EHB 211E: Basics of Electrical Circuits

Circuit Theorems

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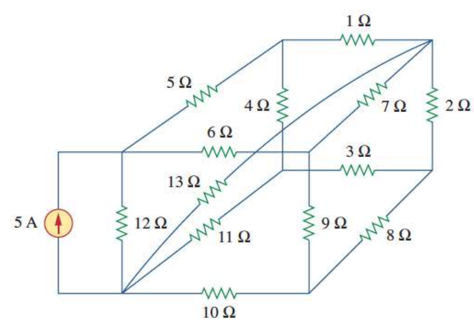
Mesh Analysis

Mesh analysis provides another general procedure for analyzing circuits, using mesh currents as the circuit variable.

-> nodal analysis applied KCL to find unknown voltages
-> mesh analysis applies KVL to find unknown currents
mesh analysis is only applicable to PLANAR circuits!

STEPS:

1. Assign mesh currents $i_1, i_2, \ldots, i_n$ to the n meshes
2. Apply KVL to all meshes. Use Ohm's law to express voltages in terms of mesh currents
3. Solve the resulting n simultaneous equations to get mesh currents



Non-planar circuit: A circuit that can not be drawn on a single planar surface, without branches crossing each other.

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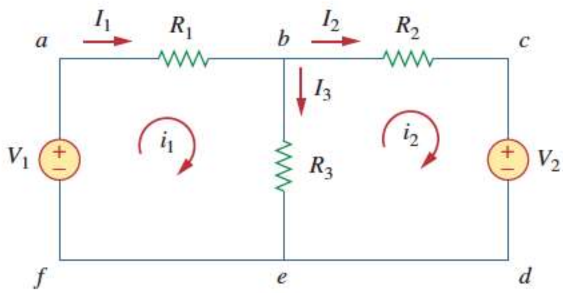
Exercise

Find mesh and branch currents for the below circuit



$$I_1 = i_1, \qquad I_2 = i_2, \qquad I_3 = i_1 - i_2$$

KVL for each mesh:

$$-V_1 + R_1 i_1 + R_3(i_1 - i_2) = 0$$

$$(R_1 + R_3)i_1 - R_3 i_2 = V_1 \quad (1)$$

$$R_2 i_2 + V_2 + R_3(i_2 - i_1) = 0$$

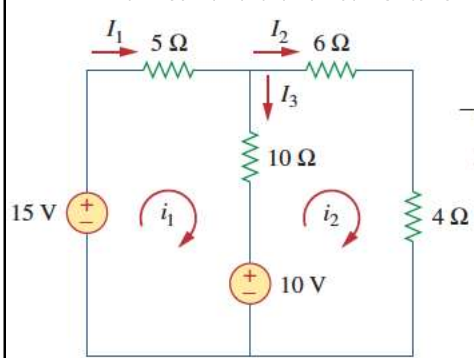
$$-R_3 i_1 + (R_2 + R_3)i_2 = -V_2 \quad (2)$$

2 equations, 2 unknowns

$$\begin{bmatrix} R_1 + R_3 & -R_3 \\ -R_3 & R_2 + R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_2 \end{bmatrix}$$

Exercise

Find mesh and branch currents for the below circuit



KVL mesh 1:

$$-15 + 5i_1 + 10(i_1 - i_2) + 10 = 0$$

$$3i_1 - 2i_2 = 1$$

KVL mesh 2:

$$6i_2 + 4i_2 + 10(i_2 - i_1) - 10 = 0$$

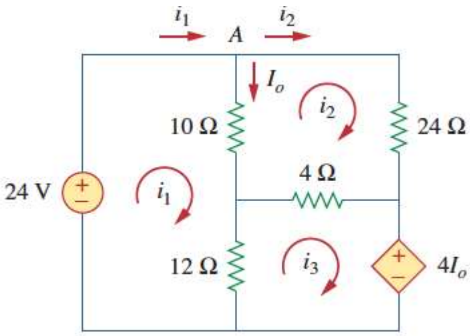
$$i_1 = 2i_2 - 1$$

$$i_1 = 1\text{ A.} \qquad i_2 = 1\text{ A}$$

$$I_1 = i_1 = 1\text{ A}, \qquad I_2 = i_2 = 1\text{ A}, \qquad I_3 = i_1 - i_2 = 0$$

Exercise

Find mesh currents for the below circuit



KVL mesh 1:

$$-24 + 10(i_1 - i_2) + 12(i_1 - i_3) = 0$$
$$11i_1 - 5i_2 - 6i_3 = 12$$

KVL mesh 2:

$$24i_2 + 4(i_2 - i_3) + 10(i_2 - i_1) = 0$$
$$-5i_1 + 19i_2 - 2i_3 = 0$$

KVL mesh 3:

$$4I_o + 12(i_3 - i_1) + 4(i_3 - i_2) = 0$$
at node A, $I_o = i_1 - i_2$
$$4(i_1 - i_2) + 12(i_3 - i_1) + 4(i_3 - i_2) = 0$$
$$-i_1 - i_2 + 2i_3 = 0$$

$$\begin{bmatrix} 11 & -5 & -6 \\ -5 & 19 & -2 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$$

3 equations, 3 unknowns

Solving 3 equations using Cramer's rule:

$$\begin{bmatrix} 11 & -5 & -6 \\ -5 & 19 & -2 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$$

Obtain determinant

$$\Delta = \begin{vmatrix} 11 & -5 & -6 \\ -5 & 19 & -2 \\ -1 & -1 & 2 \\ 11 & -5 & -6 \\ -5 & 19 & -2 \end{vmatrix}$$

$$= 418 - 30 - 10 - 114 - 22 - 50 = 192$$

$$i_1 = \frac{\Delta_1}{\Delta} = \frac{432}{192} = 2.25\text{ A}, \quad i_2 = \frac{\Delta_2}{\Delta} = \frac{144}{192} = 0.75\text{ A}$$
$$i_3 = \frac{\Delta_3}{\Delta} = \frac{288}{192} = 1.5\text{ A}$$

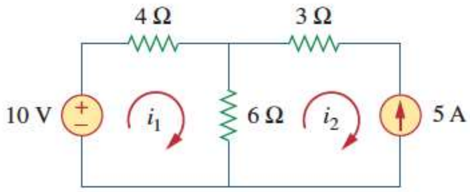
$$\Delta_1 = \begin{vmatrix} 12 & -5 & -6 \\ 0 & 19 & -2 \\ 0 & -1 & 2 \\ 12 & -5 & -6 \\ 0 & 19 & -2 \end{vmatrix} = 456 - 24 = 432$$

$$\Delta_2 = \begin{vmatrix} 11 & 12 & -6 \\ -5 & 0 & -2 \\ -1 & 0 & 2 \\ 11 & 12 & -6 \\ -5 & 0 & -2 \end{vmatrix} = 24 + 120 = 144$$

$$\Delta_3 = \begin{vmatrix} 11 & -5 & 12 \\ -5 & 19 & 0 \\ -1 & -1 & 0 \\ 11 & -5 & 12 \\ -5 & 19 & 0 \end{vmatrix} = 60 + 228 = 288$$

Mesh Analysis with Current Sources

CASE 1:
When a current source exists only in one mesh:

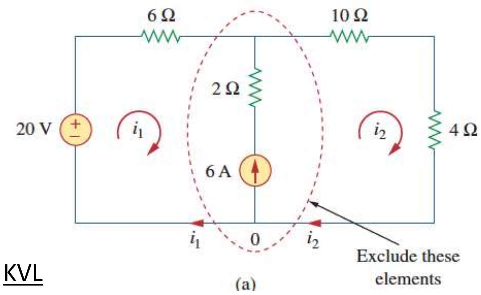


$$i_2 = -5 \text{ A}$$

$$-10 + 4i_1 + 6(i_1 - i_2) = 0$$

$$\Rightarrow \quad i_1 = -2 \text{ A}$$

CASE 2:
When a current source exists between two meshes (supermesh!):



KVL

$$-20 + 6i_1 + 10i_2 + 4i_2 = 0$$

$$6i_1 + 14i_2 = 20$$

KCL

$$i_2 = i_1 + 6$$

$$i_1 = -3.2 \text{ A}, \qquad i_2 = 2.8 \text{ A}$$

A supermesh results when two meshes have a (dependent or independent) current source in common.

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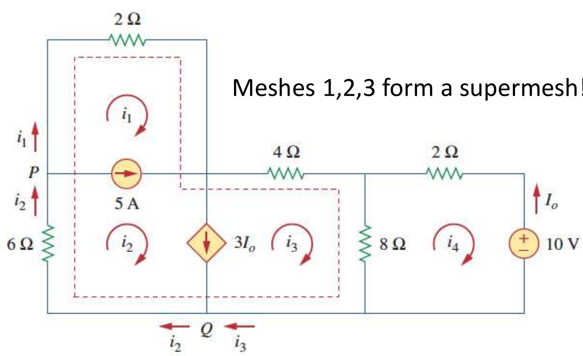
Exercise

For the below circuit, find mesh currents



Meshes 1,2,3 form a supermesh!

KVL on supermesh

$$2i_1 + 4i_3 + 8(i_3 - i_4) + 6i_2 = 0$$

$$i_1 + 3i_2 + 6i_3 - 4i_4 = 0 \quad (1)$$

KCL at node P,Q

$$i_2 = i_1 + 5 \quad (2) \qquad i_2 = i_3 + 3I_o$$

$$I_o = -i_4,$$

$$i_2 = i_3 - 3i_4 \quad (3)$$

KVL on mesh 4

$$2i_4 + 8(i_4 - i_3) + 10 = 0$$

$$5i_4 - 4i_3 = -5 \quad (4)$$

4 equations, 4 unknowns

$$i_1 = -7.5 \text{ A}, \qquad i_2 = -2.5 \text{ A}, \qquad i_3 = 3.93 \text{ A}, \qquad i_4 = 2.143 \text{ A}$$

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Nodal vs. Mesh Analysis

Circuits containing many series-connected elements -> mesh analysis
Circuits containing many parallel-connected elements -> nodal analysis

Circuits with fewer nodes than meshes -> nodal analysis
Circuits with fewer meshes than nodes -> mesh analysis

If one needs to find voltage -> nodal analysis
If one needs to find branch or mesh currents -> mesh analysis

IT IS BEST TO BE FAMILIAR WITH BOTH METHODS

Linearity

1) Homogeneity: If input is multiplied by a constant -> output is multiplied by the same constant:

$$v = iR \qquad \boxed{kiR = kv}$$

2) Additivity: Sum of inputs is the sum of the responses to each input seperately.

$$v_1 = i_1R$$

$$v_2 = i_2R$$

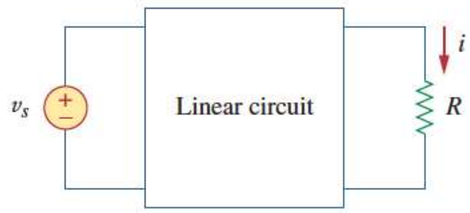
$$\boxed{v = (i_1 + i_2)R = i_1R + i_2R = v_1 + v_2}$$

A resistor is linear, because the voltage-current relationship satisfies both homogeneity and additivity properties

A linear circuit is one whose output is linearly related to its input

Linearity

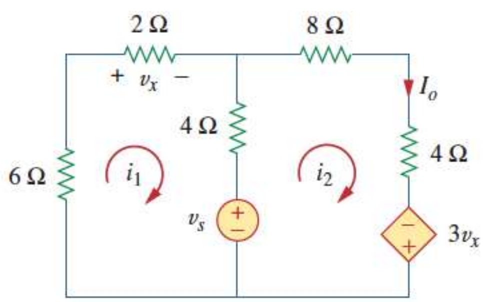
Relationship between input voltage and output power is <u>nonlinear</u>



$$\left.\begin{array}{l} p_1 = R i_1^2 \\ p_2 = R i_2^2 \end{array}\right\} \quad R(i_1 + i_2)^2 = R i_1^2 + R i_2^2 + 2R i_1 i_2 \neq p_1 + p_2$$

Exercise

For the circuit below, find I_o when v_s = 12V and v_s = 24V



KVL to 2 loops:

$$12i_1 - 4i_2 + v_s = 0 \qquad (1)$$

$$-4i_1 + 16i_2 - 3v_x - v_s = 0 \qquad v_x = 2i_1$$

$$-10i_1 + 16i_2 - v_s = 0 \qquad (2)$$

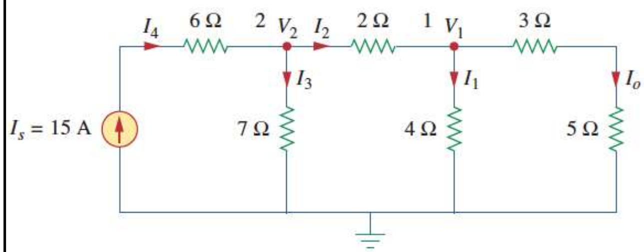
From (1) and (2), for v_s = 12V $\quad I_o = i_2 = \dfrac{12}{76}\text{A}$

for v_s = 24V $\quad I_o = i_2 = \dfrac{24}{76}\text{A}$

When source doubles, I_0 doubles

Exercise

For the circuit below, assume $I_0 = 1A$. Use linearity to find the actual value of I_0.



at node 1

$$V_1 = (3 + 5)I_o = 8 \text{ V}$$

$$I_1 = V_1/4 = 2 \text{ A}$$ Ohm's law

$$I_2 = I_1 + I_o = 3 \text{ A}$$ KCL at node 1

at node 2

$$V_2 = V_1 + 2I_2 = 8 + 6 = 14 \text{ V}$$

$$I_3 = \frac{V_2}{7} = 2 \text{ A}$$ Ohm's law

$$I_4 = I_3 + I_2 = 5 \text{ A}$$ KCL at node 2

$$I_s = 5 \text{ A}$$ Since I_s is actually 15 A

$$I_o = 3 \text{ A}$$

Superposition

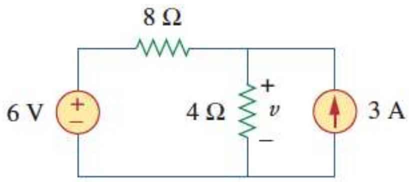
Superposition theorem states that voltage across (or current through) an element in a linear circuit is the algebraic sum of the voltage across (current through) that element due to each independent source acting alone.

STEPS:
1) Turn off all independent sources except one
   -> replace voltage source with 0V (short circuit)
   -> replace current source with 0A (open circuit)
   Find the output (voltage or current) due to that active source
2) Repeat step 1 for all independent sources
3) Find the total contribution by adding algebraically all the contributions due to independent sources

Disadvantage -> It involves more work. If the circuit has 3 independent, you'll have to analyze three simpler circuits
Advantage -> Superposition reduces a complex circuit to simpler circuits
- Superposition is based on linearity -> Can not be used for power calculation

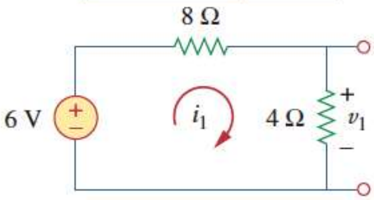
Exercise

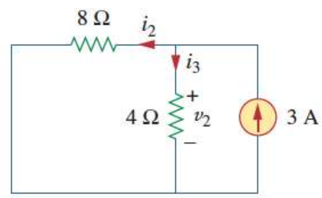
Use the superposition theorem to find v in the circuit below



$$v = v_1 + v_2$$

v_1 and v_2 are the contributions due to the 6V voltage source and 3A current source





$$12i_1 - 6 = 0 \quad \Rightarrow \quad i_1 = 0.5 \text{ A}$$

$$v_1 = 4i_1 = 2 \text{ V}$$

$$i_3 = \frac{8}{4+8}(3) = 2 \text{ A} \qquad v_2 = 4i_3 = 8 \text{ V}$$

$$\boxed{v = v_1 + v_2 = 2 + 8 = 10 \text{ V}}$$

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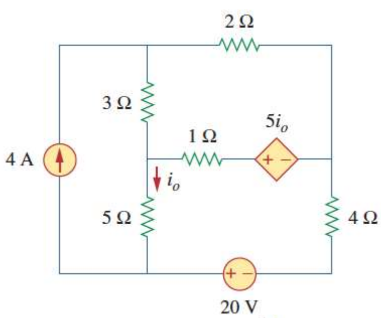
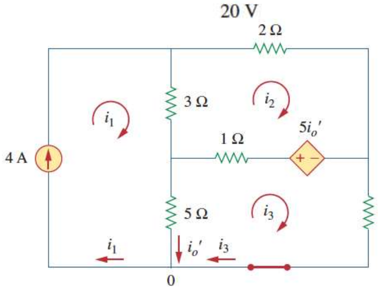
Exercise

Use the superposition theorem to find i_0 in the circuit below



$$i_o = i'_o + i''_o$$

$I_{0'}$ and i_0'' are the contributions due to the 4A current source and 20V voltage source



$i_1 = 4 \text{ A}$ *Loop 1*

$-3i_1 + 6i_2 - 1i_3 - 5i'_o = 0$ *Loop 2*

$-5i_1 - 1i_2 + 10i_3 + 5i'_o = 0$ *Loop 3*

$i_3 = i_1 - i'_o = 4 - i'_o$ *KCL at node 0*

3 loop equations, 3 unknowns
(after inserting KCL eq. into loop 2&3 equations) $i'_o = \frac{52}{17} \text{ A}$

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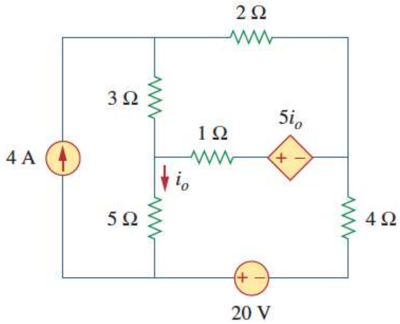
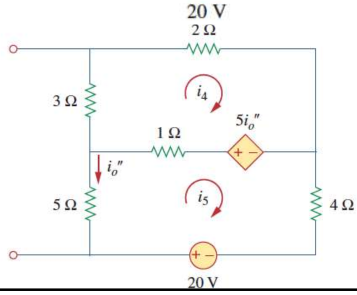
Exercise (continued…)

Use the superposition theorem to find i_0 in the circuit below



$$i_o = i'_o + i''_o$$

$I_{0'}$ and i_0'' are the contributions due to the 4A current source and 20V voltage source



$$6i_4 - i_5 - 5i''_o = 0$$ *Loop 4*

$$-i_4 + 10i_5 - 20 + 5i''_o = 0$$ *Loop 5*

$$i_5 = -i''_o.$$

2 loop equations, 2 unknowns
(after inserting last eq. into loop 4&5 equations)

$$i'_o = \frac{52}{17}\text{ A}$$

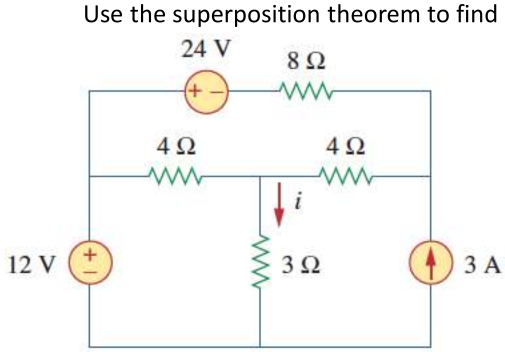
$$i_o = i'_o + i''_o$$

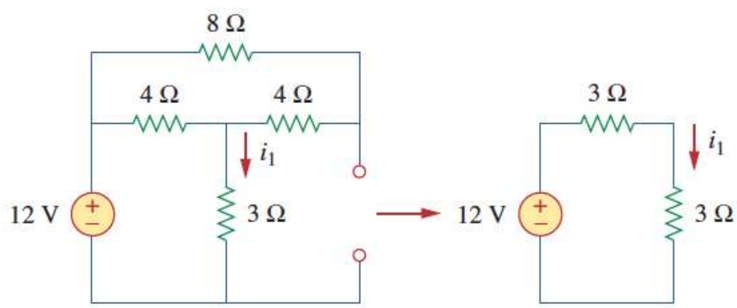
$$i''_o = -\frac{60}{17}\text{ A}$$

$$\boxed{i_o = -\frac{8}{17} = -0.4706\text{ A}}$$

Exercise

Use the superposition theorem to find *i* in the circuit below



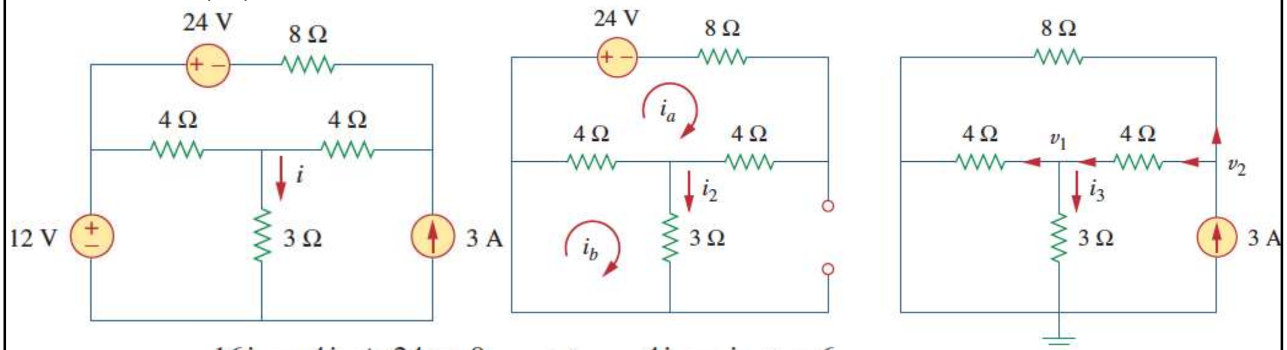


$$i = i_1 + i_2 + i_3$$

$$i_1 = \frac{12}{6} = 2\text{ A}$$

Exercise (continued…)

Use the superposition theorem to find i in the circuit below



$$16i_a - 4i_b + 24 = 0 \quad \Rightarrow \quad 4i_a - i_b = -6$$

$$7i_b - 4i_a = 0 \quad \Rightarrow \quad i_a = \frac{7}{4} i_b$$

$$i_2 = i_b = -1$$

$$3 = \frac{v_2}{8} + \frac{v_2 - v_1}{4}$$

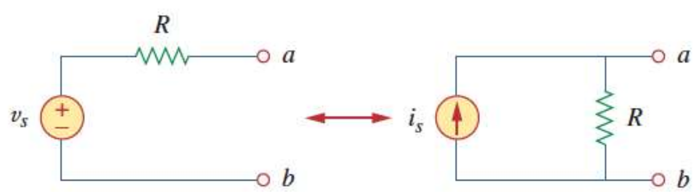
$$\frac{v_2 - v_1}{4} = \frac{v_1}{4} + \frac{v_1}{3}$$

$$i_3 = \frac{v_1}{3} = 1 \text{ A}$$

$$\boxed{i = i_1 + i_2 + i_3 = 2 - 1 + 1 = 2 \text{ A}}$$

Source Transformation

Source transformation is a technique to simplify circuits:



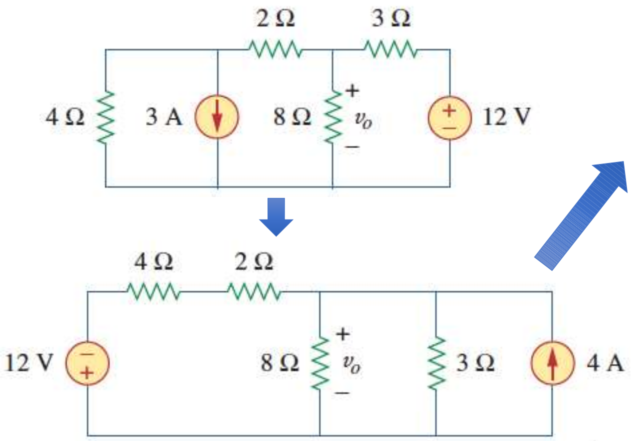
Source transformation is the process of replacing a voltage source (dependent or independent) with a resistor, by a current source (dependent or independent) with the same resistor in parallel.

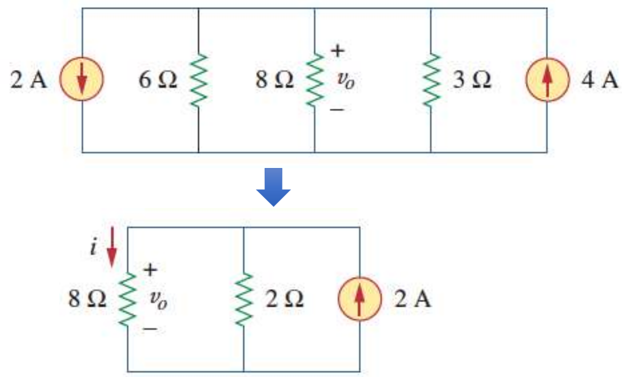
Two circuits are equivalent and replaceable, provided that they have the same voltage-current relationship at terminals a-b.
-> If the sources are turned off, the equivalent resistance at terminal a-b is R for both circuits
-> Short circuit current at terminal a-b is:
$i_{sc} = v_s/R$ for the circuit on the left and $i_{sc} = i_s$ for the circuit on the right. Source transformation requires:

$$v_s = i_s R \qquad \text{or} \qquad i_s = \frac{v_s}{R}$$

Exercise

Use source transformation to find v_o



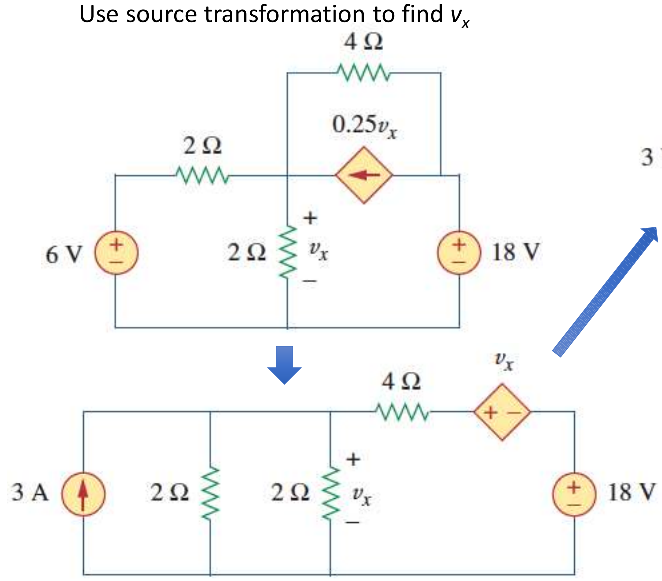


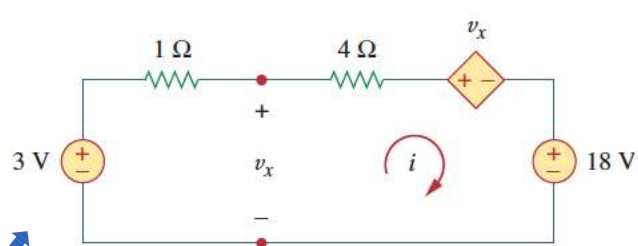
$$i = \frac{2}{2+8}(2) = 0.4 \text{ A}$$

$$\boxed{v_o = 8i = 8(0.4) = 3.2 \text{ V}}$$

Exercise

Use source transformation to find v_x





$-3 + 5i + v_x + 18 = 0$ *Entire loop*

$-3 + 1i + v_x = 0$ *Left loop*

2 equations and 2 unknowns:

$i = -4.5$ A

$v_x = 3 - i = 7.5$ V.