Numerical Methods HW1

Name-Surname: Ö. Malik Kalembaşı

Student ID: 150180112

1.

$$f_1(x_0, h) = sin(x_0 + h) - sin(x_0)$$

$$f_2(x_0, h)$$

$$sin(\emptyset) - sin(\psi) = 2\,cos\left(\frac{\emptyset + \psi}{2}\right) sin\left(\frac{\emptyset - \psi}{2}\right)$$

a.

$$\emptyset = x_0 + h$$

$$\psi = x_0$$

Replacing $\emptyset$ and ψ according to given formula,

$$sin(x_0 + h) - sin(x_0) = 2\,cos\left(\frac{2x_0 + h}{2}\right) sin\left(\frac{h}{2}\right)$$

$$f_2(x_0, h) = 2\,cos\left(\frac{2x_0 + h}{2}\right) sin\left(\frac{h}{2}\right)$$

b.

$$f'(x) = \frac{f(x_0 + h) - f(xo)}{h}$$

Changing the numerator with the given formula,

$$f'(x) = \frac{2\cos\left(x_0 + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

2. Considering the linear system $\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $z = x + y$,

a. If we solve the linear system as shown in 2.b,

$$x = \frac{a}{a^2 - b^2}$$

$$y = \frac{b}{b^2 - a^2}$$

So,

$$x + y = \frac{a - b}{a^2 - b^2} = \frac{1}{a + b}$$

Which is will be cause numerical difficulties.

b.

$$ax + by = 1$$

$$bx + ay = 0$$

If we sum these two equations,

$$ax + ay + bx + by = 1$$

$$a(x + y) + b(x + y) = 1$$

$$(a + b)(x + y) = 1$$

$$\boldsymbol{x + y = \frac{1}{a + b}}$$

c. Statement: "When $a \approx b$, the problem of solving the linear system is ill-conditioned but the problem of computing $x + y$ is not ill-conditioned."

If we solve the linear system,

$$ax + by = 1$$

$$x = \frac{-ay}{b}$$

$$-a\left(\frac{ay}{b}\right) + by = 1$$

$$-a^2y + b^2y = b$$

$$y(b^2 - a^2) = b$$

$$\boldsymbol{y = \frac{b}{b^2 - a^2}}$$

Replacing y in first equation with y we found,

$$\boldsymbol{x = \frac{a}{a^2 - b^2}}$$

Small changes in a and b,

If a = 1.00001 and $b = 1$, $x \approx$ 50000.25 and $y \approx$ -49999.75

If a = 1.00002 and $b = 1$, $x \approx$ 25000.25 and $y \approx$ -24999.75

As we saw, small changes in a and b causes big differences in x and y. **The problem of solving the linear system is ill-conditioned.**

We computed $x + y$ in 2.b,

$$x + y = \frac{1}{a+b}$$

Small changes in a and b,

If a = 1.00001 and $b = 1$, $\quad x + y \approx 0.5$

If a = 1.00002 and $b = 1$, $\quad x + y \approx 0.5$

As we saw, small changes in in a and b does not occur significant difference in result $x + y$. **The problem of computing $x + y$ is not ill-conditioned.**

Statement is true.

3. $x^y = e^{y\,ln\,x}$, assuming $x > 0$.

$$fl(x^y) = e^{y fl(ln\,x)}$$

$$= e^{y\left((ln\,x)(1+\epsilon)\right)}$$

$$= e^{y(ln\,x + \epsilon\,ln\,x)}$$

$$= e^{y\,ln\,x} \cdot e^{y\epsilon\,ln\,x}$$

$$= x^y \cdot e^{y\epsilon\,ln\,x}$$

To calculate relative error,

$$\frac{|x^y - fl(x^y)|}{|x^y|} = \frac{\left|x^y - x^y \cdot e^{y\epsilon \ln x}\right|}{|x^y|}$$

$$= \left|1 - e^{y\epsilon\,ln\,x}\right|$$

The relative error is unbounded.