EHB 211E: Basics of Electrical Circuits

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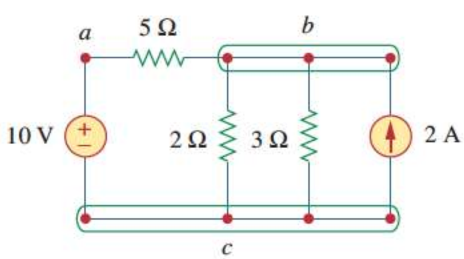
1

Network Topology: Nodes, Branches, and Loops

Elements of an electrical circuit can be interconnected in several ways. Some terminology is given below:

- A **branch** represents a single (*two-terminal*) element such as a voltage source or a resistor below circuit has 5 branches.
- A **node** is the point of connection between two or more branches (if a short circuit, a connection wire, connects two nodes, the two nodes constitute a single node) below circuit has 3 nodes (a, b, c).
- A **loop** is any closed path in a circuit. (starting from a node, and returning to the same node without passing through any node more than once). A loop is *independent*, if it contains at least one branch which is not part of any other independent loop. Independent loops (l=3 for the circuit below) result In independent set of equations.



A network of b branches, n nodes, and l independent loops will satisfy the fundamental theorem of network topology:

$$\boxed{b = l + n - 1}$$

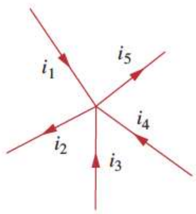
2

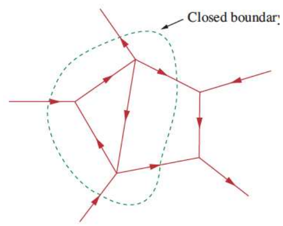
Kirchhoff's Laws - KCL

Kirchhoff's Current Law (KCL) states that algebraic sum of currents entering a node (or closed boundary) is zero.

$$\sum_{n=1}^{N} i_n = 0$$

Where, N is the number of branches entering the node, i_n is the n_{th} current entering the node





Kirchhoff's Laws - KCL

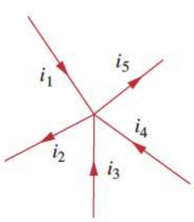
The algebraic sum of currents at a node:

$$i_T(t) = i_1(t) + i_2(t) + i_3(t) + \cdots$$

integrate

$$q_T(t) = q_1(t) + q_2(t) + q_3(t) + \cdots$$

Conservation of charge: Sum of electrical charge at a node must not change $\rightarrow i_T(t) = 0$

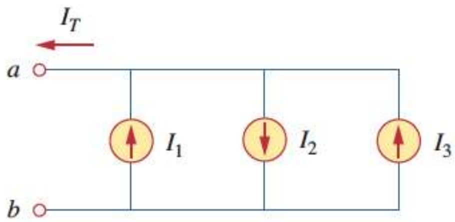


$$i_1 + (-i_2) + i_3 + i_4 + (-i_5) = 0$$

$$i_1 + i_3 + i_4 = i_2 + i_5$$

KCL (alternative): The sum of currents entering a node is equal to the sum of currents leaving a node.

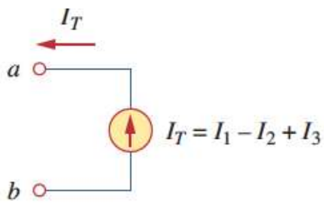
Kirchhoff's Laws - KCL



$$I_T + I_2 = I_1 + I_3$$

$$I_T = I_1 - I_2 + I_3$$

Equivalent circuit



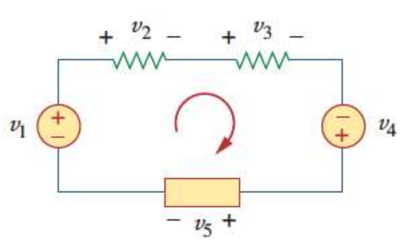
5

Kirchhoff's Laws - KVL

Kirchhoff's Voltage Law (KVL) states that algebraic sum of all voltages around a closed path (loop) is zero.

$$\boxed{\sum_{m=1}^{M} v_m = 0}$$

Where, M is the number of voltages in the loop, and v_m is the m_{th} voltage.



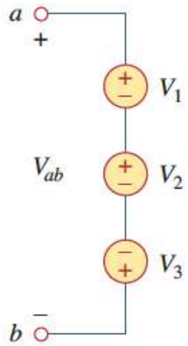
$$-v_1 + v_2 + v_3 - v_4 + v_5 = 0$$

$$v_2 + v_3 + v_5 = v_1 + v_4$$

Sum of voltage drops = sum of voltage rises.

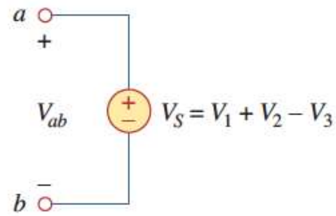
6

Kirchhoff's Laws - KVL



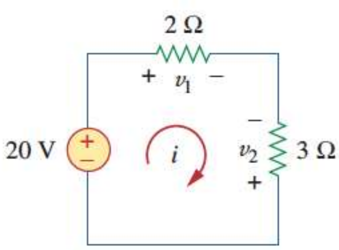
$$-V_{ab} + V_1 + V_2 - V_3 = 0$$

$$V_{ab} = V_1 + V_2 - V_3$$



Exercise

For the circuit below, Find v_1 and v_2



$v_1 = 2i, \quad v_2 = -3i$ Ohm's Law

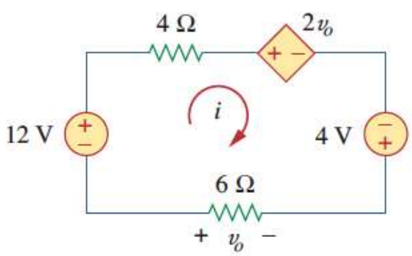
$-20 + v_1 - v_2 = 0$ KVL

$-20 + 2i + 3i = 0 \quad \Rightarrow \quad i = 4$ A

$v_1 = 8$ V, $\quad v_2 = -12$ V

Exercise

For the circuit below, Find v_0 and i



$$-12 + 4i + 2v_o - 4 + 6i = 0$$ KVL

$$v_o = -6i$$ Ohm's Law

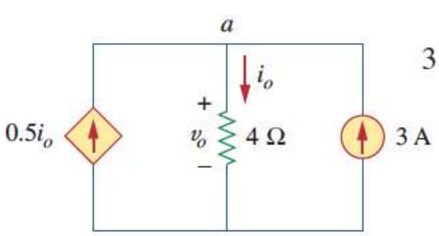
$$-16 + 10i - 12i = 0 \quad \Rightarrow \quad i = -8 \text{ A}$$

$$v_o = 48 \text{ V.}$$

9

Exercise

For the circuit below, Find v_0 and i_0



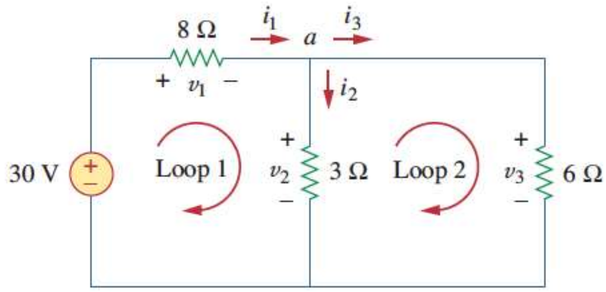
$$3 + 0.5i_o = i_o \quad \Rightarrow \quad i_o = 6 \text{ A}$$ KCL

$$v_o = 4i_o = 24 \text{ V}$$ Ohm's Law

10

Exercise

For the circuit below, Find the voltages and currents



$$v_1 = 8i_1, \qquad v_2 = 3i_2, \qquad v_3 = 6i_3$$

$$i_1 - i_2 - i_3 = 0 \qquad (1)$$

$$-30 + v_1 + v_2 = 0$$

$$-30 + 8i_1 + 3i_2 = 0$$

$$i_1 = \frac{(30 - 3i_2)}{8} \qquad (2)$$

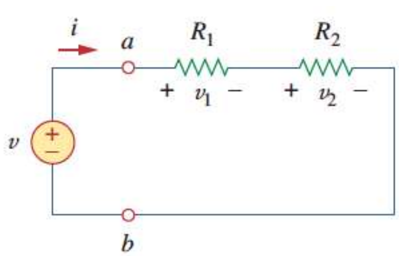
$$-v_2 + v_3 = 0 \quad \Rightarrow \quad v_3 = v_2$$

$$6i_3 = 3i_2 \quad \Rightarrow \quad i_3 = \frac{i_2}{2} \qquad (3)$$

$$\frac{30 - 3i_2}{8} - i_2 - \frac{i_2}{2} = 0 \qquad (1,2,3) \qquad i_2 = 2 \text{ A}$$

$$i_1 = 3 \text{ A}, \quad i_3 = 1 \text{ A}, \quad v_1 = 24 \text{ V}, \quad v_2 = 6 \text{ V}, \quad v_3 = 6 \text{ V}$$

Series Resistors and Voltage Division



$$v_1 = iR_1, \qquad v_2 = iR_2$$

$$v = v_1 + v_2 = i(R_1 + R_2) \qquad i = \frac{v}{R_1 + R_2}$$

$$v = iR_{eq}$$

$$R_{eq} = R_1 + R_2$$

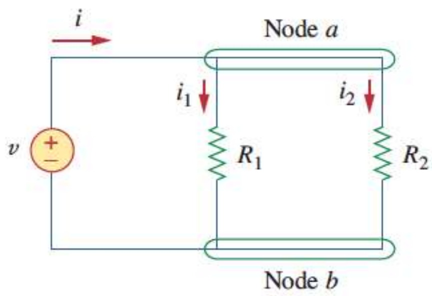
VOLTAGE DIVISION

Equivalent resistance of any number of serially connected resistors is the sum of the individual resistances

$$v_1 = \frac{R_1}{R_1 + R_2} v, \qquad v_2 = \frac{R_2}{R_1 + R_2} v$$

$$R_{eq} = R_1 + R_2 + \cdots + R_N = \sum_{n=1}^{N} R_n$$

Parallel Resistors and Current Division



$$v = i_1 R_1 = i_2 R_2$$

$$i_1 = \frac{v}{R_1}, \qquad i_2 = \frac{v}{R_2}$$

$$i = i_1 + i_2$$

$$i = \frac{v}{R_1} + \frac{v}{R_2} = v\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{v}{R_{eq}}$$

$$\boxed{R_{eq} = \frac{R_1 R_2}{R_1 + R_2}}$$

CURRENT DIVISION

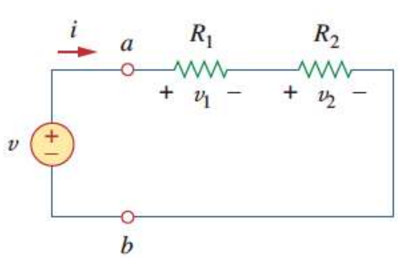
Equivalent resistance of two parallel resistors is equal to the product of their resistances divided by their sum

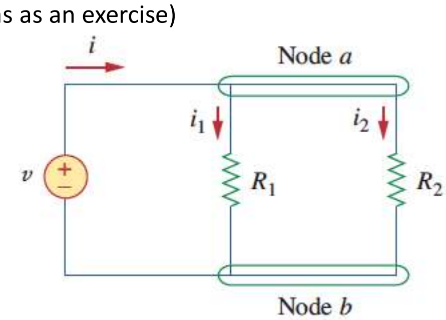
$$\boxed{i_1 = \frac{R_2\, i}{R_1 + R_2}, \qquad i_2 = \frac{R_1\, i}{R_1 + R_2}}$$

$$\boxed{\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N}}$$

Equivalent Conductance

equivalent conductance for series / parallel conductance shows opposite behavior to the calculation of equivalent resistance (derive the formulas as an exercise)



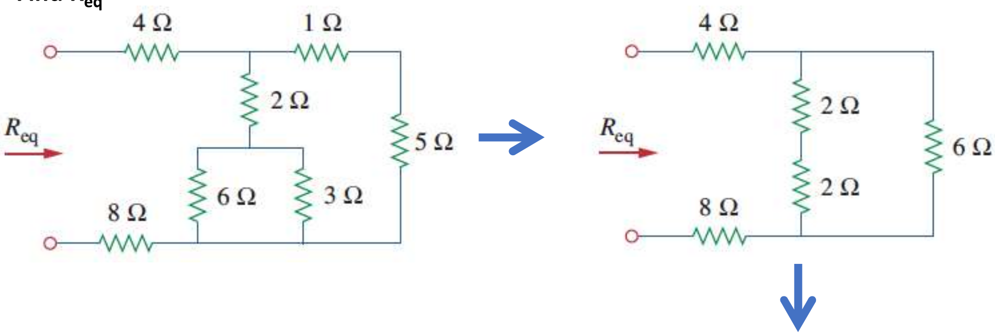


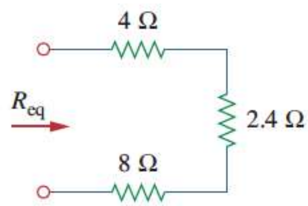
$$\boxed{\frac{1}{G_{eq}} = \frac{1}{G_1} + \frac{1}{G_2} + \frac{1}{G_3} + \cdots + \frac{1}{G_N}}$$

$$\boxed{G_{eq} = G_1 + G_2 + G_3 + \cdots + G_N}$$

Exercise

Find R_{eq}



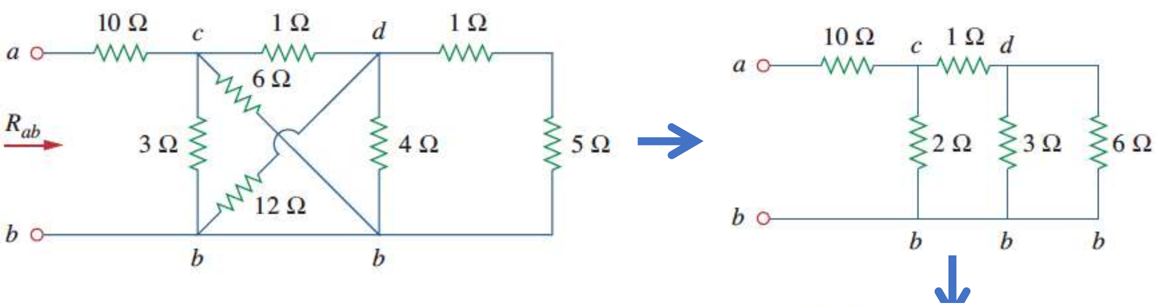


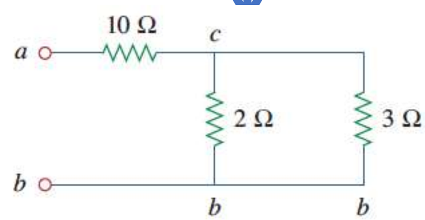
$$R_{eq} = 4\ \Omega + 2.4\ \Omega + 8\ \Omega = 14.4\ \Omega$$

15

Exercise

Find R_{ab}



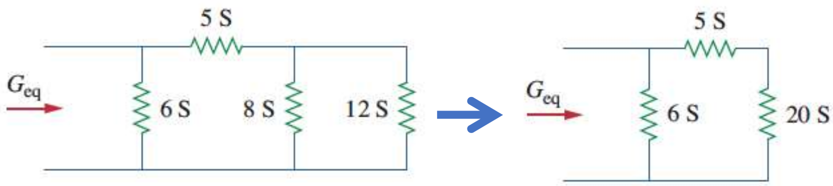


$$R_{ab} = 10 + 1.2 = 11.2\ \Omega$$

16

Exercise

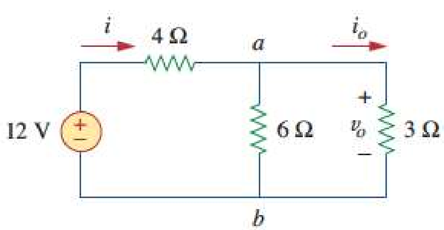
Find G_{eq}



G_{eq} = (20*5)/(20+5) + 6 = **10 S**

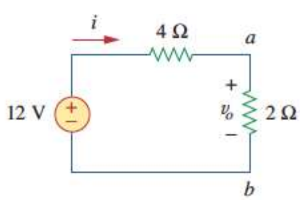
Exercise

Find v_0, i_0. Calculate the power dissipated by the 3 Ω resistor.



$$v_o = \frac{2}{2+4}(12\text{ V}) = 4\text{ V}$$

Voltage division



$$v_o = 3i_o = 4 \quad \Rightarrow \quad i_o = \frac{4}{3}\text{ A}$$

Ohm's Law

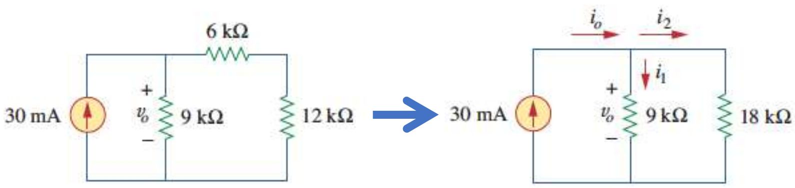
$$i_o = \frac{6}{6+3}i = \frac{2}{3}(2\text{ A}) = \frac{4}{3}\text{ A}$$

Alternative way of finding i_0: current division

$$p_o = v_o i_o = 4\left(\frac{4}{3}\right) = 5.333\text{ W}$$

Exercise

Find v_0, power supplied by the current source, power absorbed by each resistor



$$i_1 = \frac{18{,}000}{9{,}000 + 18{,}000}(30 \text{ mA}) = 20 \text{ mA}$$

$$i_2 = \frac{9{,}000}{9{,}000 + 18{,}000}(30 \text{ mA}) = 10 \text{ mA}$$

$$v_o = 9{,}000 i_1 = 18{,}000 i_2 = 180 \text{ V}$$

Alternatively: $v_0 = 30 * R_{eq} = 180$ V

Power supplied by the source:

$$p_o = v_o i_o = 180(30) \text{ mW} = 5.4 \text{ W}$$

Power absorbed by the 12-kΩ resistor is

$$p = iv = i_2(i_2 R) = i_2^2 R = (10 \times 10^{-3})^2\,(12{,}000) = 1.2 \text{ W}$$

Power absorbed by the 6-kΩ resistor is

$$p = i_2^2 R = (10 \times 10^{-3})^2\,(6{,}000) = 0.6 \text{ W}$$

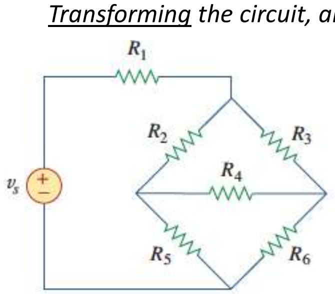
Power absorbed by the 9-kΩ resistor is

$$p = v_o i_1 = 180(20) \text{ mW} = 3.6 \text{ W}$$

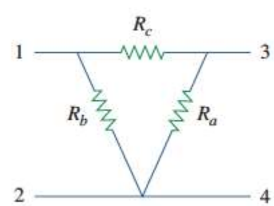
Tellegen's theorem is satisfied!
Power supplied = total power dissipated

Wye-Delta Transformation

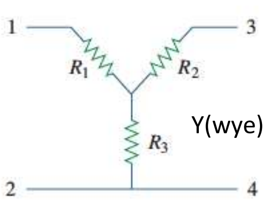
Conditions may occur (like below), where resistors are neither in series nor parallel!
<u>Transforming</u> the circuit, allows finding equivalent resistance.



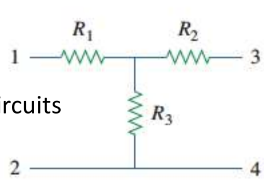
Δ (Delta) – Π (pi) circuits

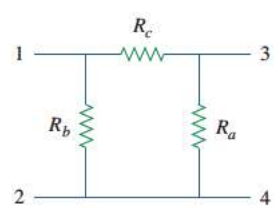


(a)

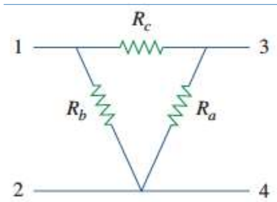


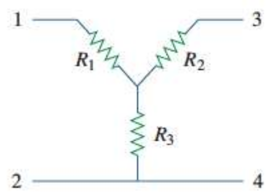
Y(wye) – T circuits





Delta-Wye conversion





$R_{12}(Y) = R_1 + R_3$

$R_{12}(\Delta) = R_b \parallel (R_a + R_c)$

$R_{12}(Y) = R_{12}(\Delta)$

$$R_{12} = R_1 + R_3 = \frac{R_b(R_a + R_c)}{R_a + R_b + R_c} \quad (1)$$

$$R_{13} = R_1 + R_2 = \frac{R_c(R_a + R_b)}{R_a + R_b + R_c} \quad (2)$$

$$R_{34} = R_2 + R_3 = \frac{R_a(R_b + R_c)}{R_a + R_b + R_c} \quad (3)$$

(1-3)

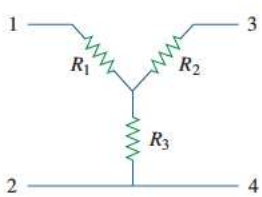
$$R_1 - R_2 = \frac{R_c(R_b - R_a)}{R_a + R_b + R_c}$$

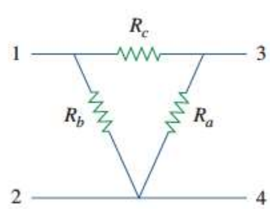
$$\boxed{R_1 = \frac{R_b R_c}{R_a + R_b + R_c}}$$

$$\boxed{R_2 = \frac{R_c R_a}{R_a + R_b + R_c}} \quad \boxed{R_3 = \frac{R_a R_b}{R_a + R_b + R_c}}$$

21

Wye-Delta conversion





(1)
$$R_1 = \frac{R_b R_c}{R_a + R_b + R_c}$$

(2)
$$R_2 = \frac{R_c R_a}{R_a + R_b + R_c}$$

(3)
$$R_3 = \frac{R_a R_b}{R_a + R_b + R_c}$$

$$R_1 R_2 + R_2 R_3 + R_3 R_1 = \frac{R_a R_b R_c (R_a + R_b + R_c)}{(R_a + R_b + R_c)^2} = \frac{R_a R_b R_c}{R_a + R_b + R_c}$$

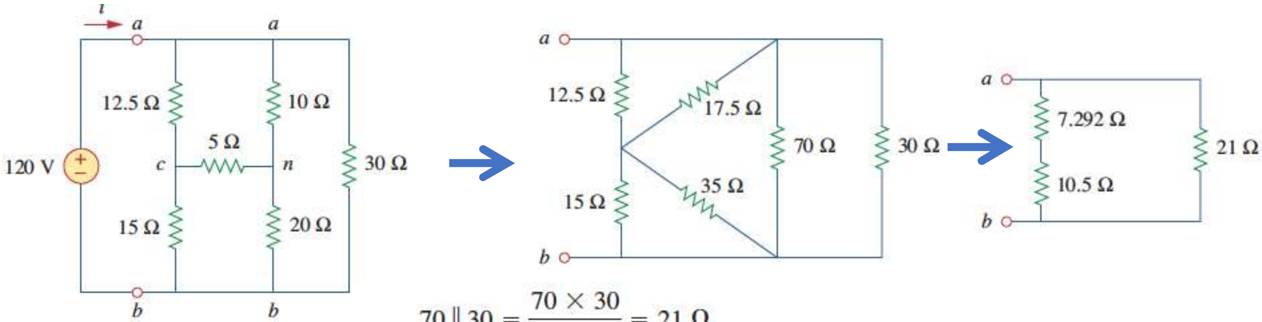
Divide by eq. 1,2,3

$$\boxed{R_a = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_1}} \quad \boxed{R_b = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_2}} \quad \boxed{R_c = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_3}}$$

22

Exercise

Obtain R_{ab} for below circuit.



$$70 \parallel 30 = \frac{70 \times 30}{70 + 30} = 21\ \Omega$$

$$12.5 \parallel 17.5 = \frac{12.5 \times 17.5}{12.5 + 17.5} = 7.292\ \Omega$$

$$15 \parallel 35 = \frac{15 \times 35}{15 + 35} = 10.5\ \Omega$$

$$R_{ab} = (7.292 + 10.5) \parallel 21 = \frac{17.792 \times 21}{17.792 + 21} = \mathbf{9.632\ \Omega}$$

Nodal Analysis

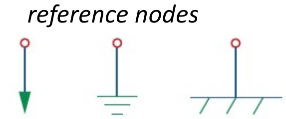
Nodal analysis provides a general procedure for analyzing circuits using nodes.
Steps to determine node voltages are:

1) Select a node as the reference node. Assign v_1, v_2,…, v_{n-1}.
 The voltages are referenced with respect to the reference node.

2) Apply KCL to each n-1 non-reference node.
 Use Ohm's law to express branch currents in terms of node voltages

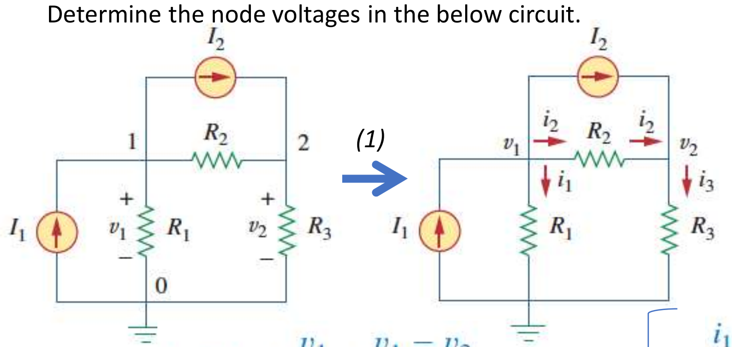
3) Solve the resulting simultaneous equations to obtain the unknown node voltages

reference nodes



Exercise

Determine the node voltages in the below circuit.



(2) $I_1 = I_2 + i_1 + i_2$

$I_2 + i_2 = i_3$

$$\boxed{i = \frac{v_{\text{higher}} - v_{\text{lower}}}{R}}$$

$$i_1 = \frac{v_1 - 0}{R_1} \quad \text{or} \quad i_1 = G_1 v_1$$

$$i_2 = \frac{v_1 - v_2}{R_2} \quad \text{or} \quad i_2 = G_2(v_1 - v_2)$$

$$i_3 = \frac{v_2 - 0}{R_3} \quad \text{or} \quad i_3 = G_3 v_2$$

$$I_1 = I_2 + \frac{v_1}{R_1} + \frac{v_1 - v_2}{R_2}$$

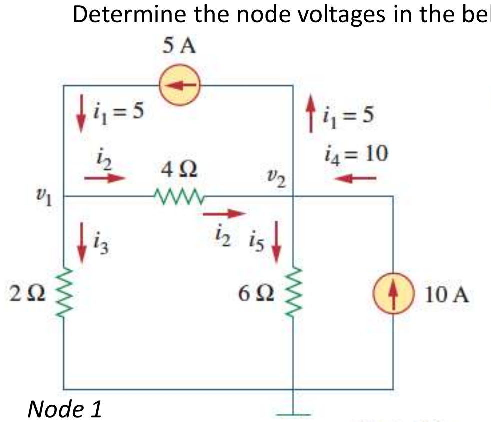
$$I_2 + \frac{v_1 - v_2}{R_2} = \frac{v_2}{R_3}$$

$$I_1 = I_2 + G_1 v_1 + G_2(v_1 - v_2)$$

$$I_2 + G_2(v_1 - v_2) = G_3 v_2$$

Exercise

Determine the node voltages in the below circuit.



Node 2

$$i_2 + i_4 = i_1 + i_5 \quad \Rightarrow \quad \frac{v_1 - v_2}{4} + 10 = 5 + \frac{v_2 - 0}{6}$$

(x 12) $3v_1 - 3v_2 + 120 = 60 + 2v_2$

$-3v_1 + 5v_2 = 60$ *(2)*

Using Equations 1 & 2

$4v_2 = 80 \quad \Rightarrow \quad v_2 = 20 \text{ V}$

$3v_1 - 20 = 20 \quad \Rightarrow \quad v_1 = \frac{40}{3} = 13.333 \text{ V}$

Node 1

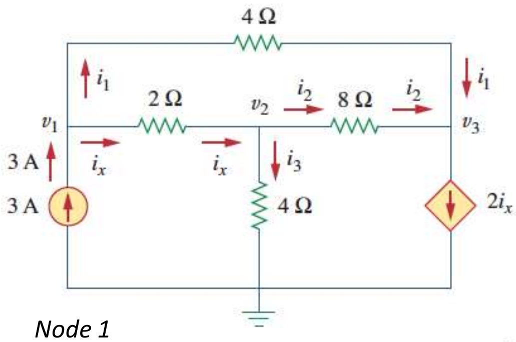
$$i_1 = i_2 + i_3 \quad \Rightarrow \quad 5 = \frac{v_1 - v_2}{4} + \frac{v_1 - 0}{2}$$

$20 = v_1 - v_2 + 2v_1$ *(x 4)*

$3v_1 - v_2 = 20$ *(1)*

Exercise

Determine the node voltages in the below circuit.



Node 1

$$3 = i_1 + i_x$$

$$3 = \frac{v_1 - v_3}{4} + \frac{v_1 - v_2}{2}$$

$$3v_1 - 2v_2 - v_3 = 12 \qquad \textit{(x 4)}$$

Node 2

$$i_x = i_2 + i_3 \quad \Rightarrow \quad \frac{v_1 - v_2}{2} = \frac{v_2 - v_3}{8} + \frac{v_2 - 0}{4}$$

$$-4v_1 + 7v_2 - v_3 = 0 \qquad \textit{(x 8)}$$

Node 3

$$i_1 + i_2 = 2i_x \quad \Rightarrow \quad \frac{v_1 - v_3}{4} + \frac{v_2 - v_3}{8} = \frac{2(v_1 - v_2)}{2}$$

$$2v_1 - 3v_2 + v_3 = 0$$

3 equations, 3 unknowns

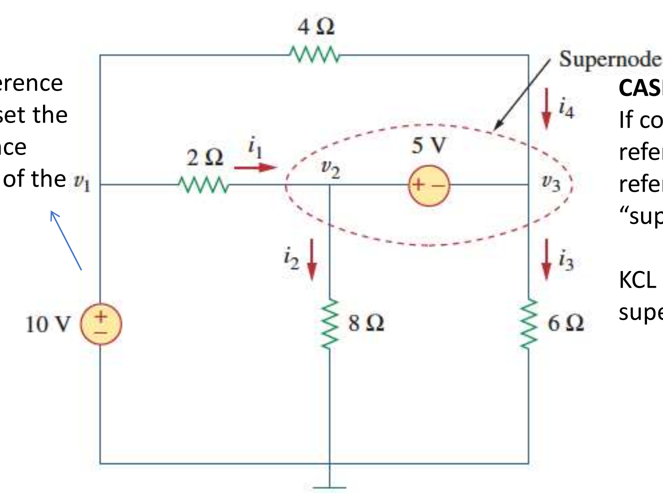
$$v_1 = 4.8 \text{ V}, \qquad v_2 = 2.4 \text{ V}, \qquad v_3 = -2.4 \text{ V}$$

27

Nodal Analysis with Voltage Sources

CASE 1:
If connected between reference and non-reference node, set the voltage of the non-reference node equal to the voltage of the voltage source

$$v_1 = 10 \text{ V}$$



CASE 2:
If connected between two non-reference nodes, the two non-reference node forms a "supernode".

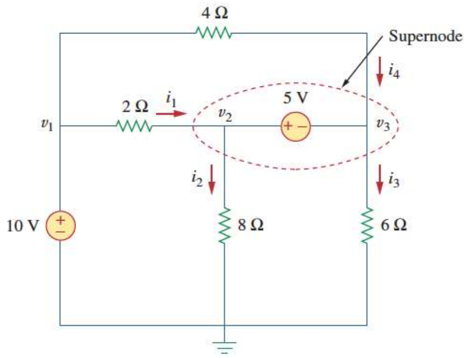
KCL must be satisfied at the supernode

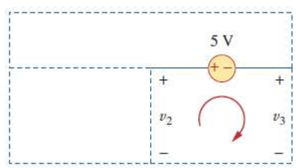
A supernode is formed by enclosing a (dependent or independent) voltage source connected between two nonreference nodes and any elements connected in parallel with it.

28

Nodal Analysis with Voltage Sources

Determine the node voltages in the below circuit.





KCL at the supernode

$$i_1 + i_4 = i_2 + i_3$$

$$\frac{v_1 - v_2}{2} + \frac{v_1 - v_3}{4} = \frac{v_2 - 0}{8} + \frac{v_3 - 0}{6}$$

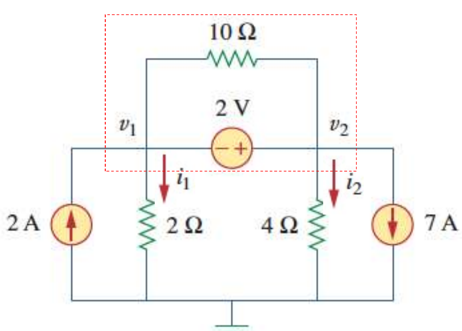
KVL at the supernode

$$-v_2 + 5 + v_3 = 0 \quad \Rightarrow \quad v_2 - v_3 = 5$$

Solution: v_1 is already known
v_2 and v_3 are dependent
1 KCL equation, 1 unknown

Nodal Analysis with Voltage Sources

Determine the node voltages in the below circuit.



KCL at the supernode

$$2 = i_1 + i_2 + 7$$

$$2 = \frac{v_1 - 0}{2} + \frac{v_2 - 0}{4} + 7 \quad \Rightarrow \quad 8 = 2v_1 + v_2 + 28$$

$$v_2 = -20 - 2v_1 \quad (1)$$

KVL at the supernode

$$-v_1 - 2 + v_2 = 0 \quad \Rightarrow \quad v_2 = v_1 + 2 \quad (2)$$

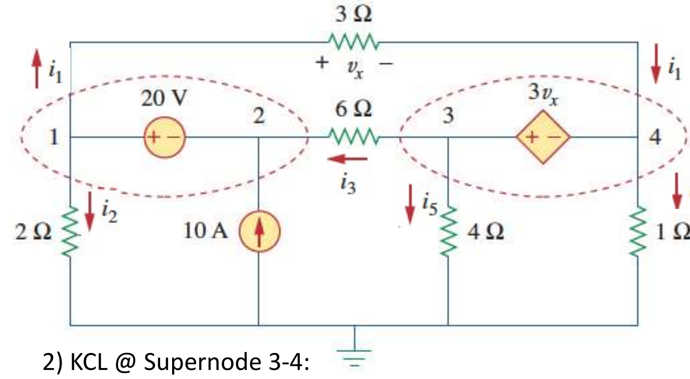
From 1 & 2:

$$v_1 = -7.333 \text{ V}$$

$$v_2 = v_1 + 2 = -5.333 \text{ V}$$

Nodal Analysis with Voltage Sources

Determine the node voltages in the below circuit.



2 supernodes!

1) KCL @ Supernode 1-2:

$$i_3 + 10 = i_1 + i_2$$

$$\frac{v_3 - v_2}{6} + 10 = \frac{v_1 - v_4}{3} + \frac{v_1}{2}$$

$$5v_1 + v_2 - v_3 - 2v_4 = 60$$ (re-arrange)

2) KCL @ Supernode 3-4:

$$i_1 = i_3 + i_4 + i_5 \quad \Rightarrow \quad \frac{v_1 - v_4}{3} = \frac{v_3 - v_2}{6} + \frac{v_4}{1} + \frac{v_3}{4}$$

$$4v_1 + 2v_2 - 5v_3 - 16v_4 = 0$$

3)KVL for 3 independent loops

$$v_1 - v_2 = 20$$

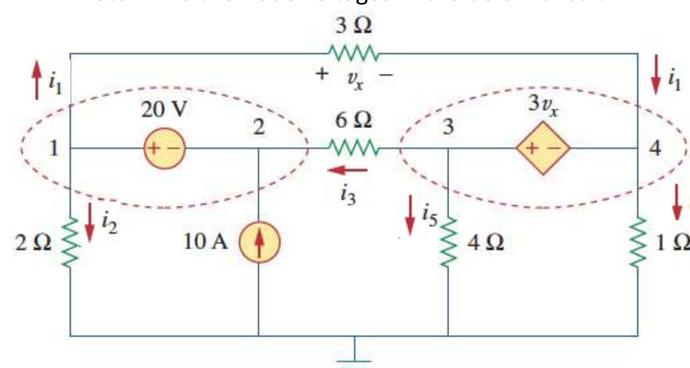
$$-v_3 + 3v_x + v_4 = 0$$

$$v_x = v_1 - v_4$$

$$3v_1 - v_3 - 2v_4 = 0$$

Nodal Analysis with Voltage Sources

Determine the node voltages in the below circuit.



3)KVL for 3 independent loops

i) $v_1 - v_2 = 20$

ii) $-v_3 + 3v_x + v_4 = 0$

$$v_x = v_1 - v_4$$

$$3v_1 - v_3 - 2v_4 = 0$$

iii) $v_x - 3v_x + 6i_3 - 20 = 0$

$$6i_3 = v_3 - v_2$$

$$v_x = v_1 - v_4$$

$$-2v_1 - v_2 + v_3 + 2v_4 = 20$$

4 unknowns, 5 equations (2 equations from KCL, 3 from KVL)

1 equation is redundant / dependent