EHB 211E: Basics of Electrical Circuits

Graph Theory

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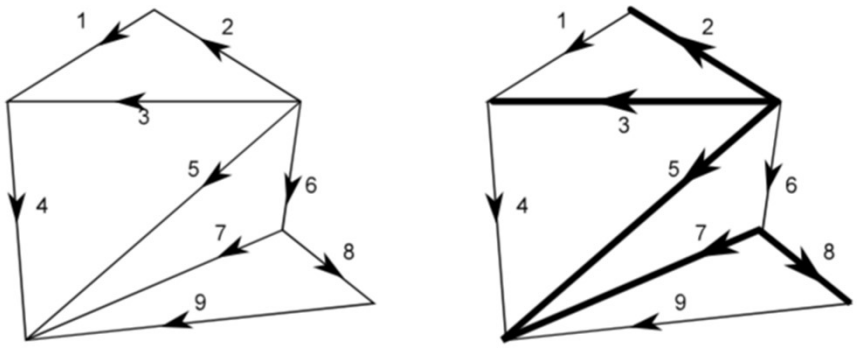
Example1 – Fundamental Cut-sets



Fundamental cut sets of the tree $G_T = \{2, 3, 5, 7, 8\}$ are $G_{C2} = \{2, 1\}$, $G_{C3} = \{3, 1, 4\}$, $G_{C5} = \{5, 4, 6\}$, $G_{C7} = \{7, 6, 9\}$, $G_{C8} = \{8, 9\}$.

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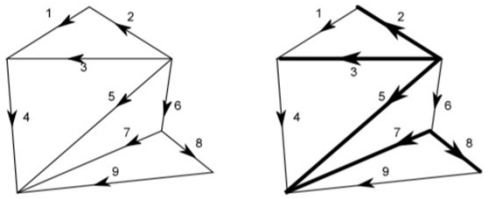
Example1 – Fundamental Cut-set Equations



KCL equations based on Fundamental cut sets

$$\left[\begin{array}{cccc|ccccc} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} i_1 \\ i_4 \\ i_6 \\ i_9 \\ -- \\ i_2 \\ i_3 \\ i_5 \\ i_7 \\ i_8 \end{array}\right] = 0$$

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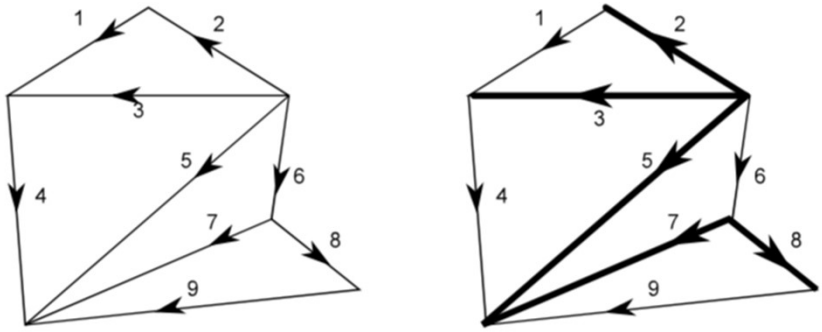
Example1 - Fundamental Loops



Fundamental Loop sets of the tree $G_T = \{2, 3, 5, 7, 8\}$ are $G_{L1} = \{1, 2, 3\}$, $G_{L4} = \{$ **4,3,5** $\}$, $G_{L6} = \{$ **6,3,7** $\}$, $G_{L9} = \{9, 7, 8\}$.

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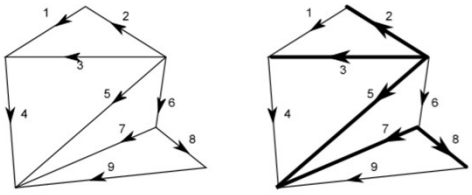
Example1 - Fundamental Loop Equations



KVL equations based on the Fundamental loops

$$\left[\begin{array}{cccc|ccccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \end{array}\right] \left[\begin{array}{c} V_1 \\ V_4 \\ V_6 \\ V_9 \\ -- \\ V_2 \\ V_3 \\ V_5 \\ V_7 \\ V_8 \end{array}\right] = 0$$

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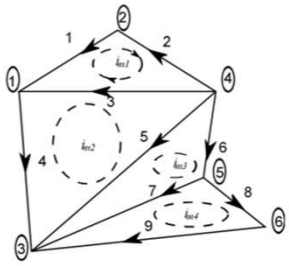
Example1 – KVL Equations for the nodes



KVL equations for the nodes

$$\left[\begin{array}{c} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \\ V_9 \end{array}\right] = \left[\begin{array}{cccccc} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} V_{n1} \\ V_{n2} \\ V_{n3} \\ V_{n4} \\ V_{n5} \\ V_{n6} \end{array}\right]$$

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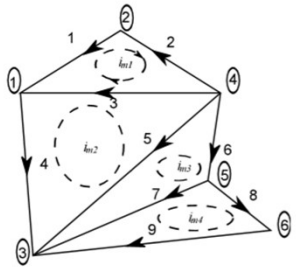
Example1 – KCL Equations for the nodes



KCL equations for the nodes

$$\begin{bmatrix} -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \\ i_8 \\ i_9 \end{bmatrix} = 0$$

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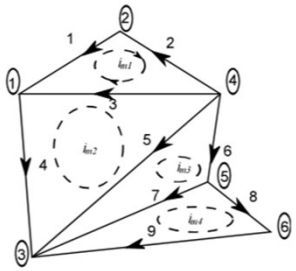
Example1 – Mesh Equations



Mesh equations

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \\ i_8 \\ i_9 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_{m1} \\ i_{m2} \\ i_{m3} \\ i_{m4} \end{bmatrix}$$

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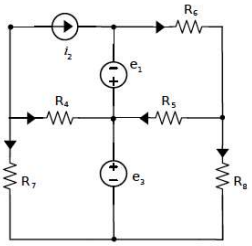
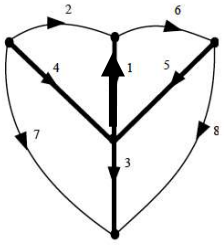
Graph Theory : Chord Current Method

Solving linear circuits with two-terminal resistors / independent sources:

1) Pick a proper tree that includes voltage sources. Current sources are placed on co-trees.
2) Write fundamental loop equations, which do not correspond to current sources
3) Write v-i relations of the resistors in v=r.i form
4) Substitute voltages found in step 3 -> equations found in step 2
5) Write fundamental cut-set equations that do not correspond to voltage sources
6) Substitute the fundamental loop equations in step 4 -> equations found in step 5
7) Present the equation in the form: $Bi_{l'} + Qi_s + Mv_s = 0$

Exercise





1) Pick a proper tree that includes voltage sources. Current sources are placed on co-trees.
2) Fundamental loop equations, which do not correspond to current sources

$$\begin{aligned} V_6 + V_5 + V_1 &= 0 \\ V_7 - V_4 - V_3 &= 0 \\ V_8 - V_3 - V_5 &= 0 \end{aligned}$$

3-4) Write v-i relations of resistors and substitute

$$\begin{aligned} R_6 i_6 + R_5 i_5 + V_1 &= 0 \\ R_7 i_7 - R_4 i_4 - V_3 &= 0 \\ R_8 i_8 - V_3 - R_5 i_5 &= 0 \end{aligned}$$

5) Fundamental cut-set equations
(that do not correspond to voltage sources)

$$\begin{aligned} i_4 + i_2 + i_7 &= 0 \qquad i_4 = -i_2 - i_7 \\ i_5 + i_8 - i_6 &= 0 \qquad i_5 = -i_8 + i_6 \end{aligned}$$

6) Substitute 5 into step 4
(all current are represented in terms of chord currents)

$$\begin{aligned} R_6 i_6 + R_5(-i_8 + i_6) + V_1 &= 0 \\ R_7 i_7 - R_4(-i_2 - i_7) - V_3 &= 0 \\ R_8 i_8 - V_3 - R_5(-i_8 + i_6) &= 0 \end{aligned}$$

$$\begin{bmatrix} i_1 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} i_2 \\ i_6 \\ i_7 \\ i_8 \end{bmatrix}$$

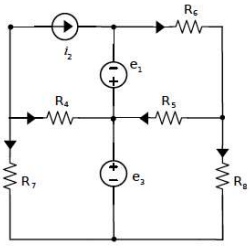
$$\begin{bmatrix} R_6 + R_5 & 0 & -R_5 \\ 0 & R_7 + R_4 & 0 \\ -R_5 & 0 & R_8 + R_5 \end{bmatrix} \begin{bmatrix} i_6 \\ i_7 \\ i_8 \end{bmatrix} + \begin{bmatrix} 0 \\ R_4 \\ 0 \end{bmatrix} i_2 + \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_3 \end{bmatrix} = 0$$

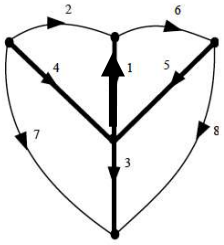
Graph Theory : Chord Current Method

Solving linear circuits with two-terminal resistors / independent sources:

1) Pick a proper tree that includes voltage sources. Current sources are placed on co-trees.
2) Write fundamental loop equations, which do not correspond to current sources
3) Write v-i relations of the resistors in v=r.i form
4) Substitute voltages found in step 3 -> equations found in step 2
5) Write fundamental cut-set equations that do not correspond to voltage sources
6) Substitute the fundamental loop equations in step 4 -> equations found in step 5
7) Present the equation in the form: $Bi_{l'} + Qi_s + Mv_s = 0$

Exercise





1) Pick a proper tree that includes voltage sources. Current sources are placed on co-trees.
2) Fundamental loop equations, which do not correspond to current sources

$$\begin{aligned} V_6 + V_5 + V_1 &= 0 \\ V_7 - V_4 - V_3 &= 0 \\ V_8 - V_3 - V_5 &= 0 \end{aligned}$$

3-4) Write v-i relations of resistors and substitute

$$\begin{aligned} R_6 i_6 + R_5 i_5 + V_1 &= 0 \\ R_7 i_7 - R_4 i_4 - V_3 &= 0 \\ R_8 i_8 - V_3 - R_5 i_5 &= 0 \end{aligned}$$

5) Fundamental cut-set equations
(that do not correspond to voltage sources)

$$\begin{aligned} i_4 + i_2 + i_7 &= 0 \qquad i_4 = -i_2 - i_7 \\ i_5 + i_8 - i_6 &= 0 \qquad i_5 = -i_8 + i_6 \end{aligned}$$

6) Substitute 5 into step 4
(all current are represented in terms of chord currents)

$$\begin{aligned} R_6 i_6 + R_5(-i_8 + i_6) + V_1 &= 0 \\ R_7 i_7 - R_4(-i_2 - i_7) - V_3 &= 0 \\ R_8 i_8 - V_3 - R_5(-i_8 + i_6) &= 0 \end{aligned}$$

$$\begin{bmatrix} i_1 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} i_2 \\ i_6 \\ i_7 \\ i_8 \end{bmatrix}$$

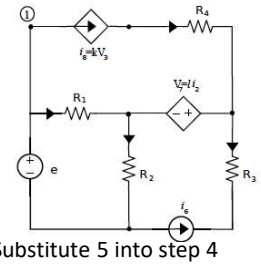
$$\begin{bmatrix} R_6 + R_5 & 0 & -R_5 \\ 0 & R_7 + R_4 & 0 \\ -R_5 & 0 & R_8 + R_5 \end{bmatrix} \begin{bmatrix} i_6 \\ i_7 \\ i_8 \end{bmatrix} + \begin{bmatrix} 0 \\ R_4 \\ 0 \end{bmatrix} i_2 + \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_3 \end{bmatrix} = 0$$

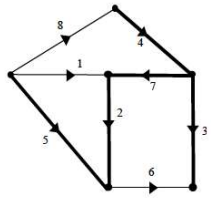
Graph Theory : Generalized Chord Current Method

Solving linear circuits with two-terminal resistors / independent + dependent sources:

- Follow the same steps, treat dependent sources as independent sources
- Place the dependent voltage source in the tree, and within the co-tree if it is a current source.
- Using v-i relations of the dependent sources, new unknown variables are written in terms of link currents, voltage and current sources.

Exercise





1) Pick a proper tree that includes voltage sources. Current sources are placed on co-trees.

2) Fundamental loop equations, which do not correspond to current sources (link 1)

$$V_1 + V_2 - V_5 = 0$$

3-4) Find v-i relations of resistors

$$R_1 i_1 + R_2 i_2 - e = 0$$

5) Fundamental cut-set equations (that do not correspond to voltage sources)

$$i_2 - i_8 - i_1 - i_6 = 0$$

6) Substitute 5 into step 4 (all current are represented in terms of chord currents)

$$R_1 i_1 + R_2(i_8 + i_1 + i_6) - e = 0$$

Express dependent source terms of independent source

$i_8 = kV_3 \quad i_8 = kR_3 i_3$

$i_8 = -kR_3 i_6$ Write in terms of link current (i3 term is replaced by i6 term)

$$(R_1 + R_2) i_1 + (R_2 - kR_3) i_6 - e = 0$$

If the circuit contains multi-terminal elements -> consider it as an independent source

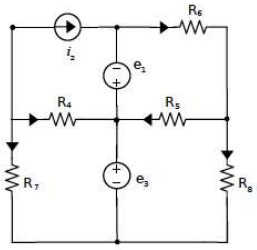
Graph Theory : Branch voltages method

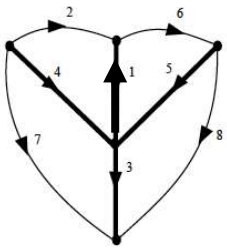
Solving linear circuits with two-terminal resistors

1) Pick a proper tree that includes voltage sources. Current sources are placed on co-trees.
2) Write fundamental cut-set equations for the branches (exclude branches having voltage sources)
3) Write v-i relations of the resistors in i=g.k form
4) Substitute the currents found in step 3 -> equations found in step 2
5) Write fundamental loop equations that do not correspond to links having current sources
6) Substitute the fundamental loop equations in step 5 -> equations found in step 4
7) Present the equation in the form:

$$MV_{b'} + Qi_s + Mv_s = 0$$

Exercise





1) Pick a proper tree that includes voltage sources. Current sources are placed on co-trees.

2) Fundamental cut set equations for branches 4-5 which do not correspond to voltage sources

$$\begin{aligned} i_4 + i_2 + i_7 &= 0 \\ i_5 - i_6 + i_8 &= 0 \end{aligned}$$

3-4) Write i-v relations of resistors and replace into 2

$$\begin{aligned} G_4 V_4 + i_2 + G_7 V_7 &= 0 \\ G_5 V_5 - G_6 V_6 + G_8 V_8 &= 0 \end{aligned}$$

5) Fundamental loop equations, without current sources

$$\begin{aligned} V_6 + V_5 + V_1 &= 0 \quad & V_6 &= -V_5 - V_1 \\ V_7 - V_4 - V_3 &= 0 \quad & V_7 &= V_4 + V_3 \\ V_8 - V_3 - V_5 &= 0 \quad & V_8 &= V_3 + V_5 \end{aligned}$$

$$\begin{aligned} G_4 V_4 + i_2 - G_7(V_4 + V_3) &= 0 \\ G_5 V_5 - G_6(-V_5 - V_1) + G_8(V_3 + V_5) &= 0 \end{aligned}$$

(everything is represented in terms of branch voltages)

Chord voltages from branch voltages

$$\begin{bmatrix} V_2 \\ V_6 \\ V_7 \\ V_8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix}$$

$$\begin{bmatrix} G_4 - G_7 & 0 \\ 0 & G_5 + G_6 + G_8 \end{bmatrix} \begin{bmatrix} V_4 \\ V_5 \end{bmatrix} + \begin{bmatrix} 0 & -G_7 \\ G_6 & G_8 \end{bmatrix} \begin{bmatrix} V_1 \\ V_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i_2 = 0$$

Branch voltages — Voltage sources — current source

Graph Theory : Nodal Analysis

Solving linear circuits with two-terminal resistors

1) Draw the circuit graph and choose a reference node which is connected to the maximum number of voltage sources.
2) Write fundamental cut-set equations for the nodes which do not correspond to node of a voltage sources.
3) Write v-i relations of the resistors in i=g.V form
4) V is written in terms of node voltages.
5) Present the equation in the form:

$$A_{11} G_R A_{11}^T V_d + A_{12} i_k = 0$$

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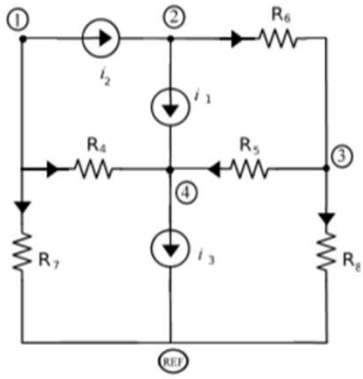
Exercise



2. The fundamental cut-set equations for the nodes:

$$\begin{aligned} i_4 + i_2 + i_7 &= 0 \\ i_1 + i_6 - i_2 &= 0 \\ i_5 - i_6 + i_8 &= 0 \\ i_3 - i_1 - i_5 - i_4 &= 0 \end{aligned}$$

3. Resistors in the circuit:

$$i_k = G_k V_k \quad k = \{4,5,6,7,8\}$$

4. Substitute terminal equations of the resistors into the previous equation

$$\begin{aligned} G_4 V_4 + i_2 + G_7 V_7 &= 0 \\ i_1 - G_6 V_6 - i_2 &= 0 \\ G_5 V_5 - G_6 V_6 + G_8 V_8 &= 0 \\ i_3 - i_1 - G_5 V_5 - G_4 V_4 &= 0 \end{aligned}$$

5. Terminal voltage is written in terms of the node voltages:

$$\begin{aligned} V_1 &= \\ V_2 &= \\ V_3 &= \\ V_4 &= \\ V_5 &= \\ V_6 &= \\ V_7 &= \\ V_8 &= \end{aligned}$$

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Exercise

6. Substitute the equation in step 5 into step 4

$$\begin{aligned} G_4(V_{d1}-V_{d4})+i_2+G_7V_{d1} &= 0\\ i_1-G_6(V_{d2}-V_{d3})-i_2 &= 0\\ G_5(V_{d3}-V_{d4})-G_6(V_{d2}-V_{d3})+G_8V_{d3} &= 0\\ i_3-i_1-G_5(V_{d3}-V_{d4})-G_4(V_{d1}-V_{d4}) &= 0 \end{aligned}$$

Equations are presented in the matrix form

$$\begin{bmatrix} G_4+G_7 & 0 & 0 & -G_4 \\ 0 & G_6 & -G_6 & 0 \\ 0 & -G_6 & G_6+G_5+G_8 & -G_5 \\ -G_4 & 0 & -G_5 & G_4+G_5 \end{bmatrix} \begin{bmatrix} V_{d1} \\ V_{d2} \\ V_{d3} \\ V_{d4} \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = 0$$

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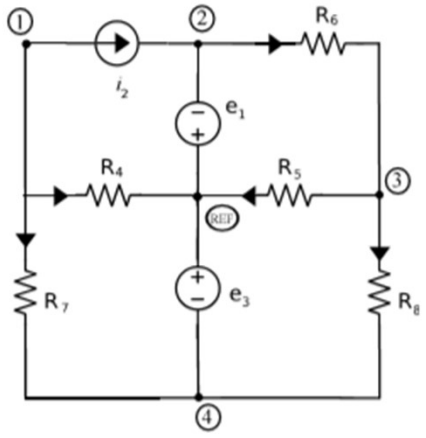
Exercise – Generalized Nodal Analysis



2.the fundamental cut-set equations for the nodes

$$\begin{aligned} i_4+i_2+i_7 &= 0\\ i_5-i_6+i_8 &= 0 \end{aligned}$$

3. Write the $v-i$ relations of the resistors:

$$i_k = G_kV_k \quad k=\{4,5,6,7,8\}$$

4. Substitute the equations in Step 3 into the equations in Step 2.

$$\begin{aligned} G_4V_4+i_2+G_7V_7 &= 0\\ G_5V_5-G_6V_6+G_8V_8 &= 0 \end{aligned}$$

5. Terminal voltage is written in terms of the node voltages:

$$\begin{aligned} V_1 &= -V_{d2}\\ V_2 &= V_{d1}-V_{d2}\\ V_3 &= -V_{d4}\\ V_4 &= V_{d1}\\ V_5 &= V_{d3}\\ V_6 &= V_{d2}-V_{d3}\\ V_7 &= V_{d4}-V_{d1}\\ V_8 &= V_{d3}-V_{d4} \end{aligned}$$

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Exercise - Generalized Nodal Analysis

6. Substitute the equation in step 5 into step 4:

$$\begin{aligned} G_4 V_{d1} + i_2 + G_7(V_{d4} - V_{d1}) &= 0 \\ G_5 V_{d3} - G_6(V_{d2} - V_{d3}) + G_8(V_{d3} - V_{d4}) &= 0 \end{aligned}$$

Voltage sources are written in the terms of the node valtages

$$\begin{aligned} V_{d2} &= -e_1 \\ V_{d4} &= -e_3 \end{aligned}$$

Equations are presented in the matrix form:

$$\begin{bmatrix} G_4 - G_7 & 0 \\ 0 & G_5 + G_6 + G_8 \end{bmatrix} \begin{bmatrix} V_{d1} \\ V_{d3} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i_2 + \begin{bmatrix} 0 & -G_7 \\ G_6 & G_8 \end{bmatrix} \begin{bmatrix} e_1 \\ e_3 \end{bmatrix} = 0$$

Graph Theory : Loop (Mesh) Current Method

1. Draw the circuit graph and assign the loop currents.
2. Write mesh equations are obtained from KVL which is applied to each of the loop: $Bv = 0$ It can also be written such as

$$Bv = B_1 v_R + B_2 v_k = 0$$

where v_k and v_R voltages of independent voltage sources and resistors, respectively.

3. Write the ohm law for the resistors: $v_R = \mathbf{R} i_R$
4. Substitute the equations in Step 3 into the equations in Step 2.

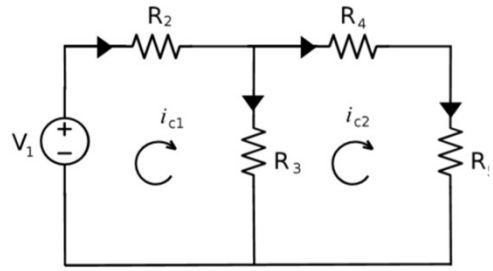
$$[B_1 \;\; B_2]v = B_1 \mathbf{R} i_R + B_2 v_k = 0$$

5. Resistors's currents are written in terms of the mesh currents:

$$i_R = B_1^T i_c$$

6. Substitute the equation in step 5 into step 4:

Exercise



2. Mesh equations:

$$\begin{aligned} V_2 + V_3 - V_1 &= 0 \\ V_4 + V_5 - V_3 &= 0 \end{aligned}$$

3. Substitute the $v_R = Ri_R$ equations into the equations in Step 2:

$$\begin{aligned} R_2 i_2 + R_3 i_3 - V_1 &= 0 \\ R_4 i_4 + R_5 i_5 - R_3 i_3 &= 0 \end{aligned}$$

4. Resistors's currents are written in terms of the mesh currents:

$$\begin{aligned} i_2 &= i_{c1} \\ i_3 &= i_{c1} - i_{c2} \\ i_4 &= i_{c2} \\ i_5 &= i_{c2} \end{aligned}$$

5. Substitute the equation in step 4 into step 3:

$$\begin{aligned} R_2 i_{c1} + R_3(i_{c1} - i_{c2}) - V_1 &= 0 \\ R_4 i_{c2} + R_5 i_{c2} - R_3(i_{c1} - i_{c2}) &= 0 \end{aligned}$$

6. In matrix form:

$$\begin{bmatrix} R_2 + R_3 & -R_3 \\ -R_3 & R_3 + R_4 + R_5 \end{bmatrix} \begin{bmatrix} i_{c1} \\ i_{c2} \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} V_1 = 0$$

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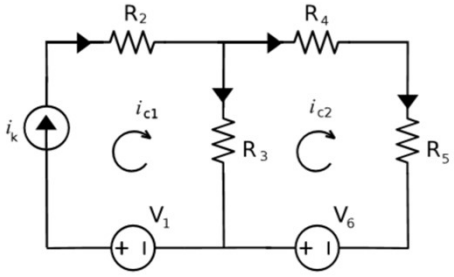
Exercise - Generalized Mesh Current Analysis



2. Mesh equations:

$$\begin{aligned} V_2 + V_3 - V_1 + V_k &= 0 \\ V_4 + V_5 - V_6 - V_3 &= 0 \end{aligned}$$

3. Substitute the $v_R = Ri_R$ equations into the equations in Step 2:

$$\begin{aligned} R_2 i_2 + R_3 i_3 - V_1 + V_k &= 0 \\ R_4 i_4 + R_5 i_5 - V_6 - R_3 i_3 &= 0 \end{aligned}$$

4. Resistors's currents are written in terms of the mesh currents:

$$\begin{aligned} i_2 &= i_{c1} \\ i_3 &= i_{c1} - i_{c2} \\ i_4 &= i_{c2} \\ i_5 &= i_{c2} \end{aligned}$$

5. Substitute the equation in step 5 into step 4:

$$\begin{aligned} R_2 i_{c1} + R_3(i_{c1} - i_{c2}) - V_1 + V_k &= 0 \\ R_4 i_{c2} + R_5 i_{c2} - V_6 - R_3(i_{c1} - i_{c2}) &= 0 \end{aligned}$$

Additional equation:

$$i_k = i_{c1}$$

6. Unknown mesh current:

$$(R_4 + R_5 + R_3) i_{c2} - R_3 i_k - V_6 = 0$$

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