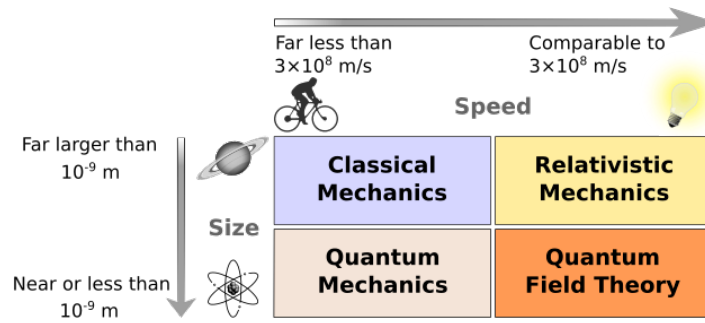


Physics



Classical Mechanics

Newton's Laws of Motion

Published in his 1687 paper Principia, these laws describe the motion of objects **in inertial reference frames** and continue to serve as the foundations of classical mechanics in the modern day.

1. An object remains at rest or in motion at a constant speed unless acted on by an external force. (aka the principle of inertia)
2. The resultant force acting on a body is the rate of change of the momentum of the object:

$$F = \frac{dP}{dt} = m\mathbf{a}$$

3. Every action results in an equal and opposite reaction, for example the weight acting on an object also acts on the Earth. This can also be used to show the conservation of linear momentum.

Kinematics

The equation of motion for a point mass in Cartesian Coordinates given its velocity function can be calculated by solving the differential equation:

$$\vec{r}(t) = \vec{r}(t_0) + \int_{t_0}^t \vec{v}(t') dt'$$

Where the initial position $\vec{r}(t_0)$ is needed due to the constant of integration. Integration only returns the total distance / velocity gained / lost or the displacement.

Acceleration Formulae

Acceleration is the rate of change of velocity, can be found by taking the derivative of velocity:

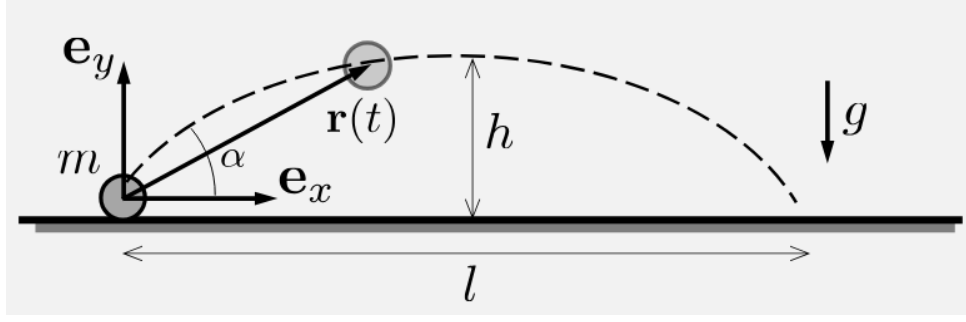
$$\vec{a} = \dot{\vec{v}} = \frac{d(\dot{x}e_x + \dot{y}e_y + \dot{z}e_z)}{dt} = \ddot{x}e_x + \ddot{y}e_y + \ddot{z}e_z$$

In cylindrical coordinates:

$$\begin{aligned} \vec{v} &= \dot{\rho}e_\rho + \rho\dot{\phi}e_\phi + \dot{z}e_z \\ \vec{a} &= (\ddot{\rho} - \rho\dot{\phi}^2)e_\rho + (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})e_\phi + \ddot{z}e_z \end{aligned}$$

Projectile Motion

Consider the following projectile with negligible air resistance:



$$\dot{\mathbf{r}}(0) = v_0(\cos(\alpha)\mathbf{e}_x + \sin(\alpha)\mathbf{e}_y)$$

$$\ddot{\mathbf{r}}(t) = -g\mathbf{e}_y$$

$$\dot{\mathbf{r}}(t) = (v_0 \cos(\alpha))\mathbf{e}_x + (v_0 \sin(\alpha) - gt)\mathbf{e}_y$$

Solving for the positions, taking the starting position as the origin:

$$\mathbf{r}(t) = (v_0 \cos(\alpha)t)\mathbf{e}_x + \left(v_0 \sin(\alpha)t - \frac{g}{2}t^2\right)\mathbf{e}_y$$

The maximum distance is achieved at $r_y(t) = 0$ and is at its maximum when the initial angle is 45° to the ground.

The maximum height is when $\dot{r}_y(t) = 0$. The solved time can then be inserted into the position equation.

Angular Velocity

In a polar coordinate system, we define the angular velocity ω as the rate of change of the φ coordinate, which is given as a component in the \mathbf{e}_z direction, allowing us to store, allowing us to calculate the velocity of a particle using the cross product:

$$\dot{\mathbf{r}}(t) = \vec{\omega}(t) \times \vec{\rho}(t)$$

Usually anti-clockwise rotation is attributed to a positive angular velocity and vice versa.

Circular Motion

Consider a particle moving around a circular path (constrained to a plane) with a constant radius r and angular velocity ω :

$$\rho(t) = r$$

$$\varphi(t) = \omega t + \varphi_0$$

Expressing its equation of motion using Cartesian coordinates:

$$\vec{\mathbf{r}}(t) = \begin{pmatrix} r \cos(\omega t + \varphi_0) \\ r \sin(\omega t + \varphi_0) \end{pmatrix}$$

$$\dot{\mathbf{r}}(t) = \omega r \begin{pmatrix} -\sin(\omega t + \varphi_0) \\ \cos(\omega t + \varphi_0) \end{pmatrix}$$

$$\ddot{\mathbf{r}}(t) = -\omega^2 r \begin{pmatrix} \cos(\omega t + \varphi_0) \\ \sin(\omega t + \varphi_0) \end{pmatrix} = -\omega^2 \vec{\mathbf{r}}(t)$$

Where the velocity and current position vector are always perpendicular to each other (inner product of sin and cos is always 0). We can see that the acceleration (and resultant force) needed to sustain such motion is always in the opposite direction of the position vector: $-\vec{e}_\rho$.

The speed of the particle can also be calculated as $v = r\omega$, therefore the acceleration for a desired speed is:

$$\omega = \frac{v}{r}$$

$$\ddot{r} = -r \left(\frac{v}{r} \right)^2 e_\rho = -\frac{v^2}{r} e_\rho$$

Furthermore, the period for a single orbit is given by:

$$T = \frac{2\pi}{\omega} = \frac{2\pi r}{v}$$

Particle accelerators operate upon these principles.

Angular Acceleration

We define the angular acceleration as the rate of change of angular velocity:

$$\alpha = \dot{\omega} = \ddot{\varphi}$$

In cylindrical coordinates the general formulae for velocity and acceleration are:

$$v = \dot{\rho}e_\rho + \rho\dot{\varphi}e_\varphi + \dot{z}e_z$$

$$a = (\ddot{\rho} - \rho\dot{\varphi}^2)e_\rho + (2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi})e_\varphi + \ddot{z}e_z$$

Momentum

Consider a system of n particles, where the only forces are those exerted by the particles on each other. The sum of momentum of this system is:

$$\sum p = P$$

The rate change of a single particle's momentum is the resultant force:

$$\dot{p} = \sum_{i=1}^n F_{1i}$$

Since differentiation is a linear operation, the rate of change of the total momentum is:

$$\dot{P} = \sum_{j=1}^n \sum_{i \neq j}^n F_{ji} = 0$$

This always equal to 0 due to Newton's 3rd Law of Motion: $F_{ji} = -F_{ij}$

Hence total momentum is constant and conserved in an **isolated system** (no external forces do any work), which is a useful fact for solving collision problems.

Newton's 3rd law and thus the conservation of momentum does not hold when relativistic effects are apparent and therefore also in the case of magnetic fields (which is a consequence of special relativity):

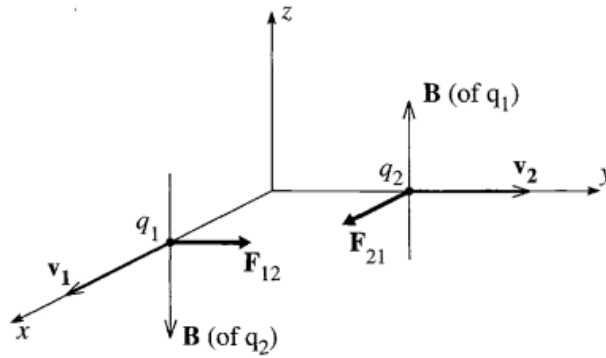


Figure 1.8 Each of the positive charges q_1 and q_2 produces a magnetic field that exerts a force on the other charge. The resulting magnetic forces \mathbf{F}_{12} and \mathbf{F}_{21} do not obey Newton's third law.

However, unless the particles are moving near the speed of light, the electrostatic forces far outweigh momentum losses due to magnetism.

Impulse

Consider a constant resultant force applied on a point mass for Δt seconds. The change in momentum is called impulse:

$$\Delta P = F \Delta t$$

This often leads to an easier approach for calculating resulting velocities due to conservation of momentum.

Angular Momentum

Momentum and Newton's 2nd Law can be defined with respect to a **fixed** point as angular momentum:

$$\vec{l} = \vec{r} \times \vec{p}$$

Where \vec{r} is the vector from the point to the particle and \vec{p} is its linear momentum.

Newton's 2nd Law is defined accordingly - the resultant moment of forces applied on the particle with respect to the **fixed** point is equal to its rate of change of angular momentum with respect to the same point:

$$\vec{M} = \dot{\vec{l}}$$

For example, the angular momentum of a centripetal system (orbiting planet) with respect to the origin stays constant as the force has no perpendicular component.

The total angular momentum of a classical system with respect to any point is conserved, just like linear momentum.

Center of Mass

Unfortunately we do not work with point masses in real life, but luckily the same linear and angular momentum conservation laws apply to the center of mass of objects.

The resultant force applied at any point of a rigid body is proportional to the acceleration of its center of mass, $F = ma$.

Furthermore, the axis through the center of mass of an object is also valid as a fixed point for angular momentum, even if it is accelerating (and non-inertial). The conservation law and $\vec{M} = \dot{\vec{L}}$ also apply in this case.

Moment of Inertia

The moment of inertia I is the rotational motion equivalent of mass and a property of an object together with the axis around which it's spinning - a measure of how much torque is needed to change an objects angular momentum:

$$L = I\omega$$

$$M = I\dot{\omega}$$

For a **rigid** body with density dm , the total angular momentum with respect to the axis of rotation can be calculated using $\vec{v} = \vec{r} \times \vec{\omega}$

$$\begin{aligned} L &= \iiint \vec{r} \times \vec{v} dm = \iiint r \vec{e}_r \times (r \vec{e}_r \times \omega \vec{e}_\omega) dm \\ &= \omega \iiint r^2 \vec{e}_r \times \vec{e}_\omega dm = \omega \iiint r^2 \vec{e}_\omega dm \\ &= \omega I \end{aligned}$$

Therefore the moment of inertia relative to an axis can be calculated using:

$$I = \iiint r^2 dm$$

Taylor's "Example 3.4 Sliding and Spinning Dumbbell" is a good summary of these ideas.

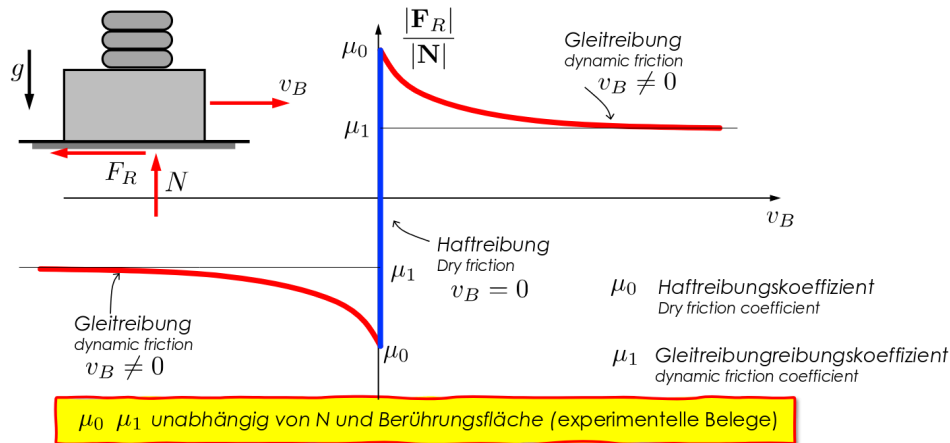
Friction

Friction is the phenomenon of irregularities in the surfaces of materials resisting their relative motion. It is completely a braking force, acting in the opposite direction of the velocity and converting energy away from the system into heat. Friction is responsible for 20% of our energy expenditure!

However it is more often than not a useful force, which is for example needed to exert lift in airplane wings and to keep vehicles from crashing as they take a turn.

Dry Friction

Of course, calculating the exact frictional force of each irregularity in a surface is not a good idea. Experiments show that the force needed to accelerate and then maintain an object on a frictional surface at a certain velocity (ie it becomes equal to the current friction) is initially higher than after it is in motion, where it becomes linearly proportional to the current normal force (and therefore the weight of the object).



In practice we can ignore the changing constant of proportionality and simply define two constants:

- μ_0 - Dry static friction (Haftreibung):

$$|F| \leq \mu_0 |N|$$

The force is \leq because it would only be at full effect if the object is on the verge of sliding.

Considering for example a box on a ramp, when the angle of the ramp is 0, no friction is acting on it yet. $|F| = \mu_0 |N|$ when the so called **critical angle** of the ramp is reached. This also helps us define a range for an external, tangential force within which the object remains at rest.

- μ_1 - Dry kinetic friction (Gleitreibung):

$$|F| = \mu_1 |N|$$

This is called Coulomb's Frictional Law. As a vector:

$$\vec{F} = -\mu_1 |N| \frac{\vec{v}}{|\vec{v}|}$$

Here are some typical coefficient values:

Materialkombination	Haftreibungskoeffizient μ_0	Gleitreibungskoeffizient μ_1
Stahl / Stahl	0.1 bis 0.5	0.1 bis 0.4
Stahl / Eis	0.02 bis 0.03	0.014 bis 0.015
Stahl / Teflon	0.04	0.04
Aluminium / Aluminium	1	1
Leder / Metall	0.4 bis 0.6	0.2 bis 0.3
Holz / Metall	0.5 bis 0.65	0.2 bis 0.5
Autoreifen / Strasse (trocken)	0.7 bis 0.9	0.5 bis 0.8
Ski / Schnee	0.1 bis 0.3	0.04 bis 0.2

They depend on both materials as seen above.

Stoke's Law

The drag force on a small sphere with radius r moving through a viscous fluid is given by:

$$\vec{F} = (-6\pi\eta r)\vec{v}$$

Where η is the viscosity of the fluid.

Newton's Law of Gravitation

This is an empirical approximation of the gravitational force acting on point mass m_1 caused by point mass m_2 :

$$\vec{F}_{12} = \frac{Gm_1m_2\vec{e}_{r_{12}}}{|\vec{r}_{12}|^2}$$

Unlike Coulomb's law, it is always attractive.

If this is the only force acting on m_1 , its acceleration is given by:

$$\begin{aligned} m_1\ddot{\vec{r}} &= \frac{Gm_1m_2\vec{e}_{r_{12}}}{|\vec{r}_{12}|^2} \\ \ddot{\vec{r}} &= \frac{Gm_2\vec{e}_{r_{12}}}{|\vec{r}_{12}|^2} \end{aligned}$$

On the surface of the Earth, this is a constant for all objects given the name $g \approx 9.81$. m_1 exerts an equal and opposite on m_2 too, but in the case of objects on the Earth, the force is too small to cause any measurable acceleration of the Earth.

Normal Force

Consider a particle on the surface of the Earth. Although the gravitational force always acts in the direction of the Earth's center, the resulting normal force is only equal and opposite to the excess weight remaining after the "fake" centripetal force required to keep it in contact with the surface has been taken care of.

Since the particle is performing circular motion around the cross-section of the Earth at its current latitude, the centripetal force is given by:

$$F = m\omega^2 r \cos(\varphi)$$

Where r is the radius of the sphere and $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is the latitude the particle is at, where $\varphi = 0$ is at the equator. **Important:** It acts towards the axis of rotation, not the center of the sphere.

In the case of the Earth, the centripetal acceleration (depending on the latitude) required is less than g , therefore objects stay on the surface and the resulting normal force is given by:

$$\begin{aligned} F_{\text{Normal}} &= -(F_{\text{Weight}} - F_{\text{Centripetal}}) \\ &= -m \left(\begin{pmatrix} g \cos(r) \\ g \sin(r) \end{pmatrix} - \begin{pmatrix} \omega^2 r \cos(\varphi) \\ 0 \end{pmatrix} \right) \\ F_{\text{Normal}} &= m \sqrt{(g \cos(r) - \omega^2 r \cos(\varphi))^2 + (g \sin(r))^2} \\ &= -m \sqrt{g^2 - \omega^2 r \cos \varphi (2g \cos r - \omega^2 r \cos \varphi)} \end{aligned}$$

This confirms as expected that the normal force is at its greatest at the North / South poles:

$$\begin{cases} F_N < mg & \text{if } |\varphi| < \frac{\pi}{2} \\ F_N = mg & \text{if } \varphi = \frac{\pi}{2} \end{cases}$$

Energy

Consider a particle of mass m traveling at speed v with initial kinetic energy T :

$$T = \frac{1}{2}mv^2$$

$$\frac{dT}{dt} = mav = Fv$$

$$dT = Fvdt = Fds$$

This proves $\Delta T = Fd$ the **Work-KE theorem**, as a line integral:

$$\Delta T = \int_C \mathbf{F} \cdot d\mathbf{s}$$

Where F is the resultant force acting on the particle.

Conservative Forces

The 4 fundamental forces are conservative due to the conservation of energy, meaning that they satisfy the following criteria:

- They depend only on relative positions, not on velocity, time or any other variable.
- The work done by the force as test particle moves between two points is the same, regardless which path it takes

Macroscopic forces which exhibit this behavior can also be defined as conservative, for example the elastic force of a spring.

The total **mechanical energy** of a particle that only interacts with conservative forces is conserved and constant:

$$E = T + U(\vec{r})$$

Where T is the convention for kinetic energy and $U(\vec{r})$ is the potential energy.

Forces such as friction are not fundamental and are called **effective forces**; approximations of the huge number of underlying fundamental forces, which do not account for all work done (some of it is lost to heat), for example moving a table between two points on the floor requires more work if the path chosen is longer.

The second condition can be checked using curl (rather than comparing every possible path):

$$\nabla \times \mathbf{F} = 0 \text{ everywhere} \Leftrightarrow \text{The work done is path independent}$$

Proof:

1. If the work done is path independent, the work done over any closed loop is 0. Consider a path from the reference potential to a point, **any** path back to the reference potential does the same work on the particle and hence:

$$\oint \mathbf{F} \cdot d\mathbf{s} = 0$$

2. Stoke's Theorem states:

$$\oint \mathbf{F} \cdot d\mathbf{s} = \iint (\nabla \cdot \mathbf{F}) \cdot d\mathbf{A}$$

If $\nabla \times \mathbf{F} = 0$ everywhere then the surface integral and subsequently the closed line integral are also 0, proving that work done is path independent ■.

For example in the case of the electrostatic / gravitational field around a point charge:

$$\nabla \times \frac{k\vec{r}}{|\vec{r}|^3} = \left| \begin{pmatrix} e_x & e_y & e_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{pmatrix} \right| = \begin{pmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ -\frac{\partial F_z}{\partial x} + \frac{\partial F_x}{\partial z} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{pmatrix}$$

All of the partial derivatives evaluate to 0 (see Taylor's example 4.5 for calculations), showing that those forces are conservative and that the potential only depends on the radial distance (since the work done is path independent) + the negative potential gradient is equal to the force.

Potential Energy

Let r_0 be the point where potential is set as 0. A **conservative force's** potential energy at a point r is defined as:

$$U(\vec{r}) := -W(\vec{r}_0, \vec{r}) = - \int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}') \cdot d\vec{r}'$$

Alternatively the **change** in potential energy can be calculated between two points.

Intuitively, change in potential energy is the change in energy stored as the force does work on a particle from one point to another. If the conservative force acts in the same direction as the particle's movement, it accelerates the particle and increases its kinetic energy, reducing that force's potential energy store, thus the negative change in potential. If the particle moves against the conservative force, it slows down and the lost kinetic energy is stored as potential $\Delta U > 0$.

Furthermore, the negative sign ensures mechanical energy (of an object only acted upon by conservative forces) is constant:

$$\Delta T = \int_C F \cdot d\vec{s} = -\Delta U$$

$$\Delta T + \Delta U = 0 = \Delta E$$

The total mechanical energy E stays constant and energy is conserved. The maximum possible kinetic energy is when the potential energies are at their minimum:

$$E = T + \sum U_i$$

If there are non-conservative forces acting on the object, the mechanical energy (usually) stolen / introduced by these forces is:

$$\Delta E = W_{nc}$$

Gradient of Potential Energy

Solving the equation $U = - \int_C F \cdot d\vec{r}$, a **conserved force** can be derived from the gradient of its potential energy at a point:

$$\vec{F} = -\nabla U(r) = - \begin{pmatrix} \frac{\partial U(r)}{\partial x} \\ \frac{\partial U(r)}{\partial y} \\ \frac{\partial U(r)}{\partial z} \end{pmatrix}$$

Because the total potential energy is simply the sum of potential energies, this also applies to the resultant force **only if** work is done completely by conserved forces:

$$\overrightarrow{F_{\text{Resultant}}} = -\nabla U(r)_{\text{Total}}$$

In the example of a box on a frictionless slope, the normal reaction force is always perpendicular to the direction of movement, hence work is only done by **a component** of the gravitational force acting on the box - the change in potential graph of a roller coaster shows its changing height and negative gradient represents the resultant force at each point (down the slope).

IMPORTANT: This is not the same as the graph of its gravitational potential, which remains largely constant as the changes in height are relatively small, rather the graph of work done by the gravitational force (which depends on the constraints of the track). TODO: Double check

Furthermore, when considering multiple isolated particles which exert conservative forces on one another and Newton's 3rd Law, we can derive the forces acting on each of them from a **single** potential energy function:

$$F_{12} = -\nabla_1 U$$

$$F_{21} = -\nabla_2 U$$

Where ∇_1 is the gradient with respect to the current coordinates of particle 1.

This can be generalized to a system of n particles:

$$T = \sum_i^n \frac{1}{2} m_i v_i^2$$

$$U = \sum_j^n \sum_{i>j}^n U_{ji} + \sum_i^n U_i^{\text{ext}}$$

$$F_{\text{Resultant}} = -\nabla_i U$$

One-Dimensional Systems

A force acting on a particle whose position is constrained to 1 degree of freedom must only satisfy the first condition (the force only depends on position) to be classified as a conservative force and the second condition (work done independent of path) is guaranteed to be true. This is because the work done by any combinations of going forwards and backwards along the path to the 2nd point cancel each other out.

Both minima and maxima represent points where the resultant force is 0, however minima are more stable as a small deviation pushes the object back to minimum potential. Points of inflection (or saddle points in 2D) are considered as unstable in one direction.

If an object starts in between two hills and its total mechanical energy is lower than the peak of one of the hills, it's "trapped in the energy well".

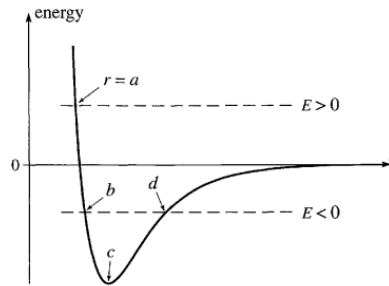


Figure 4.12 The potential energy for a typical diatomic molecule such as HCl, plotted as a function of the distance r between the two atoms. If $E > 0$, the two atoms cannot approach closer than the turning point $r = a$, but they can move apart to infinity. If $E < 0$, they are trapped between the turning points at b and d and form a bound molecule. The equilibrium separation is $r = c$.

Equation of Motion

Given a function of the potential energy $U(x)$ and the initial mechanical energy, we can solve for the equation of motion $x(t)$:

$$T = E - U(x) = \frac{1}{2}m\dot{x}^2$$

$$\dot{x} = \sqrt{\frac{2(E - U(x))}{m}}$$

Using separation of variables TODO: Review after analysis 2:

$$t = \int_{x_0}^x \frac{1}{\dot{x}(x')} dx' = \sqrt{\frac{m}{2}} \int_{x_0}^x \frac{1}{\sqrt{E - U(x')}} dx'$$

Solving this integral (may only be possible numerically) gives us $t(x)$, which can be rearranged to $x(t)$.

Elastic Collisions

An elastic collision is a collision between two particles that interact through a conservative force that tends towards 0 as their separation increases, for example between two protons approaching each other from ∞ . The steady state initial and final potential can be taken as 0, meaning that:

$$T_{\text{Init}} = T_{\text{Final}}$$

In reality this is only possible at low energies, as relativity tells us that new particles may be created.

An useful property of elastic collisions between two particles of the same mass is that they exit perpendicular to each other.

Rigid Bodies

Within rigid bodies, the distances between particles is constant and they exert **circular** electrostatic forces on each other (meaning they only dependent on the constant distances between each other). Hence, the total **internal** potential energy of the body stays constant and can be ignored - energy is only exchanged between U^{ext} and T^{Total} where:

$$T^{\text{Total}} = \frac{1}{2}mv_{\text{CM}}^2 + \frac{1}{2}I\omega^2$$

TODO: Rotational kinetic energy

Oscillations

Excellent simulator: <https://www.falstad.com/harmonicosc/>

Consider a system where some particle has an oscillating potential energy, for example a swinging pendulum. Let $U(x)$ be a function of the particle's potential energy for a given displacement x . The first 3 terms of a Taylor Series serves as a good approximation provided that the displacement is small:

$$U(x) \approx U(0) + U'(0)x + \frac{1}{2}U''(0)x^2 + \dots$$

- $U(0) = 0$ - The equilibrium can be defined as the ground potential
- $U'(0)$ - This is equal to the magnitude of the resultant force (potential well), which is 0 at the equilibrium position $x = 0$ by definition.
- $U''(0) > 0$ because the equilibrium at zero displacement is stable and a local minima

This leads to the familiar parabolic potential energy of an oscillating system:

$$\begin{aligned} U(x) &\approx \frac{1}{2}U''(0)x^2 \\ &\approx \frac{1}{2}kx^2 \end{aligned}$$

Where k is a positive constant specific to this system, a measure of how "stable" the equilibrium position is.

Simple Harmonic Motion

Recall $F_{\text{Resultant}} = -\nabla U(x)$ for a body upon which only conservative forces do work. In a one dimensional oscillating system:

$$F = -\frac{dU(x)}{dx} = -kx$$

Giving us Hooke's law.

Such motion is called simple, because the resultant force is directly proportional to displacement, and harmonic because frequency and amplitude (initial displacement) of oscillations are independent (allowing one to make musical harmonies).

Applying Newton's 2nd law:

$$\begin{aligned} \ddot{x} &= -\frac{k}{m}x \\ &= -\omega^2 x \end{aligned}$$

Where $\omega = \sqrt{\frac{k}{m}}$, which ends up being the angular velocity. x is not necessarily a linear displacement and can represent a displacement angle, for example in a swinging pendulum.

Two possible independent solutions to this 2nd order linear differential equation are:

$$x(t) = e^{\pm i\omega t}$$

Furthermore, because differentiation is a linear operation, any linear combination of these solutions is also a solution (it can all be expressed as matrix multiplication):

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$

Where the coefficients must be chosen so that $x(t)$ is real.

Expanding Euler's formula gives us more insight into the underlying functions:

$$\begin{aligned}x(t) &= C_1(\cos(\omega t) + i \sin(\omega t)) + C_2(\cos(\omega t) - i \sin(\omega t)) \\&= (C_1 + C_2) \cos(\omega t) + (C_1 - C_2)i \sin(\omega t) \\&= B_1 \cos(\omega t) + B_2 \sin(\omega t)\end{aligned}$$

By inspecting the behavior of the system we can determine the meaning of the coefficients:

$$\begin{aligned}\dot{x} &= -B_1\omega \sin(\omega t) + B_2\omega \cos(\omega t) \\x(0) &= B_1 = \text{Initial displacement} \\\dot{x}(0) &= B_2\omega = \text{Initial velocity}\end{aligned}$$

Therefore an oscillation starting at initial velocity v_0 at displacement x_0 :

$$x(t) = x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t)$$

Where ω is the angular frequency of the oscillation and hence:

$$\begin{aligned}\omega &= \frac{2\pi}{T} \\T &= \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{k}{m}}}\end{aligned}$$

A more compact solution can be expressed using a phase shift, derived through the angle sum trigonometric identity:

$$\begin{aligned}x(t) &= A \cos(\omega t - \varphi) \\\dot{x}(t) &= -A\omega \sin(\omega t - \varphi)\end{aligned}$$

Where A is the maximum displacement and φ is the initial angular displacement $\frac{\pi}{2} \cdot \frac{x}{A}$ when $t = 0$.

Energy

By substituting either of these solutions into the formulae for potential and kinetic energy:

$$\begin{aligned}E &= T + U = \frac{1}{2}(m\dot{x}^2 + kx^2) \\&= \frac{1}{2}kA^2(\sin^2(\omega t - \varphi) + \cos^2(\omega t - \varphi))\end{aligned}$$

This reveals the following intuitive properties:

- The kinetic and potential energy are in antiphase, the velocity is at its greatest at the equilibrium position and 0 at the maximum displacement.
- The total mechanical energy is constant $= \frac{1}{2}kA^2$ (taking the equilibrium potential as 0)

2D Oscillations

Isotropic oscillation is the simplest form of 2D oscillation, where the restoring has the **same constant of proportionality** (and hence the same angular frequency) in all directions:

$$\vec{F} = -k\vec{r}$$

Examples of such an oscillation include:

- A ball attached to identical springs around the origin

- A ball rolling around the bottom of a spherical bowl
- An atom vibrating around its equilibrium position in a symmetric crystal

By considering each component separately we arrive at a set of parametric equations, for example in 2D:

$$x(t) = A_x \cos(\omega t - \delta_x)$$

$$y(t) = A_y \cos(\omega t - \delta_y)$$

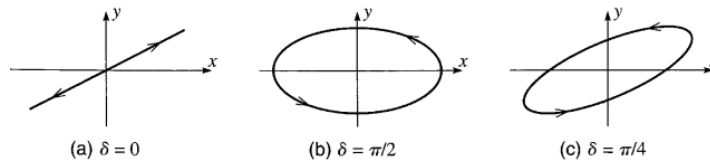


Figure 5.8 Motion of a two-dimensional isotropic oscillator as given by (5.20). **(a)** If $\delta = 0$, then x and y execute simple harmonic motion in step, and the point (x, y) moves back and forth along a slanting line as shown. **(b)** If $\delta = \pi/2$, then (x, y) moves around an ellipse with axes along the x and y axes. **(c)** In general (for example, $\delta = \pi/4$), the point (x, y) moves around a slanted ellipse as shown.

An **anisotropic oscillation** may have different constants of proportionality in different directions and hence different angular frequencies. This can give rise to cool paths called **Lissajous curves**:

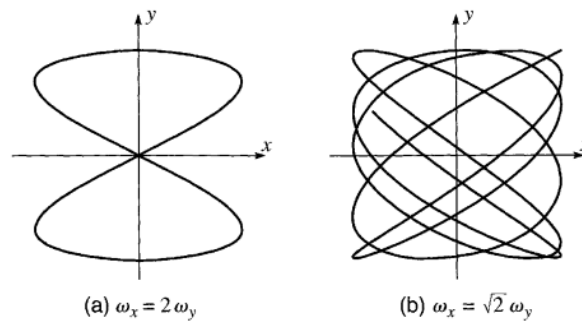


Figure 5.9 **(a)** One possible path for an anisotropic oscillator with $\omega_x = 2$ and $\omega_y = 1$. You can see that x goes back and forth twice in the time that y does so once, and the motion then repeats itself exactly. **(b)** A path for the case $\omega_x = \sqrt{2}$ and $\omega_y = 1$ from $t = 0$ to $t = 24$. In this case the path never repeats itself, although, if we wait long enough, it will come arbitrarily close to any point in the rectangle bounded by $x = \pm A_x$ and $y = \pm A_y$.

Damped Oscillations

Consider a damping force $F = -b\dot{x}$ proportional to velocity (for example air resistance). Newton's 2nd law reads:

$$m\ddot{x} + b\dot{x} + kx = 0$$

The same differential equation also applies to the voltages in an LRC circuit:

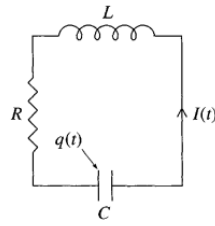


Figure 5.10 An LRC circuit.

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0$$

This can be rewritten as:

$$\ddot{x} + 2\beta\dot{x} + \omega_n^2 x = 0$$

Where $2\beta = \frac{b}{m}$ and $\omega_n = \sqrt{\frac{k}{m}}$.

- β is named the damping constant which characterizes the strength of damping in this system.
- ω_n is named the natural frequency, at which the system would oscillate if $\beta = 0$

Solving this 2nd order linear ODE:

$$\begin{aligned} x(t) &= e^{rt} \\ \dot{x}(t) &= re^{rt} \\ \ddot{x}(t) &= r^2 e^{rt} \\ r^2 e^{rt} + 2\beta r e^{rt} + \omega_n^2 e^{rt} &= 0 \\ \therefore r^2 + 2\beta r + \omega_n^2 &= 0 \\ r &= -\beta \pm \sqrt{\beta^2 - \omega_n^2} \\ x(t) &= C_1 e^{-\beta t + \sqrt{\beta^2 - \omega_n^2} t} + C_2 e^{-\beta t - \sqrt{\beta^2 - \omega_n^2} t} \\ &= e^{-\beta t} \left(C_1 e^{\sqrt{\beta^2 - \omega_n^2} t} + C_2 e^{-\sqrt{\beta^2 - \omega_n^2} t} \right) \end{aligned}$$

Whose validity can be checked by setting $\beta = 0$, giving us the solutions to undamped simple harmonic motion. We can observe that the amplitude decreases exponentially depending on β

Weak Damping

In the case of $\beta < \omega_n$, the constant $\sqrt{\beta^2 - \omega_n^2}$ is complex and can be rewritten as $i\sqrt{\omega_n^2 - \beta^2} = i\omega'$ giving a very similar equation to undamped oscillation:

$$x(t) = e^{-\beta t} (C_1 e^{i\omega' t} + C_2 e^{-i\omega' t})$$

$$\text{Ensuring a real output:} = A e^{-\beta t} \cos(\omega' t - \varphi)$$

This reveals:

- The amplitude exponentially tends towards 0
- The frequency of oscillation is constant throughout and is slightly lower than the natural frequency ω_n

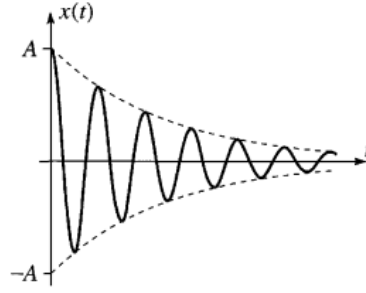


Figure 5.11 Underdamped oscillations can be thought of as simple harmonic oscillations with an exponentially decreasing amplitude $Ae^{-\beta t}$. The dashed curves are the envelopes, $\pm Ae^{-\beta t}$.

Strong Damping

If $\beta > \omega_n$, the constant $\sqrt{\beta^2 - \omega_n^2}$ is real and $e^{\omega' t}$, $\omega' \in \mathbb{R}$ is simply exponential increase, there is no more oscillation. The displacement time graph looks something like this:

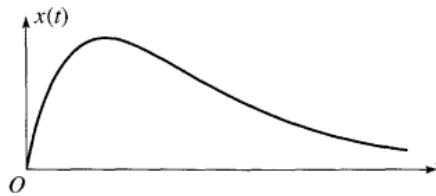


Figure 5.12 Overdamped motion in which the oscillator is kicked from the origin at $t = 0$. It moves out to a maximum displacement and then moves back toward O asymptotically as $t \rightarrow \infty$.

Critical Damping

When $\beta = \omega_n$, we get the following solutions:

$$\begin{aligned} x(t) &= C_1 e^{-\beta t} + C_2 t e^{-\beta t} \\ &= e^{-\beta t} (C_1 + C_2 t) \end{aligned}$$

TODO: Is it even possible to find the second solution methodically?

An interesting property of damped oscillations is how quickly they decay. The **decay parameter** is the factor of exponential decay in the equation of motion, here are the decay parameters of the different types of damping and a plot:

damping	β	decay parameter
none	$\beta = 0$	0
under	$\beta < \omega_0$	β
critical	$\beta = \omega_0$	β
over	$\beta > \omega_0$	$\beta - \sqrt{\beta^2 - \omega_0^2}$

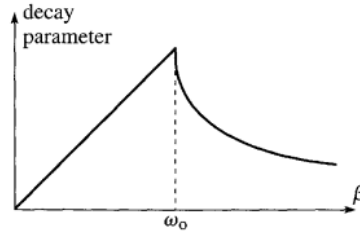


Figure 5.13 The decay parameter for damped oscillations as a function of the damping constant β . The decay parameter is biggest, and the motion dies out most quickly, for critical damping, with $\beta = \omega_0$.

The greatest decay parameter arises under critical damping, meaning that it stops oscillation in the shortest possible time, useful for applications like vehicle suspension. The intuition behind this is that a damping force with $\beta > \omega_n$ will slow the movement down so much that it actually takes longer to reach equilibrium in the first place (for example an oscillator in an overly viscous fluid).

Driven-Damped Oscillations

To keep damped oscillations oscillating, a driving force must be introduced to add energy back into the system. This can be written as a non-homogeneous ODE ($F(t)$ does not depend on x):

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

Which can be tidied up as before to:

$$\ddot{x} + 2\beta\dot{x} + \omega_n^2 x = f(t)$$

Where $f(t) = \frac{F(t)}{m}$.

Let us assume the case that the driving force is a sinusoidal function of time (of course in reality is not always a perfect sine wave, however any periodic driving force can be assembled from sinusoids thanks to Fourier transforms):

$$f(t) = f_0 \cos(\omega_d t)$$

We can find a solution to this non-homogeneous ODE thanks to the fact that its differential operator $D(f)$ is linear. We already know the solutions if it were homogeneous $D(x_h) = 0$ where x_h are the solutions to damped oscillation, and can find a **particular solution** $D(x_p) = f(t)$ by making the RHS complex with the help of Euler's formula, finding a complex solution and taking its real part (See Taylor 5.5):

$$A = \frac{f_0}{\sqrt{(\omega_n^2 - \omega_d^2)^2 + 4\beta^2\omega_d^2}}, \varphi = \arctan\left(\frac{2\beta\omega_d}{\omega_n^2 - \omega_d^2}\right)$$

$$x_p(t) = A \cos(\omega_d t - \varphi)$$

We have found a particular solution, therefore applying linearity:

$$D(x_h + x_p) = D(x_h) + D(x_p) = 0 + f(t) = f(t)$$

$x = x_h + x_p$ is also a solution. Furthermore, we know that a second order linear differential operator must have 2 dimensions in its null space, which x_h already spans. Therefore $x_h + x_p$ contains every possible solution, where x_p can be thought of as an adjustment term to fix the basis when $f(t)$ is introduced into the linear operator.

Finally:

$$x(t) = A \cos(\omega_d t - \varphi) + \left[e^{-\beta t} \left(C_1 e^{\sqrt{\beta^2 - \omega_n^2} t} + C_2 e^{-\sqrt{\beta^2 - \omega_n^2} t} \right) \right]$$

Where ω_n is the natural angular frequency (no damping or oscillation) and ω_d is the driving force angular frequency.

The terms inside of the square brackets are called **transients** because they decay over time and play no role in the steady state of the system.

This leads to the following insights as expected:

- The steady amplitude of the system is given by A . It is directly proportional to the driving force $m f_0$.
- The steady angular frequency is that of the driving force ω_d .

These steady state conditions are the same regardless of initial displacement or velocity.

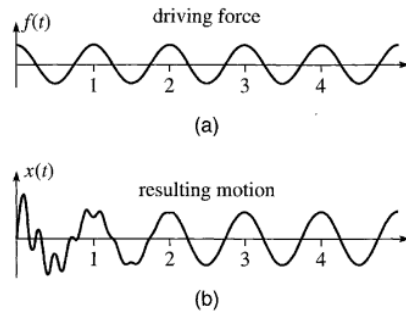


Figure 5.15 The response of a damped, linear oscillator to a sinusoidal driving force, with the time t shown in units of the drive period. **(a)** The driving force is a pure cosine as a function of time. **(b)** The resulting motion for the initial conditions $x_0 = v_0 = 0$. For the first two or three drive cycles, the transient motion is clearly visible, but after that only the long-term motion remains, oscillating sinusoidally at exactly the drive frequency. As explained in the text, the sinusoidal motion after $t \approx 3$ is called an *attractor*.

Resonance

As seen above, the amplitude of driven oscillations depends on several variables:

$$A = \frac{f_0}{\sqrt{(\omega_n^2 - \omega_d^2)^2 + 4\beta^2\omega_d^2}}$$

It is greatest when $\omega_n = \omega_d$ (the natural and driving frequency are the same) and the damping coefficient β is low. Therefore ω_n is also called the **resonant frequency**. If there is no damping and the oscillator is being driven at the resonant frequency, $A \rightarrow \infty$.

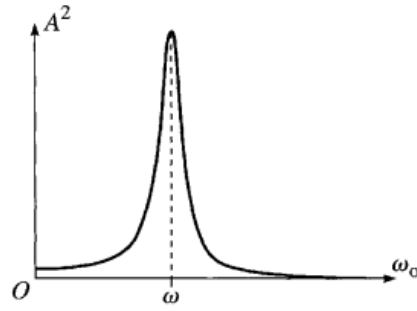


Figure 5.16 The amplitude squared, A^2 , of a driven oscillator, shown as a function of the natural frequency ω_0 , with the driving frequency ω fixed. The response is dramatically largest when ω_0 and ω are close.

An application of this is when tuning a radio, the resonant frequency of an LRC circuit is adjusted so that it is close to the desired radio station's signal and that specific induced signal has the largest amplitude.

Q Factor

Reducing the damping constant β not only increases the amplitude but also makes the resonance peak narrower, focusing the oscillator on a narrow range of driving frequencies:

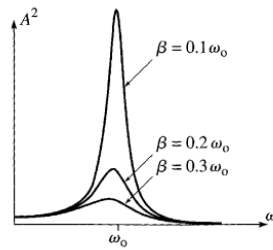


Figure 5.17 The amplitude for driven oscillations as a function of the driving frequency ω for three different values of the damping constant β . Note how as β decreases the resonance peak gets higher and sharper.

The **Q Factor** is a measure of how narrow the resonance for an oscillator is:

$$Q = \frac{\omega_n}{2\beta}$$

Where the resonance width is inversely proportional to β , hence a high Q value indicates a narrow width. A crystal oscillator should have a large Q value as clocks depend on its resonance frequency being constant.

Phase Shift

Since both the driving force and steady-state displacement were modeled using \cos functions, with the driving force starting at $t = 0$, the φ variable indicates the phase shift between driving force and oscillator displacement:

$$\varphi = \arctan\left(\frac{2\beta\omega_d}{\omega_n^2 - \omega_d^2}\right)$$

We can observe:

- $\omega_d \ll \omega_n \Rightarrow \varphi \rightarrow 0$
- $\omega_d = \omega_n \Rightarrow \varphi \rightarrow \frac{\pi}{2}$
- $\omega_d \gg \omega_n \Rightarrow \varphi \rightarrow \frac{\pi}{2} \rightarrow \pi$ - Purely mathematically, $\varphi = -\frac{\pi}{2} \rightarrow 0$ as soon as the driving frequency exceed the natural frequency, however, this is due to the restricted domain of the inverse tan function. To correct for this and keep the phase-shift continuous, $\varphi + \pi$ is added as soon as it becomes negative.

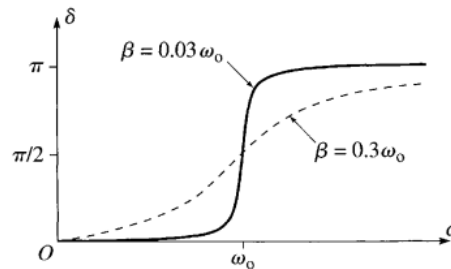


Figure 5.19 The phase shift δ increases from 0 through $\pi/2$ to π as the driving frequency ω passes through resonance. The narrower the resonance, the more suddenly this increase occurs. The solid curve is for a relatively narrow resonance ($\beta = 0.03\omega_0$ or $Q = 16.7$), and the dashed curve is for a wider resonance ($\beta = 0.3\omega_0$ or $Q = 1.67$).

Fourier Solution

Thanks to the linearity of the driven-damped differential operator $D(x(t))$ and the Fourier series of any periodic driving force, we can find a general solution:

$$D(x_1) = f_1, D(x_2) = f_2, \dots$$

$$D(x_1 + x_2 + \dots) = f_1 + f_2 + \dots = \sum_{i=0}^{\infty} c_i e^{ij\omega_d t}$$

Where c_i is the complex Fourier value for harmonic i , ω_d is the fundamental angular frequency of the driving force and $j = i$ the imaginary unit.

Since we are only interested in the steady-state solution, we can ignore the transient terms and sum the particular solutions of each frequency in the driving force Fourier series:

$$A_i = \frac{|c_i|}{\sqrt{(\omega_n^2 - i^2\omega_d^2)^2 + 4i^2\beta^2\omega_d^2}}, \varphi_i = \arctan\left(\frac{2i\beta\omega_d}{\omega_n^2 - i^2\omega_d^2}\right)$$

$$x(t) = x_1 + x_2 + \dots = \sum_{i=0}^{\infty} A_i \cos(i\omega_d t - \varphi_i)$$

TODO: It's possible to simplify A_i with complex notation?

In practice, conveniently few terms are needed for an accurate approximation of the motion as driving frequencies near the resonant frequency ω_n have by far the largest amplitudes.

Bode Plots & RMS

Bode plot - Frequency response graph, a sweep of driving frequencies can be used to find the resonant frequency and Q factor of an oscillator.

When constructing a Bode plot of an oscillator, the amplitude tells us the peak kinetic energy of an oscillator, however it doesn't account for different waveforms in its motion which may occur in

more complicated oscillators. Therefore the RMS is a more accurate representation of the energy transferred by a driving force:

$$x_{\text{RMS}} = \sqrt{\langle x^2 \rangle}$$

Where $\langle x^2 \rangle$ represents the average squared value over time (to account for negative values), it is sufficient to calculate this over one period:

$$\langle x^2 \rangle = \frac{1}{T} \int_0^T (x(t))^2 dt$$

If the equation of motion is the sum of solutions from the Fourier series of the driving force, **Parseval's Theorem** allows us to significantly reduce the computation in calculating this integral:

$$\langle x^2 \rangle = A_0^2 + \frac{1}{2} \sum_{i=1}^{\infty} A_i^2$$

Here is an example Bode plot of an oscillator, demonstrating the RMS of the resonant frequency and subsequent harmonics $\omega_d = n\omega_n$:

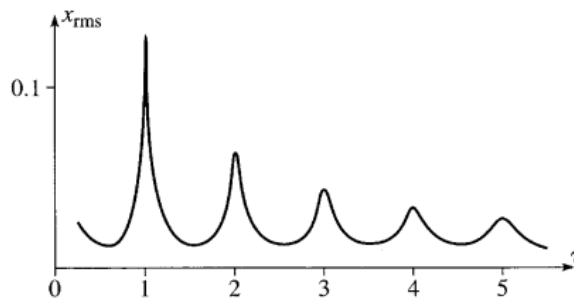


Figure 5.26 The RMS displacement of a linear oscillator, driven by periodic rectangular pulses, as a function of the drive period τ — calculated using the first six terms of the Parseval expression (5.100). The horizontal axis shows τ in units of the natural period τ_0 . When τ is an integral multiple of τ_0 , the response is especially strong.

Waves

Propagating Wave - A disturbance throughout a medium which transfers energy without transferring matter.

Longitudinal - Oscillations confined parallel to the direction of propagation

Transverse - Oscillations perpendicular to the direction of propagation

Mechanical Waves

Mechanical Wave - A sequence of coupled oscillators, energy is transferred through matter as particles exert forces on neighboring particles to transfer an oscillation around their equilibrium positions, such matter can be modeled as a row of particles with springs in between them. Examples: Sound, string vibrations

- Excitation results in a wave traveling in all (possible) directions outwards, however this is often restricted due to the driving forces on one end
- The **(phase) velocity** of a mechanical wave (velocity at which any given phase of the wave propagates through space) is dictated purely by the medium, a string with high tension and low density carries waves faster. By considering a string with tension T and uniform density ρ as a

series of infinitesimal coupled oscillating masses, one can derive the following approximation for its wave propagation velocity using the wave equation:

$$v = f\lambda \approx \sqrt{\frac{T}{\rho}}$$

- Wave shape is dictated by the motion of excitation (if dispersion is negligible)

A wave with negligible dispersion can be described mathematically as a time-dependent displacement field $\xi(\vec{r}, t) : (\mathbb{R}^3, \mathbb{R}) \rightarrow \mathbb{R}^3$. Restricted to one dimension:

$$\xi(x, t) = f(x \pm vt)$$

- This returns the displacement from equilibrium (whether this is a physical displacement vector, electric field or scalar in the case of longitudinal waves)
- A constant time $\xi(x, t = t_0)$ can be chosen to take a “picture” of the current displacements through space and the shape of the wave given by $f(x)$
- A constant position $\xi(x = x_0, t)$ shows the oscillation over time at the specified position
- The \pm represents in which direction the wave propagates over time, $-vt$ means that the wave propagates along the positive x-axis at velocity v

Harmonic Waves

A periodic wave whose shape is sinusoidal is known as a **harmonic wave** and its amplitude is given by the following equation:

$$\begin{aligned}\xi(x, t) &= \xi_0 \sin\left(\frac{2\pi}{\lambda}(x \pm vt)\right) \\ &= \xi_0 \sin\left(\omega t \pm 2\pi \frac{x}{\lambda}\right)\end{aligned}$$

- The factor $\frac{2\pi}{\lambda}$ scales the propagation of the wave as a fraction of one full angular period so that the amplitude oscillates at the correct rate for a given speed and wavelength and is sometimes denoted as k , the so-called wave number
- $2\pi \frac{x}{\lambda}$ represents the phase displacement for a given position
- To simplify calculations, Euler’s formula is often used instead:

$$\xi(x, t) = \text{Re}\left(\xi_0 e^{i(\omega t \pm 2\pi \frac{x}{\lambda})}\right)$$

Wave Equation

All waves can be characterized by the following **constraining** partial differential equation, which their amplitude equation must be a solution to:

$$\frac{\partial^2 \xi(x, t)}{\partial t^2} = v^2 \Delta \xi(x, t)$$

Where Δ represents the so-called **Laplace operator** (∇ operator applied twice):

$$\Delta := \nabla^2 = \begin{pmatrix} \frac{\partial^2}{\partial x^2} \\ \frac{\partial^2}{\partial y^2} \\ \frac{\partial^2}{\partial z^2} \end{pmatrix}.$$

This is known as the wave equation as its solutions contain all possible waves.

Intuitively, this states that the acceleration in the direction of displacement $\frac{\partial^2 \xi(x,t)}{\partial t^2}$ is directly proportional to the rate of change of gradient of the wave shape at that point $\Delta \xi(x,t)$.

At a peak, the Laplace operator is at its greatest absolute value, meaning that the acceleration towards equilibrium is also at its maximum, and since v^2 is the constant of proportionality, a faster propagating wave clearly needs faster oscillating displacement to transfer wave-fronts quicker.

The general one-dimensional solution is:

$$\xi(x,t) = f(x - vt) + g(x + vt)$$

Which represents two arbitrary waveforms f and g propagating in opposite directions.

Example: Waves Through Solids

Here is an example of a wave satisfying the wave equation.

Consider a solid cuboid with length l and cross sectional area A . Modeling it as a continuous material, a stress σ leads to strain ε :

$$\sigma = \frac{\overrightarrow{F}_\perp}{A}$$

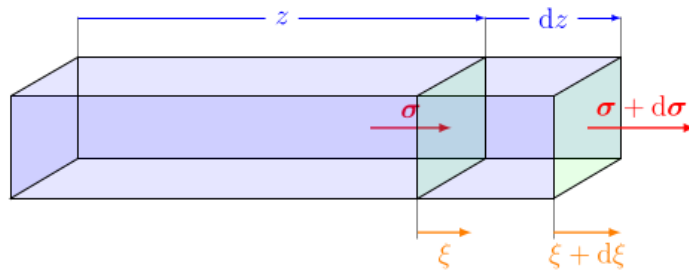
$$\varepsilon = \frac{\Delta l}{l}$$

In the elastic region, these quantities are linearly proportional to one another, related by Young's modulus E :

$$E = \frac{\sigma}{\varepsilon}$$

Furthermore, the material returns to its original shape after a stress within this range is no longer applied, allowing longitudinal (and sometimes transverse) waves to propagate throughout it.

Consider the cuboid positioned as follows along the z -axis:



We analyze a longitudinal wave between two infinitesimally small cross-sections at z and $z + dz$ at a specific point in time, with the goal of expressing the derivatives involved in the wave equation. As the wave travels through the cuboid, a changing displacement is present between the two cross-sections:

$$\xi(z) = \xi_0$$

$$\xi(z + dz) = \xi_0 + \frac{\partial \xi}{\partial z} dz$$

The change in displacement between these two points can be used to calculate the strain and therefore the stress exerted on the first cross-section with $\xi(z) = \xi_0$ due to Hooke's Law:

$$\varepsilon = \frac{\sigma}{E} = \xi + \frac{\partial \xi}{\partial z} dz - \xi = \frac{\partial \xi}{\partial z} dz$$

$$\sigma = E \frac{\Delta l}{L} = E \frac{\frac{\partial \xi}{\partial z} dz}{dz} = E \frac{\partial \xi}{\partial z}$$

Where E is the Young's Modulus of the material.

The tension force acting on the entire first cross-section is therefore:

$$F = A\sigma$$

Furthermore, there is a tension force acting on the right face; σ since tension forces act throughout the whole continua as well as an additional $d\sigma$ due to its difference in displacement:

$$F = A(\sigma + d\sigma) = A\left(\sigma + \frac{\partial \sigma}{\partial z} dz\right) = A\left(\sigma + E \frac{\partial^2 \xi}{\partial z^2} dz\right)$$

Considering the infinitesimal volume element and applying Newton's 2nd Law:

$$F = ma = A(\sigma + d\sigma - \sigma) = AE \frac{\partial^2 \xi}{\partial z^2} dz$$

The mass can be written in terms of density: $m = dV\rho = Adz\rho$, therefore:

$$Adz\rho a = AE \frac{\partial^2 \xi}{\partial z^2} dz$$

$$a = \frac{E}{\rho} \frac{\partial^2 \xi}{\partial z^2}$$

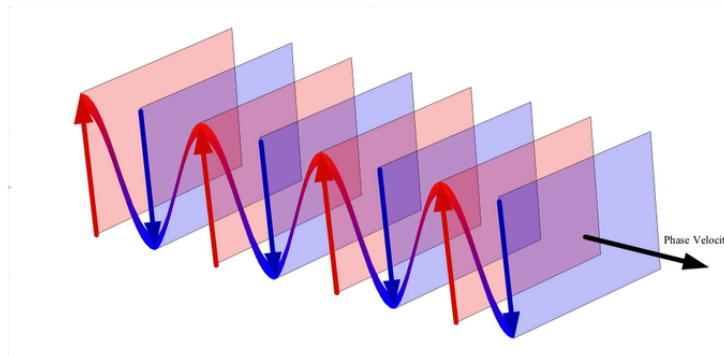
The acceleration at z is equivalent to the rate of change of displacement ξ as we have only considered the resultant force along the z -axis, hence applying the wave equation:

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 \xi}{\partial z^2}$$

$$v = \sqrt{\frac{E}{\rho}}$$

Plane Wave

This is a special case of the wave equation where wave fronts travel as planes through space. In other words, at planes normal to the direction of propagation, the displacement is equal at all points on the plane:



Since only the progress along the line of propagation is needed, such waves can be expressed mathematically as a function of two scalars, time and progress:

$$\xi : \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$\xi(\vec{r}, t) := f(\vec{r} \cdot \vec{n}, t)$$

Where \vec{n} is a unit vector in the direction of propagation of the wave.

Spherical Waves

A point wave source in space emits spherical waves, which satisfy the wave equation if we use the spherical coordinate Laplace operator:

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2 \sin \theta} \left(\cos \theta \frac{\partial f}{\partial \theta} + \sin \theta \frac{\partial^2 f}{\partial \theta^2} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \varphi^2},$$

Since the displacement only depends on the radius r (it is similar to a planar wave where every point on the same sphere has the same displacement) $\frac{\partial \xi}{\partial \theta} = \frac{\partial \xi}{\partial \varphi} = 0$, so this simplifies to:

$$\frac{\partial^2 \xi}{\partial t^2} = v^2 \Delta \xi$$

$$\frac{\partial^2 \xi}{\partial t^2} = v^2 \left(\frac{\partial^2 \xi}{\partial r^2} + \frac{2}{r} \frac{\partial \xi}{\partial r} \right)$$

Where \vec{e}_r is the radius unit vector at that point in space, always pointing normal to the spheres' surface.

The general solution is:

$$\vec{\xi}(\vec{r}, t) = \frac{\vec{\xi}_0}{r} f(\vec{r} \cdot \vec{e}_r, t)$$

Where \vec{e}_r is the (changing) unit vector normal to the sphere's surface at any point. The $\frac{1}{r}$ amplitude indicates that the displacement decreases further from the point source.

Polarization

Transverse waves exhibit 2 DOF in their amplitude (the only requirement is that they are normal to the direction of propagation, ie no component in that direction eliminating the 3rd DOF), hence why their wave function may return a 3-dimensional vector.

However, this is often confined to 1 DOF with oscillations constantly as a scalar of the same vector normal to propagation; the **transverse** wave is said to be polarized. In this case the 3D wave equation can be reduced to 1D by encapsulating direction in the amplitude, for example a polarized harmonic wave:

$$\vec{\xi}(\vec{r}, t) = \begin{pmatrix} \xi_{0x} \\ \xi_{0y} \\ \xi_{0z} \end{pmatrix} \sin \left(\omega t \pm 2\pi \frac{\vec{r} \cdot \vec{n}}{\lambda} \right)$$

Usually it is convenient to set one of the axes in the direction of propagation, reducing the need for a dot product.

Polarized waves may be superposed, resulting in a circular wave if they are normal to each other, coherent and at a $\frac{\pi}{2}$ phase difference. Non-coherent waves may result in patterns similar to Lissajous curves - mathematically they are very similar to coupled oscillators.

TODO: Understand 45 degree polar. TODO: Quarter wave polarization, relate to spin TODO: Spiraling wave, elliptic etc

LTD: Waves with normal + parallel components in their displacement also exist, see Fresnel Equations, Rayleigh waves

Energy

We can calculate the energy density throughout a Hookean material as a wave passes through it, for example a longitudinal wave passing through a solid rod. The kinetic energy of an infinitesimal section is given by:

$$\begin{aligned} dT &= \frac{1}{2} dm \left(\frac{\partial \xi(\vec{r}, t)}{\partial t} \right)^2 \\ &= \frac{1}{2} \rho dV \left(\frac{\partial \xi(\vec{r}, t)}{\partial t} \right)^2 \\ \frac{dT}{dV}(\vec{r}, t) &= \frac{1}{2} \rho \left(\frac{\partial \xi(\vec{r}, t)}{\partial t} \right)^2 \end{aligned}$$

This maths is questionable

We can also express the elastic potential energy density using Hooke's law:

$$\frac{dU}{dV}(\vec{r}, t) = \frac{1}{2} E \left(\frac{\partial \xi(\vec{r}, t)}{\partial x} \right)^2$$

Therefore the total energy density:

$$\frac{dW}{dV}(\vec{r}, t) = \frac{1}{2} \left(E \left(\frac{\partial \xi(\vec{r}, t)}{\partial x} \right)^2 + \rho \left(\frac{\partial \xi(\vec{r}, t)}{\partial t} \right)^2 \right)$$

For a one dimensional wave $\xi(x, t) = f(x - vt)$, we can simplify the derivatives:

$$\frac{dW}{dV}(x, t) = \frac{1}{2} \left(E \left(\frac{df}{du} \right)^2 + \rho \left(-v \frac{df}{du} \right)^2 \right)$$

Where $u = x - vt$ following from the chain rule. Since $v^2 = \frac{E}{\rho}$ as derived before, this simplifies to:

$$\begin{aligned} \frac{dW}{dV}(x, t) &= \frac{1}{2} \left(E \left(\frac{df}{du} \right)^2 + \rho \frac{E}{\rho} \left(\frac{df}{du} \right)^2 \right) \\ \frac{dW}{dV}(x - vt) &= E \left(\frac{df}{du}(u) \right)^2 = \rho v^2 \left(\frac{df}{du}(u) \right)^2 \end{aligned}$$

WARNING this specifically applies to a single mechanical wave throughout a Hookean material, and that the superposition principle does not apply to wave energy (unlike displacement).

- Although the total energy of the wave is constant, we observe that it depends on the speed of the wave
- The energy at a point in the solid depends on the waveform's progress throughout it. For example, in the case of a harmonic wave the energy is highest when a particle is at the equilibrium position, where the first derivative is at its maximum.
- The total energy is always ≥ 0 due to the \cdot^2
- The energy density of a transverse rope with strain $S = \frac{F}{A}$ and velocity $v^2 = \frac{S}{\rho}$ is nearly identical:

$$\frac{dW}{dV}(x - vt) = S \left(\frac{df}{du}(u) \right)^2 = \rho v^2 \left(\frac{df}{du}(u) \right)^2$$

For example, a harmonic, mechanical, one-dimensional wave has the energy density:

$$\begin{aligned} \xi(x, t) &= A \cos\left(\frac{2\pi}{\lambda}(x - vt)\right) \\ \frac{dW}{dV}(x - vt) &= \rho v^2 \left(-A \frac{2\pi}{\lambda} \sin\left(\frac{2\pi}{\lambda}(x - vt)\right) \right)^2 \\ &= \rho v^2 A^2 \left(\frac{2\pi}{\lambda} \right)^2 \sin^2\left(\frac{2\pi}{\lambda}(x - vt)\right) \\ &= \rho A^2 \omega^2 \sin^2\left(\frac{2\pi}{\lambda}(x - vt)\right) \end{aligned}$$

An interesting property to note is that a single position does not reveal in which direction a periodic wave is flowing through it. The direction is determined based on the movement relative to particles around it; positions just before a peak moving upwards indicate that the peak (and therefore the waves energy) is traveling towards them.

LTD: I would prefer to derive the energy more generally from the wave equation somehow

Intensity

The average energy density over a period of mechanical, periodic wave gives us insight into its independent variables:

$$\left\langle \frac{dW}{dV} \right\rangle = \frac{1}{T} \int_0^T \frac{dW}{dV}(x, t) dt$$

In the case of a mechanical, harmonic wave this amounts to:

$$\begin{aligned} \left\langle \frac{dW}{dV} \right\rangle &= \frac{1}{T} \left[\frac{1}{2} \rho \omega^2 A^2 t - \frac{1}{4} \rho \omega A^2 \sin \left(2 \left(\frac{2\pi}{\lambda} x - \omega t \right) \right) \right]_0^T \\ \omega T &= 2\pi \\ \therefore \left\langle \frac{dW}{dV} \right\rangle &= \frac{1}{2} \rho \omega^2 A^2 \end{aligned}$$

The anti-derivative I have computed is likely to be slightly wrong but the final result is correct.

- As expected the average energy over a period is constant at all points through which the wave propagates; energy is conserved

Because energy density is both position and time dependent (it also satisfies the wave equation), it makes sense to consider its flow as the so-called **energy flux density** vector (or the Poynting vector in the context of EM waves):

$$\begin{aligned} \vec{S} &= \frac{dW}{d\vec{a} dt} \\ [\vec{S}] &= W m^{-2} \end{aligned}$$

Which represents the energy per unit area per unit time in the direction of the wave propagation.

The Euclidean norm of the energy flux density vector is known as the **intensity** of the wave (also position and time dependent). In the case of a harmonic wave, the average intensity over a period is given by:

$$\langle I \rangle = \frac{1}{2} \rho \omega^2 A^2 v$$

Which is once again constant throughout space.

The total power through a finite surface can be integrated and is known as the **energy flux**:

$$\frac{dW}{dt} = \iint_A \vec{S} \cdot d\vec{A}$$

As expected, integrating as above over a sphere surrounding a spherical wave source gives a constant power, regardless of the sphere's radius (conservation of energy):

$$\dot{W} = 2\pi \rho \omega^2 A_0^2 v$$

Where A_0 is the source amplitude (we saw $A = \frac{1}{A_0}$ for a spherical wave).

LTD: I suspect Gauss's law will also be applicable here to any closed surface.

Superposition

If multiple waves are passing through a medium, the displacement at any point is equal to the sum of their displacement functions:

$$\xi(x, t) = \sum_{i=1}^n \xi_i(x, t)$$

The result also satisfies the wave equation, as it is a linear differential operator and hence its solutions are a vector space.

Frames of Reference

Frame of Reference - A coordinate system whose origin and basis are specified in space.

A frame of reference itself can be in motion, for example it may be useful to use a reference with the origin fixed inside a moving car, allowing us to ignore the movement of the car (if its moving at a constant velocity) and focus on movement relative to it, for example a coin being tossed into the air.

Inertial Reference Frame - A reference frame in which objects obey the principle of inertia (Newton's 3rd Law); the frame itself is at rest or moving at a constant velocity in relation to any other inertial reference frame. The Earth's surface is a good approximation of an inertial reference frame which we are accustomed to thinking in terms of, due its very small angular velocity.

Non-Inertial Reference Frame - A frame which is accelerating in some way (for example a frame based on an observer in an accelerating car). Newton's Laws do not hold true in such frames and "fictitious" forces are needed to account for the motion of the frame to accurately apply the laws of motion.

Galilean Transformation

The coordinates of two inertial frames of reference can be transformed between one another, such that time is shared between them. This approximation is accurate when considering systems with velocities significantly slower than the speed of light (non-relativistic).

Transformations between reference frames can also be represented as a matrix allowing easier vector calculations.

Inertial Force

We, the observer, are in the inertial reference frame K . Consider a reference frame K' who's origin is given by position vector $R(t)$ with respect to K and may not necessarily be inertial (might be accelerating) and who's basis vectors remain constant over time.

Given a position vector \vec{r}' of a particle in K' , we can calculate the position, velocity and acceleration relative to the inertial frame K as follows:

$$\vec{r} = \vec{r}' + \vec{R}$$

$$\dot{\vec{r}} = \dot{\vec{r}}' + \dot{\vec{R}}$$

$$\ddot{\vec{r}} = \ddot{\vec{r}}' + \ddot{\vec{R}}$$

Newton's Laws hold true in K , hence:

$$m\ddot{\vec{r}} = F = m(\ddot{\vec{r}}' + \ddot{\vec{R}})$$

If we want to take advantage of Newton's 2nd Law with respect to the non-inertial reference frame K' we can rearrange to:

$$m\ddot{\vec{r}}' = F - m\ddot{\vec{R}}$$

Where $-m\ddot{\vec{R}}$ is the so-called **inertial force** that "acts" on all objects in the accelerating, non rotating frame K' so Newton's Laws remain accurate (it must be included when calculating resultant forces, for example a coin tossed in the air in an accelerating car would only remain at rest / constant velocity if an equal and opposite (to the inertial force) external force acted upon it). An example of this is the force that pushes you back into your seat whilst an airplane is taking off, acting in the opposite direction to acceleration.

Rotating Frames

Euler's Rotation Theorem - Any displacement of a rigid body in 3D such that one of its points stays fixed must be a rotation around an axis of momentary stationary points. Only the angular velocity vector is needed to describe the rotation of a body.

Now consider that the frame K' is also rotating with angular velocity $\vec{\Omega}$ relative to the inertial frame (the direction of $\vec{\Omega}$ is usually bound to one of the principle axes of K' , for example the z-axis) and allows us to calculate the direction of K' 's basis vectors relative to K .

Given a position vector of a particle in K' , for example $\vec{r}' = x'e_{x'} + y'e_{y'} + z'e_{z'}$, we can calculate the position and velocity relative to K as follows:

$$\vec{r} = \vec{R} + \vec{r}'$$

$$\vec{v} = \dot{\vec{R}} + \dot{\vec{r}}' = \dot{\vec{R}} + (\dot{x}'e_{x'} + \dot{y}'e_{y'} + \dot{z}'e_{z'}) + (x'\dot{e}_{x'} + y'\dot{e}_{y'} + z'\dot{e}_{z'})$$

The time-dependent base vectors of K' can be calculated using the cross product with the angular velocity of K' , hence:

$$\dot{\vec{e}}_i = \dot{\vec{R}} + \dot{\vec{r}}' + \vec{\Omega} \times \vec{r}'$$

$$\dot{\vec{v}} = \ddot{\vec{R}} + \ddot{\vec{r}}' + \vec{\Omega} \times \dot{\vec{r}}'$$

The product rule is applied again when differentiating to get the acceleration with respect to K , leading to:

$$\ddot{r} = a = \ddot{R} + \ddot{r}' + 2\Omega \times \dot{r}' + \Omega \times (\Omega \times r')$$

Applying Newton's 2nd Law and rearranging to find the fictitious forces relative to K' :

$$\begin{aligned} F = m\ddot{r} &= m(\ddot{R} + \ddot{r}' + 2\Omega \times \dot{r}' + \Omega \times (\Omega \times r')) \\ m\ddot{r}' &= F - m(\ddot{R} + 2\Omega \times \dot{r}' + \Omega \times (\Omega \times r')) \\ &= F - m\ddot{R} - 2m\Omega \times \dot{r}' - m\Omega \times (\Omega \times r') \end{aligned}$$

The first term $-m\ddot{R}$ is the familiar inertial force, the last two fictitious forces arise due to the rotation of K' and are named:

1. $-2m\Omega \times \dot{r}'$ - Coriolis Force - This is largest when the particle is moving perpendicular to the axis of rotation at a high speed and acts in the direction that pushes the particle in the opposite direction of angular direction to Ω . However, it is negligible when objects are at rest or moving slowly relative to K' , usually we ignore it for objects on Earth moving at $\ll 500ms^{-1}$. An experiment called the **Foucault Pendulum** demonstrates its effect over a long period of time, allowing one to calculate their latitude to a reasonable accuracy. A free-falling object curves slightly to the East as it falls.
2. $-m\Omega \times (\Omega \times r')$ - Centrifugal Force - This acts perpendicularly away from the axis of rotation (rather than from the center of a sphere) and is the force felt when taking a corner in a vehicle. For objects on Earth, it is greatest near the equator as the South-North pole is taken as the axis of rotation and Ω and r' would point in the same direction. When the perpendicular component is subtracted from g , this results in a $\sim 0.034ms^{-2}$ smaller free-fall acceleration for objects at the equator compared to at one of the poles.

These are sometimes rearranged to be positive using the identity $a \times b = -b \times a$. Provided we take into account these 3 **fictitious forces** (one if the frame is not rotating but accelerating), we can continue to utilize Newton's Laws in any non-inertial frames.

Lagrangian Mechanics

TODO: Try Brachistochrone problem for practice

Lagrangian mechanics is a formulation of classical mechanics that uses the calculus of variations. It is independent of choice of coordinates (whose acceleration functions can get very complicated) and constraint forces (which are usually unknown), which greatly simplifies the analysis of certain systems compared to Newtonian mechanics.

The independent variable in Lagrangian problems is time and the dependent, **bijective** coordinates in a system with n degrees of freedom are called **generalized coordinates** and denoted as:

$$q_1(t), q_2(t), \dots, q_n(t)$$

For example a system of N particles free to move in 3D space has $3N$ generalized coordinates.

Unconstrained Motion

Consider a particle moving unconstrained in 3D, subject to a **conservative** resultant force. Its energies are:

$$\begin{aligned} U(r) &= U(x, y, z) \\ T(\dot{r}) &= \frac{1}{2}m\dot{r}^2 \\ E &= T + U \end{aligned}$$

The **Lagrangian** is defined as:

$$\mathcal{L}(r, \dot{r}, t) = T(\dot{r}) - U(r)$$

With the following partial derivatives:

$$\frac{\partial \mathcal{L}}{\partial r} = -\frac{\partial U}{\partial r} = \vec{F}$$
$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial T}{\partial \dot{r}} = m\dot{r} = \vec{p}$$

Applying Newton's 2nd Law we can express the following PDE called the **Lagrange Equation**:

$$F = \dot{p}$$
$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right)}{dt}$$
$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right)}{dt} = 0$$

TODO: Check if partial notation or ∇ is correct here

The remarkable thing is that this PDE in the same form as the Euler-Lagrange equation for finding stationary paths:

$$\frac{\partial f}{\partial y} - \frac{d\left(\frac{\partial f}{\partial y'}\right)}{dx} = 0$$

Where $f(y(x), y'(x), x) = \mathcal{L}(r, \dot{r}, t)$, showing that if Newton's 2nd Law holds true, the actual equation of motion for the particle $r(t)$ is a stationary path, which can be solved using the Euler-Lagrange equation **only** from the energies of the particle in the Lagrangian - no constraint force or acceleration vector was needed.

This is summarized in **Hamilton's Principle**, which states that the actual path taken between two points is such that the action integral is stationary, we do not need to check that it's explicitly a minimum:

$$S = \int_1^2 \mathcal{L} dt$$

TODO: Non-inertial reference frames? TODO: Generalized force and momentum

Electromagnetism

Special Relativity

Spacetime - A 4-dimensional representation of the universe as 3D space + time. Classical mechanics treats time as a uniform quantity throughout the universe with a constant rate of passage. However relativistic effects mean that time passes at different rates in different frames of reference, hence a 4th dimension is introduced.

TODO: Minkowski space

TODO: Michelson-Morsley experiment, Lorentzian electrodynamics and the aether

After the failed Michelson-Morsley experiment, a new theory was needed to explain the speed of light. Special relativity is a theory published in 1905 (On the Electrodynamics of Moving Bodies) by

Albert Einstein, accurately modelling motion through spacetime when gravitational and quantum effects are negligible.

In special relativity, time and distances become relative to the velocity of particles.

Postulate - Something given as true in a theory.

It is based on 2 postulates:

1. The laws of physics are invariant in all inertial frames of reference. This is known as the principle of relativity.
2. The speed of light is the same for all observers, regardless of all motion.

Lorentz Transformation

Two inertial spacetimes can be transformed between one another with the following relationships:¹

$$\begin{aligned}t' &= \gamma \left(t - \frac{vx}{c^2} \right) \\x' &= \gamma (x - vt) \\y' &= y \\z' &= z\end{aligned}$$

Where γ represents the Lorentz factor, which appears in many equations from Classical mechanics adjusted for relativistic effects:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Implications

The second postulate leads to many extremely important implications in spacetime:

- Time dilation - Time passes at a different rate in different frames of reference; slower the closer to the speed of light an inertial frame is translating with respect to another. The intuition for this is that time must be slower to account for the added velocity of an inertial frame itself to the speed of light (which must remain constant across all frames).
- Length contraction - Lengths in a relativistic frame of reference are shorter with comparison to another frame at rest.

Conservation of momentum and energy also lead to the following implication:

- Relativistic mass - Observed mass increases when an object's speed approaches the speed of light:
 $m = \gamma m_0$ where m_0 is the object's rest mass.
- The same can be applied to kinetic energy and momentum.

Bohr's Atomic Model

The model was introduced in 1911 by Niels Bohr to explain the Rydberg formula, Photoelectric effect and the stability of the atom. It does not explain other quantum phenomena such as wave particle duality. Nonetheless, it's much more accessible than Schrodinger's wave function and still remains a useful tool.

TODO: Rydberg Formula

¹The derivation is mathematically very simple and a great exercise in thought. Consider a pulse of light being emitted from a torch in a frame of reference moving at velocity v , the distance travelled by the light and the time taken with respect to each frame of reference can be expressed using the Pythagorean theorem. Due to the 2nd postulate, time and distance are different in both frames so that the speed of light remains constant.

It is based on the following postulates:

1. Electrons orbit the nucleus at shells fixed distances away called energy levels, where each shell contains a fixed number of electrons called the electron configuration.
2. The energy levels are integer multiples of Planck's constant and the possible energies are therefore quantized. Electrons further from the nucleus have more energy (less negative). At these stable energy levels, the acceleration of electrons (as they exhibit circular motion) does not lead to radiation / energy loss. The lowest energy level is called the *ground state* and the highest is called the *valence band*.
3. Electrons may lose / gain energy by jumping between energy levels when emitting / absorbing light, whose frequency is determined by the Planck relation.

TODO: PEP and why electrons stay in their energy levels

TODO: Electron configurations

TODO:

- Thermodynamics and Entropy
- Accelerating charge emitting radiation, Larmor formula, electrons losing energy in orbitals
- Zeeman Effect
- Field & thermionic emission / photoelectric effect
- Wave equation
- De Broglie equation
- Angular momentum
- Spin
- Lagrangian & Hamiltonian mechanics
- Nuclear mass defect