Analysis 1

Contents

- *Analysis 1 ITET, F Ziltener* - https://metaphor.ethz.ch/x/2024/hs/401-0231-10L/Ziltener_Notizen_Analysis_1_ITET_RW.pdf
- *Analysis für Informatik, M Struwe* - https://people.math.ethz.ch/~struwe/Skripten/InfAnalysis-bbm-8-11-2010.pdf

Logik

Aussage - Eine Aeusserung, die entweder wahr oder falsch ist
Luegner Paradox - Das ist keine Aussage: “Dieser Satz ist falsch”

Menge (Set) - eine ungeordnete Zusammenfassung verschiedener Objekte zu einem Ganzen

$\wedge$ - and
$\vee$ - or
$\dot{\vee}$ (XOR) - either … or …
Materiale Aequivalenz ($\Leftrightarrow$)
Logische Aequivalenz ($\equiv$) $A \Leftrightarrow B \equiv (A \Rightarrow B) \wedge (B \Rightarrow A)$ - Sie haben die gleichen Wahrheitstabellen

$A \Leftrightarrow B$ - A genau dann wenn B
$A \Rightarrow B$ - Wenn A, dann B
$\neg B \Rightarrow \neg A$ - Kontraposition
$A \Rightarrow B \equiv \neg B \Rightarrow \neg A$

Zum Beispiel:
Es hat geregnet $\Rightarrow$ die Strasse ist nas
Kontraposition: Die Strasse ist nicht nass $\Rightarrow$ Es hat nicht geregnet
Das ist genauso wahr aufgrund der Physik.

Wahr: $0 < 0 \Rightarrow 1 + 1 = 2$
Falsch: $0 < 0 \Leftrightarrow 1 + 1 = 2$

Distributive:

$$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$$
$$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$$

Proofs

Beweis - eine Herleitung einer Aussage aus den Axiomen
Satz - eine Bewiesene Aussage
Lemma (oder Hilfssatz) - ein Satz, der dazu dient, einen anderen Satz zu beweisen

q.e.d. ($\blacksquare$) - end of proof

Beweiss formalisieren - Express a proof formally in terms of symbols and Limmas, can be checked by a computer.
Divide et impera - divide and conquer *Zermelo + Fraenkel Axioms* - Foundational axioms of all proofs

Beweis Methode

Modus ponens - Wird (meistens mehrmals) verwendet, um etwas zu beweisen:
$A :=$ Es hat geregnet (Premise)
Wenn es geregnet hat, dann ist die Strasse nass (Regel: $A \Rightarrow B$)
$B :=$ Die Strasse ist nass (Konklusion)

Kontraposition - Prove the Kontraposition, which subsequently proves the original statement (they are logically equivalent)
Beweisen, dass $\sqrt{2} < \sqrt{3}$:

$$A := \sqrt{2} \geq \sqrt{3} \equiv \neg\sqrt{2} < \sqrt{3}$$

Monotonie des Quadrierens:

$$x, y \geq 0$$
$$\text{Wenn } x \leq y, \text{dann ist } x^2 \leq y^2$$
$$\text{Laut der Monotonie des Quadrierens, } B := 2 \geq 3 \text{ ist wahr}$$
$$A \Rightarrow B \equiv \neg B \Rightarrow \neg A \equiv 2 < 3 \Rightarrow \sqrt{2} < \sqrt{3} \blacksquare$$

Widerspruch beweis
Um A zu beweisen, nehmen wir an, dass A falsch ist.
Widerspruch finden - das beweist die Aussage A

Zum Beispiel:
Beweis des Satzes $\sqrt{2} < \sqrt{3}$
Nehmen wir an, dass $\sqrt{2} \geq \sqrt{3}$ wahr ist
Lemma (Monotonie des Quadrierens): $\sqrt{2} \geq \sqrt{3} \Rightarrow 2 \geq 3$
Widerspruch: $2 \geq 3$ ist falsch, deshalb ist $\sqrt{2} \geq \sqrt{3}$ auch falsch.
$\neg\left(\sqrt{2} \geq \sqrt{3}\right) \equiv \sqrt{2} < \sqrt{3}$ ■

It is more rigorous to prove / rewrite something through Contraposition, because we start with a false statement in contradiction.

Vollstaendige Induktion
$n \in N_0, P(n)$ ist eine Aussage
$P(0)$ ist wahr
Wenn $\forall k \in N_0$ gilt $P(k) \Rightarrow P(k+1)$
Dann ist $\forall n \in N_0, P(n) \equiv$ wahr
Zum Beispiel:

$$\text{Satz: } \forall n \in N_0, P(n) := \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
$$P(0) = \frac{0(1)}{2} = 0$$
$$\text{Sei } P(k) = \frac{k(k+1)}{2}$$
$$\text{Zu zeigen } P(k+1) = \frac{(k+1)((k+1)+1)}{2}$$
$$P(k+1) = P(k) + k + 1 = \frac{k(k+1)}{2} + k + 1$$
$$= 2k^2 + 3k + 1 = \frac{k^2 + \frac{3}{2}k + \frac{1}{2}}{2}$$
$$= \frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2}$$

Vollstaendige Induktion gibt, dass $\forall n \in N_0, P(n)$ wahr ist. ■

Mengenlehre

Eine ungeordnete Zusammenfassung von Elemente.

$\emptyset$ - Leere Menge, hat keine Elemente
$\{\emptyset\}$ hat genau ein Element

Aussageform $\{x \mid P(x)\}$ or $\{x; P(x)\}$ - die Menge aller x, fuer die $P(x)$ gilt
Example: $\{x \mid x \in \mathbb{N}_0, x \text{ ist gerade}\}$

Russelsche Antonomie - $\{x \mid x \in X, x \notin x\}$ ist ein Paradox
Loesung: Es muss immer so definiert werden $\{x \in X \mid P(x)\}$, wo X eine Menge ist.

$A \cap B - \{x \mid x \in A \wedge x \in B\}$ - Intersection
$A \cup B - \{x \mid x \in A \vee x \in B\}$ - Union
$A \setminus B - \{x \in A \mid x \notin B\}$ - Without
$A \subseteq B$ - Jedes Element von A liegt in B (between two sets, unlike $x \in A$ which describes a single element x being inside the set A)
$A \subset B$ - Jedes Element von A liegt in B und A enthaelt weniger Elemente als B
$A \subseteq X, A^{\mathsf{C}} = X \setminus A$, wo X die Grundmenge ist, die jeder Element die wir betrachten enthaelt.

Distributive:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$(1, 2, 3)$ - *Tuple* - Ordered set

Kartesische Product / Potenz - $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$
Example:

$$X := \{0, 1\}, Y := \{\alpha, \beta\}$$
$$X \times Y := \{(0, \alpha), (0, \beta), (1, \alpha), (1, \beta)\}$$
$$|X \times Y| = |X| \times |Y|$$

$\mathbb{R}^n$:= n-dimensionalen Koordinatenraum
$\mathbb{R}^2 = X \times Y$
$\mathbb{R}^3 = X \times Y \times Z$

Interval Notation

$$[a, b] - a \leq x \leq b$$
$$(a, b) - a < x < b$$

Open bounds cannot be the maximum / minimum of a set, as they are not contained in the set (and $0.\dot{9} \equiv 1$ etc.).

Let $A \subseteq \mathbb{R}$
Supremum

$$\sup A = \begin{cases} \text{Smallest upper bound} & \text{if A has an upper bound} \\ \infty & \text{if A doesn't have an upper bound} \\ -\infty & \text{if } A = \emptyset \end{cases}$$

Infimum - Largest lower bound

$$\inf A = \begin{cases} \text{Largest lower bound} & \text{if A has a lower bound} \\ -\infty & \text{if A doesn't have a lower bound} \\ \infty & \text{if } A = \emptyset \end{cases}$$

Infinity cannot be a Supre/Infimum, becuase $\infty \notin \mathbb{R}$

De Morgan's Laws

Also apply to boolean logic, where $A, B := 1, 0$

$$(A \cap B)^{\complement} = A^{\complement} \cup B^{\complement}$$
$$(A \cup B)^{\complement} = A^{\complement} \cap B^{\complement}$$

Quantoren

They cannot simply be swapped! See the largest natural number problem in script.

$\exists$ - Existenzquantor - Es gibt
$\forall$ - Allquantor - Fuer alle
$\exists!$ - Es gibt genau ein element

$\neg(\forall x \in X |\ P(x)) = \exists x \in X \mid \neg P(x)$
$\neg(\exists x \in X |\ P(x)) = \forall x \in X \mid \neg P(x)$

Goethe Prinzip - When a variable is renamed correctly, a statement is still logically equivalent

Funktionen

Eine Funktion ist ein Tripel $f = (X, Y, G)$, wobei X und Y Mengen sind und $G \subseteq X \times Y$, sodass $\forall x \in X \exists y \in Y$, sodass $(x, y) \in G$

Domain - Set of possible inputs for a function
Codomain (Range) - Set of possible outputs of a function

Example:
Both are Quadratic funktions but are not equal:
$X := Y := \mathbb{R}, G = \{(x, x^2) \mid x \in \mathbb{R}^2\}$
$X := \mathbb{R}, Y :=]0, \infty[, G = \{(x, x^2) \mid x \in \mathbb{R}^2\}$

$X \to X, \mathrm{id}(x) := x$ - *Identitaets Funktion*

Bild und Urbild - Muss nicht bijektiv sein
$\mathrm{im}(X) := f(X)$ - *Bild von f*
$f : X \to \alpha, f^{-1}(Y) := \{x \in X \mid f(x) \in Y\}$ - *Urbild von y unter f*

Surjektiv - $\forall y \in Y \exists x \in X : f(x) = y$ - Es gibt fuer jeder Ausgang einige dazugehoerige Eingange
Injektiv - $\forall x, x' \in X : x \neq x' \Rightarrow f(x) \neq f(x')$ - Es gibt genau eine Ausgang fuer jeder Eingang in dem Definitionsbereich
Bijektiv - Es ist Surjektiv und Injektiv, weshalb es eine Inverse hat

Umkehrfunktion

Sei $f : X \to Y$ eine Bijektive funktion, $f^{<-1>} := Y \to X$ - *Umkehr Funktion*
The inverse can ONLY be defined when the function is Bijektiv, unlike the Urbild. When $X = Y = \mathbb{R}$ it is the reflection of the original function over the line $y = x$. It is sometimes notated as f^{-1} when the context is clear.

Do not forget to consider the given domain / range when considering if a function is bijektiv!

Zum Beispiel:

$$f : \mathbb{R} \to \mathbb{R}, f(x) := x^2$$
$$\mathrm{im}(f) = f(\mathbb{R}) = [0, \infty]$$
$$f^{-1}([-\infty, 4]) = [-2, 2]$$

The inverse can be only be defined if f is Bijektiv:

$$f : [0, \infty] \to [0, \infty], f(X) := x^2$$
$$f^{<-1>} = \sqrt{X}$$

$g \circ f := g(f(x))$ - Only possible if the $\mathrm{codom}(f) = \mathrm{dom}(g)$

Zahlen und Vektoren

$\mathbb{N}_0 := \{0, 1, 2, ...\}$
$\mathbb{N} := \{1, 2, 3, ...\}$
$\mathbb{Z} := \{..., -1, 0, 1, ...\}$
$\mathbb{Q} := \left\{\frac{m}{n} \mid m \in Z \land n \in N\right\}$
$\mathbb{N}_0 \subseteq \mathbb{Z} \subseteq \mathbb{Q}$

There are infinite gaps in the number line of rational numbers. These can be filled with $\mathbb{R} \setminus \mathbb{Q}$ - Irrational numbers, for example $\sqrt{2}, \pi, e$. For example: $\nexists s \in \mathbb{Q} \mid s^2 = 2$.

Reelen Zahlen

Dedekind Cut

A Dedekind cut is a way of representing the real numbers using the rational numbers by cutting the number line into two sections around a "gap" represented by an irrational number. Let $x \subset \mathbb{Q}$ (x contains less elements than $\mathbb{Q}$), the following properties describe the cut:

$$x \notin \emptyset$$
$$\forall r \in x \forall s \in \mathbb{Q} : s > r \Rightarrow s \in x$$
$$\forall r \in x \exists s_0 \in x : s_0 < r$$

This definition can of course include $\mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$ and therefore the entire $\mathbb{R}$ set.

The elementary number operations (addition, subtraction, multiplication, inequalities etc.) can be defined in terms of Dedekind cuts, precisely defining our understanding of arithmetic. $\mathbb{R}$ (und deshalb auch $\mathbb{Q}$) ist eine sogennante "total geordneter Koerper".

Dedekind Completeness - Every nonempty subset of $\mathbb{R}$ with an upper / lower limit has a smallest / largest upper / lower limit.
This proves that the irrational numbers are not complete: $\{r \in Q \mid r^2 < 2\}$ has no smallest upper limit.

b-adischer Bruch

This is the formal name of the place value system which is defined for all bases ≥ 2. The values of the digits before the radix point are nb, and $\frac{1}{nb}$ after the radix.

Youngsche Ungleichung

$$x, y, c \in \mathbb{R}$$
$$c > 0$$
$$2|xy| \leq cx^2 + \frac{y^2}{c}$$

Cardinality (Mächtigkeit)

Two sets have the same cardinality if they have the same size and therefore a bijective mapping between them exists (see Cantor's Diagonalmethod).

$$|\mathbb{N}_0| = |\mathbb{Z}| = |\mathbb{Q}| \neq |\mathbb{R}|$$

Complex Numbers

The Real numbers contain no solution for $x^2 = -1$, which is why the imaginary number $i = \sqrt{-1}$ was introduced, first considered by Cardano. They can be used to solve real world problems throughout electrical engineering, particularly for oscillations because powers of i^n have a repetitive nature.

Complex addition is identical to real addition $+_{\mathbb{R}^2}$.

Complex multiplication is defined as:

$$\cdot_{\mathbb{C}} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2, \begin{pmatrix} r \\ m \end{pmatrix} \cdot_{\mathbb{C}} \begin{pmatrix} r' \\ m' \end{pmatrix} := \begin{pmatrix} rr' - mm' \\ rm' + r'm \end{pmatrix}$$

$$\begin{aligned}(r + mi)(r' + m'i) &= rr' + rm'i + mr'i + mm'i^2 \\ &= rr' - mm' + (rm' + r'm)i\end{aligned}$$

Therefore the complex body is defined as a tuple with the operations:

$$\mathbb{C} := (\mathbb{R}^2, +_{\mathbb{R}^2}, \cdot_{\mathbb{C}})$$

$$i \in \mathbb{C}, i := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

It is not a complete body as it doesn't contain any definitions for inequalities, like $\mathbb{R}$.

The following injective, non surjective function maps real numbers to complex numbers:

$$\mathbb{R} \to \mathbb{C} : x \in \mathbb{R}, \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathbb{C}$$

There exists a root for −1 in the complex body:

$$i^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot_{\mathbb{C}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\sqrt{-1} = \pm i$$

The complex conjugate is defined as follows:

$$z := a + bi$$

$$\mathfrak{R}(z) = a$$

$$\mathfrak{I}(z) = b$$

$$\overline{z} = a - bi$$

The euclidian norm is defined as:

$$|z| = \sqrt{|z_1|^2 + |z_2|^2 + \ldots + |z_n|^2}$$

General identities:

$$z\overline{z} = |z|^2$$

$$\overline{z + z'} = \overline{z} + \overline{z'}$$

$$\overline{zz'} = \overline{z} \cdot \overline{z'}$$

The function cis is defined to handle complex numbers in polar form:

$$\text{cis}(\theta) = \cos(\theta) + i\sin(\theta)$$
$$\text{cis}(\theta)\text{cis}(\varphi) = \text{cis}(\theta + \varphi)$$
$$\text{cis}\left(k\frac{\pi}{2}\right) = i^k \forall k \in \mathbb{Z}$$
$$z = |z|\ \text{cis}(\varphi) = |z|e^{i\varphi}$$
$$zz' = |z||z'|\ \text{cis}(\varphi + \varphi')$$
$$\overline{z} = |z|\text{cis}(-\varphi)$$

De Moivre's Theorem:

$$z^k = |z|^k\ \text{cis}(k\varphi) = |z|^k e^{ik\varphi}$$

k'th Roots

For a complex number z, the k'th roots w are straightforward to determine:

$$w^k = z$$
$$w_j = |z|^{\frac{1}{k}}\ \text{cis}\left(\frac{\varphi + 2j\pi}{k}\right), j := 0, 1, ..., k - 1$$

Any of these roots to the power of k is equal to z, as well the product of all of them together. If $j \geq k$ the angle completes a full circle and the same roots are found.

The roots of $z = 1$ are called roots of unity, these will be important later in Fourier transforms:

$$w^k = 1$$
$$\zeta_k(j) = e^{\frac{2j\pi i}{k}}, j := 0, 1, ..., k - 1$$

Fundamental Theroem of Algebra - Every non-constant single variable polynomial contains at least 1 complex root.

Sequences and Series

Sequence - A function that maps a natural index $n \in \mathbb{N}_0 \to \mathbb{C}$
Series - Sequence of partial sums of the terms in a sequence

Taylor Series - A series of derivatives of a function at a point, that converges towards the value of the function at that same point, more on this later...

Geometric Sequence - $n \in \mathbb{N}_0, a_n \to z^n$ - Converges towards 0 when $z < 1$
Geometric Series - $n \in \mathbb{N}_0, a_n \to \sum_{k=0}^{n} z^k$

Harmonic Sequence - $n \in \mathbb{N}_0, a_n \to \frac{1}{n}$ - Converges towards 0

Archimedes' Axiom

$$\forall x \in \mathbb{R} \exists n \in \mathbb{N}_0, x \leq n$$

Triangle Inequality

$$|x + y| \leq |x| + |y|$$
$$|x - y| \geq |x| - |y|$$

Convergence

$$\text{A sequence converges towards } A \Leftrightarrow \exists A \in \mathbb{C} \forall \varepsilon \in (0, \infty) \exists n_0 \in \mathbb{N}_0 \forall n \in \mathbb{N}_0 : n \geq n_0, |a_n - A| \leq \varepsilon$$
$$a_n \to A \text{ (converges towards A)}$$

We can also express this as a limit:

$$\lim_{n \to \infty} a_n = A$$

Note: The index n cannot be set as ∞, as infinity is not a natural number.

Divergence can be proved by proving the conjugate of the definition of convergence:

$$\text{A sequence diverges} \Leftrightarrow \forall A \in \mathbb{C} \exists \varepsilon \in (0, \infty) \forall n_0 \in \mathbb{N}_0 \exists n \in \mathbb{N}_0 : n \geq n_0, |a_n - A| > \varepsilon$$

Convergence Criteria

TODO: Read corresponding bridging course and understand divergence proof