Analysis 1

Contents

Literature:
- Alessio Figalli's 2023 Lecture Notes
- Analysis 1 by H. Amann & J. Escher

What I would like to take notes of:
1. State the theorem / definition, expand with some intuition / memory aids
2. Write the proof by myself if deemed useful for an exam
3. Name some examples only if very helpful

These notes should serve as condensed revision material - only the minimal, important facts to remember before solving problems

Note on order: Many scripts used for teaching introduce concepts as they become relevant. My goal is to build a revision reference, not a learning resource, therefore over time the order will be rearranged to group relevant definitions and theorems together.

Proofs heavily involve decomposition; to progress, smaller Lemmas need to be brought in along the way and proven (or taken as true since someone else proved them). However first of all, you need to understand and remember the axioms (rules of the game). Intuition is helpful but doesn't prove anything unless it can be formulated as a series of statements a computer can verify

Mathematics - Abstracting enough to focus on the matter

Contradiction is a useful tool for linking statements about $>$ and $\geq$. When in doubt, assume by contradiction

Conjecture - A conclusion formed on the basis of incomplete information Prove uniqueness through trichotomy, existence by completeness axiom.

TODO: Read Einsiedler einführung

1. Fundamentals

1.1.1. Definition - Set

An **unordered** collection of **distinct** ($\{x, x\} \equiv \{x\}$) elements such that:

1. It is defined by the elements it contains
2. It is not an element of itself, this prevents Russell's Paradox: $\{x \mid x \notin x\}$
3. Its elements can be filtered by a series of statements which hold true, for example the set of even integers:

$$\{n \in \mathbb{Z} \mid \exists m \in \mathbb{Z} : n = 2m\}$$

   Where | and : both mean "such that".

4. The empty set $\emptyset$ contains no elements

1.1.2. Definition - Cartesian Product

For the sets X, Y, the Cartesian product is the set of tuples (ordered lists): $X \times Y := \{(x, y) \mid x \in X, y \in Y\}$

- The number of elements in the set is:

$$|X \times Y| = |X| \cdot |Y|$$

Example:

$$X := \{0, 1\}, Y := \{\alpha, \beta\}$$
$$X \times Y := \{(0, \alpha), (0, \beta), (1, \alpha), (1, \beta)\}$$

1.1.3. Definition - Subset

A set whose elements are entirely contained in a parent set with the following notation:

- $P \subseteq Q$ - P is a subset of Q and they may be equal
- $P \subsetneq Q$ - P is a **proper** subset of Q; Q has at least 1 additional element
- $P \not\subset Q$ - There is at least one element in P that is not in Q
- The same applies in reverse using sup(er)set notation $\supseteq$
- The symbols $\subset$ and $\supset$ are ambiguous in meaning
- Two sets can be shown to be equal if $P \subseteq Q \land Q \subseteq P$ holds true

1.1.4. Definition - Power Set

The power set of a set X is the set of all subsets:

$$\mathcal{P}(X) := \{\text{Set } Q \mid Q \subseteq X\}$$

Example:

$$X = \{0, 1, 2\}$$
$$\mathcal{P}(X) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$$

1.1.5. Definition - Interval Notation

Interval notation allows us to succinctly express common sets of real numbers between limits $a, b \in \mathbb{R}$:

- Closed interval

$$[a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

- Open interval

$$(a, b) := \{x \in \mathbb{R} \mid a < x < b\}$$

- Half-open interval

$$[a, b) := \{x \in \mathbb{R} \mid a \leq x < b\}$$

- Unbounded interval

$$[a, \infty) := \{x \in \mathbb{R} \mid a \leq x\}$$

- For a non-empty interval, the **length** is defined as $b - a$
- Sometimes inverted square brackets are used to specify an open bound, ex. $[a, b[$
- The intersection of a finite number of intervals is also an interval, such that the lower bound is the smallest lower bound and vice versa for the upper bound
- Sets aren't doors, they don't need to be either open or closed.

1.1.6. Definition - Set Operations

This allows us to construct common sets from component sets P, Q:

- Intersection:

$$P \cap Q := \{x \in P \mid x \in Q\}$$

- Union:

$$P \cup Q := \{x \in P \vee x \in Q\}$$

- Relative Complement:

$$P \setminus Q := \{x \in P \mid x \notin Q\}$$

- Complement of a Subset:

$$R \subseteq X$$
$$R^{\complement} := \{x \in X \mid x \notin R\}$$

- Symmetric Difference:

$$P \triangleright Q := (P \cup Q) \setminus (P \cap Q)$$

- Addition:

$$P + Q := \{p + q \mid p \in P \wedge q \in Q\}$$

- Multiplication:

$$P \cdot Q := \{p \cdot q \mid p \in P \wedge q \in Q\}$$

- They are distributive:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- Let $\mathbb{A}$ be a set of sets, we can also define:

$$\bigcap_{A \in \mathbb{A}} A := \{x \mid \exists A \in \mathbb{A} \mid x \in A\}$$
$$\bigcup_{A \in \mathbb{A}} A := \{x \mid \forall A \in \mathbb{A} \mid x \in A\}$$

1.1.7. Theorem - De Morgan's Laws

This states:

$$(A \cap B \cap ...)^{\mathrm{C}} = A^{\mathrm{C}} \cup B^{\mathrm{C}} \cup ...^{\mathrm{C}}$$
$$(A \cup B \cup ...)^{\mathrm{C}} = A^{\mathrm{C}} \cap B^{\mathrm{C}} \cap ...^{\mathrm{C}}$$

It is commonly applied to Boolean logic, where $A, B \subsetneq \{0, 1\}$:

$$\overline{A \land B \land ...} \equiv \overline{A} \lor \overline{B} \lor ...$$

1.1.8. Definition - Maxima and Minima

The maximum of a set is the smallest upper bound, which is **contained** in the set:

$$X \subseteq$$
$$\max(X) \coloneqq m \in X \mid \forall x \in Xx \leq m$$

The **minimum** is defined analogously:

$$\min(X) \coloneqq m \in X \mid \forall x \in Xx \geq m$$

- An open bound has no maximum or minimum defined as there is always some number slightly larger / smaller than a number we can express inside it. An open bound itself is not in the set
- The maximum / minimum is unique. Proof: Let m_1, m_2 be 2 maxima of the set. It follows $m_1 \leq m_2$ and $m_2 \leq m_1$, therefore $m_1 = m_2$ (trichotomy)

1.1.9. Definition - Supremum and Infimum

Let $B = \{b \in \mathbb{R} \mid \forall x \in Xx \leq (\geq)b\}$ be the set of upper (lower) bounds for the set X. The supremum (infimum) is defined as the smallest (largest) such bound:

$$\sup(X) \coloneqq \min(B)$$
$$\inf(X) \coloneqq \max((B))$$

- Due to the $\leq$ $(\geq)$ the supremum infimum may be the same as the maximum / minimum for a closed bound
- An alternative characterization states there is no smaller bound, anything smaller is not a bound of X:

$$\forall x \in Xx \leq \sup(X), t \leq \sup(X) \Rightarrow \exists x' \in X : t < x'$$

- The supremum / infimum does **not** exist for an unbounded or empty set, as this would be infinitely large / small, and $\infty \notin \mathbb{R} \therefore \infty \notin B$
- For all bounded, non-empty sets X, the supremum / infimum exists.
  Proof:
  The set of bounds $B = \{b \in \mathbb{R} \mid \forall x \in Xx \leq (\geq)b\} \neq \emptyset$
  We need to show that $\exists \sup(X) \in \mathbb{R} \mid \forall b \in B, \sup(X) \leq b$
  Lemma: Completeness Axiom $\forall x \in X \forall b \in B, x \leq b \Rightarrow \exists c \in \mathbb{R} \mid x \leq c \leq b \forall x \in X \forall b \in B$ This c is an upper bound **and** minimum of B, therefore it is the supremum ■

Let X, Y be non-empty sets with an upper bound:
- $\sup(X \cup Y) = \max(\sup(X), \sup(Y))$
- $\sup(X \cap Y) = \min(\sup(X), \sup(Y)) \mid (X \cap Y) \neq \emptyset$
- $\sup(X + Y) = \sup(X) + \sup(Y)$
- $\sup(X \cdot Y) = \sup(X) \cdot \sup(Y) \mid \forall x \in X \forall y \in Yx, y \geq 0$ (two "large" negative elements can make a larger supremum)

TODO: Review proof 2.59

1.1.10. Definition - Identity Function

This function simply outputs its input and is needed to define the inverse of a function:

$$\text{id} : X \to X$$
$$\text{id}(x) := x$$

1.1.11. Definition - Characteristic Function

$X \subseteq Y$, the characteristic / indicator function $\chi_X : Y \to \{0, 1\}$ indicates whether an element is part of a set:

$$\chi_X(x) := \begin{cases} 1 \text{ if } x \in X \\ 0 \text{ otherwise} \end{cases}$$

1.1.12. Definition - Restriction Function

A new function can be defined with a smaller domain, which is useful for drawing conclusions from its properties over that domain:

$$f : X \to Y, A \subseteq X$$
$$f \mid_A : A \to Y$$

2. Topology

2.1.1. Definition - Ball / Disk

A topological ball with radius r and center $x_0 \in \mathbb{R}^d$ in dimension $\mathbb{R}^d$ is defined as the set of points:

$$B_r^d(x_0) = \{x \in \mathbb{R}^d \mid |x - x_0| < r\} - \text{Open ball}$$
$$\overline{B_r^d}(x_0) = \{x \in \mathbb{R}^d \mid |x - x_0| \leq r\} - \text{Closed ball}$$
$$S_r^{d-1}(x_0) = \{x \in \mathbb{R}^d \mid |x - x_0| = r\} - \text{Sphere (boundary of ball)}$$

Where $|x - x_0|$ is the length of the vector from $x_0 \to x$ ie the radius. This can also be defined using complex numbers and the complex absolute function. It follows:

$$B_0(x_0) = \emptyset$$
$$\overline{B_0}(x_0) = \{x_0\}$$
$$B_\infty^d(x_0) = \overline{B_\infty^d}(x_0) = \mathbb{R}^d$$

- Man muss immer am B_r^d bleiben!

The sphere has dimensions $d - 1$ because its points only form a subspace in the dimension below the ball which it is enclosing:

- S_r^1 - Is the line of points around a circle ie 1 dimensional
- S_r^2 - Every point in the surface of a 3D ball can be reached with linear combinations of two basis vectors (such that they stay within the subspace).

2.1.2. Definition - Inner Point

A point $x \in S^n$ is inner $\Leftrightarrow \exists r \in (0, \infty) \mid B_r^n(x) \subseteq S$ - there is an open ball with a radius > 0 around x such that it is entirely a subset of / equal to S.

2.1.3. Definition - Interior

- The interior of a set is the set of all its inner points:

$$\text{Int } S := \{\text{inner points of S}\}$$

- $\text{Int } S \subset S$ is always true.
- Alternatively, the interior can be defined as the union of open balls:

$$\text{Int } S := \bigcup_{B_r^n(x) \mid B \subseteq S} B$$

2.1.4. Definition - Open Set

A set which is equal to its interior: $S = \text{Int } S$. In other words, it is defined with $>$ or $<$ relations.
- It has no maximum / minimum, only an infimum / supremum.
- Every point of an open ball is an inner point, hence making the ball "open".

$$\text{Int } \overline{B_r^d}(x_0) = B_r^d(x_0)$$
$$x \in \mathbb{R}, \{x\} \text{ is not open}$$

- The union of arbitrarily many open sets is open (the outer boundaries will remain open no matter what)
- The intersection of finitely many open sets is also open

2.1.5. Definition - Closed Set

The definition is built upon that of an open set: Let $A \subseteq \mathbb{R}^n$:

$$A \text{ is closed} \Leftrightarrow (\mathbb{R}^n \setminus A) \text{ is open}$$

For example, $[a, b] \mid a < b$ can instead be expressed as $(-\infty, a) \cup (b, \infty)$, which is open.
- $\{x\}$ is closed
- $\emptyset, \mathbb{R}^n$ are both open and closed, since $\emptyset^\complement = \mathbb{R}^n$ and $\mathbb{R}^\complement = \emptyset$
- $[a, b) \subsetneq \mathbb{R}$ is neither open or closed

2.1.6. Definition - Closure

The closure of $\overline{S}$ is the smallest possible closed set which entirely includes the set S, this can be formed using the intersection of all possible closed balls with different radii and centers, as long as they entirely contain S:

$$\overline{S} := \text{clos}(S) := \bigcup_{\text{All } \overline{B}_r^n(c) \mid S \subseteq \overline{B}} \overline{B}$$

For example:

$$S := (0, 1]$$
$$\overline{S} = [0, 1]$$

- $S \subseteq \overline{S}$ - The closure of a set contains the set itself
- A topological set can only be called **closed** if it is equal to its closure

2.1.7. Definition - Boundary

The boundary of a set ∂S is:

$$\partial S := \overline{S} \setminus \text{Int } S$$

Characterized more fundamentally:

$$\partial S := \{x \in \mathbb{R}^n \mid (\forall r \in (0, \infty) \mid (B_r^n(x) \cap S) \neq \emptyset \neq B_r^n(x) \setminus S)\}$$

The boundary of a set S is the set of points such that:

- A ball with increasing radius (starting just above 0) always continues to overlap with some elements of S ($(B_r^n(x) \cap S) \neq \emptyset$), ie the ball must be actually in or right next to S
- The points themselves are part of $B_r^n(x) \setminus S$, which is never equal to the empty set, ie the point itself is never in S

By definition, a topological sphere is the boundary of a ball:

$$\partial B_r^n(x) = \partial \overline{B_r^n}(x) = S_r^{n-1}(x)$$

Furthermore, a boundary is a closed set:

$$\left(\mathbb{R}^n \setminus \partial S = \left(\text{Int } S \cup \mathbb{R}^n \setminus \overline{S}\right)\right) \text{ which is open} \Rightarrow \partial S \text{ is closed}$$

2.1.8. Definition - Bounded

A set which is a subset of a closed set (other than $\mathbb{R}^n$). In other words, the set of bounds $B = \{b \in \mathbb{R}^n \mid \forall x \in X x \leq (\geq) b\} \neq \emptyset$.

2.1.9. Definition - Compact

A set which is closed **and** bounded
- A closed ball is by definition compact.
- $\mathbb{R}^n$ is not compact, because it is an infinitely large (albeit open & closed) set.

2.1.10. Definition - Neighborhood (Umgebung)

A subset $U \subseteq X$ is considered a **neighborhood** of a point x_0 relative to a set X if:

$$x_0 \in O \subseteq U \subseteq X$$

Where O is a non-empty open set.
- For example, there are many possible neighborhoods around a point in the middle of a non-empty set.
- Points on the boundary of X have no neighborhood U as no non-empty open set contains only points which remain in X

3. Axioms of The Real Numbers

An axiomatic approach to defining the set of real numbers $\mathbb{R}$.

3.1.1. Definition - Group

A **non-empty** set G endowed with an operation $\star$ which satisfies the following criteria $\forall a, b, c \in G$:
1. *Associativity* - $a \star (b \star c) = (a \star b) \star c$
2. $\exists$ *Neutral Element* n - $a \star n = n \star a = a$ - Examples:
   - $a + 0 = 0 + a = a$
   - $a \cdot 1 = 1 \cdot a = a$
3. $\forall a \exists$ *Inverse Element* i - $a \star i = i \star a = n$ - Examples:
   - $a + (-a) = (-a) + a = 0$
   - $a \neq 0 \Rightarrow a \cdot a^{-1} = a^{-1} \cdot a = 1$
4. If $a \star b = b \star a$ it is a **commutative group**, although this is not required.

Properties:
- The *Neutral Element* is unique. Proof:

$$\text{Let } n, n' \in G \text{ be neutral elements}$$
$$n \star n' = n = n'$$

- There is unique *Inverse Element* for all elements. Proof:

$$\begin{aligned} &\text{Let } i, i' \in G \text{ be inverse elements for } a \\ &\qquad i \star (a \star i') = (i \star a) \star i' \\ &\qquad\qquad i \star n = n \star i' \\ &\qquad\qquad\quad i = i' \end{aligned}$$

Examples:
- The non-zero rational numbers $\mathbb{Q} := \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z} : p, q \neq 0 \right\}$ with operation $\cdot$ is a group, where $n = \frac{1}{1}$ and $i\left(\frac{p}{q}\right) = \frac{q}{p}$
- The natural numbers $\mathbb{N}$ with operation $+$ is **not** a group, as there are no negative inverse elements

3.1.2. Definition - Ring

A non-empty set R with operations $+$ and $\cdot$.
1. Addition is **always commutative** with $n = 0, i = -a$
2. Multiplication is not necessarily commutative, for example a matrix ring
3. If multiplication is commutative, it is a **commutative ring** and has neutral element $n = 1$
4. It is **not necessarily** a group for multiplication as 0 may be included and has no inverse element $0 \cdot i \neq 1$

3.1.3. Definition - Field (Körper)

A commutative ring K (Körper) where $\forall a \in K \mid k \neq 0$ the inverse element for multiplication exists.
1. Addition: $n = 0, i = -a, -(-a) = a$
2. Multiplication: $n = 1, i = a^{-1}, \left(a^{-1}\right)^{-1} = a \mid a \neq 0$

Examples:
- $\mathbb{Z}$ is a ring but not a field as there is no multiplicative inverse element for all non-zero elements
- The complete set of rational numbers $\mathbb{Q} := \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z} : q \neq 0 \right\}$ is a commutative ring and a field.
- $0 \cdot a = 0$. Proof:

$$0 = 0 \cdot a - 0 \cdot a = (0 - 0) \cdot a = 0 \cdot a$$

3.1.4. Definition - Relationship

A relationship on X is the subset $\mathfrak{R} := \{(a, b) \in X \times X \mid a \sim b\}$ where ~ is an operator for expressing conditions called a relation and may have the following properties if the corresponding condition holds true $\forall x, y, z \in X$:
- Reflexive - $x \sim x$ - Example: $\leq$
- Transitive - $x \sim y \land y \sim z \Rightarrow x \sim z$ - Example: $<$
- Symmetric - $x \sim y \Rightarrow y \sim x$ - Example: $=, \neq$
- Anti-Symmetric - $x \sim y \land y \sim x \Rightarrow x = y$ - Example: $\leq$ - Although such relations are often reflexive too, this is not a requirement, consider $<$, which is anti-symmetric (no such x, y exist) but not reflexive.
- A relation is called **equivalence relation** if it is reflexive, transitive and symmetric. For example $=$ is an equivalence relation, $\leq$ is not (not symmetric).
- A relation is called **order relation** if it is reflexive, transitive and anti-symmetric. For example $\leq$

3.1.5. Definition - Ordered Field

This extends the definition of a field K with the relation $\leq$, which is denoted as $(K, \leq)$, under which all elements $x, y, z \in K$ satisfy the following:
1. Linearity of the order:

$$x \leq y \vee y \leq x$$

2. Compatibility of order and addition:

$$x \leq y \Rightarrow x + z \leq y + z$$

3. Compatibility of order and multiplication:

$$0 \leq x \wedge 0 \leq y \Rightarrow 0 \leq x \cdot y$$

   This can be combined with the Inverse Element of addition (which exists in all fields) to make statements about multiplication of negative numbers.

Axioms 2 and 3 **also** apply to the relation $<$, which significantly simplifies proofs. Proof: https://math.stackexchange.com/a/3271338
These conditions allow us to define conventions such as:
- Positive $:= x > 0$
- Non-negative $:= x \geq 0$
- $(x \leq y = z) \equiv (x \leq y \wedge y = z)$
- An example of an ordered field is the set of rational numbers $\mathbb{Q}$.
- An example of a non-ordered field is the set of complex numbers $\mathbb{C}$

The conditions of an ordered field lead to many properties we take as given. Here are some interesting proofs:
- $(x < y \wedge y \leq z) \Rightarrow x < z$ - Proof:

$$x < y \Rightarrow x \leq y. \leq \text{ is a transitive relation, hence } x \leq z$$
$$\text{We must now show that } x < z.$$
$$\text{Assume by contradiction that } \neg(x < z) \equiv x \geq z \text{ holds true}$$
$$\text{Due to } x \leq z \wedge x \geq z, x = z$$
$$\text{Recalling } x < y \text{ this implies } z < y \text{ which contradicts } y \leq z$$
$$\therefore x < z \blacksquare$$

- If $x \neq 0$, $x^2 > 0$ holds true. Proof:
  As $x \neq 0$ there are 2 cases:
  - $x > 0$
  - $x < 0$

  The ring is only guaranteed to be valid for the relation $\leq$, so we will prove $x^2 \geq 0$ first.
  If $x > 0$, $x \geq 0$ also holds true and also $x^2 \geq 0$ per condition 3.
  If $x < 0$, $x \leq 0$ also holds true. Applying condition 2, $(x - x \leq 0 - x) \equiv (0 \leq -x)$. Applying condition 3, $-x \cdot -x = x^2 \geq 0$.
  Lastly, we must show that $x^2 \geq 0 \Rightarrow x^2 > 0$. Assume by contradiction that $\exists x \neq 0 : x^2 \leq 0 \Rightarrow x^2 < 0$. This contradicts $x^2 \geq 0$, which we have proven for all $x \neq 0$ in the field. Hence $x^2 > 0$ must also be true $\blacksquare$
- $0 < 1$. Proof:
  Lemma: $0 \neq 1$ (Neutral Elements of addition and multiplication are not the same)
  Lemma: If $x \neq 0$, $x^2 > 0$ holds true. Therefore $1^2 = 1 > 0$ $\blacksquare$

Based on the fact that $0 < 1$ and the compatibility + inverse element of addition, it is clear that the integers $\mathbb{Z} := \ldots, < -1 < 0 < 1 < \ldots$ are a subset of any ordered field. Furthermore, the rational numbers are defined from the set of integers, which are also a subset of all ordered fields K:

$$\mathbb{Z} \subsetneq \mathbb{Q} \subseteq K$$

3.1.6. Definition - Absolute Function

A function $|x| : K \to K_+$ defined on every **ordered field** such that:

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

- $(|x| \leq y) \equiv (-y \leq x \leq y)$
- $|xy| \equiv |x||y|$

3.1.7. Definition - Sign Function

A function $|x| : K \to \{-1, 0, 1\}$ defined on every **ordered field** such that:

$$\operatorname{sgn}(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

- Every $x \in K$ can be expressed as $x = \operatorname{sgn}(x) \cdot |x|$

3.1.8. Theorem - Triangle Inequality

It holds $\forall x, y \in K$ elements of an ordered field that:

$$|x + y| \leq |x| + |y|$$

The name stems from considering a triangle spanned by two vectors. It is clear that the length of their vector sum is $\leq$ the sum of both side lengths. Proof (on an ordered field):
Lemma: $|x| \Rightarrow -|x| \leq x \leq |x|$
Therefore we can state the following:

$$-|x| \leq x \leq |x|$$
$$-|y| \leq y \leq |y|$$

Lemma: $x \leq y \Rightarrow x + z \leq y + z$:

$$-|x| + -|y| \leq x + -|y|$$
$$-|y| + x \leq y + x$$

Lemma: $x \leq y \wedge y \leq z \Rightarrow x \leq z$

$$\therefore -(|x| + |y|) \leq x + y$$

Applying the same procedure to $x \leq |x|$ and $y \leq |y|$ we get:

$$-(|x| + |y|) \leq x + y \leq |x| + |y|$$

Lemma: $(|x| \leq y) \equiv (-y \leq x \leq y)$

$$\therefore |x + y| \leq |x| + |y| \blacksquare$$

An alternative, the **inverse triangle inequality** can also be useful:

$$|x - y| \geq ||x| - |y||$$

3.1.9. Definition - Completeness Axiom

The definition of an **ordered field** so far is unsuitable as we need to “fill in the gaps”. The completeness axiom is an alternative but equivalent approach to Dedekind cuts (which define the cuts first and then operations in terms of cuts) which defines a **complete ordered field** if the completeness axiom holds true:

1. Let $X, Y \subsetneq K \mid X, Y \neq \emptyset : \forall x \in X \forall y \in Y$ the inequality $x \leq y$ holds true. If there exists $c \in K \mid x \leq c \leq y$ for all such subsets X and Y, the ordered field is complete.

- The field of real numbers $\mathbb{R}$ is a completely ordered field
- The reason subsets are checked instead of individual elements x, y is because subsets can be defined in terms of inequalities. For example, consider checking the existence of $\sqrt{2}$ in $\mathbb{Q}$. The set of rational numbers is **dense**, therefore no matter which lower bound x we choose, there is **always** a rational number closer to $\sqrt{2}$ and therefore the check $x \leq c \leq y$ holds true (although $\sqrt{2}$ is not a member of $\mathbb{Q}$). On the other hand if we choose the subset $X = \left\{x \in \mathbb{Q} \mid x \leq \sqrt{2}\right\}$, this contains the true infimum of $\sqrt{2}$ and checks **completeness** rather than **density**. Of course, both approaches would involve checking infinitely many elements but luckily we can arrive at such an inequality from the axioms of an ordered set.

3.1.10. Definition - Compactification

The reals can be extended to be compact (closed an bounded) with $-\infty, \infty$ for certain purposes, such as defining the supremum / infimum of an unbounded / empty set:

$$\overline{R} = \mathbb{R} \cup \{-\infty, \infty\}$$
$$\forall x \in \mathbb{R}, -\infty < x < \infty$$

Certain conventions are defined, however these are ambiguous and should be used sparingly:

$$\infty + x = \infty$$
$$-\infty + x = -\infty$$
$$x \cdot \infty = \infty \mid x > 0$$
$$\sup(\emptyset) = -\infty$$
$$\inf(\emptyset) = \infty$$

3.1.11. Definition - Archimedean Principle

For every $x \in \mathbb{R}$ there exists **exactly one** $n \in \mathbb{Z} \mid n \leq x < n + 1$. In simpler words, $\forall x \in \mathbb{R} \exists z \in \mathbb{Z} \mid z > x$

3.1.12. Definition - Integer / Fractional Part

The integer part of any $r \in \mathbb{R}$ is given by the floor function $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ which returns the lower n which exists due to the Archimedean principle.
The fractional part is given by $r - \lfloor r \rfloor \in [0, 1)$

3.1.13. Corollary - $\frac{1}{n}$ is arbitrarily small

$\forall \varepsilon \in \mathbb{R} \mid \varepsilon > 0 \exists n \in \mathbb{Z} \mid n \geq 1 \land \frac{1}{n} < \varepsilon.$
Proof:
If $\varepsilon > 1$ this holds true with $n = 1$.
For $\varepsilon \leq 1$, $\frac{1}{\varepsilon} \geq 1$. The Archimedian principle states that there always exists a $n \geq 1 \mid n > \frac{1}{\varepsilon}$, which becomes $\frac{1}{n} < \varepsilon ■$

3.1.14. Definition - Cardinality

The cardinality of two sets describes their relative "sizes".

- $X \sim Y$ - We say two sets X and Y have the same cardinality (the same number of elements) if there exists a **bijective** mapping $f : X \to Y$. Surjectivity guarantees that $|Y| \geq |X|$ and injectivity guarantees $|X| \geq |Y|$, which leads to$|X| = |Y|$ (trichotomy).
- $X \lesssim Y$ - X is **smaller than or equal to** Y if there exists an injective mapping $f : X \to Y$
- $X \lesssim Y \land Y \lesssim X \Rightarrow X \sim Y$ - One can find a bijective mapping (Schröder-Bernstein Theorem)

- $|\emptyset| = 0$
- $\exists f : X \to \{1, 2, ..., n\}$ is bijective $\Rightarrow |X| = n, X$ is finite
- $|\mathbb{N}| = \aleph_0$ - A set which has the same cardinality as $\mathbb{N}$ is called **countable**

3.1.15. Theorem - Cantor's Theorem

The power set $\mathcal{P}(X)$ of any (infinite too) non-empty set X is larger than and not equal to X.
This reveals that $\mathcal{P}(\mathbb{N}) > \mathbb{N} \land \mathcal{P}(\mathbb{N}) \neq \mathbb{N}$ which is useful for showing that other sets are larger than or equal to $\mathcal{P}(\mathbb{N})$ ($\exists$ injection) and therefore also uncountable.
Proof:
Although this may seem obvious, when dealing with infinity it is easier to write a formal proof than find logical reasoning behind the intuition.
First we must show that there is an injective mapping $i : X \to \mathcal{P}(X)$, which indeed exists: $x \in X \to \{x\}$.
Now we show that there is **no** surjective mapping. Assume by contradiction that such a mapping $s : X \to \mathcal{P}(X)$ exists.
We will demonstrate its absurdity by defining the subset:

$$B = \{x \in X \mid x \notin f(x)\} \subseteq X$$

For every $x \in X$ there are two cases:
1. $x \in s(x)$, therefore $x \notin B$ and $s(x) \neq B$ because x would need to be a member of B for them to be equal
2. $x \notin s(x)$, therefore $x \in B$ and $s(x) \neq B$ because x would need to be a member of $s(x)$ for them to be equal

We have shown that $\nexists x \in X \mid s(x) = B$ and because $B \in \mathcal{P}(X)$, there exists no surjective mapping $s : X \to \mathcal{P}(X) \blacksquare$

3.1.16. Theorem - $\mathbb{R}$ is Uncountable

To prove this, we can find an injection $i : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$, which is given from the decimal expansion of reals TODO: Understand Cantor diagonalization

4. Sequences of Real Numbers

4.1.1. Definition - Sequence

A sequence is a function $a : \mathbb{N} \to \mathbb{R}$ which is often written as $(a_n)_{n \in \mathbb{N}}$
- A sequence is called **constant** if $\forall n, m \in \mathbb{N}, a_n = a_m$ and **eventually constant** if $\exists N \in \mathbb{N} \mid \forall n, m \geq N, a_n = a_m$

4.1.2. Definition - Convergence

A sequence is said to **converge towards** A if:

$$\exists A \in \mathbb{R} \forall \varepsilon \in (0, \infty) \exists N \in \mathbb{N} \mid \forall n \in \mathbb{N} : n \geq N, |a_n - A| < \varepsilon$$
$$\lim_{n \to \infty} a_n = A$$

- **Divergence** can be proved by proving the conjugate:

$$\forall A \in \mathbb{R} \exists \varepsilon \in (0, \infty) \forall N \in \mathbb{N}_0 \exists n \in N : n \geq N, |N - A| > \varepsilon$$

- A convergent sequence has **only one** limit. Proof:
  Let $A_1, A_2 \in \mathbb{R}$ be two limits of the sequence a_n.
  Due to the convergence criteria:

$$\exists N_1, N_2 \in \mathbb{N} \mid \forall n \geq \max(N_1, N_2), |a_n - A_1| < \varepsilon \land |a_n - A_2| < \varepsilon, \forall \varepsilon \in (0, \infty)$$
$$0 < \varepsilon - |a_n - A_1|, 0 < \varepsilon - |a_n - A_2|$$
$$\therefore |a_n - A_1| + |a_n - A_2| < 2\varepsilon$$

Applying the Lemma $|a + b| \leq |a| + |b|$:

$$a_n - A_1 - (a_n - A_2) = A_2 - A_1$$
$$0 \leq |A_2 + (-A_1)| \leq |a_n - A_1| + |-(a_n - A_2)| < 2\varepsilon$$

Since $|-x| = |x| \geq 0$, and this is true $\forall \varepsilon > 0$:

$$|A_2 - A_1| = 0 = A_2 - A_1$$
$$A_2 = A_1 \blacksquare$$

- The sequence $a_n = \frac{1}{n}$ converges to 0, because $\forall \varepsilon > 0, \exists n \in \mathbb{Z} \mid \frac{1}{n} < \varepsilon$, satisfying the criteria of convergence $\left|\frac{1}{n} - 0\right| < \varepsilon$

4.1.3. Definition - Subsequence

A subsequence of a_n is any sequence obtained by keeping certain elements a_{n_i} indexed by $i_{k_{k \in \mathbb{N}}} \mid \forall k \in \mathbb{N}, i_{k+1} > i_k$

- It follows that $i_k \geq k$ (proof by induction invoking the property of natural numbers $x > y \Rightarrow x \geq y + 1$
- A sequence can have convergent subsequences **without** itself converging, for example $a_n = (-1)^n$ does not converge but the subsequences a_{2n}, a_{2n+1} are constant and convergent

4.1.4. Lemma - Subsequences of a Convergent Sequence are Convergent to the Same Limit

Proof:
Let a_{n_i} (indexed by $i_{k_{k \in \mathbb{N}}}$) be a subsequence of a_n, which converges to $A \in \mathbb{R}$, ie $\exists N \in \mathbb{N} \mid \forall n > N, |a_n - A| < \varepsilon$.
$i_k \geq k \Rightarrow j \geq n$ is a term of i_k, which satisfies the convergence condition for the same A, along with all subsequent elements.

4.1.5. Definition - Accumulation Point

A point $A \in \mathbb{R}$ is called an **accumulation point** of a sequence a_n if:

$$\forall \varepsilon > 0 \forall N \in \mathbb{N} \exists n \in \mathbb{N} \mid n \geq N \land |a_n - A| < \varepsilon$$

This no longer requires that every $n \geq N$ is close to A, just that such an n can be chosen for every minimum N (which is a similar notion to choosing the terms that make up a converging subsequence). For example, both 1 and -1 are accumulation points of $a_n = (-1)^n$ but not limits.
The following corollaries apply:

- A is an accumulation point of a sequence $a_n \Leftrightarrow$ There exists a subsequence of a_n which converges towards A
- $\forall \varepsilon > 0$ there are infinitely many elements of the sequence a_n near an accumulation point $(A - \varepsilon, A + \varepsilon)$. This follows from the fact that there is a subsequence that converges to A and all elements of the subsequence after N are both close to A **and** elements of the parent sequence.
- A convergent sequence's limit is its **unique** accumulation point. The Lemma states: "All subsequences of a convergent sequence are convergent to the same limit" and applying the first corollary proves that they all correspond to the same accumulation point.

4.1.6. Definition - Ring of Sequences

Sequences $\in \mathbb{R}$ form a commutative ring together with point wise addition and multiplication and the constant sequences 0_n and 1_n as neutral elements:

$$a_n + b_n = (a+b)_n$$
$$a_n \cdot b_n = (a \cdot b)_n$$
$$\alpha \cdot b_n = (\alpha \cdot b)_n$$

- They do not form a field under pointwise multiplication, as a non-zero sequence may still contain $0 \in \mathbb{R}$ in it, which under real multiplication has no inverse $0 \cdot i \neq 1$.

4.1.7. Theorem - Operations on Limits

Operations on the sequence x_n which converges to X and y_n which converges to Y have the following effects on their limits:

1. $(x_n + y_n)_n \to X + Y$ Proof:
   We can say the following about these sequences:
   $$\forall \varepsilon > 0 \exists n \in \mathbb{N} \mid n \geq \max(N_x, N_y) \wedge |x_n - X| < \varepsilon \wedge |y_n - Y| < \varepsilon$$
   To show that $(x_n + y_n)_n$ converges to $X + Y$, we need to show $|(x_n + y_n) - (X + Y)| < \varepsilon$ for increasing n. Due to $0 \leq |x|$ and the compatibility of addition in the ordered field $\mathbb{R}$, we can add these inequalities:
   $$|x_n - X| + |y_n - Y| < 2\varepsilon$$
   Applying the triangle inequality:
   $$|x_n - X + y_n - Y| = |(x_n + y_n) - (X + Y)| < 2\varepsilon \blacksquare$$
   Similar proofs for 2, 3 and 4
2. $(x_n \cdot y_n)_n \to X \cdot Y$
3. $\forall \alpha \in \mathbb{R}, \alpha \cdot x_n \to \alpha X$
4. $\forall n \in \mathbb{N}, x_n \neq 0 \wedge X \neq 0 \Rightarrow (x^{-1})_n \to X^{-1}$

Furthermore:

1. $X < Y \Rightarrow \exists N \in \mathbb{N} \mid \forall n > N, x_n < y_n$
   Proof:
   Since both sequences converge, there exists:
   $$\exists N \mid \forall \varepsilon > 0, \forall n > N, |x_n - X| < \varepsilon \wedge |y_n - Y| < \varepsilon$$
   We want to consider the case of $X < Y$ and look for terms that "surround" this inequality. These inequalities can be rewritten as:
   $$-\varepsilon + X < x_n < \varepsilon + X$$
   $$-\varepsilon + Y < y_n < \varepsilon + Y$$
   $$x_n - \varepsilon < X$$
   $$y_n + \varepsilon > Y$$
   $$x_n - \varepsilon < X < Y < y_n + \varepsilon$$
   $$x_n - \varepsilon < y_n + \varepsilon$$
   Since ε can be chosen to tend towards 0, $x_n < y_n \forall n > N \blacksquare$

2. $(\exists N \in \mathbb{N} \mid \forall n > N, x_n < y_n) \Rightarrow X \leq Y$. This can be proved by contradiction using the previous Lemma. We cannot say $X < Y$, for example the two sequences $x_n = -\frac{1}{n}, y_n = \frac{1}{n}$ approach their limit 0 from different sides.

These can be very useful to calculate the limits of complicated expressions, for example:

$$\lim_{n\to\infty} \frac{7n^4 + 15}{3n^4 + n^3 + n - 1} = \lim_{n\to\infty} \frac{7 + 15n^{-4}}{3 + n^3 n^{-4} + nn^{-4} - 1n^{-4}} = \lim_{n\to\infty} \frac{7 + 15n^{-4}}{3 + n^{-1} + n^{-3} - n^{-4}}$$
$$x \geq 1, \lim_{n\to\infty} n^{-x} = 0$$
$$\therefore \frac{7}{3}$$

Care must be taken to not divide by 0 or ∞ when simplifying such limits.

4.1.8. Lemma - Sandwich Lemma

Consider 3 sequences, such that $\exists N \in \mathbb{N} \mid \forall n > N \mid x_n \leq y_n \leq z_n$. If x_n and z_n converge to the same limit, the sequence y_n also converges to the same limit. The proof follows from the previous Lemma $(\exists N \in \mathbb{N} \mid \forall n > N, x_n < y_n) \Rightarrow X \leq Y$ applied for both x_n and z_n and the principle of trichotomy.

4.1.9. Definition - Bounded Sequence

A sequence is bounded if $\exists M \in \mathbb{R} \mid M \geq 0, |x_n| \leq M \forall n \in \mathbb{N}$. This is different from a limit as the sequence may oscillate between negative and positive.
A sequence is unbounded if $\forall M \in \mathbb{R} \mid M \geq 0, \exists n \in \mathbb{N} \mid |x_n| \geq M$

- Every convergent sequence is bounded (but not every bounded sequence is convergent). Proof: The bound M is $\max(|A|, |x_1|, |x_2|, ..., |x_{N-1}|)$, where N is finite.
- Bounded sequences have at least 1 accumulation point / a convergent subsequence.

4.1.10. Definition - Monotone Sequence

A sequence x_n is called (strictly) monotonically increasing / decreasing if:

$$(\forall m, n \in \mathbb{N} \mid m > n) \Rightarrow x_m \geq (>) x_n$$
$$(\forall m, n \in \mathbb{N} \mid m > n) \Rightarrow x_m \leq (<) x_n$$

- Consider a monotone sequence. It is bounded $\Leftrightarrow$ it converges, such that:

$$\text{Montonically increasing: } \lim_{n\to\infty} x_n = \sup\{x_n \mid n \in \mathbb{N}\}$$
$$\text{Montonically decreasing: } \lim_{n\to\infty} x_n = \inf\{x_n \mid n \in \mathbb{N}\}$$

Proof:
I state the following facts:

$$\text{Monotonically increasing: } (\forall m, n \in \mathbb{N} \mid m > n) \Rightarrow x_m \geq x_n$$
$$\text{Bounded: } \exists M \in \mathbb{R} \mid M \geq 0, \forall b \in \mathbb{N}, |x_b| \leq M, -M \leq x_b \leq M$$
$$\text{Supremum:} \min\{b \in \mathbb{R} \mid \forall x \in X, x \leq b\}$$

I aim to show convergence by combining these definitions towards: $\exists A \in \mathbb{R} \forall \varepsilon \in (0, \infty) \exists N \in \mathbb{N} \mid \forall n \in \mathbb{N} : n \geq N, |a_n - A| < \varepsilon$.
The existence of a bound M shows that the bound set is **not** empty and a supremum “on the sequence” exists (although it may be smaller to M). Let $A \in \mathbb{R}$ be such a supremum:

$$\forall x_n, -A \leq x_n \leq A$$
$$\therefore x_n - A \leq 0$$

Since $\varepsilon > 0$, we can rearrange this to:

$$x_n - A < \varepsilon$$

We now wish to show $-\varepsilon < x_n - A$. It is given $x_n \geq x_{n-1}$. Furthermore, x_{n-1} also respects the bound A. Hence:

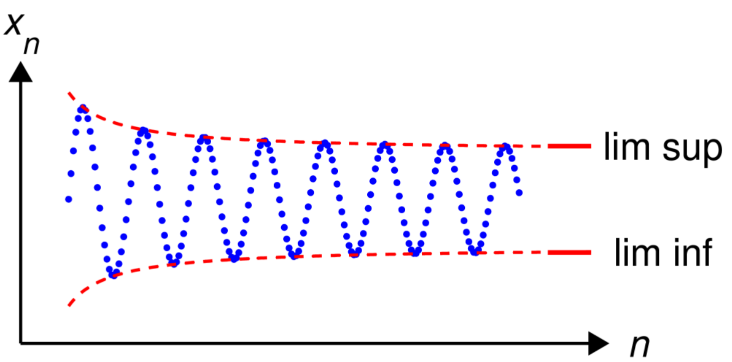
$$-A \leq x_{n-1} \leq x_n$$

TODO: This proof is taking too long :(but I think I am almost there. Hoping to apply the definition of the absolute function, $|x_n - A| < \varepsilon$. I also need to show that there is an N after which this is valid, will try again another time…

4.1.11. Definition - Superior / Inferior Limits

These can be thought of as the steady-state bounds of a sequence. Consider the sequence $s_n = \sup\{x_k \mid k \geq n\}$ based on the sequence x_n. As the starting term to be included n gets larger, the supremum can only stay the same or get smaller (monotonically decreasing) because $m > n, \{x_k \mid k \geq m\} \subsetneq \{x_k \mid k \geq n\}$, ie. the starting terms get excluded and $\therefore s_m \leq s_n$. **If the sequence x_n is bounded**, s_n is also bounded as the first (and subsequent) suprema are a real, non-infinite number. Therefore, s_n converges to $\inf\{s_n \mid n \in \mathbb{N}\}$ and vice versa for the inferior limit i_n, such that:

$$\forall n \in \mathbb{N}, i_n \leq x_n \leq s_n$$
$$(\limsup)_{n\to\infty} x_n = \lim_{n\to\infty} (\sup\{x_k \mid k \geq n\}) = \inf\{\sup\{x_k \mid k \geq n\} \mid n \in \mathbb{N}\}$$
$$(\liminf)_{n\to\infty} x_n = \lim_{n\to\infty} (\inf\{x_k \mid k \geq n\}) = \sup\{\inf\{x_k \mid k \geq n\} \mid n \in \mathbb{N}\}$$
$$(\liminf)_{n\to\infty} x_n \leq (\limsup)_{n\to\infty} x_n$$



- The superior and inferior limits of all **bounded** sequences are accumulation points, and therefore have convergent subsequences:

$$A = \text{Set of accumulation points}$$
$$(\limsup)_{n\to\infty} x_n = \max(A)$$
$$(\liminf)_{n\to\infty} x_n = \inf(A)$$

4.1.12. Theorem - Squeeze Theorem

This is known as the **squeeze theorem** as the bounds squeeze towards the limit from either side and is an alternative criteria for convergence:

$$\text{A sequence converges} \Leftrightarrow (\liminf)_{n\to\infty} x_n = (\limsup)_{n\to\infty} x_n$$

Proof:
Let $s_n = \sup\{x_k \mid k \geq n\}$, $i_n = \inf\{x_k \mid k \geq n\}$, we know that $\forall n \in \mathbb{N}, i_n \leq x_n \leq s_n$.
It is given that s_n and i_n converge to the same limit, therefore x_n also converges to this limit (sandwich lemma).
To show convergence $\Rightarrow \limsup = \liminf$, as $n \to \infty$, the minimum index for convergence N will have been reached and:

$$\forall n > N \forall \varepsilon \in \mathbb{R} \mid \varepsilon > 0, -\varepsilon < x_n - A < \varepsilon$$
$$A - \varepsilon < x_n < A + \varepsilon$$
$$A - \varepsilon < i_n \leq s_n < A + \varepsilon$$

This can be rearranged to convergence criteria for s_n and i_n, showing that their limits are equal:

$$\forall \varepsilon > 0, A - \varepsilon < i_n < A + \varepsilon \Rightarrow |i_n - A| < \varepsilon$$
$$|s_n - A| < \varepsilon \blacksquare$$

4.1.13. Definition - Cauchy Sequence
A sequence is a **Cauchy Sequence** if:

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \mid \forall n, m > N, |x_n - x_m| < \varepsilon$$

Important: It is only a Cauchy sequence if $|x_n - x_m| < \varepsilon$ **for all** $n, m > N$. For example, the sequence $1, 1 + \frac{1}{2}, 2, 2 + \frac{1}{3}, 2 + \frac{2}{3}, 3, \ldots$ satisfies $|x_n - x_{n-1}| < \varepsilon \forall n > N$, but it is not a Cauchy sequence and is unbounded.
- They are bounded
- A sequence converges $\Leftrightarrow$ it is a Cauchy sequence.

Proof:
Let x_n be a sequence converging to A:

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \mid \forall n > N, |x_n - A| < \varepsilon$$
$$m > N \Rightarrow |x_m - A| < \varepsilon$$
$$|x_n - A| + -|x_m - A| < \varepsilon$$

Applying the triangle inequality:

$$|x_n - x_m| < \varepsilon$$

Since ε is arbitrary, this shows that it is a Cauchy sequence. Now we must show that this implies that it converges, let x_n be a Cauchy sequence. Because it is bounded, there exists a subsequence x_{n_k} which converges to the bound A as $n \to \infty$. Elements of this subsequence are also elements of x_n and can be included in the Cauchy inequality as $n \to \infty$:

$$\left|x_n - x_{n_k}\right| < \varepsilon$$
$$\left|x_{n_k} - A\right| < \varepsilon$$
$$|x_n - A| \leq \left|x_n - x_{n_k}\right| + \left|x_{n_k} - A\right| < 2\varepsilon$$

Therefore the entire sequence converges to $A \blacksquare$

4.1.14. Definition - Divergence

We say that a sequence x_n diverges to ∞ $(-\infty)$ if:

$$\forall M \in \mathbb{R} \mid M > (<)0, \exists N \in \mathbb{N} \mid \forall n > N, x_n > (<)M$$
$$\lim_{n\to\infty} x_n = \infty(-\infty)$$

The limits ∞ and $-\infty$ are called **improper**.

- An unbounded sequence doesn't necessarily diverge to ∞, for example $(-1)^n n$ oscillates.
- An unbounded sequence always has a subsequence which diverges to ∞ or $-\infty$.
  Proof:
  The definition of an unbounded sequence is very similar to that of divergence to ∞ or $-\infty$:

  $$\forall M \in \mathbb{R} \mid M \geq 0, \exists k \in \mathbb{N} \mid |x_k| \geq M$$
  $$x_k > M \vee x_k < -M$$

  We now need to show that there are infinitely many such elements with index greater than k. Assume by contradiction that $\nexists i > k \mid x_i > M \vee x_k < -M$. This violates the criteria for an unbounded sequence, as it implies there are finitely ($\mathbb{N}$ is not dense) many elements $x_k > M$ but this is required for **infinitely** many $M > 0$.
  This contradicts the assumption and shows:

  $$\forall M \in \mathbb{R} \mid M > (<)0, \exists N \in \mathbb{N} \mid \forall n > N, x_{n_k} > (<)M\blacksquare$$

- The superior / inferior limits of an unbounded sequence are the improper limits ∞ or $-\infty$ depending on if it has an upper or lower bound.

4.1.15. Definition - Complex Sequences

We can study the limits of the real and imaginary parts of a complex number individually. A sequence z_n converges to $A + Bi$ if $\mathrm{Re}(z_n) \to A, \mathrm{Im}(z_n) \to B$. Since complex numbers are not ordered, we check divergence using the absolute function $|z_n| \to \infty$.

5. Complex Numbers

5.1.1. Definition - Complex Numbers

The set of complex numbers $\mathbb{C}$ is defined from the Cartesian coordinates, where the $+$ can be thought of as a substitute for the comma in a tuple, and i is called the **complex unit**:

$$z = a + bi \in \mathbb{C} := \mathbb{R}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}$$
$$a = \mathrm{Re}(z)$$
$$b = \mathrm{Im}(z)$$

Complex addition $+_{\mathbb{C}}$ and multiplication $\cdot_{\mathbb{C}}$ are defined such that $\mathbb{C}$ is a field (the operations follow the conditions for a ring excluding division by 0) **and** $i^2 = -1$ holds:

$$(a + bi) +_{\mathbb{C}} (c + di) = (a + b) + (c + d)i$$
$$(a + bi) \cdot_{\mathbb{C}} (c + di) = ac + adi + bci + bdi^2$$
$$= (ac - bd) + (ad + bc)i$$

- Addition has Neutral Element $(0, 0) = 0$ and Inverse Element $(-a, -b) = -a - bi$
- Multiplication has Neutral Element $(1, 0) = 1$ and (non-zero) Inverse Element $\left(\frac{a}{a^2+b^2}, -\frac{b}{a^2+b^2}\right)$
- The same notation as in $\mathbb{R}$ is normally used, for example $-(i) = (-0, -1) = -i$ and $i^{-1} = (0, -1) = -i$

- An order relation cannot be defined in a way that satisfies the ordered field axioms. Proof:

$$\text{Let } 0 \leq i, \text{condition 3. implies } 0 \leq i \cdot i = -1 \text{ which is false} \therefore i \leq 0$$
$$\text{Applying condition 2. } i + -i \leq 0 + -i \Rightarrow 0 \leq -i$$
$$\text{Applying condition 3. } 0 \leq -i \cdot -i = i^2 = -1 \text{ which is also false and contradicts } i \leq 0 \blacksquare$$

Nevertheless, they satisfy a generalization of the completeness axiom and we can perform calculus on them.

5.1.2. Definition - Complex Conjugate

The mapping $\overline{\cdot} : \mathbb{C} \to \mathbb{C}$ of a complex number $z = a + bi \in \mathbb{C}$ is denoted as $\overline{z}$ and defined:

$$\overline{z} := a - bi$$

It has the following properties $\forall z, w \in \mathbb{C}$:

- $z \cdot \overline{z} \in \mathbb{R} \geq 0$
- $\overline{z + w} = \overline{z} + \overline{w}$
- $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$
- $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$
- $\mathrm{Re}(z) = \frac{z + \overline{z}}{2}$
- $\mathrm{Im}(z) = \frac{z - \overline{z}}{2i}$

5.1.3. Definition - Complex Absolute Function

The complex absolute function $|\cdot| : \mathbb{C} \to \mathbb{R}$ is defined as:

$$|z = a + bi| := \sqrt{z \cdot \overline{z}} = \sqrt{a^2 + b^2}$$

- $|z \cdot w| = \sqrt{(z \cdot w) \cdot \overline{z \cdot w}} = \sqrt{z \cdot \overline{z} \cdot w \cdot \overline{w}} = \sqrt{z \cdot \overline{z}} \cdot \sqrt{w \cdot \overline{w}} = |z| \cdot |w|$
- It has the same notion of length when complex numbers are plotted on an Argand diagram

5.1.4. Theorem - Cauchy-Schwartz Inequality

$\forall z = x_1 + y_1 i, w = x_2 + y_2 i \in \mathbb{C}$:

$$x_1 x_2 + y_1 y_2 \leq |z \cdot w|$$

Proof:
Through algebraic rearrangement, we can show that:

$$|z \cdot w| - x_1 x_2 + y_1 y_2 = (x_1 y_2 - x_2 y_1)^2$$

Lemma: $x^2 \geq 0$
Therefore $|z \cdot w| - x_1 x_2 + y_1 y_2 \geq 0$
By applying the compatibility of addition in an ordered field (although $\mathbb{C}$ is not an ordered field, $|x \cdot w|$ can be expressed in terms of the component real numbers), we arrive at:

$$|z \cdot w| \geq x_1 x_2 + y_1 y_2 \blacksquare$$

5.1.5. Theorem - Complex Triangle Inequality

We can show that the triangle inequality also holds true $\forall z = x_1 + y_1 i, w = x_2 + y_2 i \in \mathbb{C}$:

$$|z + w| \leq |z| + |w|$$

Proof:
Through algebraic rearrangement, we can show:

$$|z+w|^2 = |z|^2 + |w|^2 + 2(x_1 x_2 + y_1 y_2)$$

Applying the Cauchy-Schwarz Inequality:

$$|z|^2 + |w|^2 + 2(x_1 x_2 + y_1 y_2) \leq |z|^2 + |w|^2 + 2|z \cdot w|$$
$$|z+w|^2 \leq (|z| + |w|)^2$$
$$\therefore |z+w| \leq |z| + |w|$$

6. Functions of One Real Variable

6.1.1. Definition - Function

A function $f : X \to Y$ is a mapping from a domain X (not just the natural numbers like sequences) to range / codomain Y. For now we only discuss single-valued **real** functions: $X, Y \subseteq \mathbb{R}$. It **may** have the following properties:

1. *Injective* - $\forall x, x' \in X : x \neq x' \Rightarrow f(x) \neq f(x')$ - Assigns each element of X a **unique** element in Y
2. *Surjective* - $\forall y \in Y \exists x \in X : f(x) = y$ - Every element in the range is a possible output of the function
3. *Bijective* - It is both injective and surjective, and therefore an inverse function can be defined
4. Two functions are **equal** $\Leftrightarrow X_1 = X_2 \land Y_1 = Y_2 \land \forall x \in X, f_1(x) = f_2(x)$

6.1.2. Definition - Image and Preimage (Urbild) of a Function

Consider a function $f : X \to Y$.

- The Image $f(A)$ of $A \subseteq X$ under f is defined as:

$$f(A) := \{y \in Y \mid \exists x \in A : f(x) = y\}$$
$$f(A) \subseteq Y$$

- The Preimage (Urbild) $f^{-1}(B)$ of $B \subseteq Y$ under f is defined as:

$$f^{-1}(B) := \{x \in X \mid \exists y \in B : f(x) = y\} = \{x \in X \mid f(x) \in B\}$$
$$f^{-1}(B) \subseteq X$$

- A function $f : X \to Y$ is surjective $\Leftrightarrow$ The set $f(X) = Y$, because the image can only contain domain elements which map to Y by definition.
- For example consider $f : \mathbb{R} \to \mathbb{R} := x \to 0$:
  - $f(\mathbb{R}) = \{0\}$ - It is not surjective
  - $f^{-1}(\mathbb{R}) = f^{-1}(\{0\}) = \mathbb{R}$
  - $f^{-1}(\{1\}) = \emptyset$

There is an interesting property of finite sets; consider $f : X \to Y$, where X and Y are **finite** sets with the same number of elements n:

$$f \text{ is injective} \Leftrightarrow f \text{ is surjective}$$

Proof:
If f is injective, the image $f(X)$ has at least n distinct elements so every distinct $x \in X$ has its own $y \in f(X)$.
Lemma: A function $f : X \to Y$ is surjective $\Leftrightarrow$ The set $f(X) = Y$
We are given that Y has n elements, and since $f(X) \subseteq Y \Rightarrow f(X) = Y$ showing that it must also be surjective.
We must now show that surjectivity $\Rightarrow$ injectivity. If f is surjective, $f(X) = Y$ (Lemma), therefore

$f(X)$ has n elements.
Consider two elements $x_1, x_2 \in X$. Since X is a set, they are distinct $x_1 \neq x_2$.
If $f(x_1) = f(x_2)$ for any two elements, they would "validate" the same member of $f(X)$, leaving out at least one element of Y (deterministic, another input cannot have 2 outputs to make up for it) meaning $f(X)$ would have $n - 1$ elements, which contradicts the lemma about surjectivity, therefore f must also be injective ■
This is **not** necessarily true for infinite sets, for example $f : \mathbb{N} \to \mathbb{N}, f(x) := x + 1$ is injective but not surjective.

6.1.3. Definition - Square Root

This is the bijective function $\sqrt{\cdot} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\forall a \in \mathbb{R}_{\geq 0} (\sqrt{a})^2 = a$, whose existence is only possible due to the real numbers being a **complete ordered field**. TODO: Various proofs Exercise 2.27 Figalli's script

6.1.4. Definition - Ring of Functions

For a domain X, we can define a **commutative** ring (not a field, there is no inverse element for multiplication) on the set of real valued functions $\mathcal{F}(X) := \{f \mid f : X \to \mathbb{R}\}$ with the following operations:

$$f_1, f_2 \in \mathcal{F}(X)$$
$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$
$$(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x)$$
$$\alpha \in \mathbb{R}, (\alpha \cdot f_1)(x) = \alpha \cdot f_1(x)$$

The constant function is defined as $\forall x \in X, f(x) = a$.
- Neutral elements - Addition: $f(x) = 0$, Multiplication: $f(x) = 1$

An order relation $\leq$ is defined:

$$f_1 \leq f_2 \Leftrightarrow \forall x \in X, f_1(x) \leq f_2(x)$$

6.1.5. Definition - Bounded Functions

As with many function definitions, they are very similar to the definitions for sequences. A function $f : X \to Y$ is bounded if:

$$\exists M \in \mathbb{R} \mid M > 0, \forall x \in X, |f(x)| < M$$

This can be separated into "bounded from above" $f(x) < M$ and below $f(x) < -M$

6.1.6. Definition - Monotone Functions

A function $f : X \to Y$ is called (strictly) monotonically increasing / decreasing if $\forall m, n \in X$:

$$m > n \Rightarrow f(m) \geq (>) f(n)$$
$$m > n \Rightarrow f(m) \leq (<) f(n)$$

- The rounding function $\lfloor x \rfloor$ is monotonically increasing but not strictly
- A function is constant $\Leftrightarrow$ A function is both monotonically increasing and decreasing. Proof: Trichotomy
- A **strictly** monotone function is always injective. Proof:
  Assume by contradiction that it is not injective. $\exists x_1 \neq x_2 \mid f(x_1) = f(x_2$. However, since $x_1 \neq x_2$, they must be either $>$ or $<$ each other and therefore $f(x_1) \neq f(x_2)$ (monotone) which proves that they are injective by contradiction ■

6.1.7. Definition - $\varepsilon\delta$ Continuity

Intuitively, a function is continuous over an interval if we can draw it without lifting the pencil. A function is continuous at a point $x_0 \in X$ if:

$$\forall\varepsilon > 0 \exists\delta > 0 \mid \forall x \in X, |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

It is continuous over a set X if it is continuous $\forall x_0 \in X$, the definition can be amended to $\forall x_1, x_2 \in X, |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$. If there is a jump in the function at the point x, then there exists a small enough ε, so that no matter how close x is to x_0, the max error ε will never be satisfied.

- The **Dirichlet Function** based on the characteristic function is non-continuous at every point:

$$1_{\mathbb{Q}} : \mathbb{R} \to \{1, 0\}$$
$$1_{\mathbb{Q}} := \chi_{\mathbb{Q}}$$

  This is because there are irrational, real numbers around every rational number so for $0 < \varepsilon < 1 \nexists\delta$ such that points next to each other have an output $< \varepsilon$. This also demonstrates why the $\forall x \in X$ is necessary, otherwise one could simply pick two rational numbers within the δ interval (it is a dense set).
- Constant functions are continuous. Proof:
  $\forall x, x_0 \in X, f(x) - f(x_0) = 0 < \varepsilon$ therefore there always exists such a δ■
- The function $f(x) = x$ is continuous. Proof:
  We need to find a δ such that $\forall x_1, x_2 \in X, |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$.
  Because $|f(x_1) - f(x_2)| = |x_1 - x_2|$ and $\delta, \varepsilon > 0$, δ can always be chosen such that the second inequality also holds true $\forall\varepsilon > 0$■
- The absolute function $f(x) = |x|$ is continuous. Proof:
  Inverse triangle inequality: $|f(x_1) - f(x_2)| = ||x_1| - |x_2|| \leq |x_1 - x_2| < \delta = \varepsilon$■
- x^2 is continuous TODO

6.1.8. Lemma - Operations on Continuous Functions

Let $f_1, f_2 : D \to \mathbb{R}$ be continuous functions at a point $x_0 \in D$. The following functions are also continuous at x_0:

1. $(f_1 + f_2)(x)$ - Proof:
   We are given:

$$\forall\varepsilon > 0 \forall x \in X$$
$$\exists\delta_1 > 0 \mid |x - x_0| < \delta_1 \Rightarrow |f_1(x) - f_1(x_0)| < \varepsilon$$
$$\exists\delta_2 > 0 \mid |x - x_0| < \delta_2 \Rightarrow |f_2(x) - f_2(x_0)| < \varepsilon$$

   Setting $\delta = \min\{\delta_1, \delta_2\}$:

$$\exists\delta > 0 \mid |x - x_0| < \delta \Rightarrow 0 \leq |f_1(x) - f_1(x_0)| < \varepsilon \land 0 \leq |f_2(x) - f_2(x_0)| < \varepsilon$$
$$|f_1(x) - f_1(x_0)| + |f_2(x) - f_2(x_0)| < 2\varepsilon$$

   Since functions form a ring, we want to show $\exists\delta \mid |x - x_0| < \delta \Rightarrow |f_1(x) + f_2(x) - f_1(x_0) + f_2(x_0)|$. Applying the triangle inequality gives:

$$|f_1(x) + f_2(x) + (-(f_1(x_0) + f_2(x_0)))| \leq |f_1(x) - f_1(x_0)| + |f_2(x) - f_2(x_0)| < 2\varepsilon$$

   Therefore $(f_1 + f_2)(x)$ is also continuous at x_0■

2. $(f_1 \cdot f_2)(x)$ - Proof:
   Following the previous definitions, we wish to show that $\exists\delta \mid |x - x_0| < \delta \Rightarrow |f_1(x) \cdot f_2(x) - f_1(x_0) \cdot f_2(x_0)| < \varepsilon$, which we can achieve as follows:

$$|f_2(x)||f_1(x) - f_1(x_0)| < |f_2(x)|\varepsilon$$
$$|f_1(x_0)||f_2(x) - f_2(x_0)| < |f_1(x_0)|\varepsilon$$
$$|f_2(x)f_1(x) - f_2(x)f_1(x_0)| + |f_1(x_0)f_2(x) - f_1(x_0)f_2(x_0)| < \varepsilon(|f_1(x)| + |f_2(x)|)$$
$$|f_1(x) \cdot f_2(x) - f_1(x_0) \cdot f_2(x_0)| < \varepsilon(|f_1(x)| + |f_2(x)|)$$

   Since $\varepsilon > 0$ is arbitrary, this proves the continuity of the product of continuous functions ■
3. $\forall\alpha \in \mathbb{R}, (\alpha \cdot f_2)(x)$ - The proof follows from setting $f_1(x)$ in the previous Lemma to the constant function $f_1(x) = \alpha$, which has been shown to be continuous.

This can be extended to continuity over a common subset if they are both continuous over that set.

6.1.9. Corollary - Polynomials are continuous

All polynomials can be constructed from a linear combination of $f(x) = x$ and constant functions $f(x) = a$, which were both shown to be continuous $\in \mathbb{R}$. Hence polynomials are also continuous for all points in $\mathbb{R}$.

6.1.10. Definition - Composition of Functions

Functions can be passed as arguments into one another:

$$f : X \to Y, g : Y \to Z$$
$$g \circ f : X \to Z$$
$$g \circ f := g(f(x))$$

- Composition of functions is associative and brackets are irrelevant

6.1.11. Theorem - Composition of Continuous Functions

Let $f : X \to Y$ which is continuous at x_0 and $g : Y \to Z$ continuous at $f(x_0)$ such that $X, Y, Z \subseteq \mathbb{R}$. $g \circ f$ is also continuous at x_0.
Proof:
The following properties apply $\forall\varepsilon > 0$:

$$\exists\delta_1 \mid \forall x \in X, |x - x_0| < \delta_1 \Rightarrow |f(x) - f(x_0)| < \varepsilon$$
$$\exists\delta_2 \mid \forall f(x) \in Y, |f(x) - f(x_0)| < \delta_2 \Rightarrow |g(f(x)) - g(f(x_0))| < \varepsilon$$

To show continuity of $g \circ f$ at x_0, I will show $\exists\delta \mid \forall x \in X, |x - x_0| < \delta \Rightarrow |g(f(x)) - g(f(x_0))| < \varepsilon$. By choosing $\delta = \min(\delta_1, \delta_2)$, this is clearly the case as:

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \Rightarrow |g(f(x)) - g(f(x_0))|$$

The surjectivity of $f(x)$ does not matter, as the final inequality only applies if the intermediate $|f(x) - f(x_0)| < \varepsilon$ is true ■
As usual, this extends to the entire domain if both f and g are continuous functions.

6.1.12. Theorem - Sequential Continuity

This is an alternative characterization of a continuous function. The function $f : X \to \mathbb{R}$ is continuous at $x_0 \Leftrightarrow$ For every sequence $x_n \subsetneq X$ converging to x_0 (this always exists, for example $x_n = x_0$), the sequence $f(x_n)$ converges to $f(x_0)$.
Proof:
The following holds true for all sequences converging to x_0:

$$\forall \delta > 0 \exists N_1 \mid \forall n > N_1, |x_n - x_0| < \delta$$

If the sequence $f(x_n)$ converges to $f(x_0)$:

$$\forall \varepsilon > 0 \exists N_2 \mid \forall n > N_2, |f(x_n) - f(x_0)| < \varepsilon$$

Choosing $N = \max\{N_1, N_2\}$ both conditions hold true:

$$\forall \varepsilon > 0 \exists N \mid \forall n > N, |x_n - x_0| < \delta \land |f(x_n) - f(x_0)| < \varepsilon$$

TODO: Somehow this is equivalent to saying $\forall x \in X$:

$$\forall \varepsilon > 0 \exists \delta \mid \forall x \in X |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \blacksquare$$

6.1.13. Theorem - Intermediate Value Theorem (Zwischenwertsatz)

Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $a \leq b$. $\forall c \in [f(a), f(b)] \exists x \in [a, b] \mid f(x) = c$.
In simpler words, a continuous function f takes on every value between $[f(a), f(b)]$ at least once - it is **surjective** over $[f(a), f(b)]$. This is useful to show surjectivity or to prove that an injective, continuous function is **strictly monotone** over an interval (assume it is not and choose a point that violates the monotonicity, there exists two points on either side with the same output due to this theorem).
Proof:
Let c be any value who want to show is in the domain of the continuous function $f : [a, b] \to X$ such that $f(a) < c < f(b)$ (one can prove for $f(b) < f(a)$ similarly by working with the set $U = \{x \in [a, b] \mid c < f(x)\}$).
Consider the set $L = \{x \in [a, b] \mid f(x) < c\}$. The set is not-empty ($a \in L$) and is bounded, therefore it has a supremum s. We will now show that $s \neq a \land s \neq b$.
Due to the continuity of f at a, we can keep $f(x)$ any $\varepsilon > 0$ away (at most) from $f(a)$ such that $|x - a| < \delta$. Choose $\varepsilon < c - f(a)$ (to pull the values away from a towards whichever value outputs c).
The x's which satisfy this ε must be in L ($\forall x \in [a, b] \land f(x) - f(a) < \varepsilon = c - f(a) \Rightarrow f(x) < c$) but not equal to a (because $\varepsilon \neq 0$), therefore L contains at least one element greater than $a \Rightarrow$ the supremum $s > a$. The same can be argued about the continuity at b, any elements close to b are greater than c (due to $c < f(b)$) $\Rightarrow$ not in L, therefore the supremum $s < b$.
We now know that $s \in (a, b)$ and want to show $f(s) = c$. Since f is continuous over $[a, b]$, it is also continuous at s:

$$\forall \varepsilon > 0 \exists \delta > 0 | \ \forall x \in [a, b], |x - s| < \delta \Rightarrow |f(x) - f(s)| < \varepsilon$$

Considering δ for any ε, there must exist some $x_0 \mid -\varepsilon + f(s) < f(x_0) < \varepsilon + f(s)$ which is in the interval $x_0 \in (s - \delta, s]$ (because s is the supremum). Since $f(x_0) < c$, this leads to:

$$-\varepsilon + f(s) < c$$
$$f(s) < c + \varepsilon$$

Furthermore, there exists an $x_1 \in [s, s + \delta) \mid f(x_1) \geq c$ (otherwise it would've been in L), and due to the continuity of f at s:

$$-\varepsilon + f(s) < f(x_1) < \varepsilon + f(s)$$
$$c < \varepsilon + f(s)$$
$$-\varepsilon + c < f(s)$$
$$-\varepsilon + c < f(s) < c + \varepsilon$$

Choosing $\varepsilon \to 0$, it is clear that $f(s) = c$ whilst $s \in [a, b]$■
Summary:
- Define set $L = \{x \in [a, b] \mid f(x) < c\}$
- Show that its supremum $s \in (a, b)$ due to continuity at both of those points
- Exploit the continuity at s along with points $x_0 \in (s - \delta, s], \in L \Rightarrow f(x_0) < c$ and $x_1 \in [s, s + \delta), \notin L \Rightarrow f(x_1) \geq c$ to show that $f(s) = c$ as $\varepsilon \to 0$

6.1.14. Definition - Inverse Function

Any **bijective** function $f : X \to Y$ has a corresponding inverse $f^{-1} : Y \to X$ (not to be confused with the preimage, which is defined for all functions but doesn't take account of every element in Y, not surjective) defined such that:

$$f^{-1} \circ f = f \circ f^{-1} = \text{id}$$

6.1.15. Theorem - Inverse Function Theorem

A function that is strictly monotone and continuous over an interval $I \subseteq \mathbb{R}$ is bijective and has an inverse function, which is also strictly monotone and continuous.
Proof:
Let $J = f(I)$ and consider the strictly monotone, continuous function $f : I \to J$. It is surjective by definition and injective due to its strict monotonicity, therefore the inverse $f^{-1} : J \to I$ exists, which is also strictly monotone:

$$\forall x_1, x_2 \in X$$
$$x_1 < x_2 \Leftrightarrow f(x_1) < (>) f(x_2) \Leftrightarrow f^{-1}(f(x_1)) < f^{-1}(f(x_2))$$

The continuity of the inverse can be shown using the sequential continuity criteria but is rather complicated.

6.1.16. Definition - n'th Root Function

This is defined as the inverse of $x^n : [0, \infty) \to [0, \infty)$ TODO

The range of a continuous function with / bounded to a compact domain is also compact.

The maxi-, mini-, supre- and infimum of a function are defined as expected on its range.

Every continuous function with a compact domain and therefore range possesses a maximum and minimum.

6.1.17. Topological Continuity Definitions

Continuity of a function can also be defined with the following topological criteria:

6.1.17.1. Closed / Open Sets

$$f : X \to Y \text{ is continuous} \Leftrightarrow$$
$$\text{The inverse image (Urbild) of every relatively open / closed subset in } X \text{ is also relatively open / closed.}$$

6.1.17.2. Neighbourhoods

$$f : X \to Y \text{ is continuous} \Leftrightarrow$$
$$\text{The inverse image of every neighbourhood at point } f(x_0) \text{ in } Y \text{ is also a neighbourhood of } x_0 \text{ in } X$$