Substitution Method

Q1) $T(n) = T(n-1) + n$, $T(n) \in O(n^2)$

add n to both sides

$T(n) \leq cn^2 \Rightarrow T(n-1) \leq c(n-1)^2 \Rightarrow \underbrace{T(n-1)+n}_{\text{this is now } T(n)} \leq c(n-1)^2 + n$

make this part residual

$\Rightarrow T(n) \leq cn^2 - 2cn + c + n \leq cn^2 \Rightarrow T(n) \leq cn^2 - \underbrace{(2cn - c - n)}_{\text{make residual positive}} \leq cn^2$

es: $c=2$ $n_0=1$

we know that $T(n) \leq cn^2 - (2cn - c - n)$,
if we can ensure residual is positive $T(n) \leq cn^2$ (using trasitivity: $a \leq b$ and $b \leq c \Rightarrow a \leq c$)

Q2) $T(n) = T(\frac{n}{2}) + 1$, $T(n) \in O(\log(n))$

add +1 to both sides

$T(n) \leq c\log(n) \Rightarrow T(\frac{n}{2}) \leq c\log(\frac{n}{2}) \Rightarrow \underbrace{T(\frac{n}{2}) + 1}_{\text{this is now } T(n)} \leq c\log(\frac{n}{2}) + 1$

$\Rightarrow T(n) \leq c\log(\frac{n}{2}) + 1 \leq c\log(n) \Rightarrow T(n) \leq c\log(n) - c\log(2) + 1 \leq c\log(n)$

assume/make log in base 2

make this part residual

$\Rightarrow T(n) \leq c\log(n) - c + 1 \leq c\log(n) \Rightarrow T(n) \leq c\log(n) - \underbrace{(c-1)}_{\text{make residual positive}} \leq c\log(n)$

es: $c > 1$

with transivity

$\Rightarrow T(n) \leq c\log(n)$

Q3) $T(n) = 2T(\frac{n}{2}) + n$, $T(n) \in O(n\log(n))$

★ Rather than using tranformation like above, I will directly put in place

$T(n) \leq cn\log(n) \Rightarrow T(\frac{n}{2}) \leq c\frac{n}{2}\log(\frac{n}{2}) \Rightarrow T(n) \leq 2\left(c\frac{n}{2}\log(\frac{n}{2})\right) + n \leq cn\log(n)$

assume/make log in base 2

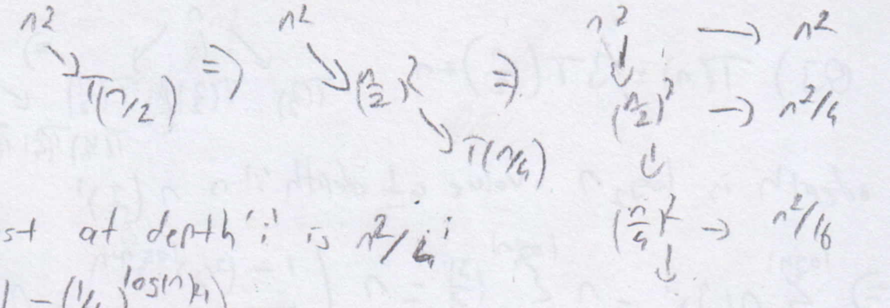
make this part residual

$\Rightarrow T(n) \leq cn(\log(n) - 1) + n \leq cn\log(n) \Rightarrow T(n) \leq cn\log(n) - cn + n \leq cn\log(n)$

$\Rightarrow T(n) \leq cn\log(n) - (cn - n) \leq cn\log(n) \Rightarrow$ as long as $c > 0$, the residual will be positive. therefore $T(n) \leq cn\log(n)$

Recursion Tree

Q1) $T(n) = T(\frac{n}{2}) + n^2$



• depth is $\log_2(n)$ • value/cost at depth 'i' is $n^2/4^i$

$$\Rightarrow \sum_{i=0}^{\log(n)} \frac{n^2}{4^i} = n^2 \sum_{i=0}^{\log(n)} \frac{1}{4^i} = n^2\left(\frac{1-(1/4)^{\log(n)+1}}{1-1/4}\right) \cong \frac{4n^2}{3}$$ (since log(n) is very big)

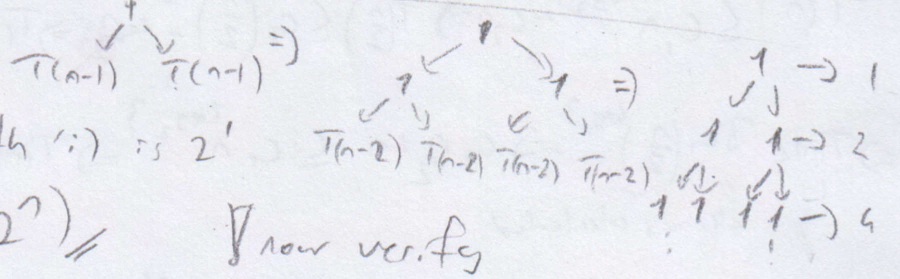
$\Rightarrow$ as we found $4n^2/3$, $T(n) \in O(n^2)$ ∀ now verify

$T(n) \le cn^2 \Rightarrow T(n/2) \le c(\frac{n}{2})^2 \Rightarrow T(n) \le c(\frac{n}{2})^2 + n^2 \le cn^2$ (add and substract $3cn^2/4$ to make cn^2)

$\Rightarrow T(n) \le \frac{cn^2}{4} + n^2 \mp \frac{3cn^2}{4} \le cn^2 \Rightarrow T(n) \le cn^2 - \underbrace{\left(\frac{3cn^2}{4} - n^2\right)}_{\text{residual}} \le cn^2$

$\Rightarrow$ as long as $c > \frac{4}{3}$, residual is positive therefore $T(n) \le cn^2$

Q2) $T(n) = 2T(n-1) + 1$



• depth is n • value at depth (i) is 2^i

$\Rightarrow \sum_{i=0}^{n} 2^i = 2^{n+1} - 1 \in O(2^n)$ ∀ now verify

$T(n) \le c2^n \Rightarrow T(n-1) \le c2^{n-1} \Rightarrow T(n) \le 2c2^{n-1} + 1 \le c2^n$

$\Rightarrow T(n) \le c2^n + 1 \le c2^n$ ∀ $c2^n + 1 > c2^n$ which is a contradiction

idea: remove lower order term

$\Rightarrow T(n) \le c_1 2^n - c_2 \Rightarrow T(n-1) \le c_1 2^{n-1} - c_2 \Rightarrow T(n) \le 2(c_1 2^{n-1} - c_2) + 1 \le c_1 2^n$

$\Rightarrow T(n) \le c_1 2^n - 2c_2 + 1 \le c_1 2^n \Rightarrow T(n) \le c_1 2^n - \underbrace{(2c_2 - 1)}_{\text{residual}} \le c_1 2^n$

$\Rightarrow$ as long as $c_2 > \frac{1}{2}$, residual is positive therefor $T(n) \le c_1 2^n$

Q3) $T(n) = 3T\left(\frac{n}{2}\right)+n$

$$n \to T\left(\tfrac{n}{2}\right)\ T\left(\tfrac{n}{2}\right)\ T\left(\tfrac{n}{2}\right) \Rightarrow \begin{array}{c} n \\ n/2 \quad n/2 \quad n/2 \\ T(n/4)\,T(n/4)\,T(n/4)\,T(n/4)\,T(n/4)\,T(n/4)\,T(n/4)\,T(n/4)\,T(n/4) \end{array} \Rightarrow \begin{array}{cl} n & \to n \\ n/2 \quad n/2 \quad n/2 & \to \frac{3n}{2} \\ n/4\ n/4\ n/4\ \ n/4\ n/4\ n/4\ \ n/4\ n/4\ n/4 & \to \frac{9n}{4} \\ \vdots & \quad \vdots \end{array}$$

• depth is $\log_2 n$ • value at depth 'i' is $n\left(\frac{3}{2}\right)^i$

$$\Rightarrow \sum_{i=0}^{\log_2 n} n\left(\frac{3}{2}\right)^i = n\sum_{i=0}^{\log_2 n}\left(\frac{3}{2}\right)^i = n\left(\frac{1-(3/2)^{\log_2 n+1}}{1-3/2}\right) = n\left(\frac{\frac{3}{2}\cdot\left(\frac{3}{2}\right)^{\log_2 n}-1}{\frac{3}{2}-1}\right) = 2n\left(\frac{3}{2}\left(\frac{3}{2}\right)^{\log_2 n}-1\right)$$

$$= 3n\left(\frac{3}{2}\right)^{\log n} - 2n = 3n\,\frac{3^{\log_2 n}}{\cancel{2^{\log_2 n}}} - 2n = \underbrace{3n^{\log 3}}_{\text{higher order}} - 2n\ ,\ T(n) \in (n^{\log_2 3})$$

∇ now verify

$$T(n) \le cn^{\log_2 3} \Rightarrow T\left(\frac{n}{2}\right) \le c\left(\frac{n}{2}\right)^{\log_2 3} \Rightarrow T(n) \le 3c\left(\frac{n}{2}\right)^{\log_2 3} + n \le cn^{\log_2 3}$$

$$\Rightarrow T(n) \le 3c\,\frac{n^{\log_2 3}}{\cancel{2^{\log_2 3}}_{3}} + n \le cn^{\log_2 3} \Rightarrow T(n) \le cn^{\log_2 3}, n \le cn^{\log_2 3}$$

∇ contradiction, remove lower order term

$$T(n) \le c_1 n^{\log_2 3} - c_2 n \Rightarrow T\left(\frac{n}{2}\right) \le c_1\left(\frac{n}{2}\right)^{\log_2 3} - \frac{c_2 n}{2} \Rightarrow T(n) \le 3\left[c_1\left(\frac{n}{2}\right)^{\log_2 3} - \frac{c_2 n}{2}\right] + n \le c_1 n^{\log_2 3}$$

$$\Rightarrow T(n) \le \underbrace{3c_1\left(\frac{n}{2}\right)^{\log_2 3}}_{\text{already calculated}} - \frac{3c_2 n}{2} + n \le c_1 n^{\log_2 3} \Rightarrow T(n) \le c_1 n^{\log_2 3} - \left(\frac{3c_2 n}{2} - n\right) \le c_1 n^{\log_2 3}$$

$\Rightarrow$ as long as $c_2 > \frac{2}{3}$, residual will be positive therefor $T(n) \in \Theta(n^{\log_2 3})$

Q4) $T(n) = T\left(\frac{n}{2}\right) + T(n/4) + T(n/8) + n$

$$n \to T\left(\tfrac{n}{2}\right)\ T(n/4)\ T(n/8) \Rightarrow \begin{array}{c} n \\ \frac{n}{2} \quad n/4 \quad n/8 \\ T\left(\frac{n}{4}\right)\,T(n/8)\,T(n/16)\,T(n/8)\,T(n/16)\,T(n/32)\,T(n/16)\,T(n/32)\,T(n/64) \end{array} \Rightarrow \begin{array}{cl} n & \to n \\ \frac{n}{2} \quad \frac{n}{4} \quad n/8 & \to 7n/8 \\ n/4\ n/8\ n/16\ \ n/8\ n/16\ n/32\ \ n/16\ n/32\ n/64 & \to \frac{45n}{64} \\ \vdots & \end{array}$$

• depth is $\log_2 n$ since the max depth will be on $T(n/2)$ side always (we use O not Θ)

• value at depth 'i' is $n\left(\frac{7}{8}\right)^i$

$$\Rightarrow \sum_{i=0}^{\log_2 n} n(7/8)^i = n\sum_{i=0}^{\log_2 n}(7/8)^i = n\left(\frac{1-\overbrace{7/8^{\log(n)+1}}^{\text{converges to } 0}}{1-7/8}\right) \cong 8n\ ,\ T(n) \in O(n)$$

$\triangle$ You can verify easily

Master Theorem

Q1) $T(n) = \underbrace{2T\left(\frac{n}{4}\right)}_{g(n)} + 1 \Rightarrow f(n) \in \Theta(1)$, $a=2, b=4 \Rightarrow g(n) = n^{\log_4 2} = \sqrt{n} \in \underbrace{\Theta(\sqrt{n})}_{\text{dominates}}$

$\Rightarrow T(n) \in \Theta(\sqrt{n})$

Q2) $T(n) = \overbrace{2T\left(\frac{n}{4}\right)}^{g(n)} + \sqrt{n} \Rightarrow f(n) \in \underline{\Theta(\sqrt{n})}$, $a=2, b=4 \Rightarrow g(n) = n^{\log_4 2} = \sqrt{n} \in \underbrace{\Theta(\sqrt{n})}_{\text{equal}}$

$\Rightarrow T(n) \in \Theta(\sqrt{n} \log n)$

Q3) $T(n) = 2T(n/4) + n^2 \Rightarrow f(n) \underbrace{\in \Theta(n^2)}_{\text{dominates}}$, $a=2, b=4 \Rightarrow g(n) = n^{\log_4 2} = \sqrt{n} \in \Theta(\sqrt{n})$

$\Rightarrow T(n) \in \Theta(n^2)$

Q4) $T(n) = T\left(\frac{7n}{10}\right) + n \Rightarrow f(n) \in \underbrace{\Theta(n)}_{\text{dominates}}$, $a=1$, $b=\frac{10}{7} \Rightarrow g(n) = n^{\overbrace{\log_{10/7} 1}^{0}} = 1 \in \Theta(1)$

$\Rightarrow T(n) \in \Theta(n)$

Q5) $T(n) = 7T(n/3) + n^2 \Rightarrow f(n) \underbrace{\in \Theta(n^2)}_{\text{dominates}}$, $a=7, b=3 \Rightarrow g(n)\ n^{\overbrace{\log_3 7}^{\sim 1.77}} < n^{\overbrace{\log_3 9}^{2}} = n^2$

$\Rightarrow T(n) \in \Theta(n^2)$

Q6) $T(n) = 7T(n/2) + n^2 \Rightarrow f(n) \in \Theta(n^2)$, $a=7, b=2 \Rightarrow \underbrace{g(n) = n^{\overbrace{\log_2 7}^{\sim 2.8}}}_{\text{dominates}} > n^{\overbrace{\log_2 4}^{2}} = n^2$

$\Rightarrow T(n) \in \Theta(n^{\log_2 7})$

Q7) $T(n) = \underbrace{4T(\sqrt{n})}_{g(n)} + \underbrace{\log(n)}_{f(n)} \Rightarrow$ let $n = 4^m \Rightarrow m = \log_4 n$, $\sqrt{n} = 2^m$

and $S(m) = T(4^m)$

$\Rightarrow T(4^m) = 4T(4^{m/2}) + m \Rightarrow S(m) = \underbrace{4S(m/2)}_{g'(m)} + \underbrace{m}_{f'(m)} \Rightarrow$ $f'(n) \in \Theta(m)$, $a'=4$, $b'=2$

$g'(m) = m^{\log_2 4} = m^2 \in \underbrace{\Theta(m^2)}_{\text{dominates}}$

put $m = \log_4 n$

$S(m) \in \Theta(m^2) \Rightarrow T(n) \in \Theta\left((\log_4 n)^2\right)$ //

Q Caesar) • Strassen algorithm is $\Theta(n^{\log_2 7}) \approx \Theta(n^{2.8074}) < \underbrace{\Theta(n^3)}_{\text{normal matrix mult}}$

$T(n) = \overbrace{a T(n/4)}^{g(n)} + \overbrace{\Theta(n^2)}^{f(n)}$

this part should be dominate since we look for max a valu

$g(n) = n^{\log_4 a} < n^{\log_2 7} = n^{\log_4 49}$

$\Rightarrow a < 49 \Rightarrow$ max a value is 48