Analysis 1

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Literature:
- Alessio Figalli's 2023 Lecture Notes
- Analysis 1 by H. Amann & J. Escher

What I would like to take notes of:
1. State the theorem / definition, expand with some intuition / memory aids
2. Write the proof by myself if deemed useful for an exam
3. Name some examples only if very helpful

These notes should serve as condensed revision material - only the minimal, important facts to remember before solving problems

Note on order: Many scripts used for teaching introduce concepts as they become relevant. My goal is to build a revision reference, not a learning resource, therefore over time the order will be rearranged to group relevant definitions and theorems together.

Proofs heavily involve decomposition; to progress, smaller Lemmas need to be brought in along the way and proven (or taken as true since someone else proved them). However first of all, you need to understand and remember the axioms (rules of the game). Intuition is helpful but doesn't prove anything unless it can be formulated as a series of statements a computer can verify

Mathematics - Abstracting enough to focus on the matter

Contradiction is a useful tool for linking statements about $>$ and $\geq$.

Conjecture - A conclusion formed on the basis of incomplete information Prove uniqueness through trichotomy, existence by completeness axiom.

TODO: Read Einsiedler einführung

1. Fundamentals

1.0.1. Definition - Set

An **unordered** collection of **distinct** ($\{x, x\} \equiv \{x\}$) elements such that:

1. It is defined by the elements it contains
2. It is not an element of itself, this prevents Russell's Paradox: $\{x \mid x \notin x\}$
3. Its elements can be filtered by a series of statements which hold true, for example the set of even integers:

   $$\{n \in \mathbb{Z} \mid \exists m \in \mathbb{Z} : n = 2m\}$$

   Where $|$ and $:$ both mean "such that".
4. The empty set $\emptyset$ contains no elements

1.0.2. Definition - Cartesian Product

For the sets X, Y, the Cartesian product is the set of tuples (ordered lists): $X \times Y \coloneqq \{(x, y) \mid x \in X, y \in Y\}$

- The number of elements in the set is:

  $$|X \times Y| = |X| \cdot |Y|$$

Example:

$$X \coloneqq \{0, 1\}, Y \coloneqq \{\alpha, \beta\}$$
$$X \times Y \coloneqq \{(0, \alpha), (0, \beta), (1, \alpha), (1, \beta)\}$$

1.0.3. Definition - Subset

A set whose elements are entirely contained in a parent set with the following notation:

- $P \subseteq Q$ - P is a subset of Q and they may be equal
- $P \subsetneq Q$ - P is a **proper** subset of Q; Q has at least 1 additional element
- $P \not\subset Q$ - There is at least one element in P that is not in Q
- The same applies in reverse using sup(er)set notation $\supseteq$
- The symbols $\subset$ and $\supset$ are ambiguous in meaning
- Two sets can be shown to be equal if $P \subseteq Q \land Q \subseteq P$ holds true

1.0.4. Definition - Power Set

The power set of a set X is the set of all subsets:

$$\mathcal{P}(X) \coloneqq \{\text{Set } Q \mid Q \subseteq X\}$$

Example:

$$X = \{0, 1, 2\}$$
$$\mathcal{P}(X) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$$

1.0.5. Definition - Interval Notation

Interval notation allows us to succinctly express common sets of real numbers between limits $a, b \in \mathbb{R}$:

- Closed interval

  $$[a, b] \coloneqq \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

- Open interval

$$(a, b) := \{x \in \mathbb{R} \mid a < x < b\}$$

- Half-open interval

$$[a, b) := \{x \in \mathbb{R} \mid a \leq x < b\}$$

- Unbounded interval

$$[a, \infty) := \{x \in \mathbb{R} \mid a \leq x\}$$

- For a non-empty interval, the **length** is defined as $b - a$
- Sometimes inverted square brackets are used to specify an open bound, ex. $[a, b[$
- The intersection of a finite number of intervals is also an interval, such that the lower bound is the smallest lower bound and vice versa for the upper bound
- Sets aren't doors, they don't need to be either open or closed.

1.0.6. Definition - Set Operations

This allows us to construct common sets from component sets P, Q:

- Intersection:

$$P \cap Q := \{x \in P \mid x \in Q\}$$

- Union:

$$P \cup Q := \{x \in P \vee x \in Q\}$$

- Relative Complement:

$$P \setminus Q := \{x \in P \mid x \notin Q\}$$

- Complement of a Subset:

$$R \subseteq X$$
$$R^{\complement} := \{x \in X \mid x \notin R\}$$

- Symmetric Difference:

$$P \vartriangle Q := (P \cup Q) \setminus (P \cap Q)$$

- Addition:

$$P + Q := \{p + q \mid p \in P \wedge q \in Q\}$$

- Multiplication:

$$P \cdot Q := \{p \cdot q \mid p \in P \wedge q \in Q\}$$

- They are distributive:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- Let $\mathbb{A}$ be a set of sets, we can also define:

$$\bigcap_{A \in \mathbb{A}} A := \{x \mid \exists A \in \mathbb{A} \mid x \in A\}$$
$$\bigcup_{A \in \mathbb{A}} A := \{x \mid \forall A \in \mathbb{A} \mid x \in A\}$$

1.0.7. Theorem - De Morgan's Laws

This states:

$$(A \cap B \cap \ldots)^{\mathrm{C}} = A^{\mathrm{C}} \cup B^{\mathrm{C}} \cup \ldots^{\mathrm{C}}$$
$$(A \cup B \cup \ldots)^{\mathrm{C}} = A^{\mathrm{C}} \cap B^{\mathrm{C}} \cap \ldots^{\mathrm{C}}$$

It is commonly applied to Boolean logic, where $A, B \subsetneq \{0, 1\}$:

$$\overline{A \land B \land \ldots} \equiv \overline{A} \lor \overline{B} \lor \ldots$$

1.0.8. Definition - Maxima and Minima

The maximum of a set is the smallest upper bound, which is **contained** in the set:

$$X \subseteq$$
$$\max(X) \coloneqq m \in X \mid \forall x \in X x \leq m$$

The **minimum** is defined analogously:

$$\min(X) \coloneqq m \in X \mid \forall x \in X x \geq m$$

- An open bound has no maximum or minimum defined as there is always some number slightly larger / smaller than a number we can express inside it. An open bound itself is not in the set
- The maximum / minimum is unique. Proof: Let m_1, m_2 be 2 maxima of the set. It follows $m_1 \leq m_2$ and $m_2 \leq m_1$, therefore $m_1 = m_2$ (trichotomy)

1.0.9. Definition - Supremum and Infimum

Let $B = \{b \in \mathbb{R} \mid \forall x \in X x \leq (\geq) b\}$ be the set of upper (lower) bounds for the set X. The supremum (infimum) is defined as the smallest (largest) such bound:

$$\sup(X) \coloneqq \min(B)$$
$$\inf(X) \coloneqq \max((B))$$

- Due to the $\leq$ $(\geq)$ the supremum infimum may be the same as the maximum / minimum for a closed bound
- An alternative characterization states there is no smaller bound, anything smaller is not a bound of X:

$$\forall x \in X x \leq \sup(X), t \leq \sup(X) \Rightarrow \exists x' \in X : t < x'$$

- The supremum / infimum does **not** exist for an unbounded or empty set, as this would be infinitely large / small, and $\infty \notin \mathbb{R} \therefore \infty \notin B$
- For all bounded, non-empty sets X, the supremum / infimum exists.
  Proof:
  The set of bounds $B = \{b \in \mathbb{R} \mid \forall x \in X x \leq (\geq) b\} \neq \emptyset$
  We need to show that $\exists \sup(X) \in \mathbb{R} \mid \forall b \in B, \sup(X) \leq b$
  Lemma: Completeness Axiom $\forall x \in X \forall b \in B, x \leq b \Rightarrow \exists c \in \mathbb{R} \mid x \leq c \leq b \forall x \in X \forall b \in B$ This c is an upper bound **and** minimum of B, therefore it is the supremum ■

Let X, Y be non-empty sets with an upper bound:
- $\sup(X \cup Y) = \max(\sup(X), \sup(Y))$
- $\sup(X \cap Y) = \min(\sup(X), \sup(Y)) \mid (X \cap Y) \neq \emptyset$
- $\sup(X + Y) = \sup(X) + \sup(Y)$
- $\sup(X \cdot Y) = \sup(X) \cdot \sup(Y) \mid \forall x \in X \forall y \in Y x, y \geq 0$ (two “large” negative elements can make a larger supremum)

TODO: Review proof 2.59

2. Topology

2.0.1. Definition - Ball / Disk

A topological ball with radius r and center $x_0 \in \mathbb{R}^d$ in dimension $\mathbb{R}^d$ is defined as the set of points:

$$B_r^d(x_0) = \{x \in \mathbb{R}^d \mid |x - x_0| < r\} - \text{Open ball}$$
$$\overline{B_r^d}(x_0) = \{x \in \mathbb{R}^d \mid |x - x_0| \leq r\} - \text{Closed ball}$$
$$S_r^{d-1}(x_0) = \{x \in \mathbb{R}^d \mid |x - x_0| = r\} - \text{Sphere (boundary of ball)}$$

Where $|x - x_0|$ is the length of the vector from $x_0 \to x$ ie the radius. This can also be defined using complex numbers and the complex absolute function. It follows:

$$B_0(x_0) = \emptyset$$
$$\overline{B_0}(x_0) = \{x_0\}$$
$$B_\infty^d(x_0) = \overline{B_\infty^d}(x_0) = \mathbb{R}^d$$

- Man muss immer am B_r^d bleiben!

The sphere has dimensions $d - 1$ because its points only form a subspace in the dimension below the ball which it is enclosing:
- S_r^1 - Is the line of points around a circle ie 1 dimensional
- S_r^2 - Every point in the surface of a 3D ball can be reached with linear combinations of two basis vectors (such that they stay within the subspace).

2.0.2. Definition - Inner Point

A point $x \in S^n$ is inner $\Leftrightarrow \exists r \in (0, \infty) \mid B_r^n(x) \subseteq S$ - there is an open ball with a radius > 0 around x such that it is entirely a subset of / equal to S.

2.0.3. Definition - Interior

- The interior of a set is the set of all its inner points:

$$\text{Int } S := \{\text{inner points of S}\}$$

- Int $S \subset S$ is always true.
- Alternatively, the interior can be defined as the union of open balls:

$$\text{Int } S := \bigcup_{B_r^n(x) \mid B \subseteq S} B$$

2.0.4. Definition - Open Set

A set which is equal to its interior: $S = \text{Int } S$. In other words, it is defined with $>$ or $<$ relations.
- It has no maximum / minimum, only an infimum / supremum.
- Every point of an open ball is an inner point, hence making the ball “open”.

$$\text{Int } \overline{B_r^d}(x_0) = B_r^d(x_0)$$
$$x \in \mathbb{R}, \{x\} \text{ is not open}$$

- The union of arbitrarily many open sets is open (the outer boundaries will remain open no matter what)
- The intersection of finitely many open sets is also open

2.0.5. Definition - Closed Set

The definition is built upon that of an open set: Let $A \subseteq \mathbb{R}^n$:

$$A \text{ is closed} \Leftrightarrow (\mathbb{R}^n \setminus A) \text{ is open}$$

For example, $[a, b] \mid a < b$ can instead be expressed as $(-\infty, a) \cup (b, \infty)$, which is open.

- $\{x\}$ is closed
- $\emptyset, \mathbb{R}^n$ are both open and closed, since $\emptyset^\complement = \mathbb{R}^n$ and $\mathbb{R}^\complement = \emptyset$
- $[a, b) \subsetneq \mathbb{R}$ is neither open or closed

2.0.6. Definition - Closure

The closure of $\overline{S}$ is the smallest possible closed set which entirely includes the set S, this can be formed using the intersection of all possible closed balls with different radii and centers, as long as they entirely contain S:

$$\overline{S} \coloneqq \operatorname{clos}(S) \coloneqq \bigcup_{\text{All } \overline{B_r^n}(c) \mid S \subseteq \overline{B}} \overline{B}$$

For example:

$$S \coloneqq (0, 1]$$
$$\overline{S} = [0, 1]$$

- $S \subseteq \overline{S}$ - The closure of a set contains the set itself
- A topological set can only be called **closed** if it is equal to its closure

2.0.7. Definition - Boundary

The boundary of a set ∂S is:

$$\partial S \coloneqq \overline{S} \setminus \text{Int } S$$

Characterized more fundamentally:

$$\partial S \coloneqq \{x \in \mathbb{R}^n \mid (\forall r \in (0, \infty) \mid (B_r^n(x) \cap S) \neq \emptyset \neq B_r^n(x) \setminus S)\}$$

The boundary of a set S is the set of points such that:

- A ball with increasing radius (starting just above 0) always continues to overlap with some elements of S ($(B_r^n(x) \cap S) \neq \emptyset$), ie the ball must be actually in or right next to S
- The points themselves are part of $B_r^n(x) \setminus S$, which is never equal to the empty set, ie the point itself is never in S

By definition, a topological sphere is the boundary of a ball:

$$\partial B_r^n(x) = \partial \overline{B_r^n}(x) = S_r^{n-1}(x)$$

Furthermore, a boundary is a closed set:

$$\left(\mathbb{R}^n \setminus \partial S = \left(\text{Int } S \cup \mathbb{R}^n \setminus \overline{S}\right)\right) \text{ which is open} \Rightarrow \partial S \text{ is closed}$$

2.0.8. Definition - Bounded

A set which is a subset of a closed set (other than $\mathbb{R}^n$). In other words, the set of bounds $B = \{b \in \mathbb{R}^n \mid \forall x \in X x \leq (\geq) b\} \neq \emptyset$.

2.0.9. Definition - Compact

A set which is closed **and** bounded

- A closed ball is by definition compact.
- $\mathbb{R}^n$ is not compact, because it is an infinitely large (albeit open & closed) set.

2.0.10. Definition - Neighborhood (Umgebung)

A subset $U \subseteq X$ is considered a **neighborhood** of a point x_0 relative to a set X if:

$$x_0 \in O \subseteq U \subseteq X$$

Where O is a non-empty open set.
- For example, there are many possible neighborhoods around a point in the middle of a non-empty set.
- Points on the boundary of X have no neighborhood U as no non-empty open set contains only points which remain in X

3. Axioms of The Real Numbers

An axiomatic approach to defining the set of real numbers $\mathbb{R}$.

3.0.1. Definition - Group

A **non-empty** set G endowed with an operation $\star$ which satisfies the following criteria $\forall a, b, c \in G$:
1. *Associativity* - $a \star (b \star c) = (a \star b) \star c$
2. $\exists$ *Neutral Element* n - $a \star n = n \star a = a$ - Examples:
   - $a + 0 = 0 + a = a$
   - $a \cdot 1 = 1 \cdot a = a$
3. $\forall a \exists$ *Inverse Element* i - $a \star i = i \star a = n$ - Examples:
   - $a + (-a) = (-a) + a = 0$
   - $a \neq 0 \Rightarrow a \cdot a^{-1} = a^{-1} \cdot a = 1$
4. If $a \star b = b \star a$ it is a **commutative group**, although this is not required.

Properties:
- The *Neutral Element* is unique. Proof:

$$\begin{aligned} \text{Let } n, n' \in G \text{ be neutral elements} \\ n \star n' = n = n' \end{aligned}$$

- There is unique *Inverse Element* for all elements. Proof:

$$\begin{aligned} \text{Let } i, i' \in G \text{ be inverse elements for } a \\ i \star (a \star i') = (i \star a) \star i' \\ i \star n = n \star i' \\ i = i' \end{aligned}$$

Examples:
- The non-zero rational numbers $\mathbb{Q} := \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z} : p, q \neq 0 \right\}$ with operation $\cdot$ is a group, where $n = \frac{1}{1}$ and $i\left(\frac{p}{q}\right) = \frac{q}{p}$
- The natural numbers $\mathbb{N}$ with operation $+$ is **not** a group, as there are no negative inverse elements

3.0.2. Definition - Ring

A non-empty set R with operations $+$ and $\cdot$.
1. Addition is **always commutative** with $n = 0, i = -a$
2. Multiplication is not necessarily commutative, for example a matrix ring
3. If multiplication is commutative, it is a **commutative ring** and has neutral element $n = 1$

4. It is **not necessarily** a group for multiplication as 0 may be included and has no inverse element $0 \cdot i \neq 1$

3.0.3. Definition - Field

A commutative ring K (Körper) where $\forall a \in K \mid k \neq 0$ the inverse element for multiplication exists.
1. Addition: $n = 0, i = -a, -(-a) = a$
2. Multiplication: $n = 1, i = a^{-1}, \left(a^{-1}\right)^{-1} = a \mid a \neq 0$

Examples:
- $\mathbb{Z}$ is a ring but not a field as there is no multiplicative inverse element for all non-zero elements
- The complete set of rational numbers $\mathbb{Q} := \left\{\frac{p}{q} \mid p, q \in \mathbb{Z} : q \neq 0\right\}$ is a commutative ring and a field.
- $0 \cdot a = 0$. Proof:

$$0 = 0 \cdot a - 0 \cdot a = (0 - 0) \cdot a = 0 \cdot a$$

3.0.4. Definition - Relationship

A relationship on X is the subset $\mathfrak{R} := \{(a, b) \in X \times X \mid a \sim b\}$ where $\sim$ is an operator for expressing conditions called a relation and may have the following properties if the corresponding condition holds true $\forall x, y, z \in X$:
- Reflexive - $x \sim x$ - Example: $\leq$
- Transitive - $x \sim y \land y \sim z \Rightarrow x \sim z$ - Example: $<$
- Symmetric - $x \sim y \Rightarrow y \sim x$ - Example: $=, \neq$
- Anti-Symmetric - $x \sim y \land y \sim x \Rightarrow x = y$ - Example: $\leq$ - Although such relations are often reflexive too, this is not a requirement, consider $<$, which is anti-symmetric (no such x, y exist) but not reflexive.
- A relation is called **equivalence relation** if it is reflexive, transitive and symmetric. For example $=$ is an equivalence relation, $\leq$ is not (not symmetric).
- A relation is called **order relation** if it is reflexive, transitive and anti-symmetric. For example $\leq$

3.0.5. Definition - Ordered Field

This extends the definition of a field K with the relation $\leq$, which is denoted as $(K, \leq)$, under which all elements $x, y, z \in K$ satisfy the following:
1. Linearity of the order:

$$x \leq y \lor y \leq x$$

2. Compatibility of order and addition:

$$x \leq y \Rightarrow x + z \leq y + z$$

3. Compatibility of order and multiplication:

$$0 \leq x \land 0 \leq y \Rightarrow 0 \leq x \cdot y$$

   This can be combined with the Inverse Element of addition (which exists in all fields) to make statements about multiplication of negative numbers.

Axioms 2 and 3 **also** apply to the relation $<$, which significantly simplifies proofs. Proof: https://math.stackexchange.com/a/3271338
These conditions allow us to define conventions such as:
- Positive $:= x > 0$
- Non-negative $:= x \geq 0$
- $(x \leq y = z) \equiv (x \leq y \land y = z)$
- An example of an ordered field is the set of rational numbers $\mathbb{Q}$.

- An example of a non-ordered field is the set of complex numbers $\mathbb{C}$

The conditions of an ordered field lead to many properties we take as given. Here are some interesting proofs:

- $(x < y \land y \leq z) \Rightarrow x < z$ - Proof:

$$x < y \Rightarrow x \leq y. \leq \text{ is a transitive relation, hence } x \leq z$$
$$\text{We must now show that } x < z.$$
$$\text{Assume by contradiction that } \neg(x < z) \equiv x \geq z \text{ holds true}$$
$$\text{Due to } x \leq z \land x \geq z, x = z$$
$$\text{Recalling } x < y \text{ this implies } z < y \text{ which contradicts } y \leq z$$
$$\therefore x < z\blacksquare$$

- If $x \neq 0$, $x^2 > 0$ holds true. Proof:
  As $x \neq 0$ there are 2 cases:
  - $x > 0$
  - $x < 0$

  The ring is only guaranteed to be valid for the relation $\leq$, so we will prove $x^2 \geq 0$ first.
  If $x > 0$, $x \geq 0$ also holds true and also $x^2 \geq 0$ per condition 3.
  If $x < 0$, $x \leq 0$ also holds true. Applying condition 2, $(x - x \leq 0 - x) \equiv (0 \leq -x)$. Applying condition 3, $-x \cdot -x = x^2 \geq 0$.
  Lastly, we must show that $x^2 \geq 0 \Rightarrow x^2 > 0$. Assume by contradiction that $\exists x \neq 0 : x^2 \leq 0 \Rightarrow x^2 < 0$. This contradicts $x^2 \geq 0$, which we have proven for all $x \neq 0$ in the field. Hence $x^2 > 0$ must also be true $\blacksquare$
- $0 < 1$. Proof:
  Lemma: $0 \neq 1$ (Neutral Elements of addition and multiplication are not the same)
  Lemma: If $x \neq 0$, $x^2 > 0$ holds true. Therefore $1^2 = 1 > 0$ $\blacksquare$

Based on the fact that $0 < 1$ and the compatibility + inverse element of addition, it is clear that the integers $\mathbb{Z} := ..., < -1 < 0 < 1 < ...$ are a subset of any ordered field. Furthermore, the rational numbers are defined from the set of integers, which are also a subset of all ordered fields K:

$$\mathbb{Z} \subsetneq \mathbb{Q} \subseteq K$$

3.0.6. Definition - Absolute Function

A function $|x| : K \to K_+$ defined on every **ordered field** such that:

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

- $(|x| \leq y) \equiv (-y \leq x \leq y)$
- $|xy| \equiv |x||y|$

3.0.7. Definition - Sign Function

A function $|x| : K \to \{-1, 0, 1\}$ defined on every **ordered field** such that:

$$\operatorname{sgn}(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

- Every $x \in K$ can be expressed as $x = \operatorname{sgn}(x) \cdot |x|$

3.0.8. Theorem - Triangle Inequality

It holds $\forall x, y \in K$ elements of an ordered field that:

$$|x + y| \leq |x| + |y|$$

The name stems from considering a triangle spanned by two vectors. It is clear that the length of their vector sum is $\leq$ the sum of both side lengths. Proof (on an ordered field):
Lemma: $|x| \Rightarrow -|x| \leq x \leq |x|$
Therefore we can state the following:

$$-|x| \leq x \leq |x|$$
$$-|y| \leq y \leq |y|$$

Lemma: $x \leq y \Rightarrow x + z \leq y + z$:

$$-|x| + -|y| \leq x + -|y|$$
$$-|y| + x \leq y + x$$

Lemma: $x \leq y \land y \leq z \Rightarrow x \leq z$

$$\therefore -(|x| + |y|) \leq x + y$$

Applying the same procedure to $x \leq |x|$ and $y \leq |y|$ we get:

$$-(|x| + |y|) \leq x + y \leq |x| + |y|$$

Lemma: $(|x| \leq y) \equiv (-y \leq x \leq y)$

$$\therefore |x + y| \leq |x| + |y| \blacksquare$$

An alternative, the **inverse triangle inequality** can also be useful:

$$|x - y| \geq ||x| - |y||$$

3.0.9. Definition - Completeness Axiom

The definition of an **ordered field** so far is unsuitable as we need to “fill in the gaps”. The completeness axiom is an alternative but equivalent approach to Dedekind cuts (which define the cuts first and then operations in terms of cuts) which defines a **complete ordered field** if the completeness axiom holds true:

1. Let $X, Y \subsetneq K \mid X, Y \neq \emptyset : \forall x \in X \forall y \in Y$ the inequality $x \leq y$ holds true. If there exists $c \in K \mid x \leq c \leq y$ for all such subsets X and Y, the ordered field is complete.

- The field of real numbers $\mathbb{R}$ is a completely ordered field
- The reason subsets are checked instead of individual elements x, y is because subsets can be defined in terms of inequalities. For example, consider checking the existence of $\sqrt{2}$ in $\mathbb{Q}$. The set of rational numbers is **dense**, therefore no matter which lower bound x we choose, there is **always** a rational number closer to $\sqrt{2}$ and therefore the check $x \leq c \leq y$ holds true (although $\sqrt{2}$ is not a member of $\mathbb{Q}$). On the other hand if we choose the subset $X = \left\{x \in \mathbb{Q} \mid x \leq \sqrt{2}\right\}$, this contains the true infimum of $\sqrt{2}$ and checks **completeness** rather than **density**. Of course, both approaches would involve checking infinitely many elements but luckily we can arrive at such an inequality from the axioms of an ordered set.

3.0.10. Definition - Compactification

The reals can be extended to be compact (closed an bounded) with $-\infty, \infty$ for certain purposes, such as defining the supremum / infimum of an unbounded / empty set:

$$\overline{R} = \mathbb{R} \cup \{-\infty, \infty\}$$
$$\forall x \in \mathbb{R}, -\infty < x < \infty$$

Certain conventions are defined, however these are ambiguous and should be used sparingly:

$$\infty + x = \infty$$
$$-\infty + x = -\infty$$
$$x \cdot \infty = \infty \mid x > 0$$
$$\sup(\emptyset) = -\infty$$
$$\inf(\emptyset) = \infty$$

3.0.11. Definition - Archimedean Principle

For every $x \in \mathbb{R}$ there exists **exactly one** $n \in \mathbb{Z} \mid n \leq x < n + 1$. In simpler words, $\forall x \in \mathbb{R} \exists z \in \mathbb{Z} \mid z > x$

3.0.12. Definition - Integer / Fractional Part

The integer part of any $r \in \mathbb{R}$ is given by the floor function $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ which returns the lower n which exists due to the Archimedean principle.
The fractional part is given by $r - \lfloor r \rfloor \in [0, 1)$

3.0.13. Corollary - $\frac{1}{n}$ is arbitrarily small

$\forall \varepsilon \in \mathbb{R} \mid \varepsilon > 0 \exists n \in \mathbb{Z} \mid n \geq 1 \land \frac{1}{n} < \varepsilon$.
Proof:
If $\varepsilon > 1$ this holds true with $n = 1$.
For $\varepsilon \leq 1$, $\frac{1}{\varepsilon} \geq 1$. The Archimedian principle states that there is always a $n \geq 1 \mid n > \frac{1}{\varepsilon}$, which becomes $\frac{1}{n} < \varepsilon$■

3.0.14. Definition - Cardinality

The cardinality of two sets describes their relative "sizes".

- $X \sim Y$ - We say two sets X and Y have the same cardinality (the same number of elements) if there exists a **bijective** mapping $f : X \to Y$. Surjectivity guarantees that $|Y| \geq |X|$ and injectivity guarantees $|X| \geq |Y|$, which leads to$|X| = |Y|$ (trichotomy).
- $X \underset{\sim}{<} Y$ - X is **smaller than or equal to** Y if there exists an injective mapping $f : X \to Y$
- $X \underset{\sim}{<} Y \land Y \underset{\sim}{<} X => X \sim Y$ - One can find a bijective mapping (Schröder-Bernstein Theorem)
- $|\emptyset| = 0$
- $\exists f : X \to \{1, 2, ..., n\}$ is bijective $\Rightarrow |X| = n, X$ is finite
- $|\mathbb{N}| = \aleph_0$ - A set which has the same cardinality as $\mathbb{N}$ is called **countable**

3.0.15. Theorem - Cantor's Theorem

The power set $\mathcal{P}(X)$ of any (infinite too) non-empty set X is larger than and not equal to X.
This reveals that $\mathcal{P}(\mathbb{N}) > \mathbb{N} \land \mathcal{P}(\mathbb{N}) \neq \mathbb{N}$ which is useful for showing that other sets are larger than or equal to $\mathcal{P}(\mathbb{N})$ ($\exists$ injection) and therefore also uncountable.
Proof:
Although this may seem obvious, when dealing with infinity it is easier to write a formal proof than find logical reasoning behind the intuition.
First we must show that there is an injective mapping $i : X \to \mathcal{P}(X)$, which indeed exists: $x \in X \to \{x\}$.
Now we show that there is **no** surjective mapping. Assume by contradiction that such a mapping s :

$X \to \mathcal{P}(X)$ exists.
We will demonstrate its absurdity by defining the subset:

$$B = \{x \in X \mid x \notin f(x)\} \subseteq X$$

For every $x \in X$ there are two cases:
1. $x \in s(x)$, therefore $x \notin B$ and $s(x) \neq B$ because x would need to be a member of B for them to be equal
2. $x \notin s(x)$, therefore $x \in B$ and $s(x) \neq B$ because x would need to be a member of $s(x)$ for them to be equal

We have shown that $\nexists x \in X \mid s(x) = B$ and because $B \in \mathcal{P}(X)$, there exists no surjective mapping $s : X \to \mathcal{P}(X)\blacksquare$

3.0.16. Theorem - $\mathbb{R}$ is Uncountable

To prove this, we can find an injection $i : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$, which is given from the decimal expansion of reals TODO: Understand Cantor diagonalization

4. Sequences of Real Numbers

4.0.1. Definition - Sequence

A sequence is a function $a : \mathbb{N} \to \mathbb{R}$ which is often written as $(a_n)_{n\in\mathbb{N}}$
- A sequence is called **constant** if $\forall n, m \in \mathbb{N}, a_n = a_m$ and **eventually constant** if $\exists N \in \mathbb{N} \mid \forall n, m \geq N, a_n = a_m$

4.0.2. Definition - Convergence

A sequence is said to **converge towards A** if:

$$\exists A \in \mathbb{R} \forall \varepsilon \in (0, \infty) \exists N \in \mathbb{N} \mid \forall n \in \mathbb{N} : n \geq N, |a_n - A| < \varepsilon$$
$$\lim_{n\to\infty} a_n = A$$

- **Divergence** can be proved by proving the conjugate:

$$\forall A \in \mathbb{R} \exists \varepsilon \in (0, \infty) \forall N \in \mathbb{N}_0 \exists n \in N : n \geq N, |N - A| > \varepsilon$$

- A convergent sequence has **only one** limit. Proof:
  Let $A_1, A_2 \in \mathbb{R}$ be two limits of the sequence a_n.
  Due to the convergence criteria:

$$\exists N_1, N_2 \in \mathbb{N} \mid \forall n \geq \max(N_1, N_2), |a_n - A_1| < \varepsilon \wedge |a_n - A_2| < \varepsilon, \forall \varepsilon \in (0, \infty)$$
$$0 < \varepsilon - |a_n - A_1|, 0 < \varepsilon - |a_n - A_2|$$
$$\therefore |a_n - A_1| + |a_n - A_2| < 2\varepsilon$$

  Applying the Lemma $|a + b| \leq |a| + |b|$:

$$a_n - A_1 - (a_n - A_2) = A_2 - A_1$$
$$0 \leq |A_2 + (-A_1)| \leq |a_n - A_1| + |-(a_n - A_2)| < 2\varepsilon$$

  Since $|-x| = |x| \geq 0$, and this is true $\forall \varepsilon > 0$:

$$|A_2 - A_1| = 0 = A_2 - A_1$$
$$A_2 = A_1 \blacksquare$$

- The sequence $a_n = \frac{1}{n}$ converges to 0, because $\forall \varepsilon > 0, \exists n \in \mathbb{Z} \mid \frac{1}{n} < \varepsilon$, satisfying the criteria of convergence $\left|\frac{1}{n} - 0\right| < \varepsilon$

4.0.3. Definition - Subsequence

A subsequence of a_n is any sequence obtained by keeping certain elements a_{n_i} indexed by $i_{k_{k \in \mathbb{N}}} \mid \forall k \in \mathbb{N}, i_{k+1} > i_k$

- It follows that $i_k \geq k$ (proof by induction invoking the property of natural numbers $x > y \Rightarrow x \geq y + 1$
- A sequence can have convergent subsequences **without** itself converging, for example $a_n = (-1)^n$ does not converge but the subsequences a_{2n}, a_{2n+1} are constant and convergent

4.0.4. Lemma - Subsequences of a Convergent Sequence are Convergent to the Same Limit

Proof:
Let a_{n_i} (indexed by $i_{k_{k \in \mathbb{N}}}$) be a subsequence of a_n, which converges to $A \in \mathbb{R}$, ie $\exists N \in \mathbb{N} \mid \forall n > N, |a_n - A| < \varepsilon$.
$i_k \geq k \Rightarrow j \geq n$ is a term of i_k, which satisfies the convergence condition for the same A, along with all subsequent elements.

4.0.5. Definition - Accumulation Point

A point $A \in \mathbb{R}$ is called an **accumulation point** of a sequence a_n if:

$$\forall \varepsilon > 0 \forall N \in \mathbb{N} \exists n \in \mathbb{N} \mid n \geq N \wedge |a_n - A| < \varepsilon$$

This no longer requires that every $n \geq N$ is close to A, just that such an n can be chosen for every minimum N. For example, both 1 and -1 are accumulation points of $a_n = (-1)^n$ but not limits.
The following corollaries apply:

- A is an accumulation point of a sequence $a_n \Leftrightarrow$ There exists a subsequence of a_n which converges towards A
- $\forall \varepsilon > 0$ there are infinitely many elements of the sequence a_n near an accumulation point $(A - \varepsilon, A + \varepsilon)$. This follows from the fact that there is a subsequence that converges to A and all elements of the subsequence after N are both close to A **and** elements of the parent sequence.
- A convergent sequence's limit is its **only** accumulation point. The Lemma states: All subsequences of this sequence are convergent to the same limit and applying the first corollary proves that they all correspond to the same accumulation point.

4.0.6. Definition - Ring of Sequences

Sequences $\in \mathbb{R}$ form a commutative ring together with point wise addition and multiplication and the constant sequences 0_n and 1_n as neutral elements:

$$a_n + b_n = (a + b)_n$$
$$a_n \cdot b_n = (a \cdot b)_n$$
$$\alpha \cdot b_n = (\alpha \cdot b)_n$$

- They do not form a field under pointwise multiplication, as a non-zero sequence may still contain $0 \in \mathbb{R}$ in it, which under real multiplication has no inverse $0 \cdot i \neq 1$.

4.0.7. Theorem - Operations on Limits

Operations on the sequence x_n which converges to X and y_n which converges to Y have the following effects on their limits:

1. $(x_n + y_n)_n \to X + Y$ Proof:
   We can say the following about these sequences:

$$\forall \varepsilon > 0 \exists n \in \mathbb{N} \mid n \geq \max(N_x, N_y) \wedge |x_n - X| < \varepsilon \wedge |y_n - Y| < \varepsilon$$

To show that $(x_n + y_n)_n$ converges to $X + Y$, we need to show $|(x_n + y_n) - (X + Y)| < \varepsilon$ for increasing n. Due to $0 \leq |x|$ and the compatibility of addition in the ordered field $\mathbb{R}$, we can add these inequalities:

$$|x_n - X| + |y_n - Y| < 2\varepsilon$$

Applying the triangle inequality:

$$|x_n - X + y_n - Y| = |(x_n + y_n) - (X + Y)| < 2\varepsilon \blacksquare$$

Similar proofs to 2, 3 and 4

2. $(x_n \cdot y_n)_n \to X \cdot Y$
3. $\forall \alpha \in \mathbb{R}, \alpha \cdot x_n \to \alpha X$
4. $\forall n \in \mathbb{N}, x_n \neq 0 \wedge X \neq 0 \Rightarrow (x^{-1})_n \to X^{-1}$

TODO: 2.97

5. Complex Numbers

5.0.1. Definition - Complex Numbers

The set of complex numbers $\mathbb{C}$ is defined from the Cartesian coordinates, where the $+$ can be thought of as a substitute for the comma in a tuple, and i is called the **complex unit**:

$$z = a + bi \in \mathbb{C} := \mathbb{R}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}$$
$$a = \operatorname{Re}(z)$$
$$b = \operatorname{Im}(z)$$

Complex addition $+_{\mathbb{C}}$ and multiplication $\cdot_{\mathbb{C}}$ are defined such that $\mathbb{C}$ is a field (the operations follow the conditions for a ring excluding division by 0) **and** $i^2 = -1$ holds:

$$(a + bi) +_{\mathbb{C}} (c + di) = (a + b) + (c + d)i$$
$$\begin{aligned}(a + bi) \cdot_{\mathbb{C}} (c + di) &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

- Addition has Neutral Element $(0, 0) = 0$ and Inverse Element $(-a, -b) = -a - bi$
- Multiplication has Neutral Element $(1, 0) = 1$ and (non-zero) Inverse Element $\left(\frac{a}{a^2+b^2}, -\frac{b}{a^2+b^2}\right)$
- The same notation as in $\mathbb{R}$ is normally used, for example $-(i) = (-0, -1) = -i$ and $i^{-1} = (0, -1) = -i$
- An order relation cannot be defined in a way that satisfies the ordered field axioms. Proof:

$$\text{Let } 0 \leq i, \text{condition 3. implies } 0 \leq i \cdot i = -1 \text{ which is false} \therefore i \leq 0$$
$$\text{Applying condition 2. } i + -i \leq 0 + -i \Rightarrow 0 \leq -i$$
$$\text{Applying condition 3. } 0 \leq -i \cdot -i = i^2 = -1 \text{ which is also false and contradicts } i \leq 0 \blacksquare$$

Nevertheless, they satisfy a generalization of the completeness axiom and we can perform calculus on them.

5.0.2. Definition - Complex Conjugate

The mapping $\overline{\cdot} : \mathbb{C} \to \mathbb{C}$ of a complex number $z = a + bi \in \mathbb{C}$ is denoted as $\overline{z}$ and defined:

$$\overline{z} := a - bi$$

It has the following properties $\forall z, w \in \mathbb{C}$:
- $z \cdot \overline{z} \in \mathbb{R} \geq 0$
- $\overline{z + w} = \overline{z} + \overline{w}$
- $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$
- $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$
- $\mathrm{Re}(z) = \frac{z + \overline{z}}{2}$
- $\mathrm{Im}(z) = \frac{z - \overline{z}}{2i}$

5.0.3. Definition - Complex Absolute Function

The complex absolute function $|\cdot| : \mathbb{C} \to \mathbb{R}$ is defined as:

$$|z = a + bi| := \sqrt{z \cdot \overline{z}} = \sqrt{a^2 + b^2}$$

- $|z \cdot w| = \sqrt{(z \cdot w) \cdot \overline{z \cdot w}} = \sqrt{z \cdot \overline{z} \cdot w \cdot \overline{w}} = \sqrt{z \cdot \overline{z}} \cdot \sqrt{w \cdot \overline{w}} = |z| \cdot |w|$
- It has the same notion of length when complex numbers are plotted on an Argand diagram

5.0.4. Theorem - Cauchy-Schwartz Inequality

$\forall z = x_1 + y_1 i, w = x_2 + y_2 i \in \mathbb{C}$:

$$x_1 x_2 + y_1 y_2 \leq |z \cdot w|$$

Proof:
Through algebraic rearrangement, we can show that:

$$|z \cdot w| - x_1 x_2 + y_1 y_2 = (x_1 y_2 - x_2 y_1)^2$$

Lemma: $x^2 \geq 0$
Therefore $|z \cdot w| - x_1 x_2 + y_1 y_2 \geq 0$
By applying the compatibility of addition in an ordered field (although $\mathbb{C}$ is not an ordered field, $|x \cdot w|$ can be expressed in terms of the component real numbers), we arrive at:

$$|z \cdot w| \geq x_1 x_2 + y_1 y_2 \blacksquare$$

5.0.5. Theorem - Complex Triangle Inequality

We can show that the triangle inequality also holds true $\forall z = x_1 + y_1 i, w = x_2 + y_2 i \in \mathbb{C}$:

$$|z + w| \leq |z| + |w|$$

Proof:
Through algebraic rearrangement, we can show:

$$|z + w|^2 = |z|^2 + |w|^2 + 2(x_1 x_2 + y_1 y_2)$$

Applying the Cauchy-Schwarz Inequality:

$$|z|^2 + |w|^2 + 2(x_1 x_2 + y_1 y_2) \leq |z|^2 + |w|^2 + 2|z \cdot w|$$
$$|z + w|^2 \leq (|z| + |w|)^2$$
$$\therefore |z + w| \leq |z| + |w|$$

6. Functions of One Real Variable

6.0.1. Definition - Function

A function $f : X \to Y$ is a mapping from the domain X to range / codomain Y. It **may** have the following properties:

1. *Injective* - $\forall x, x' \in X : x \neq x' \Rightarrow f(x) \neq f(x')$ - Assigns each element of X a **unique** element in Y
2. *Surjective* - $\forall y \in Y \exists x \in X : f(x) = y$ - Every element in the range is a possible output of the function
3. *Bijective* - It is both injective and surjective, and therefore an inverse function can be defined
4. Two functions are **equal** $\Leftrightarrow X_1 = X_2 \wedge Y_1 = Y_2 \wedge \forall x \in X, f_1(x) = f_2(x)$

6.0.2. Definition - Image and Preimage (Urbild) of a Function

Consider a function $f : X \to Y$.

- The Image $f(A)$ of $A \subseteq X$ under f is defined as:

$$f(A) := \{y \in Y \mid \exists x \in A : f(x) = y\}$$
$$f(A) \subseteq Y$$

- The Preimage (Urbild) $f^{-1}(B)$ of $B \subseteq Y$ under f is defined as:

$$f^{-1}(B) := \{x \in X \mid \exists y \in B : f(x) = y\} = \{x \in X \mid f(x) \in B\}$$
$$f^{-1}(B) \subseteq X$$

- A function $f : X \to Y$ is surjective $\Leftrightarrow$ The set $f(X) = Y$, because the image can only contain domain elements which map to Y by definition.
- For example consider $f : \mathbb{R} \to \mathbb{R} := x \to 0$:
  - $f(\mathbb{R}) = \{0\}$ - It is not surjective
  - $f^{-1}(\mathbb{R}) = f^{-1}(\{0\}) = \mathbb{R}$
  - $f^{-1}(\{1\}) = \emptyset$

There is an interesting property of finite sets; consider $f : X \to Y$, where X and Y are **finite** sets with the same number of elements n:

$$f \text{ is injective} \Leftrightarrow f \text{ is surjective}$$

Proof:
If f is injective, the image $f(X)$ has at least n distinct elements so every distinct $x \in X$ has its own $y \in f(X)$.
Lemma: A function $f : X \to Y$ is surjective $\Leftrightarrow$ The set $f(X) = Y$
We are given that Y has n elements, and since $f(X) \subseteq Y \Rightarrow f(X) = Y$ showing that it must also be surjective.
We must now show that surjectivity $\Rightarrow$ injectivity. If f is surjective, $f(X) = Y$ (Lemma), therefore $f(X)$ has n elements.
Consider two elements $x_1, x_2 \in X$. Since X is a set, they are distinct $x_1 \neq x_2$.
If $f(x_1) = f(x_2)$ for any two elements, they would "validate" the same member of $f(X)$, leaving out at least one element of Y (deterministic, another input cannot have 2 outputs to make up for it) meaning $f(X)$ would have $n - 1$ elements, which contradicts the lemma about surjectivity, therefore f must also be injective ■
This is **not** necessarily true for infinite sets, for example $f : \mathbb{N} \to \mathbb{N}, f(x) := x + 1$ is injective but not surjective.

6.0.3. Definition - Square Root

This is the bijective function $\sqrt{\cdot} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\forall a \in \mathbb{R}_{\geq 0} (\sqrt{a})^2 = a$, whose existence is only possible due to the real numbers being a **complete ordered field**. TODO: Various proofs Exercise 2.27 Figalli's script

TODO: **The range of a continuous function with / bounded to a compact domain is also compact.**

The maxi-, mini-, supre- and infimum of a function are defined as expected on its range.

Every continuous function with a compact domain and therefore range possesses a maximum and minimum.

6.0.4. Topological Continuity Definitions

Continuity of a function can also be defined with the following topological criteria:

6.0.4.1. Closed / Open Sets

$$f : X \to Y \text{ is continuous} \Leftrightarrow$$
$$\text{The inverse image (Urbild) of every relatively open / closed subset in } X \text{ is also relatively open / closed.}$$

6.0.4.2. Neighbourhoods

$$f : X \to Y \text{ is continuous} \Leftrightarrow$$
$$\text{The inverse image of every neighbourhood at point } f(x_0) \text{ in } Y \text{ is also a neighbourhood of } x_0 \text{ in } X$$