EHB 211E: Basics of Electrical Circuits

Graph Theory

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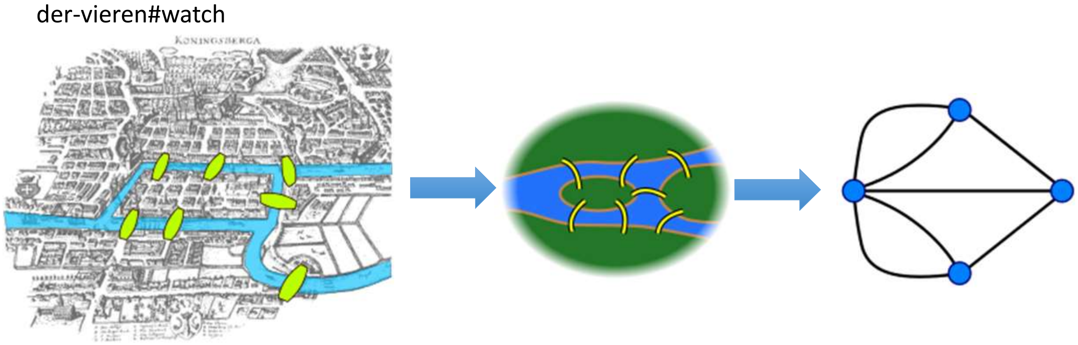
Graph Theory - History

1st scientific paper: Leonhard Euler's paper on "Seven Bridges of Königsberg", published in 1736.

-> The city of Königsberg in Prussia was set on both sides of Pregel River, including two islands.
Problem: Find a way to walk through the city that would cross each bridge only once, and once.
Difficulty is to develop a technique of analysis to establish a solution.

https://ed.ted.com/lessons/how-the-konigsberg-bridge-problem-changed-mathematics-dan-van-der-vieren#watch



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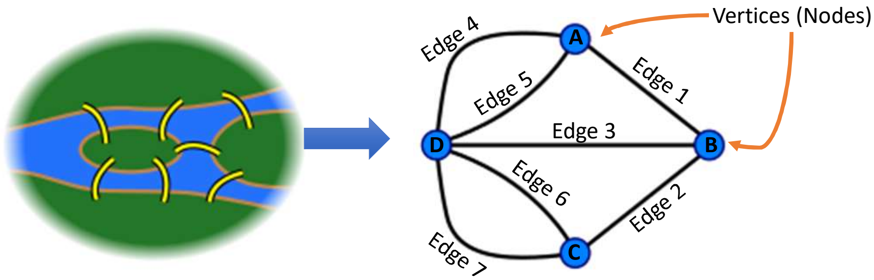
Fundamentals of Graph Theory

A graph G=(N,E) is a finite set of N nodes (or vertices or points), together with a set of E edges (arcs or lines), each of them connecting a pair of distinct nodes.

Node (Vertex) : A point

Edge : A line segment connecting a pair of distinct nodes is called an edge.



G=(N,E)
N={A,B,C,D}
E={1,2,3,4,5,6,7}

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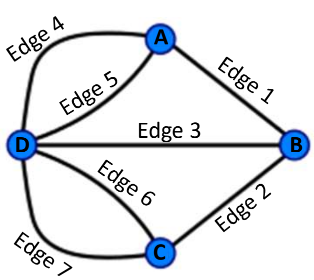
Fundamentals of Graph Theory

End nodes: The nodes n_i and n_j associated with an edge are called end nodes of the edge.

Incidence: An edge is incident to a node that is one of its end nodes.

Degree (Order) of a node: Number of edges connecting this node to other nodes.

Adjacency: Two vertices connected by an edge are adjacent.



Edge 1=(A,B)
Edge 2=(B,C)

Order of Node A = 3
Order of Node D = 5

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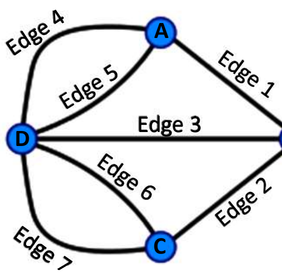
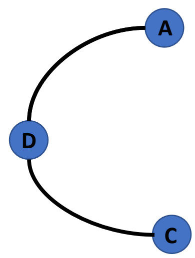
Fundamentals of Graph Theory

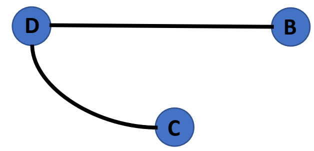
Planar graph: A planar graph is a graph which can be drawn on a plane in such a way that all its edges intersect only at their endpoints.

Subgraph: G_1 is called a subgraph of G if G_1 itself is a graph, N_1 is a subset of N and E_1 is a subset of E. In other words a subgraph is a subset of the elements of a given graph, obtained by removing some edges and/or nodes together with the corresponding edges.

Path: A path is a subgraph that can be drawn so that all of its nodes, all distinct from one another, and edges lie on a single straight line. (No node can be repeated.)







Subgraph

Graph

- V nodes
- B branches
- Each branch is incident to two nodes
- connected

Subgraph

- V\` nodes, subset of V
- B\` branches, subset of B
- Does not have to be connected

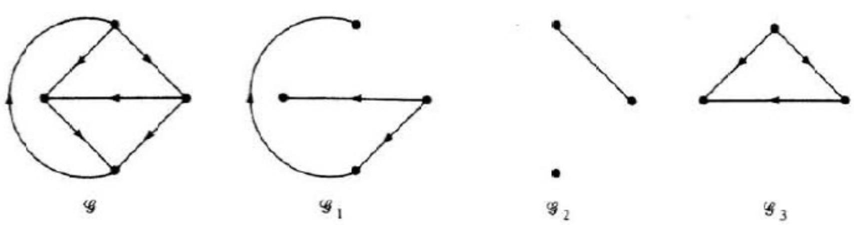


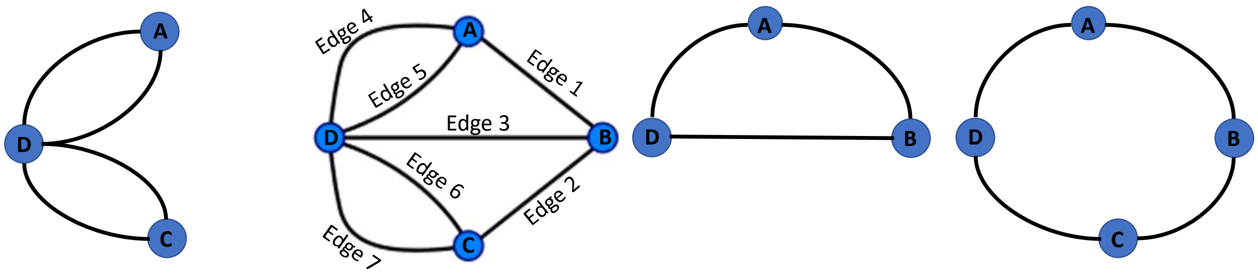
Figure 2.3 Digraph $\mathcal{G}$ and its three subgraphs.

Fundamentals of Graph Theory

Connected graph: A graph is connected when there is a path between every pair of nodes. Otherwise it is disconnected.

Hinged graph: A connected graph is hinged when it can be partitioned into two subgraphs connected by one node, called a hinge.

Loop: A subgraph containing only nodes of order 2 and a set of edges between these nodes.



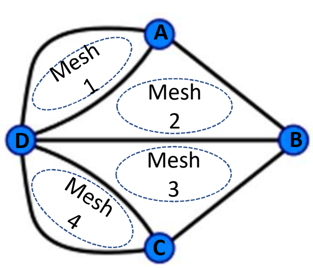
Fundamentals of Graph Theory

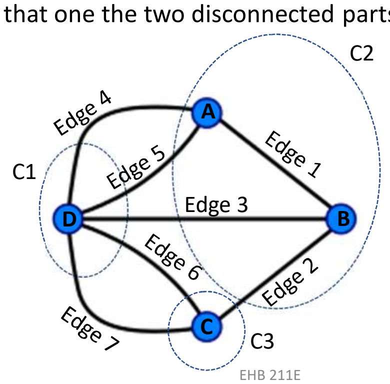
Mesh: A loop of a planar graph not containing any graph elements inside.

Cut-set: A set of edges of a graph which, when removed, makes the graph disconnected.

- The removal of all the edges of the cut-set results in a disconnected graph.
- The removal of all but any one edge of the cut-set leaves the graph connected.

Nodal cut-set: A cut-set such that one the two disconnected parts of the resulting graph is a single node.





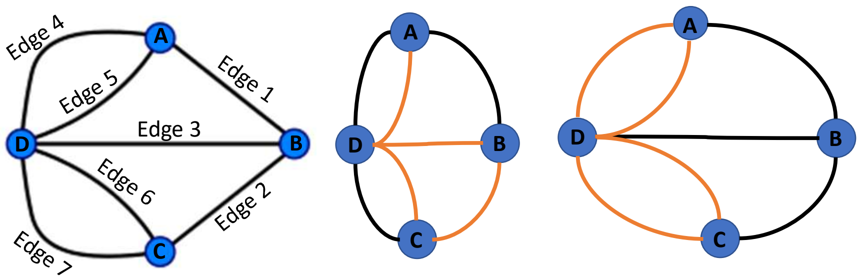
E_{C1}={3,4,5,6,7}
E_{C2}={2,3,4,5}
E_{C3}={2,,6,7}

Fundamentals of Graph Theory

Tree: A subgraph containing all the N nodes and N-1 edges of a given graph and in which any two nodes are connected by exactly one path.

- It contains all the nodes of the graph.
- It contains no loops.

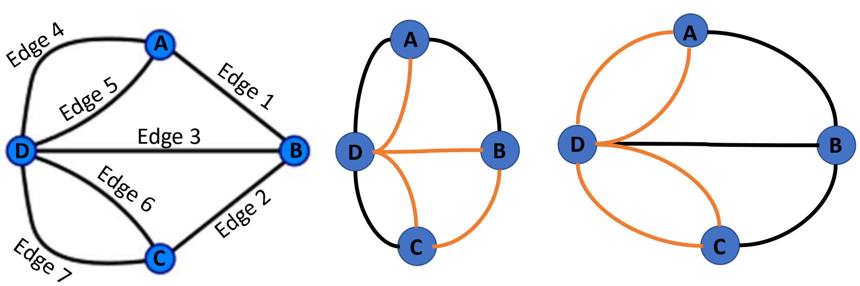
Co-tree: A subgraph associated with a tree, containing all the N nodes and the remaining E-N+1 edges of the graph not contained in the tree (compliment of the tree)



Fundamentals of Graph Theory

Theory: Given a connected graph G of N nodes and E edges, and a tree T of G, there is a unique path along the tree between any given pair of nodes.

Branch (twig): The edges of the tree are called branches.
Chord (link): The edges of the cotree are called chords.



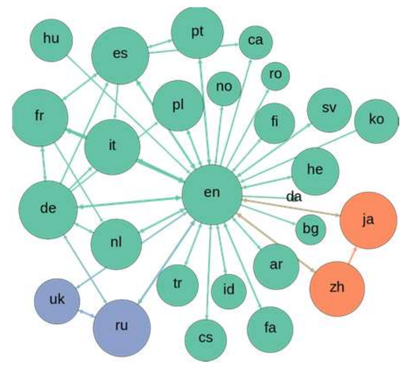
T_1={1,,4,7}
T_2={1,2,3}
G_{L1}={2,3,5,6}
G_{L2}={4,5,6,7}

Graph Theory

A graph is an ordered pair of nodes. Graphs are used to model many relations and processes in physical, biological, social, and information systems.

- In computer science, graphs are used to represent networks of communication
- In physics, graphs are used to represent connections between part of systems & dynamics of physical processes within a system
- In sociology, graphs are used in social network analysis software.
- In biology, graphs are used to model interaction between species

https://ed.ted.com/lessons/can-you-solve-the-control-room-riddle-dennis-shasha#watch



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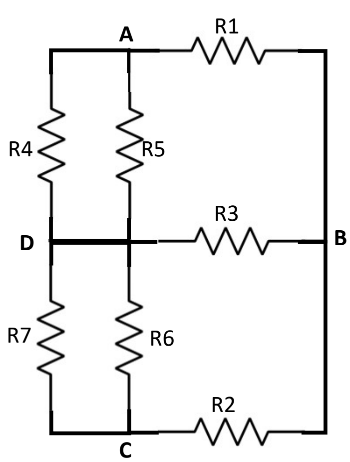
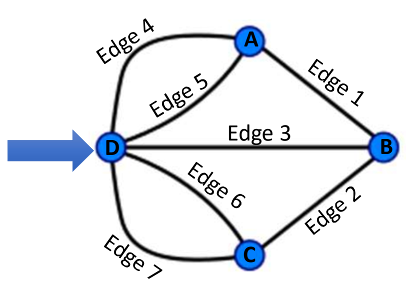
From Circuit to Graph

For a given circuit if we replace each element by its element graph, we get the circuit graph.





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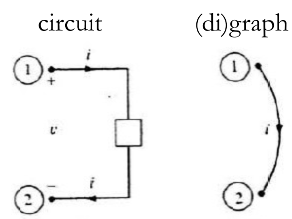
Modeling Circuit Element

A mathematical model can be developed for each circuit element after performing certain tests on the element.

The relation between the terminal variables is called terminal equation. f(v,i)=0 or f(v,I, dv/dt, di/dt)=0

Mathematical Model: The terminal graph and the terminal equation are the mathematical model of the circuit element.

- Current direction is preserved w.r.t the circuit:
  current direction points from +v sign towards –v sign
- No need to mark voltage signs in digraphs
- Circuit element is suppressed (deleted)
- power delivered to element P(t) = v(t)i(t)



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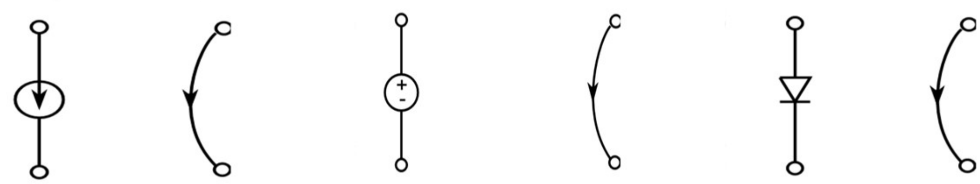
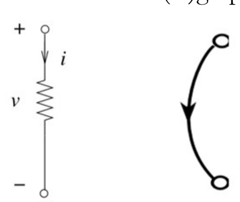
Mathematical Models

Current Source circuit | Terminal Graph (di)graph | Voltage Source circuit | Terminal Graph (di)graph | Diode circuit | Terminal Graph (di)graph



Terminal Equation: $i=i_k$ Terminal Equation: $v=v_k$ Terminal Equation: $i=I_o e^{(v/vT-1)}$

Resistor circuit | Terminal Graph (di)graph



Terminal Equation: v=iR

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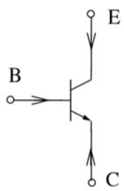
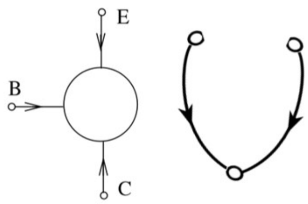
Circuit Graphs

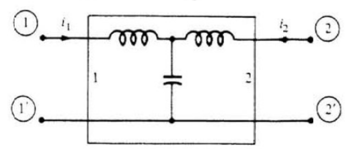
3 Terminal Element
circuit



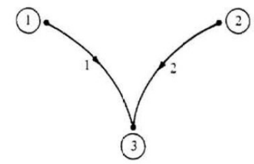
Terminal Graph
(di)graph



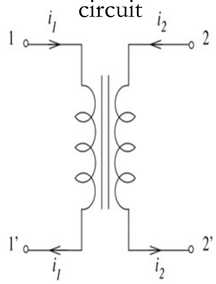
Grounded Two Port
circuit



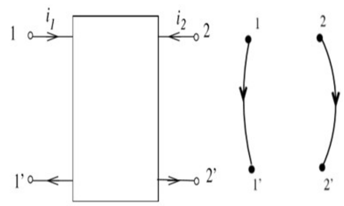
Terminal Graph
(di)graph



Two Port
circuit

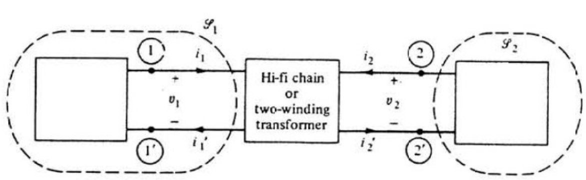


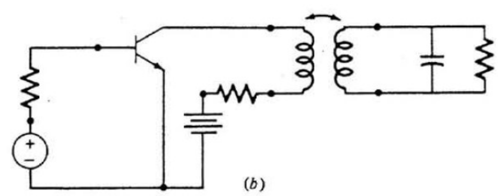
Terminal Graph
(di)graph



Circuit Graphs – Two Ports

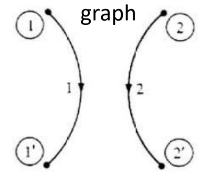
Two port is a circuit (element) with two pairs of accessible terminals:
Example: transformers, hi-fi's



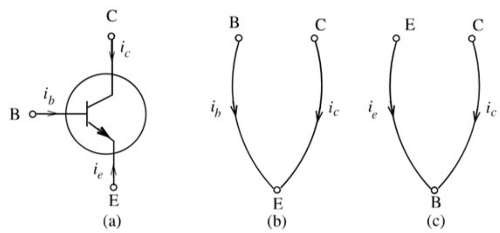


- *KCL:* -> i1 = i1` & i2 = i2`
- Power delivered: **P= v1(t)i1(t) + v2(t)i2(t)**
- The graph of a two port(4 terminal) circuit contains 2 branches, but the graph of a 1 port 4 terminal circuit contains 3 branches

graph



Mathematical Model Example - BJT



Mathematical model is given by the terminal equation

$$\begin{bmatrix} v_{bc} \\ i_c \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} i_b \\ V_{ce} \end{bmatrix}$$

and terminal graph (b). Find the terminal equation in the form

$$\begin{bmatrix} v_{eb} \\ i_c \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} i_e \\ V_{cb} \end{bmatrix}$$

if (c) is the terminal graph.

Mathematical Model Example - BJT

Terminal equations

$$\begin{aligned} v_{bc} &= h_{11} i_b + h_{12} v_{ce} \\ i_c &= h_{21} i_b + h_{22} v_{ce} \end{aligned}$$

KCL and KVL for the circuit element

$$\begin{aligned} i_c + i_e + i_b &= 0 \\ v_{ce} + v_{eb} + v_{bc} &= 0. \end{aligned}$$

New terminal variables are i_e and V_{eb} (additional to i_c and V_{cb}).
Substituting KVL and KCL Eqs. into above Eqs. we obtain

$$\begin{aligned} v_{bc} &= h_{11}(-i_c - i_e) + h_{12}(-v_{eb} + v_{cb}) \\ i_c &= h_{21}(-i_c - i_e) + h_{22}(-v_{eb} + v_{cb}) \end{aligned}$$

Mathematical Model Example - BJT

$$\begin{aligned} h_{12}v_{eb} + h_{11}i_c &= -h_{11}(i_e) + (1+h_{12})v_{cb} \\ (1+h_{21})i_c + h_{22}v_{eb} &= -h_{21}i_e + h_{22}v_{cb} \end{aligned}$$

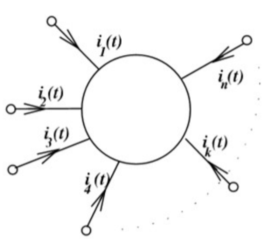
New terminal equations

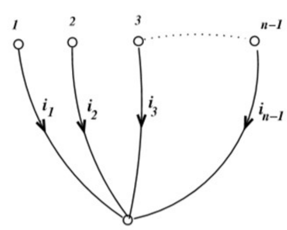
$$\begin{bmatrix} h_{12} & h_{11} \\ h_{22} & 1+h_{21} \end{bmatrix} \begin{bmatrix} v_{eb} \\ i_c \end{bmatrix} = \begin{bmatrix} -h_{11} & (1+h_{12}) \\ -h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} i_e \\ V_{cb} \end{bmatrix}$$

and terminal graph (c) will be the new mathematical model !

First Postulate of Circuit Theory

First Postulate of Circuit Theory:
All the properties of an n-terminal (or n-1 port) electrical element can be described by a mathematical relation between a set of (n-1) voltage and a set of (n-1) current variables.





Terminal variables and Terminal equation of n-terminal circuit element:

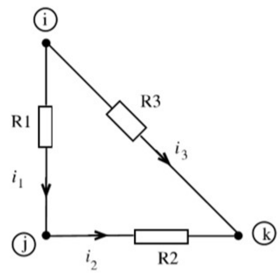
$$v = \begin{bmatrix} V_{1,n} \\ V_{2,n} \\ V_{3,n} \\ . \\ . \\ . \\ V_{n-1,n} \end{bmatrix}, i = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ . \\ . \\ . \\ i_{n-1} \end{bmatrix} \text{ and } f\left(v, i, \frac{dv}{dt}, \frac{di}{dt}, t\right) = 0$$

Power delivered at time t to the n-terminal circuit element:

$$P = \sum_{k=1}^{n} v_k i_k$$

Second Postulate of Circuit Theory: Kirchhoff’s Voltage Law (KVL)

Second Postulate of Circuit Theory: Kirchhoff’s Voltage Law (KVL)
For all lumped connected circuits, for all closed node sequences, for all times t, the algebraic sum of all node-to-node voltages around the chosen closed node sequence is equal to zero.



For the closed node sequence i-j-k-i

$V_{i,j} + V_{j,k} + V_{k,i} = 0$

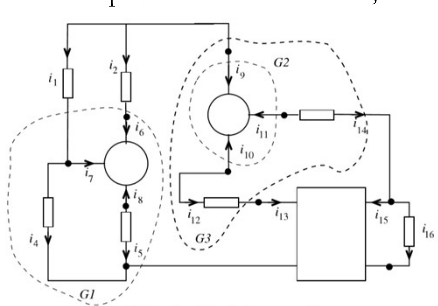
$e_i - e_j + e_j - e_k + e_k - e_i = 0$

Third Postulate of Circuit Theory: Kirchhoff’s Current Law (KCL)

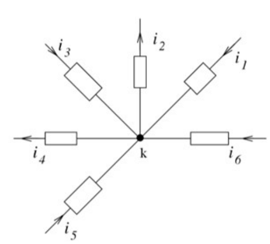
Third Postulate of Circuit Theory: Kirchhoff’s Current Law (KCL)
For all lumped circuits, for all gaussian surfaces G, for all times t, the algebraic sum of all the currents leaving the gaussian surface G at time t is equal to zero.

Gaussian surface: It is a closed surface that cuts only the wires which connect the circuit elements.

KCL (node law)
For all lumped circuits for all times t, the algebraic sum of the currents leaving any node is equal to zero.



$$
\begin{aligned}
G1 \quad & i_1 + i_2 + i_{13} = 0 \\
G2 \quad & i_9 + i_{10} + i_{11} = 0 \\
G3 \quad & i_9 - i_{12} - i_{14} = 0
\end{aligned}
$$



For the node k:

$$i_1 - i_2 + i_3 - i_4 + i_5 + i_6 = 0$$

Tellegen's Theorem

Tellegen's Theorem
The algebraic sum of power absorbed by all elements in a circuit is zero at any instant.

Tellegen's theorem asserts that

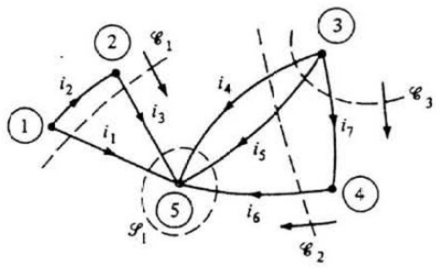
$$\sum_{k=1}^{n_e} v_k i_k = 0$$

Cut Sets and KCL

Cut set (ξ) is an important graph-theoretical concept:
ξ of a Gaussian surface is called a cut set if

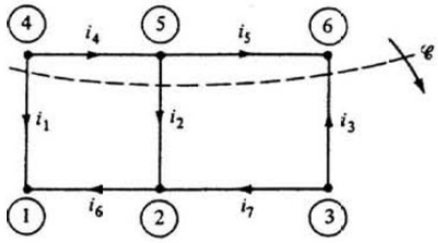
- Removal of all branches of the cut set results in an unconnected graph
- If you leave 1 branch within the cut set, the digraph stays connected

For the digraph of Fig. 5.15, $\mathscr{C}_1 = \{\beta_1, \beta_3\}$, $\mathscr{C}_2 = \{\beta_4, \beta_5, \beta_6\}$, and $\mathscr{C}_3 = \{\beta_4, \beta_5, \beta_7\}$ form cut sets. Here, β_k denotes "branch k."



Cut Sets and KCL

KCL: the sum of currents within a cut set is 0
Arrow of the cut set is its reference direction



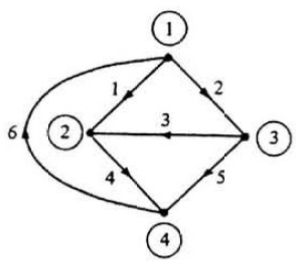
i1 + i2 – i3 = 0

Proof:
node 5: i4 – i2 – i5 = 0 (node 6: i3 = -i5)
i4 -i2 +i3 = 0 (node 4: i4 = -i1)
-i1 –i2 +i3 = 0

Cut set partitions set of nodes into 2 subsets
By writing KCL for each node and adding the result, we obtain the cut-set equation

Matrix Formulation and Independence Property - KCL

A digraph with 4 nodes and 6 branches



KCL:
Branches: 1 2 3 4 5 6

i1 + i2 – i6 = 0
-i1 -i3 +i4 = 0
- i2 +i3 + i5 = 0
- i4 – i5 + i6 = 0

n: # of nodes

Rank: # of independent equations
= n-1 -> 4 nodes -> rank: 3

Incidence matrix: A_a

A

$$\begin{array}{c} ①\rightarrow \\ ②\rightarrow \\ ③\rightarrow \\ ④\rightarrow \end{array} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

branch 1 branch 6

For an n_n-node n_b-branch connected graph G, node equations are given by

$$A_a i = 0$$

where $i = [i_1\ i_2 \ldots i_{n_b}]^T$ is called the branch current vector. A_a is called incidence matrix of the graph G and $A_a \in \{-1, 0, 1\}^{n_n \times n_b}$.

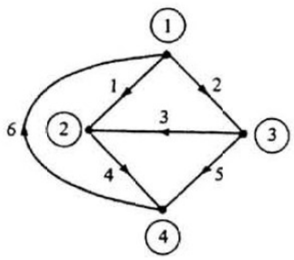
Independence property of KCL equations

For any connected grahp with n_n nodes, the KCL equations for any $n_n - 1$ of these nodes form a set of $n_n - 1$ linearly independent equations.

Reduced incidence matrix: **A**i = 0

Matrix Formulation – KCL

A digraph with 4 nodes and 6 branches



All distinct cut-sets of the Graph

$\mathscr{C}_1 = \{1, 2, 6\}$ $\quad$ $\mathscr{C}_4 = \{4, 5, 6\}$

$\mathscr{C}_2 = \{1, 3, 4\}$ $\quad$ $\mathscr{C}_5 = \{2, 3, 4, 6\}$

$\mathscr{C}_3 = \{2, 3, 5\}$ $\quad$ $\mathscr{C}_6 = \{1, 3, 5, 6\}$

Cut-set matrix
rank: n-1 = 3

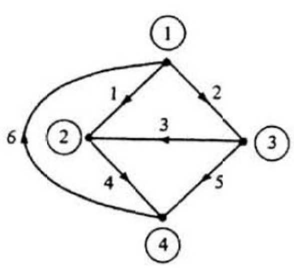
$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{Q}_a \mathbf{i} = \mathbf{0}$$

Reduced: $\quad Q_R\, I = 0$

Matrix Formulation - KVL

A digraph with 4 nodes and 6 branches



Branch voltages
v1 = e1 – e2
v2 = e1 - e3
v3 = -e2 + e3
v4 = e2
v5 = e3
v6 =-e1

$$\begin{bmatrix} v1 \\ v2 \\ v3 \\ v4 \\ v5 \\ v6 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e1 \\ e2 \\ e3 \\ e4 \\ e5 \\ e6 \end{bmatrix}$$

Matrix form $\;$ v = Me
v = A^T e
M = A^T

$$\begin{array}{c} ① \rightarrow \\ ② \rightarrow \\ ③ \rightarrow \\ ④ \rightarrow \end{array} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

branch 1 $\qquad$ branch 6

Independent KVL Equations

For an n_n-node n_b-branch connected graph G, independent KVL equations are given by

$$V = MV_n$$

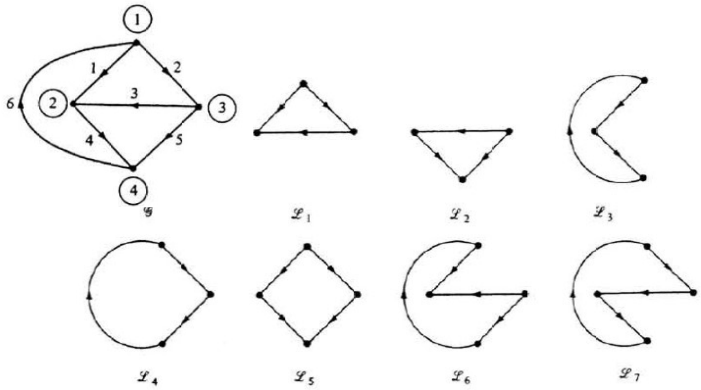
where $V = [V_1\ V_2 \ldots V_{n_e}]^T$ and $V_n = [V_{n1}\ V_{n2} \ldots V_{nn_n-1}]^T$ are called the branch voltage vector and node-to-datum voltage vector, respectively. M is a $n_e \times n_n - 1$ matrix.

Comparing the independent KVL and KCL equations, we conclude that

$$M = A^T$$

Loop and KVL

7 distinct loops of the graph



$$\begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{B}_a \mathbf{v} = \mathbf{0}$$

Linearly dependent equations!

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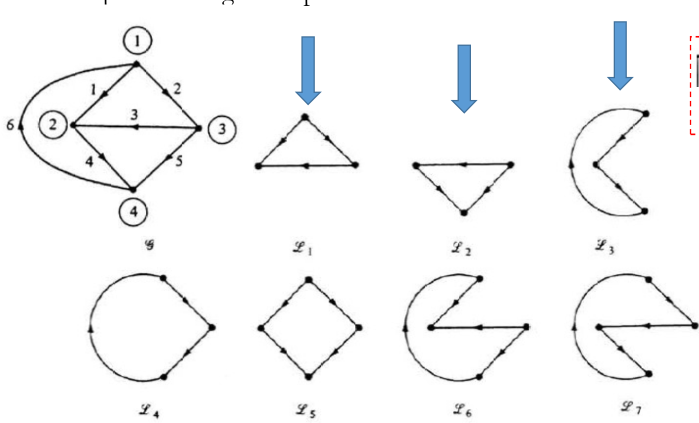
Loop and KVL

3 loops are enough to represent all nodes!



$$\begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{B}_a \mathbf{v} = \mathbf{0}$$

B_R v = 0

Linearly independent equations!

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Tree

A **Tree** is a subgraph that is
- Connected
- Contains all the nodes of the graph
- Has no loops
- Tree branches: **twigs**
- Branches that do not belong to the tree within a graph: **links & chords & cotree branches**

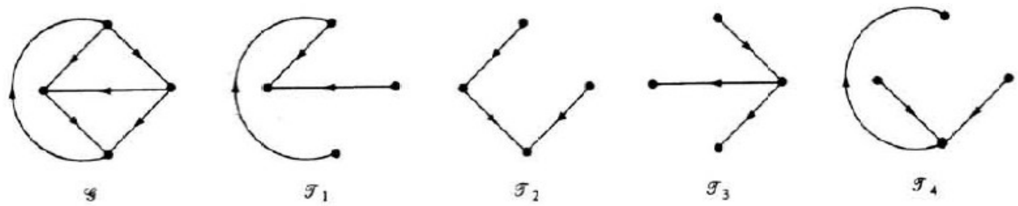


Figure 3.1 Four distinct trees of the digraph $\mathcal{G}$.

Fundamental Theorem of Graphs

1) **There is a unique path along the tree between any pairs of nodes**
   since a tree is connected
2) **There are n-1 twigs and l=b-(n-1) links**
3) **Every twig together with some links define a unique cut set, called <u>fundamental cut set</u> associated with the twig**
4) **Every link and the unique path on the tree between its two nodes constitute a unique loop called the <u>fundamental loop associated with the link</u>**

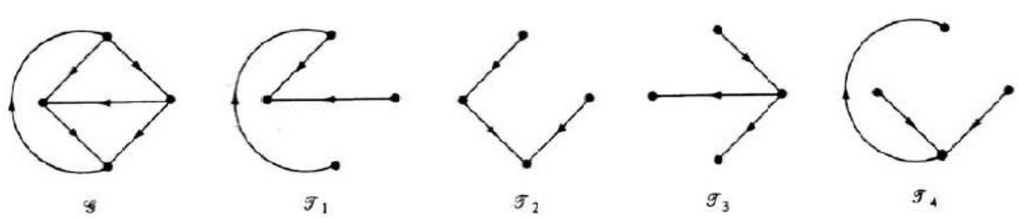


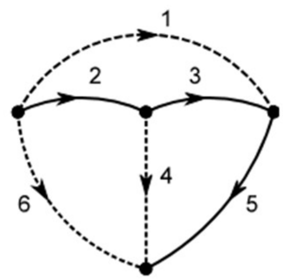
Figure 3.1 Four distinct trees of the digraph $\mathcal{G}$.

Fundamental Loop Analysis

Fundamental Loop: Every link (chord) of co-tree and the unique tree path between its nodes constitute a unique loop. This loop is called the Fundamental Loop associated with the link.

Fundamental Loop Equation: The linear algebraic equations obtained by applying KVL to each Fundamental Loop constitute a set of b-n+1 linearly independent equations.

Reference direction for the loop is taken as the direction which agrees with that of the link defining the loop.

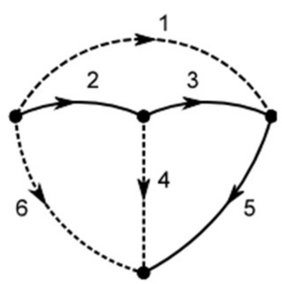


The links $G_L = \{1, 4, 6\}$ for the chosen tree $G_T = \{2, 3, 5\}$. The Fundamental loop sets are $G_{L1} = \{1, 2, 3\}$ $G_{L4} = \{4, 5, 3\}$ $G_{L6} = \{6, 2, 3, 5\}$.

If we apply KVL to the Fundamental Loops,

$$V_1 - V_3 - V_2 = 0$$
$$V_4 - V_5 - V_3 = 0$$
$$V_6 - V_5 - V_3 - V_2 = 0$$

Fundamental Loop Analysis



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$$\underbrace{\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{array}\right]}_{B = [\,I \;\; F\,]} \begin{bmatrix} V_1 \\ V_4 \\ V_6 \\ V_2 \\ V_3 \\ V_5 \end{bmatrix} \begin{matrix} \}\, V_l \\ \\ \}\, V_t \end{matrix} = 0$$

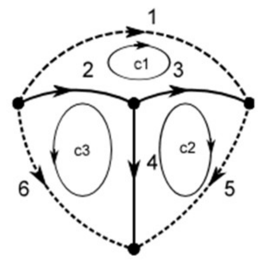
V_l Link Voltage Vector
V_t Twig Voltage Vector
B Fundamental Loop Matrix

$$V_l = -F\,V_t$$

The number of Fundamental loop equations is b-n+1 (number of links)

Mesh Analysis

Meshes are special case of the Fundamental Loops i.e., there exists a tree such that the meshes are Fundamental loops.

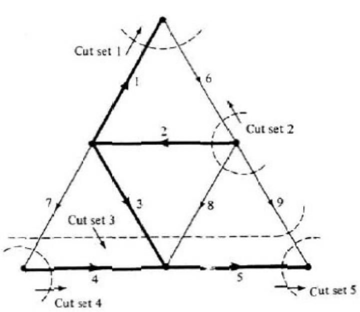


$i_1 = i_{m1}$
$i_2 = -i_{m3} - i_{m1}$
$i_3 = i_{m2} - i_{m1}$
$i_4 = -i_{m3} - i_{m2}$
$I_5 = i_{m2}$
$I_6 = i_{m3}$

$$\begin{bmatrix} i_1 \\ i_5 \\ i_6 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} i_{m1} \\ i_{m2} \\ i_{m3} \end{bmatrix}$$

There are 3 meshes. Corresponding loop sets and mesh currents (loop currents) $G_{M1} = \{1, 2, 3\}$ and i_{m1}: $G_{M2} = \{3, 4, 5\}$ and i_{m2}; $G_{M3} = \{2, 4, 6\}$ and i_{m3}.

KVL using twig voltages



Twig voltages

$v_1 = v_{t1}$
$v_2 = v_{t2}$
$v_3 = v_{t3}$
$v_4 = v_{t4}$
$v_5 = v_{t5}$

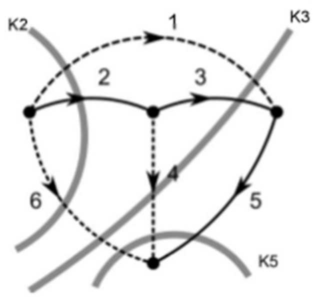
$v_6 = -v_1 - v_2 \qquad = -v_{t1} - v_{t2}$
$v_7 = v_3 - v_4 \qquad = v_{t3} - v_{t4}$
$v_8 = v_2 + v_3 \qquad = v_{t2} + v_{t3}$
$v_9 = v_2 + v_3 + v_5 \qquad = v_{t2} + v_{t3} + v_{t5}$

$$\begin{bmatrix} v_1 \\ v_2 \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ \\ v_9 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}}_{n-1} \begin{bmatrix} v_{t1} \\ v_{t2} \\ v_{t3} \\ v_{t4} \\ v_{t5} \end{bmatrix}$$

$v = Q^T v_t$

Fundamental Cut-set Analysis

Cut-set is made up of links and one twig, namely the twig which defines the cut-set.
Every twig defines a unique Fundamental cut-set.
Reference direction for the cut-set is the direction of the twig defining the cut-set.



Cut sets of the tree of $G_T = \{2, 3, 5\}$ are $G_{C2} = \{2, 1, 6\}$ $G_{C3} = \{3, 1, 4, 5\}$ $G_{C5} = \{5, 4, 6\}$.

If we apply KCL to the Fundamental cut-sets,

$$i_2 + i_1 + i_6 = 0$$
$$i_3 + i_1 + i_4 + i_6 = 0$$
$$i_4 + i_5 + i_6 = 0$$

$$Q\;\Big[\overbrace{\begin{matrix}1&0&1\\1&1&1\\0&1&1\end{matrix}}^{E}\;\overbrace{\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}}^{I}\Big]\begin{bmatrix}\left.\begin{matrix}i_1\\i_4\\i_6\end{matrix}\right\} i_l\\ \left.\begin{matrix}i_2\\i_3\\i_5\end{matrix}\right\} i_t\end{bmatrix} = 0$$

i_l Link Current Vector
i_t Twig Current Vector
Q Fundamental Loop Matrix

$$Qi = [E \quad I]\begin{bmatrix} i_l \\ i_t \end{bmatrix}$$
$$i_t = -E i_l$$

Each fundamental cut-set constitute a linearly independent equation. (n-1 linearly independent equations total)

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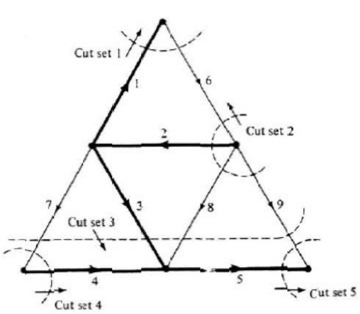
Fundamental Cut-Sets Associated with a Tree: KCL based on fundamental cut-sets



Cut set 1:	$i_1 - i_6 = 0$
Cut set 2:	$i_2 - i_6 + i_8 + i_9 = 0$
Cut set 3:	$i_3 + i_7 + i_8 + i_9 = 0$
Cut set 4:	$i_4 - i_7 = 0$
Cut set 5:	$i_5 + i_9 = 0$

n = 6, b = 9
-> 5 twigs, 4 links

Each twig defines a unique fundamental cut-set
(Fundamental Theorem of graphs # 3)

$$\underbrace{\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{n-1 \text{ twigs}}}_{1}\;\underbrace{\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\ell \text{ links}}}_{Q_l} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ \cdot \\ \cdot \\ \cdot \\ i_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Q.i = 0
Q: (n-1)*b matrix: fundamental cut-set matrix
$Q = [1_{n-1} Q_l]$

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Relationship Between B and Q

$F = -E^T$

Proof : Since they are the tree-branch voltages of the tree, the branch voltages are given by

$$V = Q^T V_n$$
$$BV = BQ^T V_n = 0$$
$$BQ^T V_n = 0$$

$$BQ^T = 0$$
$$IE^T + FI = 0$$
$$E^T + F = 0$$
$$E^T = -F$$