exists N > 0 \mid \forall n \in \mathbb{N} \mid n > N, |s_n - A| < \varepsilon \end{aligned}$$

Otherwise, it may diverge to $\pm\infty$ or oscillate just like a sequence / function.

- A series converges \Rightarrow its **sequence of summands converges** to 0.

Proof:

We are given $\lim_{n \rightarrow \infty} s_n = A$. The current a_n is given by $s_n - s_{n-1}$ which when applying the sum of limits Lemma tends to $\lim_{n \rightarrow \infty} (s_n - s_{n-1}) = A - A = 0 \blacksquare$

- A sequence converges to 0 \nRightarrow Its series converges.

If we sum the elements of such a sequence until we pass N so that it converges to 0 for a given ε , we get:

$$|a_k| + |a_{k-1}| + \dots < n\varepsilon$$

However $\lim_{\varepsilon \rightarrow 0} n\varepsilon$ is undefined, therefore we cannot conclude that the series also converges.

- The limit of linear combinations of convergent series can be calculated from the component limits since they are valid functions upon which the limit linear combination lemmas can be applied.

7.1.2. Example - Geometric Series

This is defined as:

$$q \in \mathbb{R}, s_n := \sum_{k=0}^n q^k$$

- Its n 'th term can be calculated using the following formula (provable through induction):

$$s_n = \begin{cases} \frac{1-q^{n+1}}{1-q} & \text{if } q \neq 1 \\ n & \text{otherwise} \end{cases}$$

Such n'th term formulae are very useful as they easily reveal convergence and can be found by writing out the first few + last terms of a series, where certain patterns could be found.

- Taking the limit of its n'th term shows that it converges when $|q| < 1$ and the sequence of summands does indeed converge to 0, in which case:

$$\lim_{n \rightarrow \infty} s_n = \frac{1}{1-q}$$

7.1.3. Example - Harmonic Series

Its name stems from the harmonics of a frequency, and it is defined as:

$$s_n := \sum_{k=1}^n \frac{1}{k}$$

- It does **not converge**, despite the summands converging to 0. $\sum \frac{1}{k^2}$ on the other hand **does** converge.

I will mostly discuss various convergence criteria now.

7.1.4. Theorem - Convergence of the Tail

A series is convergent \Leftrightarrow Its tail converges:

$$\forall N \in \mathbb{N} \\ \sum_{k=0}^{\infty} s_n = \sum_{k=0}^{N-1} s_n + \sum_{k=N}^{\infty} s_n$$

This is simply a sum of limits, whereby a finite sum is a constant hence the existence of the limit only depends on the tail converging ■

- Divergent tail \Leftrightarrow divergent sequence

7.1.5. Theorem - Monotone Convergence of Series

Just like a sequence, a bounded, monotone series converges to its supremum / infimum (see monotone convergence theorem proof). Such series are monotone increasing (decreasing) if the sign of its summands is consistent and they remain positive (negative).

- If it is not bounded but monotone, it diverges to $\pm\infty$

7.1.6. Theorem - Majorant-Minorant Criterion

This is similar to the Sandwich Lemma for sequences. Let $A_n = \sum a_k$ (minorant) and $B_n = \sum b_k$ (majorant) be two series such that $\forall k > N, 0 \leq a_k \leq b_k$ for some $N \in \mathbb{N}$. The following implications hold:

$$\begin{aligned} B_n \text{ convergent} &\Rightarrow A_n \text{ convergent} \\ A_n \text{ divergent to } \infty &\Rightarrow B_n \text{ divergent to } \infty \end{aligned}$$

Proof:

$a_n \geq 0$ ensures that the minorant A_n is monotone increasing. Applying the Monotone Convergence theorem shows that it converges or diverges depending on if it is bounded by $\sup(B_n)$ ■

7.1.7. Theorem - Cauchy Condensation Test

A series $\sum a_k$ whose sequence of summands is non-negative but monotonically decreasing has the following property:

$$\sum 2^k a_{2^k} \text{ converges} \Leftrightarrow \sum a_k \text{ converges}$$

This may seem like a useless fact but there are in fact many series which simplify when multiplied by 2^k , for example $\sum \frac{1}{k^i} \Rightarrow \sum \frac{2^k}{(2^k)^i} = \sum (2^{1-i})^k$, transforming into a geometric series which proves that it converges $\forall i > 1$, a useful fact for the **Riemann Zeta function** $\zeta(i) := \sum_{k=1}^{\infty} \frac{1}{k^i}$.

7.1.8. Definition - Absolute Convergence

A series $\sum a_k$ is absolute convergent if $\sum |a_k|$ converges.

- Convergent monotone series (all summands either positive or negative and bounded accordingly) are absolute convergent.
- A series which converges, but does **not** absolutely converge is called **conditionally convergent**, both positive and negative summands are necessary for it to converge.

7.1.9. Theorem - Riemann's Rearrangement Theorem

This incredible result relies on the counter-intuitive fact that an infinite sum depends on the order of summands (unlike a finite sum which is commutative) and states that the summands of a conditionally convergent series s_n can be rearranged by a bijective mapping φ (every term added once) to make it converge to any real limit:

$$\forall A \in \mathbb{R} \exists \varphi : \mathbb{N} \rightarrow \mathbb{N} \mid \sum_{k=0}^{\infty} a_{\varphi(k)} = A$$

Proof:

We will consider the indexes of the positive and negative summands separately:

$$P = \{n \in \mathbb{N} \mid a_n \geq 0\}, N = \{n \in \mathbb{N} \mid a_n < 0\}$$

The following "subseries" must both diverge to ensure that the sum of their limits $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} p_n + \lim_{n \rightarrow \infty} n_n = \infty - \infty < \infty$:

$$\begin{aligned} \lim_{n \rightarrow \infty} p_n &:= \sum_{k=0}^{\infty} a_{P_k} = \infty \\ \lim_{n \rightarrow \infty} n_n &:= \sum_{k=0}^{\infty} a_{N_k} = -\infty \end{aligned}$$

Furthermore, they must both be divergent - if they weren't $\lim_{n \rightarrow \infty} |s_n|$ would converge which contradicts the fact that it is conditionally convergent.

The mapping φ can be recursively constructed to choose a positive / negative summand as the next index depending on if the current partial sum is smaller / larger than A , adding the summands with the largest absolute values first so that $a_n \rightarrow 0$ as $n \rightarrow \infty$ (although a null sequence of summands doesn't imply convergence, we have already guaranteed it due to the $\infty - \infty < \infty$ considerations)

■

- This might be my favorite theorem so far

TODO: Leibnitz, Cauchy convergence criterion