# EHB 211E: Basics of Electrical Circuits
*Operational Amplifiers*

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## Motivation

- Two new and important passive elements will be introduced: capacitors & inductors
- Unlike resistors, which dissipate energy, capacitors and inductors store energy that can be retrieved at a later time. For this reason, these circuit elements are called *storage elements*.
- With capacitors and inductors, we will be able to analyze other practical circuits, that we can not using only resistors



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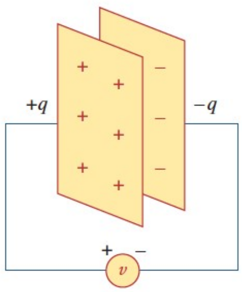
# Capacitors

- A capacitor is a passive element designed to store energy in its electric field.
- Besides resistors, capacitors are the most common electrical component.
- Capacitors are extensively used in electronics, communications, computers, and power systems.
- As an example, capacitors are used in tuning circuits of the radio receivers.

A capacitor consists of two conducting plates separated by an insulator (or dielectric)




- When a voltage source is applied to the capacitor, the source deposits a positive charge on one plate and negative charge on the other.
- Therefore, the capacitor *stores* the electric charge.
- The amount of charge stored is proportional to the voltage:

$$q = Cv$$





# Capacitors

$$q = Cv$$

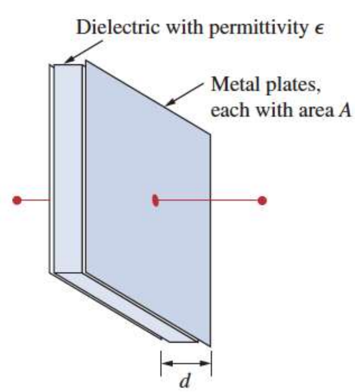
Capacitance is the ratio of the charge on one plate of the capacitor to the voltage difference between two plates, measured in farads (F). 1 farad = 1 coulomb/volt.

- Capacitance does not depend on $q$ or $v$. For the parallel plate capacitor:

$$C = \frac{\epsilon A}{d}$$

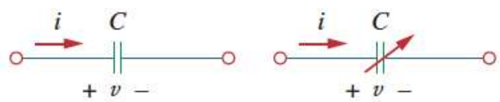
(*A is the surface area, d is the distance between plates, and ε is the permittivity of the dielectric material between the plates*)

- Typically capacitors have values in the pF (picofarad)- μF (microfarad) range.



# Capacitors

- Circuit symbol




- If v > 0 and i > 0, or if v < 0 and i < 0 , the capacitor is being charged.
- If v.i < 0 the capacitor is discharging



Polyester, ceramic and electrolytic capacitor

(a)

(b)

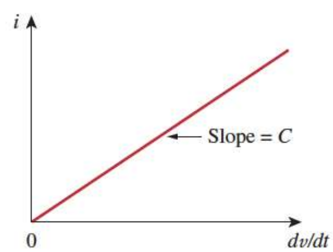
Variable capacitors
(trimmer and filmtrim type)





# Capacitors – current vs. voltage relationship

$$\boxed{q = Cv} \Rightarrow i = \frac{dq}{dt} \Rightarrow \boxed{i = C\frac{dv}{dt}}$$




$$\boxed{i = C\frac{dv}{dt}} \Rightarrow v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

$$\boxed{v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + v(t_0)}$$

- Capacitor voltage depends on the past history of the capacitor current. Therefore capacitor has a memory!

# Capacitors – power and energy

Instantaneous power: $p = vi = Cv\frac{dv}{dt}$

Energy stored: $w = \int_{-\infty}^{t} p(\tau)d\tau = C\int_{-\infty}^{t} v\frac{dv}{d\tau}d\tau = C\int_{v(-\infty)}^{v(t)} v\,dv = \frac{1}{2}Cv^2\Big|_{v(-\infty)}^{v(t)}$

v(-∞) = 0 –> capacitor is uncharged at t = -∞

$$\boxed{w = \frac{1}{2}Cv^2} \rightarrow \boxed{q = Cv} \rightarrow w = \frac{q^2}{2C}$$





# Capacitors – two important characteristics

- When the voltage across a capacitor is not changing with time
  -> the current through the capacitor is 0.
  -> **A capacitor is open to DC!**

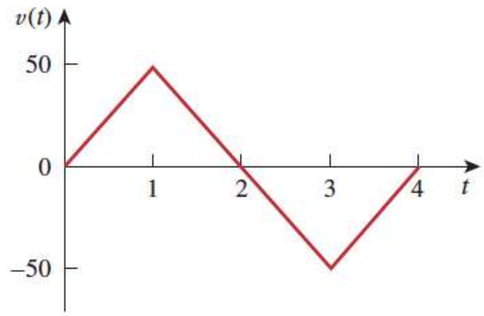
- **The voltage on a capacitor cannot change abruptly**
  -> The voltage must be continuous in time
  (discontinuous change requires infinite current)

$$\boxed{i = C\frac{dv}{dt}}$$

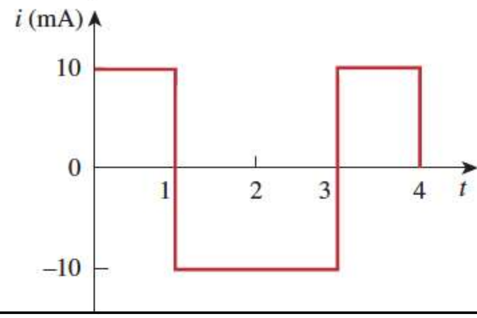
# Exercise

Determine the current through a 200 µF capacitor, whose voltage characteristics is given as ->

$$v(t) = \begin{cases} 50t \text{ V} & 0 < t < 1 \\ 100 - 50t \text{ V} & 1 < t < 3 \\ -200 + 50t \text{ V} & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases}$$




$$i(t) = \begin{cases} 10 \text{ mA} & 0 < t < 1 \\ -10 \text{ mA} & 1 < t < 3 \\ 10 \text{ mA} & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases}$$

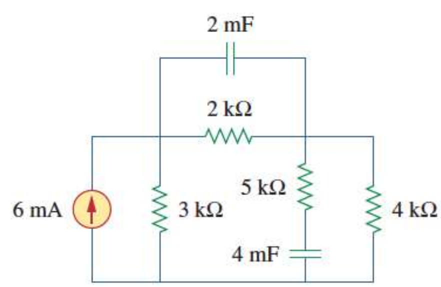




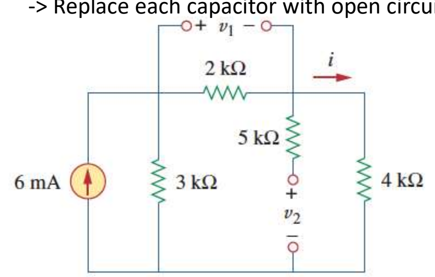



# Exercise

Obtain the energy stored in each capacitor under DC conditions




-> Replace each capacitor with open circuit!




$$i = \frac{3}{3+2+4}(6 \text{ mA}) = 2 \text{ mA}$$

$$v_1 = 2000i = 4 \text{ V} \qquad v_2 = 4000i = 8 \text{ V}$$

$$w_1 = \frac{1}{2}C_1v_1^2 = \frac{1}{2}(2 \times 10^{-3})(4)^2 = 16 \text{ mJ}$$

$$w_2 = \frac{1}{2}C_2v_2^2 = \frac{1}{2}(4 \times 10^{-3})(8)^2 = 128 \text{ mJ}$$

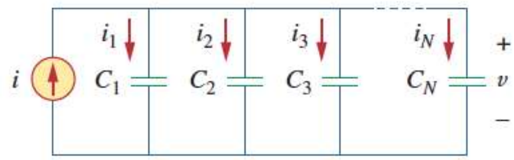
# Capacitors – Series and Parallel Connection

**Capacitors in parallel:**

$$i = i_1 + i_2 + i_3 + \cdots + i_N$$

$$i = C_1\frac{dv}{dt} + C_2\frac{dv}{dt} + C_3\frac{dv}{dt} + \cdots + C_N\frac{dv}{dt}$$

$$= \left(\sum_{k=1}^{N} C_k\right)\frac{dv}{dt} = C_{eq}\frac{dv}{dt}$$




$$\boxed{C_{eq} = C_1 + C_2 + C_3 + \cdots + C_N}$$

Capacitors in parallel combine in a similar manner as conductance in parallel (or resistors in series)





# Capacitors – Series and Parallel Connection

**Capacitors in series:**

$$v = v_1 + v_2 + v_3 + \cdots + v_N$$

$$v_k = \frac{1}{C_k}\int_{t_0}^{t} i(\tau)d\tau + v_k(t_0)$$

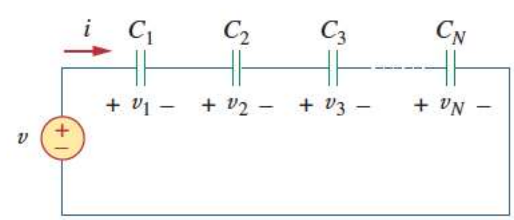
$$v = \frac{1}{C_1}\int_{t_0}^{t} i(\tau)d\tau + v_1(t_0) + \frac{1}{C_2}\int_{t_0}^{t} i(\tau)d\tau + v_2(t_0)$$

$$+ \cdots + \frac{1}{C_N}\int_{t_0}^{t} i(\tau)d\tau + v_N(t_0)$$

$$= \left(\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}\right)\int_{t_0}^{t} i(\tau)d\tau + v_1(t_0) + v_2(t_0)$$

$$+ \cdots + v_N(t_0)$$

$$= \frac{1}{C_{eq}}\int_{t_0}^{t} i(\tau)d\tau + v(t_0)$$



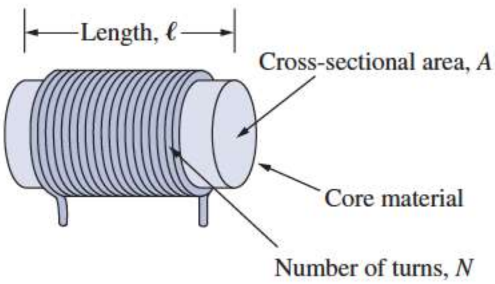

$$\boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots + \frac{1}{C_N}}$$

$$v(t_0) = v_1(t_0) + v_2(t_0) + \cdots + v_N(t_0)$$

Capacitors in series combine in a similar manner as resistors in parallel

# Inductors

An inductor is a passive element designed to store energy in its magnetic field.
An inductor consists of a coil of conducting wire




If a current is allowed to pass through an inductor, it is found that the voltage across the inductor is directly proportional to the time rate of change of the current:

$$\boxed{v = L\frac{di}{dt}}$$

*L:* inductance (units of Henry:H = 1 volt.second/ampere )



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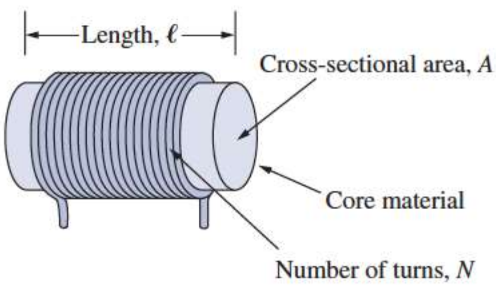
# Inductors




Inductance depends on physical dimensions and construction.

$$L = \frac{N^2 \mu A}{\ell}$$

*N:* number of turns
*μ:* permeability
*A:* cross-sectional area
*L:* length



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# Inductors - types

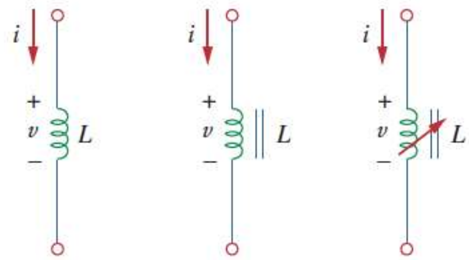
(a) solenoidal

(b) toroidal

(c) chip

Typically inductors have values in the μH (microHenry)

**Circuit symbol**




air-core    iron-core    variable iron-core



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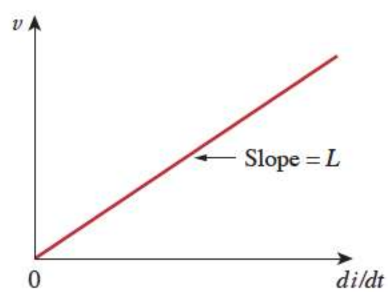
# Inductors

$$\boxed{v = L\frac{di}{dt}}$$

$$di = \frac{1}{L} v\, dt \qquad i = \frac{1}{L}\int_{-\infty}^{t} v(\tau)\, d\tau$$

$$\boxed{i = \frac{1}{L}\int_{t_0}^{t} v(\tau)\, d\tau + i(t_0)}$$



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# Inductors – stored energy

$$\boxed{v = L\frac{di}{dt}}$$

**Inductor acts like a short circuit to DC !**

**The current through an inductor cannot change instantaneously**

$$p = vi = \left(L\frac{di}{dt}\right)i$$

$$w = \int_{-\infty}^{t} p(\tau)\,d\tau = L\int_{-\infty}^{t} \frac{di}{d\tau}\, i d\tau$$

$$= L\int_{-\infty}^{t} i\,di = \frac{1}{2}Li^2(t) - \frac{1}{2}Li^2(-\infty)$$

i(-∞) = 0 –> $\boxed{w = \frac{1}{2}Li^2}$





# Exercise

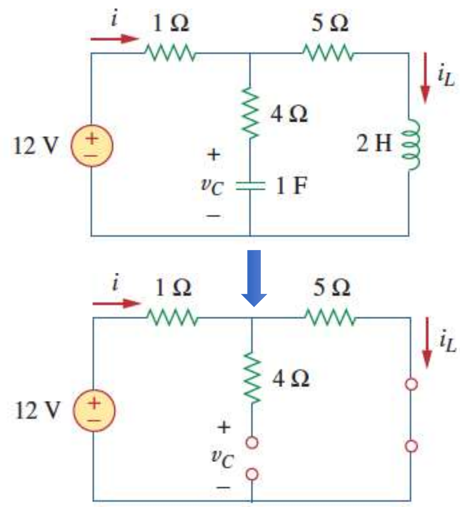
Under DC conditions, find $i$, $v_C$, and $v_L$. Also find the energy stored in the capacitor and the inductor.

$$i = i_L = \frac{12}{1+5} = 2\text{ A}$$

$$v_C = 5i = 10\text{ V}$$

$$w_C = \frac{1}{2}Cv_C^2 = \frac{1}{2}(1)(10^2) = 50\text{ J}$$

$$w_L = \frac{1}{2}Li_L^2 = \frac{1}{2}(2)(2^2) = 4\text{ J}$$



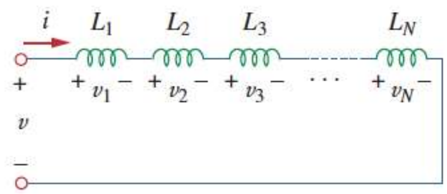
# Inductors – series connection

$$v = v_1 + v_2 + v_3 + \cdots + v_N$$

$$v = L_1\frac{di}{dt} + L_2\frac{di}{dt} + L_3\frac{di}{dt} + \cdots + L_N\frac{di}{dt}$$

$$= (L_1 + L_2 + L_3 + \cdots + L_N)\frac{di}{dt}$$

$$= \left(\sum_{k=1}^{N} L_k\right)\frac{di}{dt} = L_{eq}\frac{di}{dt}$$




$$\boxed{L_{eq} = L_1 + L_2 + L_3 + \cdots + L_N}$$

(similar to series connection of resistors)



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# Inductors – parallel connection

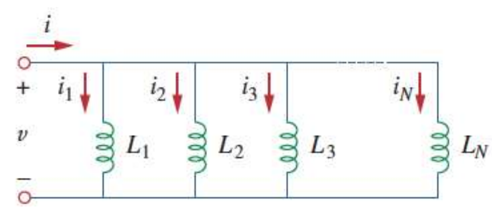
$$i = i_1 + i_2 + i_3 + \cdots + i_N$$

$$i = \frac{1}{L_1}\int_{t_0}^{t} v\,dt + i_1(t_0) + \frac{1}{L_2}\int_{t_0}^{t} v\,dt + i_2(t_0) + \cdots + \frac{1}{L_N}\int_{t_0}^{t} v\,dt + i_N(t_0)$$

$$= \left(\frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}\right)\int_{t_0}^{t} v\,dt + i_1(t_0) + i_2(t_0) + \cdots + i_N(t_0)$$

$$= \left(\sum_{k=1}^{N}\frac{1}{L_k}\right)\int_{t_0}^{t} v\,dt + \sum_{k=1}^{N} i_k(t_0) = \frac{1}{L_{eq}}\int_{t_0}^{t} v\,dt + i(t_0)$$

$$i(t_0) = i_1(t_0) + i_2(t_0) + \cdots + i_N(t_0)$$




$$\boxed{\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots + \frac{1}{L_N}}$$

(similar to parallel connection of resistors)



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# Capacitors & Inductors: Summary

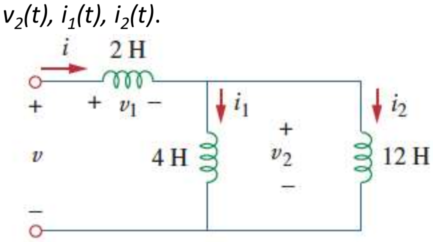
| Relation | Resistor ($R$) | Capacitor ($C$) | Inductor ($L$) |
|---|---|---|---|
| $v$-$i$: | $v = iR$ | $v = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + v(t_0)$ | $v = L\frac{di}{dt}$ |
| $i$-$v$: | $i = v/R$ | $i = C\frac{dv}{dt}$ | $i = \frac{1}{L}\int_{t_0}^{t} v(\tau)d\tau + i(t_0)$ |
| $p$ or $w$: | $p = i^2R = \frac{v^2}{R}$ | $w = \frac{1}{2}Cv^2$ | $w = \frac{1}{2}Li^2$ |
| Series: | $R_{eq} = R_1 + R_2$ | $C_{eq} = \frac{C_1C_2}{C_1 + C_2}$ | $L_{eq} = L_1 + L_2$ |
| Parallel: | $R_{eq} = \frac{R_1R_2}{R_1 + R_2}$ | $C_{eq} = C_1 + C_2$ | $L_{eq} = \frac{L_1L_2}{L_1 + L_2}$ |
| At dc: | Same | Open circuit | Short circuit |
| Circuit variable that cannot change abruptly: | Not applicable | $v$ | $i$ |





# Exercise

For the circuit, $i(t) = 4(2-e^{-10t})$mA. *If* $i_2(0) = -1$mA, find: $i_1(0)$, $v(t)$, $v_1(t)$, $v_2(t)$, $i_1(t)$, $i_2(t)$.




$$i(0) = 4(2 - 1) = 4 \text{ mA}$$

$$i_1(0) = i(0) - i_2(0) = 4 - (-1) = 5 \text{ mA}$$

$$L_{eq} = 2 + 4 \parallel 12 = 2 + 3 = 5 \text{ H}$$

$$v(t) = L_{eq}\frac{di}{dt} = 5(4)(-1)(-10)e^{-10t} \text{ mV} = 200e^{-10t} \text{ mV}$$

$$v_1(t) = 2\frac{di}{dt} = 2(-4)(-10)e^{-10t} \text{ mV} = 80e^{-10t} \text{ mV}$$

$$v_2(t) = v(t) - v_1(t) = 120e^{-10t} \text{ mV}$$

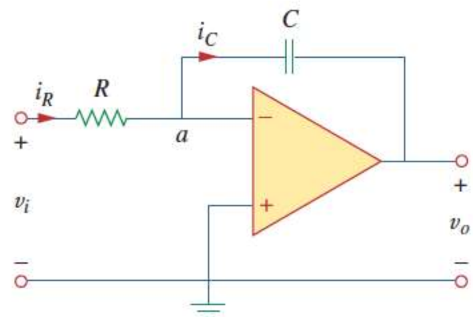
$$i_1(t) = \frac{1}{4}\int_0^t v_2\,dt + i_1(0) = \frac{120}{4}\int_0^t e^{-10t}\,dt + 5 \text{ mA}$$
$$= -3e^{-10t}\Big|_0^t + 5 \text{ mA} = -3e^{-10t} + 3 + 5 = 8 - 3e^{-10t} \text{ mA}$$

$$i_2(t) = \frac{1}{12}\int_0^t v_2\,dt + i_2(0) = \frac{120}{12}\int_0^t e^{-10t}\,dt - 1 \text{ mA}$$
$$= -e^{-10t}\Big|_0^t - 1 \text{ mA} = -e^{-10t} + 1 - 1 = -e^{-10t} \text{ mA}$$

# Applications: Integrator

$$i_R = i_C$$

$$i_R = \frac{v_i}{R}, \qquad i_C = -C\frac{dv_o}{dt}$$




$$\frac{v_i}{R} = -C\frac{dv_o}{dt}$$

$$dv_o = -\frac{1}{RC}v_i\,dt$$

$$v_o(t) - v_o(0) = -\frac{1}{RC}\int_0^t v_i(\tau)d\tau \qquad \boxed{v_o = -\frac{1}{RC}\int_0^t v_i(\tau)d\tau}$$

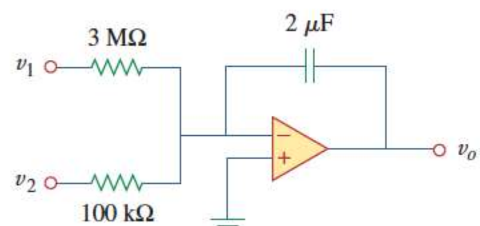
Assuming the capacitor is discharged ($v_0(0) = 0$) prior to the application of the signal



# Exercise

If $v_1(t) = 10\cos 2t$ mV, $v_2(t) = 0.5t$ mV, find $v_o(t)$.

<u>The circuit is a summing integrator!</u>




$$v_o = -\frac{1}{R_1C}\int v_1\,dt - \frac{1}{R_2C}\int v_2\,dt$$

$$= -\frac{1}{3\times 10^6 \times 2\times 10^{-6}}\int_0^t 10\cos(2\tau)d\tau$$

$$-\frac{1}{100\times 10^3 \times 2\times 10^{-6}}\int_0^t 0.5\tau d\tau$$

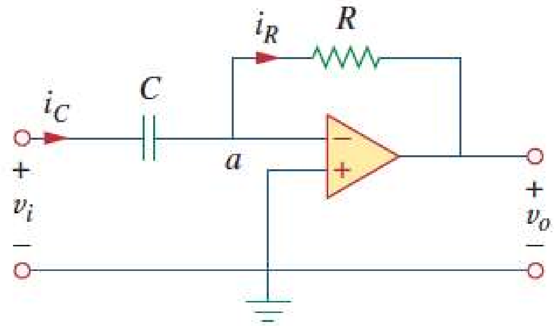
$$= -\frac{1}{6}\frac{10}{2}\sin 2t - \frac{1}{0.2}\frac{0.5t^2}{2} = -0.833\sin 2t - 1.25t^2 \text{ mV}$$

# Applications: Differentiator

$i_R = i_C$

$i_R = -\frac{v_o}{R}, \quad i_C = C\frac{dv_i}{dt}$

$$\boxed{v_o = -RC\frac{dv_i}{dt}}$$








# Exercise

For the circuit, sketch the output for the given input

$RC = 5 \times 10^3 \times 0.2 \times 10^{-6} = 10^{-3}$ s

$$v_i = \begin{cases} 2000t & 0 < t < 2 \text{ ms} \\ 8 - 2000t & 2 < t < 4 \text{ ms} \end{cases}$$

$$v_o = -RC\frac{dv_i}{dt} = \begin{cases} -2 \text{ V} & 0 < t < 2 \text{ ms} \\ 2 \text{ V} & 2 < t < 4 \text{ ms} \end{cases}$$