TODO: Vector calculus, ∇, $\nabla \cdot$, $\nabla \times$ Spherical coordinate vector operators

Analysis 2

Books:

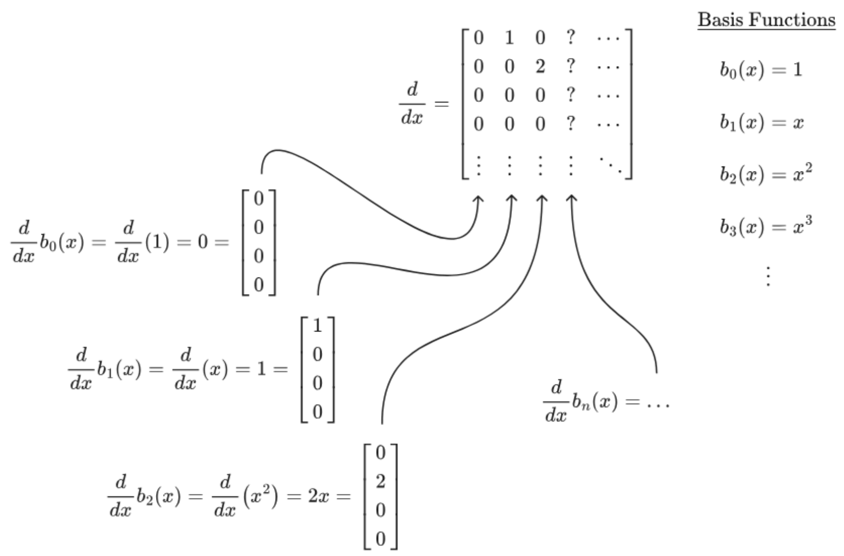
- Understanding Analysis: https://link.springer.com/book/10.1007/978-1-4939-2712-8
  Good for reviewing and developing intuition of Analysis 1
- Analysis II Amann Escher: https://link.springer.com/book/10.1007/3-7643-7402-0 https://link.springer.com/book/10.1007/978-3-7643-7478-5 too rigorous / abstract
- Zorich looks great

Linear Differential Equations

Differential equations are functions like any other - they can also be represented as linear combination of some basis, for example the infinite-dimensional Fourier basis which can represent any (long-term periodic or bounded) function.

Differential Operator

Basic differential operator - $\frac{d^n}{dx^n}$ - A linear mapping between a function and its n'th (partial) derivative within some vector space, which can be represented as a matrix if finite.



A linear differential operator is a linear combination of basic differential operators, in other words a mapping from a function to a differential equation, for example:

$$D(f) = \ddot{f} + 2\beta\dot{f} + \omega_n^2 f$$

Which is the differential operator for damped simple harmonic motion.

Differential operators (TODO: check) never have full rank, because the range is restricted to a subset of the domain linear space. **Carathéodory's existence theorem states that the kernel of an order-n differential operator has dimensions n** (and of course infinite solutions are spanned by n basis elements):

$$Dx(t) = 0$$
$$x(t) \in \text{Kernel}(D)$$

TODO: Exact conditions for Carathéodory's Theorem to hold true

Particular Solution - Single element that satisfies a differential equation

Homogeneous Differential Equation - Can be defined by a linear operator with constant scalars
TODO: Inhomogeneous equations and how to solve them (see solutions to driven-damped oscillator)

Metric Spaces

A complete metric space:
- Contains a metric (measure of distance), usually the Euclidean norm
- Is complete; formally every Cauchy sequence in the space converges to an element in the space

A metric $d : V \times V \to [0, \infty)$ must satisfy the following:
1. $\forall x, y \in V, d(x, y) = 0 \Rightarrow x = y$
2. $\forall x, y \in V, d(x, y) = d(y, x)$
3. $\forall x, y, z \in V, d(x, z) \leq d(x, y) + d(y, z)$

Most definitions of convergence / limits etc. can be easily extended by replacing the 1-dimensional $|\cdot|$ function with the metric $d(x, y)$.

Theorem A sequence in $\mathbb{R}^n$ converges $\Leftrightarrow$ Each of its coordinates converges

Theorem $\forall n \geq 1, \mathbb{R}^n$ is complete. Proof:

Each Cauchy sequence converges coordinate wise to an element in $\mathbb{R}$, therefore the sequence also converges to an element in $\mathbb{R}^n$

- $(\mathbb{Q}, d(x, y) = |x - y|)$ is an incomplete metric space.
  Counterexample: Sequence converging to $\sqrt{2}$. TODO: Find easier example

Topological definitions may also be extended through the metric.