Lineare Algebra

LGS - Lineare Gleichung System - linear system of equations

Vektoren

Lineare kombination - Summe von skalierten Vektoren
Basis - the set of base vectors $e_1 ... e_n$ that define space R^n

Vektoren werden immer als Spalten in diesem Kurs gezeichnet.
Standard vector notation:

$$\boldsymbol{a_1} = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}$$

Matrix multiplication comes from the motivation for an efficient way of representing LGSs.

Geometry of an LGS (Beispiel 1.1.0.8)

An LGS can be viewed geometrically (2D/3D) in multiple different ways:
1. A linear combination of vectors (the columns of the matrix), where we are solving for the set of scalars where the superposition of the vectors is equal to the RHS.
2. Alternatively it can be viewed as a set of line / plane equations, where each row is the normal vector to the plane (unsure if the coefficients are meaningful in ax + by=c)

Superposition:

The solution of a LGS is finding the scalars which make the linear combination of n vectors $\in R_n$ equal to the RHS vector. It is utterly NOT the same as finding the points of intersection with a plane.

In this example, one of the LHS vectors is a linear combination of the other two. This results in the LGS only being able to express vectors in a single plane rather than the entire 3D space (it doesn't contain a 3rd component).

Infinite solutions - if the RHS vector lays in the plane expressed by $\boldsymbol{a_{1-3}}$, any point in the positive / negative direction of the solution vector lays in the plane.
No solutions - the vector does not lay perfectly on the plane, the LHS vectors lack a component (not necessarily base unit vector) in its direction.

Line / Plane equations:

The solution is the point at which the lines / planes represented by the horizontal equations intersect. There are many possible arrangements which we can visualize, especially in $3D$ space.

Unique solution - Common point of intersection of n non parallel lines / planes.
Infinite solutions - Sheaf of planes or if all lines are the same.
No solution - Not all lines / planes meet at a common point, which is more likely the more equations are introduced into the system. Examples: Parallel lines, trianglular prism from 3 planes.

Gaussische Eliminationsverfahren

Ideal method for solving a $m \times n$ system of equations, easy to implement algorithmically and works for all dimensions.

Pivot - element on the diagonal of a matrix that has a non 0 coefficient

Rang / rank - number of non 0 pivots, ie (number of rows - number of Kompatibilitaetsbedingungen)
TODO: Intuitive meaning

Kompatibilitaetsbedingungen - Empty rows at the bottom of the matrix (0 coefficients in one of the equations). If their result is not 0 then there are no solutions for the system. If their result is 0 and the number of equations $\leq$ the number of variables, there are infinite solutions.
Intuition: When thinking of the LGS as superposition, each LHS vector has a 0 component in this dimension, meaning that $\forall x \in \mathbb{R}$ scalar in the Lineare Kombination satisfies the system. Viewing the system with insufficient equations as a system of planes, two planes will intersect along an entire line. In 2D, there would just be a single line, which of course has solutions along its entirety.

Any variables not accounted for due to an all 0 row / no pivot in their column are called *free variables* and can take any real value. TODO: Solidify understanding

Tips:
- Never divide / subtract in Gaussian elimination. Either multiply by $\frac{1}{x}$ or −1. Order is half of the work in maths.
- Switch rows columns carefully **before** carrying out additions.
- When switching rows to get pivots in the correct place, it is usually best to swap a line with zero pivot with the row that has the largest pivot in that place.

U - Upper (Deutsch: R - Rechts) Matrix - Matrix with 0s under the diagonal and any numbers above it
L - Lower Matrix - Matrix with 0s above the diagonal and any numbers below it
Identity Matrix - Matrix with 0s above and below the diagonal, which only contains 1s
Tridiagonal Matrix - Matrix with 3 diagonals, and otherwise 0s everywhere

Protokolmatrix (aka L / Kontrollmatrix) - Identity matrix with the same dimensions as the system matrix, used for tracking the elimination process (TODO: Expand after learning LU decomposition). The scalar by which another row was multiplied $\times -1$ is written in the position of the currently eliminated variable of the row it was added to. **Caution**: when swapping rows, do NOT forget adjusting the Protokolmatrix accordingly, by simply swapping all non diagonal values in the rows.

Homogene LGS - $\boldsymbol{Ax} = 0$ hat eine triviale Loesung $\boldsymbol{x} = 0$, unless it has free variables.

Square Matrices ($m \times n$):
Regular Matrix, Rank = n, has exactly one solution and only the trivial solution when homogenous
Singular Matrix, Rank < n, has infinite / no solutions and has infinite non trivial solutions when homogenous

TODO: Ask professor / TA regarding question from Series 1 - *Wir betrachten im Folgenden ein lineares Gleichungssystem mit m Zeilen, n Spalten und Rang r. Das Gleichungssystem ist nicht für beliebige rechte Seiten lösbar, wenn $r < m$.*
However an overdefined system may still (albeit rarely) have a unique solution if rank = n. https://en.wikipedia.org/wiki/Overdetermined_system#/media/File:3_equations_-5.JPG