BLG335e 2022 Fall Recitation 1

Erdi Sarıtaş

saritas21@itu.edu.tr

Outline

- Prerequisites
  - Proofing Techniques
  - Probability
- Sorting
  - Insertion Sort
  - Merge Sort
  - Growth of Functions
    - Asymptotic Notation
    - Comparison of Functions
  - Comparison

Proofing Techniques

- Mathematical Induction
  1. Basis Step: $P(n_0)$ is true
  2. Induction Step: $P(n_k) \rightarrow P(n_{k+1})$

$$\sum_{x=1}^{n} x = \frac{1}{2} n(n+1)$$

- Proof by Contradiction
  - Assume P(n) is false
  - Show assumption is wrong

$\sqrt{2}$ is irrational

$$\sum_{x=1}^{n} x = \frac{1}{2}\, n\,(n+1)$$

$$Verify\!: P(1) = \ 1\ (true)$$

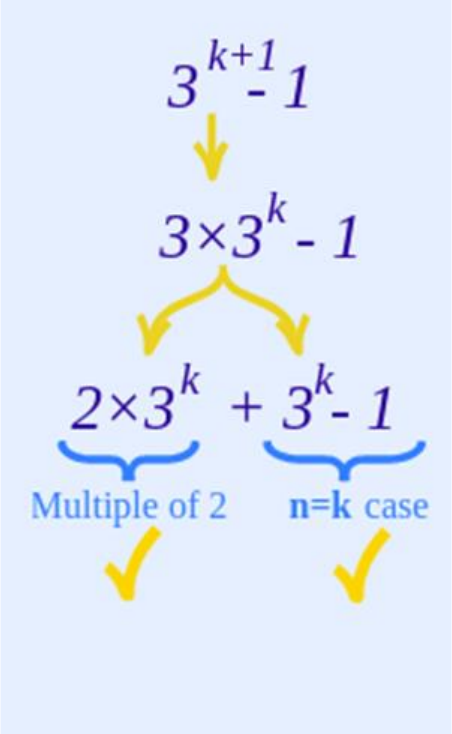
$$Induction\!: P(n)\!: 1\ +\ 2\ +\ 3\ +\ \ldots\ +\ n\ = \frac{n(n+1)}{2}$$

$$P(n+1)\!: 1 +\ 2\ +\ 3\ +\ \ldots + n\ +\ (n\ +\ 1) = \frac{(n+1)(n+2)}{2}$$

$$\frac{n(n+1)}{2} +\ (n+1)\ ?= \frac{(n+1)(n+2)}{2}\ (true)$$

- 3^k -1 is multiple of 2

n=1: 3-1 = 2 is multiple of 2



$$Let's\ assume\ \sqrt{2}\ is\ rational.$$

$$Then\ \sqrt{2}\ should\ be\ written\ as\ \frac{p}{q}\ \ and\ \ 2\ = \frac{p^2}{q^2}$$

$$2q^2 = p^2\ and\ p^2\ is\ even, thus\ p\ is\ even\ (p = 2m)$$

$$2q^2 = 4m^2\ and\ q^2 = 2m^2\ meaning\ q^2 is\ also\ even$$

$$If\ both\ p^2\ and\ q^2\ are\ even\ there\ is\ no\ possibility\ to\ get\ 2\ by\ dividing\ p^2 by\ q^2.$$

$$Thus, \sqrt{2}\ cannot\ be\ rational, it\ is\ irrational.$$

There are infinitely many prime numbers

Proof. For the sake of contradiction, suppose there are only finitely many prime numbers. Then we can list all the prime numbers as $p_1, p_2, p_3, \ldots p_n$, where $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7$ and so on. Thus p_n is the nth and largest prime number. Now consider the number $a = (p_1 p_2 p_3 \cdots p_n) + 1$, that is, a is the product of all prime numbers, plus 1. Now a, like any natural number, has at least one prime divisor, and that means $p_k \mid a$ for at least one of our n prime numbers p_k. Thus there is an integer c for which $a = cp_k$, which is to say

$$(p_1 p_2 p_3 \cdots p_{k-1} p_k p_{k+1} \cdots p_n) + 1 = cp_k.$$

Dividing both sides of this by p_k gives us

$$(p_1 p_2 p_3 \cdots p_{k-1} p_{k+1} \cdots p_n) + \frac{1}{p_k} = c,$$

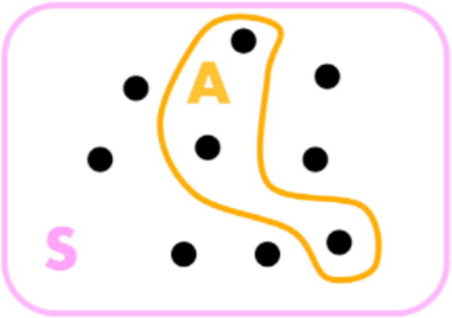
so

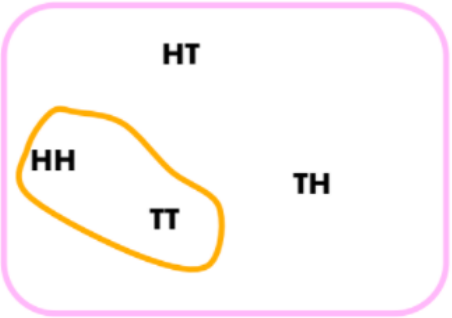
$$\frac{1}{p_k} = c - (p_1 p_2 p_3 \cdots p_{k-1} p_{k+1} \cdots p_n).$$

The expression on the right is an integer, while the expression on the left is not an integer. This is a contradiction. ■

Probability

- $Pr[A] \geq 0$ for every event A
- $Pr[S] = 1$
- $Pr[\emptyset] = 0$
- If $A \subseteq B$, then $Pr[A] \leq Pr[B]$
- $Pr[S — A] = 1 — Pr[A]$ (complement)
- $Pr[A \cup B] = Pr[A] + Pr[B] – Pr[A \cap B] \leq Pr[A] + Pr[B]$
- Mutually exclusive events $Pr[A \cap B] = 0$
- Independed events $Pr[A \cap B] = Pr[A]Pr[B]$





Insertion Sort

```
INSERTION-SORT(A)
for j = 1 to A.length
    key = A[j]
    // Insert A[j] into the sorted sequence A[1 .. j − 1].
    i = j − 1
    while i ≥ 0 and A[i] > key
        A[i + 1] = A[i]
        i = i − 1
    A[i + 1] = key
```

Merge Sort

```
MERGE-SORT(A, p, r)
if p < r
    q = ⌊(p + r)/2⌋
    MERGE-SORT(A, p, q)
    MERGE-SORT(A, q + 1, r)
    MERGE(A, p, q, r)
```

```
MERGE(A, p, q, r)
n1 = q - p + 1
n2 = r - q
let L[1 .. n1 + 1] and R[1 .. n2 + 1] be new arrays
for i = 1 to n1
    L[i] = A[p + i - 1]
for j = 1 to n2
    R[j] = A[q + j]
L[n1 + 1] = ∞
R[n2 + 1] = ∞
i = 1
j = 1
for k = p to r
    if L[i] ≤ R[j]
        A[k] = L[i]
        i = i + 1
    else A[k] = R[j]
        j = j + 1
```

Asymptotic Notation

$$f(n) = o(g(n)) \approx a < b$$
$$f(n) = O(g(n)) \approx a \leq b$$
$$f(n) = \Theta(g(n)) \approx a = b$$
$$f(n) = \Omega(g(n)) \approx a \geq b$$
$$f(n) = \omega(g(n)) \approx a > b$$

Comparison of Functions

$$O(2) < O(\log(x)) < O(x) < O(x\log(x)) < O(x^2) < O(2^x) < O(x!) < O(x^x)$$

Comparision

- Speed: Merge > Insertion
- Memory: Insertion > Merge

Case	Insertion	Merge
best	O(n)	O(nlog(n))
worst	O(n^2)	O(nlog(n))
average	O(n^2)	O(nlog(n))