Data structures and Algorithms

Induction

Overview

- **Recursion**
  - a strategy for writing programs that compute in a "divide-and-conquer" fashion
  - solve a large problem by breaking it up into smaller problems of same kind
- **Induction**
  - a mathematical strategy for proving statements about integers (more generally, about sets that can be ordered in some fairly general ways)
- **Induction** and **recursion** are intimately related.

Recursive programming

- In recursion we saw that a method must calls itself, that is , a method is defined in terms of itself.
- Each call to a method creates a new environment in which all local variables and parameters are newly defined.
- Each time a method terminates, processing returns to the method that called it (which for recursive calls, is an earlier invocation of the same method).
- To ensure program termination, the method must define both :
  - A base case and
  - A recursive case

Recursive programming

```
int factorial(int n) {
   if(n == 1)          Base case
        return 1;
   else
        return (n*factorial(n-1)); //Recursive case
}
```

Mathematical Induction

$$1 + 2 + 3 + 4 + \dots + 97 + 98 + 99 + 100 = 50 \times 101$$

101

101

101

101

$$= 5050$$

Note: $S_1 = 1 = \dfrac{1 \times (1+1)}{2}$ $\quad 5050 = \dfrac{100 \times (100+1)}{2}$

$$S_2 = 3 = \frac{2 \times (2+1)}{2}$$

$$S_3 = 6 = \frac{3 \times (3+1)}{2}$$

Note: $S_4 = S_3 + t_4$

$$10 = 6 + 4$$

$$S_4 = 10 = \frac{4 \times (4+1)}{2}$$

$$\therefore S_{k+1} = S_k + t_{k+1}$$

- Step 1

  Show that the statement is true for the initial value of n

- Step 2

  Assume that the statement is true for n = k

- Step 3

  Prove that the statement is true for n = k + 1

Mathematical Induction

Conjecture:

$$S_n = \frac{n(n+1)}{2} \quad \text{or} \quad 1+2+3+\ldots+n = \frac{n(n+1)}{2}$$

$$S_1 = 1 = \frac{1 \times (1+1)}{2}$$

Note: $t_n = n$

Assume $S_k = \frac{k(k+1)}{2}$

Show this pattern holds for the next integer after k, i.e. $k+1$

$$S_{k+1} = \frac{(k+1)\left[(k+1)+1\right]}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

Mathematical Induction

Assume $S_k = \frac{k(k+1)}{2}$

$S_{k+1} = \frac{(k+1)(k+2)}{2}$

$S_{k+1} = S_k + t_{k+1}$

$S_{k+1} = \frac{k(k+1)}{2} + (k+1)$

$S_{k+1} = \frac{k(k+1)}{2} + \frac{(k+1)2}{2}$

$S_{k+1} = \frac{k^2 + k + 2k + 2}{2}$

$S_{k+1} = \frac{k^2 + 3k + 2}{2}$

$= \frac{k^2 + 3k + 2}{2}$

$\therefore S_{k+1} = \frac{(k+1)(k+2)}{2}$

$\therefore 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

Where n can be any natural number

This method of proof is called MATHEMATICAL INDUCTION.

Steps:

1. Show the result is true for the first case, usually when $n = 1$.
2. Show if the result is true when $n = k$, it is true for $n = k + 1$.

Writing a Proof by Induction

1. State the hypothesis very clearly:

   P(n) is true for all integers $n \geq b$ – state the property P in English

2. Identify the base case

   P(b) holds because …

3. Inductive Hypothesis

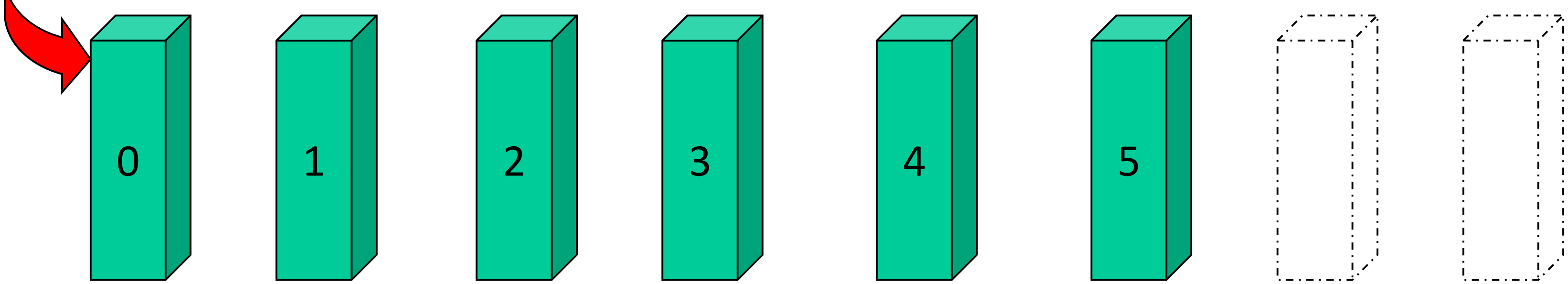
   Assume P(k)

4. Inductive Step - Assuming the inductive hypothesis P(k), prove that P(k+1) holds; i.e.,

   $P(k) \rightarrow P(k+1)$

   - Conclusion
   - By induction we have shown that P(k) holds for all $k \geq b$ (b is what was used for the base case).

Dominoes



- Assume equally spaced dominoes, where spacing between dominoes is less than domino length. Argue that dominoes fall.
- Dumb argument:
  - Domino 0 falls because we push it over.
  - Domino 1 falls: domino 0 falls; it is longer than inter-domino spacing, so it knocks over domino 1.
  - Domino 2 falls: domino 1 falls; it is longer than inter-domino spacing, so it knocks over domino 2.
  - ……
- How do we do this argument nicely?

Recursion and Induction relationship

- We see from the observable occurrence of the illustration above that there is:
  - The base case
  - Inductive case
- This therefore mean that both recursion and induction are related

Example 1: Prove

$$6 + 12 + 18 + \ldots + 6n = 3n(n+1) \quad \forall \; n \in N$$

Step 1: Verify the result is true for n = 1.

$$S_1 = 6 \qquad 3 \bullet 1(1+1) = 3 \bullet 2$$
$$= 6$$

Step 2: Assume the result is true for $n = k$ and show the result is true for $n = k + 1$.

Assume $S_k = 3k(k+1)$

Show $S_{k+1} = 3(k+1)((k+1)+1) = 3(k+1)(k+2)$

Remember $S_{k+1} = S_k + t_{k+1}$

$$\therefore S_{k+1} = 3k(k+1) + 6(k+1)$$
$$= 3k^2 + 3k + 6k + 6$$
$$= 3k^2 + 9k + 6$$
$$= 3(k^2 + 3k + 2)$$
$$= 3(k+1)(k+2)$$

same

$$\therefore 6 + 12 + 18 + \ldots + 6n = 3n(n+1) \ \forall \ n \in N$$

Example 2: Prove

$$2 + 8 + 14 + \dots + (6n - 4) = n(3n - 1) \quad \forall \; n \in N$$

Step 1: Verify the result is true for n = 1.

$$S_1 = 2 \qquad 1(3 \bullet 1 - 1) = 1 \bullet 2$$
$$= 2$$

Therefore the result is true for $n = 1$.

<u>Step 2</u>: Assume the result is true for $n = k$ and show the result is true for $n = k + 1$.

Assume $S_k = k(3k-1)$

Show $S_{k+1} = (k+1)(3(k+1)-1) = (k+1)(3k+2)$

$6(k+1)-4$

Remember $S_{k+1} = S_k + t_{k+1}$

$$\therefore S_{k+1} = k(3k-1) + (6k+2)$$

$$= 3k^2 - k + 6k + 2$$

$$= 3k^2 + 5k + 2$$

$$= (k+1)(3k+2)$$

same

$$\therefore 2 + 8 + 14 + \ldots + (6n-4) = n(3n-1) \quad \forall \; n \in N$$

Example 3: a) Conjecture a sum formula for

$$\frac{1}{1\bullet 2}+\frac{1}{2\bullet 3}+\frac{1}{3\bullet 4}+\frac{1}{4\bullet 5}+\dots$$

$$S_1=\frac{1}{1\bullet 2}=\frac{1}{2}$$

$$S_2=\frac{1}{2}+\frac{1}{2\bullet 3}=\frac{1}{2}+\frac{1}{6}=\frac{3}{6}+\frac{1}{6}=\frac{4}{6}=\frac{2}{3}$$

$$S_3=\frac{2}{3}+\frac{1}{12}=\frac{8}{12}+\frac{1}{12}=\frac{9}{12}=\frac{3}{4}$$

$$S_4=\frac{3}{4}+\frac{1}{20}=\frac{15}{20}+\frac{1}{20}=\frac{16}{20}=\frac{4}{5}$$

Sums: $\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5},\dots$ $\quad\therefore S_n=\frac{n}{n+1}$

b) Prove

$$\frac{1}{1\bullet 2}+\frac{1}{2\bullet 3}+\frac{1}{3\bullet 4}+\frac{1}{4\bullet 5}+\dots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

Step 1: Verify the result is true for n = 1.

$$S_1=\frac{1}{2} \quad \text{and} \quad \frac{1}{1+1}=\frac{1}{2}$$

Step 2: Assume the result is true for $n = k$ and show the result is true for $n = k + 1$.

$$\therefore \text{ Assume } S_k=\frac{k}{k+1} \qquad \text{Show } S_{k+1}=\frac{(k+1)}{(k+1)+1}$$

$$S_{k+1}=\frac{k+1}{k+2}$$

$\therefore$ Assume $S_k = \dfrac{k}{k+1}$ Show $S_{k+1} = \dfrac{k+1}{k+2}$

$$S_{k+1} = S_k + t_{k+1}$$

$$S_{k+1} = \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)}$$

$$S_{k+1} = \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$S_{k+1} = \frac{(k+1)^{\cancel{2}}}{\cancel{(k+1)}(k+2)}$$

$\therefore$ true for $k+1$

$$\therefore S_{k+1} = \frac{k+1}{k+2}$$

$$\therefore \frac{1}{1 \bullet 2} + \dots \frac{1}{n(n+1)} = \frac{n}{n+1}$$

When induction fails

- Sometimes, an inductive proof strategy for some proposition may fail.
- This does not necessarily mean that the proposition is wrong.
  - It just means that the inductive strategy you are trying fails.
- A different induction or a different proof strategy altogether may succeed.

Example 4: Induction fallacy.

$$1 + 6 + 11 + \ldots + (5n - 4) = n^2 + 3n - 3 \quad \forall \; n \in N$$

$S_1 = 1 \qquad 1^2 + 3 \bullet 1 - 3 = 1 + 3 - 3 = 1$

Therefore the result is true for $n = 1$.

Example 4: Induction fallacy.

$$1 + 6 + 11 + \ldots + (5n - 4) = n^2 + 3n - 3 \quad \forall\ n \in N$$

$$S_1 = 1 \qquad 1^2 + 3 \bullet 1 - 3 = 1 + 3 - 3 = 1$$

Assume $S_k = k^2 + 3k - 3$

Show $S_{k+1} = (k+1)^2 + 3(k+1) - 3$

$$= k^2 + 2k + 1 + 3k + 3 - 3$$

$$= k^2 + 5k + 1$$

$$S_{k+1} = S_k + t_{k+1}$$

$$S_{k+1} = k^2 + 3k - 3 + [5(k+1) - 4]$$

$$S_{k+1} = k^2 + 3k - 3 + 5k + 5 - 4$$

$$S_{k+1} = k^2 + 8k - 2 \qquad \therefore S_{k+1} \text{ is NOT true}$$

$$\neq k^2 + 5k + 1 \qquad \forall\ n \in N$$

Application of Mathematical Induction

- It is used to prove results about the complexity of algorithms,
- The correctness of certain types of computer programs,
- Theorems about graphs and trees, as well as a wide range of identities and inequalities.