EHB 211E: Basics of Electrical Circuits

State Space Representation

Asst. Prof. Ahmet Can Erten
(aerten@itu.edu.tr)

1

State Equations

Capacitor current and or its voltage are given by

$$C\frac{dV_C}{dt} = i_C$$

and

$$V_C(t) = \frac{1}{C}\int_{t_0}^{t} i_C(\tau)d\tau + V_C(0)$$

Inductor voltage and current are given by

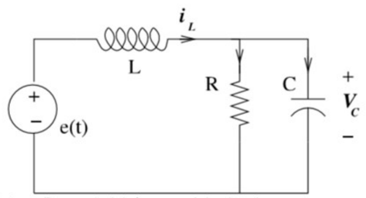
$$L\frac{di_L}{dt} = V_L$$

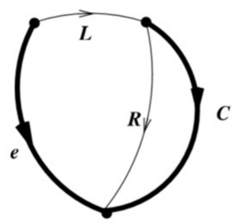
and

$$i_L(t) = \frac{1}{L}\int_{t_0}^{t} V_L(\tau)d\tau + i_L(0)$$

2

State Equations





Using Branch Voltages Method

$$i_C = i_L - i_R$$

Using the definition of L and C elements

$$\begin{aligned} C\frac{dV_C}{dt} &= \frac{1}{L}\int_{t_0}^{t} V_L(\tau)d\tau + i_L(0) - GV_R \\ &= \frac{1}{L}\left(\int_{t_0}^{t} e(\tau)d\tau - \int_{t_0}^{t} V_C(\tau)d\tau\right) + i_L(0) - GV_R \end{aligned}$$

we have an Integro-Differential Equation !.

State Equations

We can represent the same circuit by differential equations of the form

$$\begin{aligned} C\frac{dV_C}{dt} &= i_L - i_R \\ L\frac{di_L}{dt} &= e - V_C \end{aligned}$$

We can write the state equations in matrix form:

$$\begin{bmatrix} \frac{dV_C}{dt} \\ \frac{di_L}{dt} \end{bmatrix} = \begin{bmatrix} -G/C & 1/C \\ -1/L & 0 \end{bmatrix} \begin{bmatrix} V_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} e$$

where $i_R = GV_R = GV_C$. This equation can be recast into the standard form

$$\begin{aligned} \dot{X} &= AX + Bu \\ y &= CX + Du \end{aligned}$$

where X is state variable vector, y is output and u is input.

State-Space Representation

- The "state" of a system is the minimum information needed about the system in order to determine its future behavior.
- State variables are smallest set of variables that together with any input to the system is sufficient to determine the future behavior of the system.
- Each state variable has "memory" (voltage across capacitor, current through inductor)
- Each state variable has an "initial condition"
- State-space representation is a mathematical model of a physical system as a set of input, output, and state variables related by 1st order differential equations.
- State equations: Set of coupled 1st order differential equations.

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = C(t)\mathbf{x}(t) + D(t)\mathbf{u}(t)$$

x: state variable vector, y: output vector, u: input vector

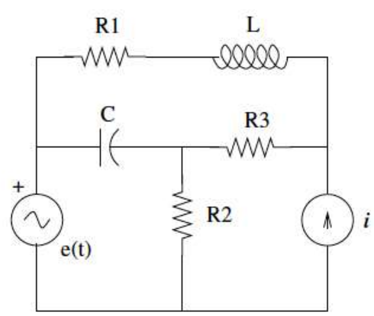
Obtaining State Equations

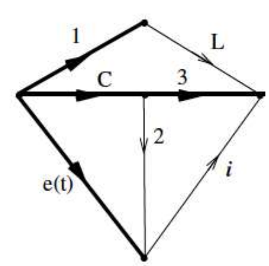
1. Pick a proper tree :
   - The voltage sources must be placed in the tree.
   - If the tree is not complete, the edges corresponding to as many capacitors as possible must be placed in the tree. If a capacitor in a loops which consisting entirely of capacitors and voltage sources. The capacitor must not be placed in the tree.
   - If the tree is not complete, the edges corresponding to the resistors must be chosen and as many resistors as possible must be included.
   - If still the tree is not completed, then, the edges corresponding to the inductors will be chosen until the tree is completed. If an inductors in a cut set which consisting entirely of inductors and current sources, the inductor must be placed in tree.
   - All the edges corresponding to the current sources must be placed in the co-tree.
2. After the selection of proper tree, the state variables are branch capacitor voltages and chord inductor currents.

Obtaining State Equations

3. Obtaining State Equations from the circuit: Express the voltage across each element corresponding to a branch and the current through each element corresponding to non-branch edge in terms of voltage sources, current sources, and state variables. * If not possible, assign a new voltage variable to a resistor corresponding to a branch and a new current variable to a resistor corresponding to a non-branch edge.
   a Apply KVL to the fundamental loop determined by each non-branch inductor.
   b Apply KCL to the fundamental cut-set determined by each branch capacitor.
   c Apply KVL to the fundamental loop determined by each resistor with a new current variable assigned in *.
   d Apply KCL to the node or super-node corresponding to the fundamental cut-set determined by each resistor with a new voltage variable assigned in *
   e Solve the simultaneous equations obtained from steps c and d for the new variables in terms of the voltage sources, current sources, and the state variables.
   f Substitute the expressions obtained in step e into the equations determined in steps a and b.

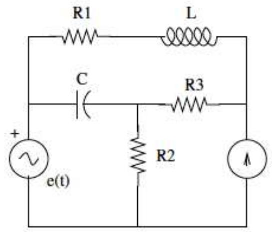
Example

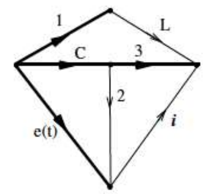




1. Graph is drawn and pick the proper tree.
2. V_C and i_L state variables.

$$\dot{V}_C = f(V_C, i_L, e(t), i(t)) \quad \dot{i}_L = f(V_C, i_L, e(t), i(t))$$





1. KVL for the fundamental loop determined by the inductor and KCL to the fundamental cut-set determined by the capacitor.

$$\begin{aligned} i_C + i_L - i_2 + i &= 0 \\ V_L - V_3 - V_C + V_1 &= 0 \end{aligned}$$

using the definition of the inductor and capacitor

$$\begin{aligned} C\frac{dV_C}{dt} &= -i_L + i_2 - i \\ L\frac{di_L}{dt} &= V_3 + V_C - V_1 \end{aligned}$$

9

9

KVL for the fundamental loop determined by R_2 and KCL to the fundamental cut-set determined by R_1 and R_3

$$\begin{aligned} R_2 i_2 &= e - V_C \\ G_1 V_1 &= i_L \\ G_3 V_3 &= -i_L - i \end{aligned}$$

Substitute the expressions

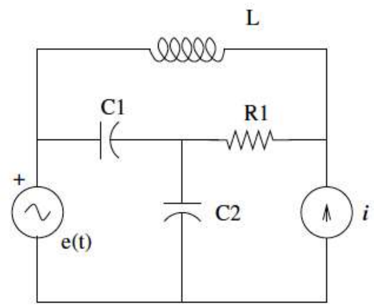
$$\frac{d}{dt}\begin{bmatrix} V_C \\ i_L \end{bmatrix} = \begin{bmatrix} \frac{-1}{R_2 C} & \frac{-1}{C} \\ \frac{1}{L} & \frac{-(R_3+R_1)}{L} \end{bmatrix}\begin{bmatrix} V_C \\ i_L \end{bmatrix} + \begin{bmatrix} \frac{1}{R_2 C} \\ 0 \end{bmatrix} e(t) + \begin{bmatrix} \frac{-1}{C} \\ -\frac{R_3}{L} \end{bmatrix} i$$

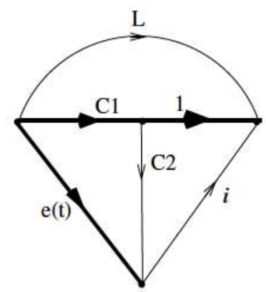
10

10

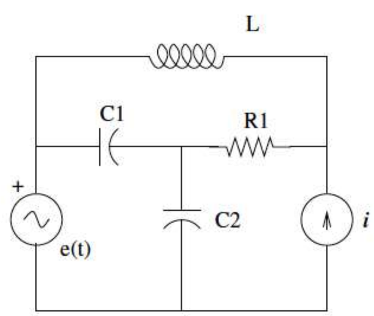
Circuit which contains any

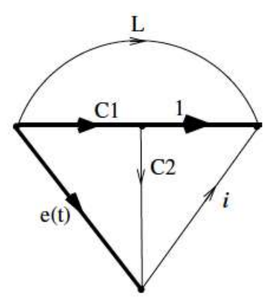
- Loops consisting entirely of capacitors and voltage sources.
- Cutsets consisting entirely of inductors and current sources.



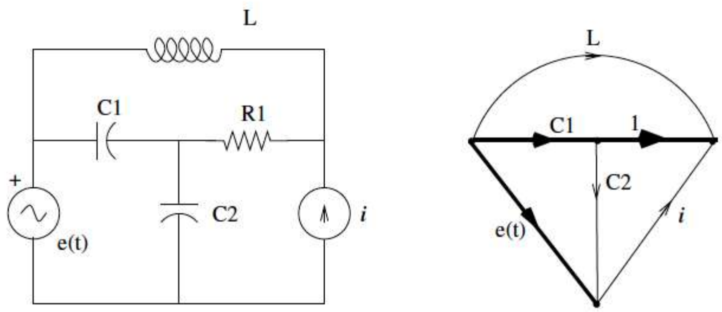


two capacitors and the voltage source make a loop.





1. $C2$ must be placed to co-tree.



2. The state variable are V_{C1} i_L.
3. KCL and KVL

$$\begin{aligned} i_{C1} + i_L - i_{C2} + i &= 0 \\ V_L - V_1 - V_{C1} &= 0 \end{aligned}$$

Using the definition of C and L elements, the state equations;

$$\begin{aligned} C_1 \frac{dV_{C1}}{dt} &= -i_L - i + i_{C2} \\ L\frac{di_L}{dt} &= V_1 + V_{C1} \end{aligned}$$

Apply KVL to the fundamental loop determined by $C2$ and KCL to the fundamental cut-set determined by $R1$

$$\begin{aligned} V_{C2} &= e - V_{C1} \\ G_1 V_1 &= -i_L - i \end{aligned}$$

In order to obtain i_{C2} in terms of the voltage sources, current sources, and the state variables, we will use the definition of capacitor ($i_{C2} = C_2 \frac{dV_{C2}}{dt}$).

$$C_2 \frac{dV_{C2}}{dt} = C_2 \frac{de}{dt} - C_2 \frac{dV_{C1}}{dt}$$

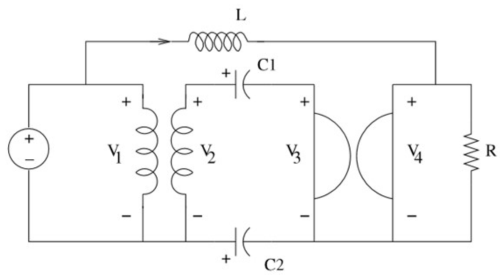
The state equation in standard form

$$\begin{aligned} C_1 \frac{dV_{C1}}{dt} &= -i_L - i + C_2 \frac{de}{dt} - C_2 \frac{dV_{C1}}{dt} \\ L\frac{di_L}{dt} &= -R_1(i_L - i) + V_{C1} \end{aligned}$$

$$\frac{d}{dt}\begin{bmatrix} V_{C1} \\ i_L \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{C_2+C_1} \\ \frac{1}{L} & \frac{-R}{L} \end{bmatrix}\begin{bmatrix} V_{C1} \\ i_L \end{bmatrix} + \begin{bmatrix} \frac{C_2}{C_1+C_2} \\ 0 \end{bmatrix}\frac{de}{dt} + \begin{bmatrix} \frac{-1}{C_1+C_2} \\ \frac{-R}{L} \end{bmatrix} i$$

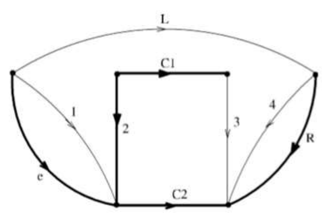
RLC and Multi-terminal Elements

All the edge corresponding to the dependent voltage source must be placed in tree. All the edge corresponding to the dependent current source must be placed in co-tree.



Transformer $V_2 = nV_1$, $i_1 = -ni_2$ and Gyrator $i_3 = -\alpha V_4$, $i_4 = \alpha V_3$

RLC and Multi-terminal Elements



1. Graph is drawn. The voltage sources e, capacitors $C1$ and $C2$ are placed to tree. The tree is not complete, edge 2 is a dependent voltage source which is placed to tree. The edges 3 and 4 are placed to co-tree.
2. V_{C1}, V_{C2} and i_L are state variable.
3. From the fundamental cut-sets and loop, we have

$$\begin{aligned} i_{C1} + i_3 &= 0 \\ i_{C2} + i_L + i_3 &= 0 \\ V_L + V_R - V_{C2} - e &= 0 \end{aligned}$$

RLC and Multi-terminal Elements

The state equations;

$$
\begin{aligned}
C_1 \frac{dV_{C1}}{dt} &= -i_3 \\
C_2 \frac{dV_{C2}}{dt} &= -i_L - i_3 \\
L \frac{di_L}{dt} &= -V_R + V_{C2} + e
\end{aligned}
$$

Express the i_3 and V_R as function of state variable and independent sources

$$
\begin{aligned}
i_R &= -i_4 + i_L = i_L - \alpha V_3 = i_L - \alpha(-V_{C1} + V_2 + V_{C2}) \\
&= i_L - \alpha(-V_{C1} + ne + V_{C2}) \\
i_3 &= \alpha V_4 = \alpha V_R = \alpha R i_R
\end{aligned}
$$

$$
\frac{d}{dt}\begin{bmatrix} V_{C1} \\ V_{C2} \\ i_L \end{bmatrix} = \begin{bmatrix} -- & -- & -- \\ -- & -- & -- \\ -- & -- & -- \end{bmatrix} \begin{bmatrix} V_{C1} \\ V_{C2} \\ i_L \end{bmatrix} + \begin{bmatrix} -- \\ -- \\ -- \end{bmatrix} e
$$

Obtaining State Equations Directly from the Circuit

Consider a dynamic circuit that does not contain any

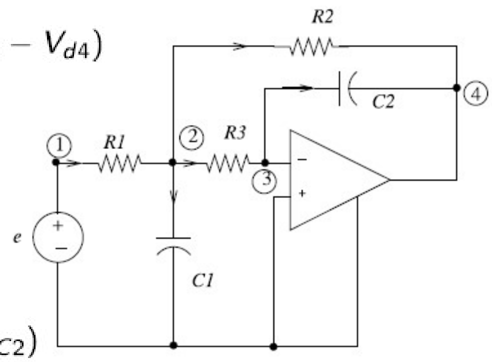
- Loops consisting entirely of capacitors and voltage sources.
- Cutsets consisting entirely of inductors and current sources.

The objective of the analysis is the express the currents of capacitors and the voltages of the inductors as a function of the voltages of the capacitors, the currents of the inductors and the independent sources.

Example

$i_{C1}=i_1-i_3-i_2$

$$\begin{aligned} C_1\frac{dV_{C1}}{dt} &= G1(V_{d1}-V_{d2}) - G_3(V_{d2}-V_{d3}) - G_2(V_{d2}-V_{d4}) \\ C_2\frac{dV_{C2}}{dt} &= G_3(V_{d2}-V_{d3} \end{aligned}$$



$$\begin{aligned} V_{d1} &= e \\ V_{d3} &= 0 \\ V_{d2} &= V_{C1} \\ V_{d4} &= -V_{C2} \end{aligned}$$

$$\begin{aligned} C_1\frac{dV_{C1}}{dt} &= G1(e-V_{C1}) - G_3(V_{C1}) - G_2(V_{C1}+V_{C2}) \\ C_2\frac{dV_{C2}}{dt} &= G_3V_{C1} \end{aligned}$$

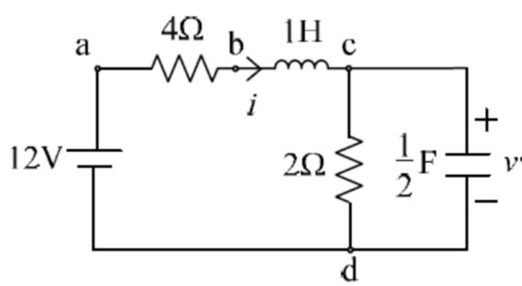
$$\frac{d}{dt}\begin{bmatrix} V_{C1} \\ V_{C2} \end{bmatrix} = \begin{bmatrix} -\frac{G_1+G_2+G_3}{C_1} & -\frac{G_2}{C_1} \\ -\frac{G3}{C_2} & 0 \end{bmatrix}\begin{bmatrix} V_{C1} \\ V_{C2} \end{bmatrix} + \begin{bmatrix} \frac{G_1}{C_1}0 \end{bmatrix} e$$

EHB 211E 19

19

Example 2

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t)$$



$i_C = i_L - i_R$ KCL at Node C

$12-V_L-V_C-4i_L= 0$ KVL

$$C\frac{dv}{dt} = i - \frac{v}{2}$$

$$L\frac{di}{dt} + v - 12V + 4i = 0$$

$$\frac{d}{dt}\begin{bmatrix} v \\ i \end{bmatrix} = \begin{bmatrix} \frac{-1}{2C} & \frac{1}{C} \\ \frac{-1}{L} & \frac{-4}{L} \end{bmatrix}\begin{bmatrix} v \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}(12V)$$

EHB 211E 20

20

Example 3

$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t)$

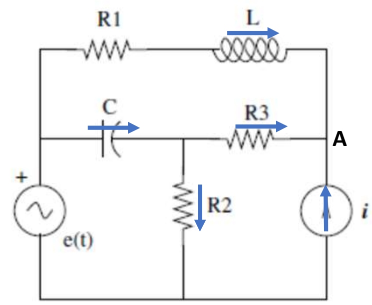
$i_3 + i_L + i = 0$

$$\begin{aligned} i_C + i_L - i_2 + i &= 0 \quad \text{KCL at node A} \\ V_L - V_3 - V_C + V_1 &= 0 \quad \text{KVL upper loop} \end{aligned}$$

$i_3 = i_C - i_2$

$$\begin{aligned} C\frac{dV_C}{dt} &= -i_L + i_2 - i \\ L\frac{di_L}{dt} &= V_3 + V_C - V_1 \end{aligned}$$

$$\begin{aligned} R_2 i_2 &= e - V_C \\ G_1 V_1 &= i_L \\ G_3 V_3 &= -i_L - i \end{aligned}$$



State variable

Input

Input

$$\frac{d}{dt}\begin{bmatrix} V_C \\ i_L \end{bmatrix} = \begin{bmatrix} \frac{-1}{R_2 C} & \frac{-1}{C} \\ \frac{1}{L} & \frac{-(R_3+R_1)}{L} \end{bmatrix} \begin{bmatrix} V_C \\ i_L \end{bmatrix} + \begin{bmatrix} \frac{1}{R_2 C} \\ 0 \end{bmatrix} e(t) + \begin{bmatrix} \frac{-1}{C} \\ -\frac{R_3}{L} \end{bmatrix} i$$

Two-terminal Elements

Two-terminal elements play a major role in electric circuits!

Two-terminal circuit elements are defined by the between basic variables which are current ($i(t)$), voltage ($v(t)$), charge ($q(t)$) and flux ($\phi(t)$). The units of them are Amperes, Volts, Coulomb and Weber, respectively.

Two pairs of the basic variables

$$i(t) = \frac{dq}{dt},$$

and

$$v(t) = \frac{d\phi}{dt},$$

are the definition.

Two-terminal Elements

Controlled circuit element (Dependent element)

If the relation between the terminal variable is given by the equation $x = h(y, t)$, this two-terminal element is called as a y controlled element e.g. voltage controlled voltage sources,...

Time-invariant two-terminal element

A two-terminal element whose variables x and y fall on some fixed curve in the $x - y$ plane at any time t is called a time-invariant circuit element e.g. Linear resistor $Vv = Ri$.

$x - y$ characteristic

The curve on the $x - y$ plane at any time t is called $x - y$ characteristic e.g. $v - i$ characteristic of linear resistor.

Two-terminal Elements

Bilateral property

A element has a $x - y$ characteristics which is not symmetric with respect to the origin of the $x - y$ plane.

Linear element

A linear element is an element with a linear relationship between its variables x and y.

Linear

$f(x)$ is a function which satisfies the following two properties:

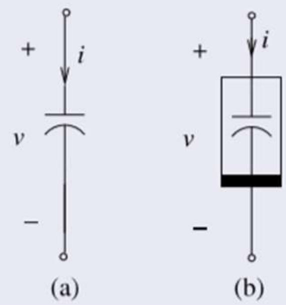
- Additivity (superposition): $f(x + y) = f(x) + f(y)$.
- Homogeneity : $f(\alpha x) = \alpha f(x)$ for all α.

Capacitor

A two-terminal element whose charge $q(t)$ and voltage $v(t)$ fall on some fixed curve in the $q-v$ plane at any time t is called a time-invariant capacitor. Linear time- invariant capacitor is represented by the equations

$$q = Cv \text{ or } i = C\frac{dv}{dt}$$

Values of capacitors are specified in ranges of farads (F).



25

Time-varying and Nonlinear Capacitor

If the $q-v$ characteristic changes with time, the capacitor is said to be time-varying. Then the mathematical model becomes

$$q = C(t)v$$

and

$$i = \frac{dC}{dt}v + C(t)\frac{dv}{dt}$$

The most general case, a time-varying nonlinear capacitor is defined by a family of time-dependent and nonlinear $q-v$ characteristics

$$f(q, v, t) = 0$$

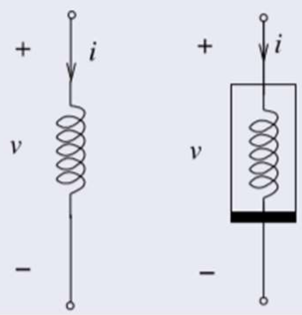
26

Inductor

A two-terminal element whose flux $\phi(t)$ and current $i(t)$ fall on some fixed curve in the $\phi - i$ plane at any time t is called a time-invariant inductor. The mathematical model of LTI inductor is

$$\phi = Li \ \text{ veya } \ \mathrm{v} = \mathrm{L}\frac{\mathrm{di}}{\mathrm{dt}}$$

Values of inductors are specified in ranges of Henry (H).



27

Time-varying and Nonlinear Inductor

If the $\phi - i$ characteristic changes with time, the inductor is said to be time-varying. Then the mathematical model becomes

$$v = L(t)i$$

and

$$v = \frac{dL}{dt}i + L(t)\frac{di}{dt}$$

The most general case, a time-varying nonlinear capacitor is defined by a family of time-dependent and nonlinear $\phi - v$ characteristics

$$f(\phi, v, t) = 0$$

.

28

Resistor

A two-terminal element will be called a resistor if its voltage v and current i satisfy the following relation:

$$\mathrm{R} = \{(v, i) | f(v, i) = 0\}$$

This relation is called the $v - i$ characteristic of the resistor and can be plotted graphically in the $v - i$ plane. The equation $f(v, i) = 0$ represents a curve in the $v - i$ plane and specifies completely the two-terminal resistor.
The linear resistor is a special case of a resistor and satisfies Ohm's law which is

$$f(v, i) = v - Ri \quad or \quad f(v, i) = Gv - i$$

It means that the voltage across resistor is proportional to the current flowing through it.

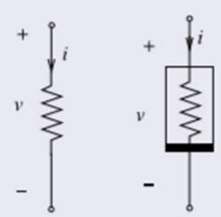
Linear and Nonlinear Resistor

Ohm's law states

$$v = Ri \quad or \quad i = Gv$$

where the constant R is the resistance of the linear resistor measured in the unit of ohms (Ω), and G is the conductance measured in the unit of Siemens (S). A resistor which is not linear is called nonlinear.

$$G = \frac{1}{R}$$



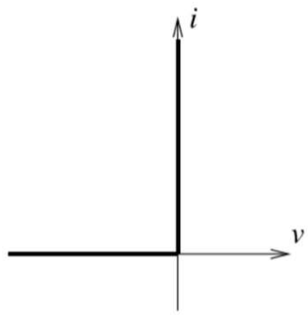
(a) (b)

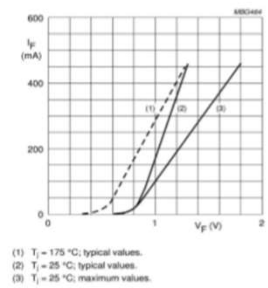
Nonlinear resistor: Diode

Ideal diode: Nonlinear resistor, whose v-i characteristics consists of two straight line segments.

$$i = I_0 e^{(v/v_T - 1)}$$

where $v_T = 0.026V$ ve I_0 μA.



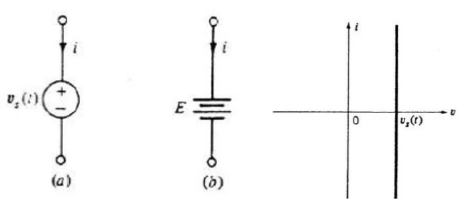


v < 0 -> i = 0 : open circuit when reverse biased
v = 0 -> i = ∞ : short circuit

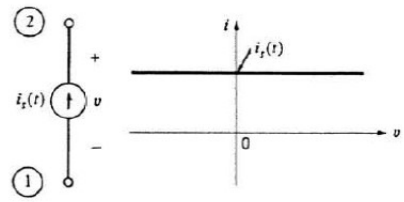
Independent sources

Independent sources: batteries, signal generators
could be either voltage or current source

Independent voltage source:
Voltage across is irrespective of current



Independent current source:
Current flowing is irrespective of voltage



Open & Short Circuit

Open circuit: i(t) = 0 for every v(t) : R -> ∞
Short circuit: v(t) = 0 for every i(t) : R -> 0

Dual elements

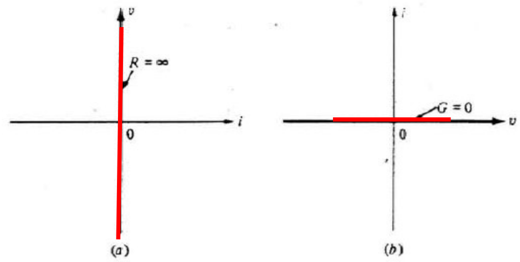


Figure 1.4 Characteristic of an open circuit.

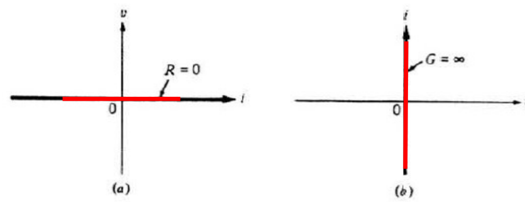
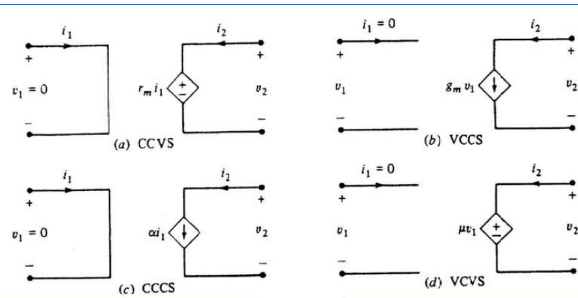


Figure 1.5 Characteristic of a short circuit.

Linear controlled sources



Current-Controlled Current Source (CCCS)

CCCS is characterized by two linear equations

$$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ v_2 \end{bmatrix}$$

where α is called the current transfer ratio.

Current-Controlled Voltage Source (CCVS)

CCVS is characterized by two linear equations

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ r & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

where r is called the transresistance.

Voltage-Controlled Current Source (VCCS)

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ g & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

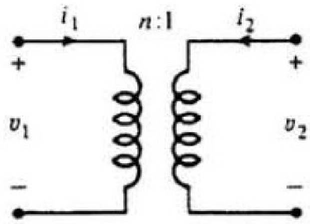
where g is called the transconductance.

Voltage-Controlled Voltage Source (VCVS)

$$\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ k & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ i_2 \end{bmatrix}$$

where k is called the voltage transfer ratio.

Ideal Transformer



$$\frac{v_1}{v_2} = n = \frac{-i_2}{i_1}$$

Ideal transformer is a two-port resistive circuit characterized by:

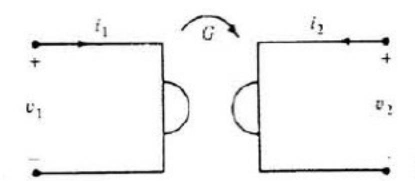
v1 = n.v2
i2 = -n.i1
n : *turns ratio*

$$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \mathbf{H}\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & n \\ -n & 0 \end{bmatrix}\begin{bmatrix} i_1 \\ v_2 \end{bmatrix}$$

- Ideal transformer neither dissipates nor stores energy (non-energetic element):
  p = v1it + v2i2 = 0

35

Ideal Gyrator



An ideal gyrator is a linear two port device which couples the current on one port to the voltage on the other and vice versa.

$$v_1 = -Ri_2 \qquad i_1 = Gv_2$$

$$i_2 = -Gv_1$$

$$\mathbf{i} = \begin{bmatrix} 0 & G \\ -G & 0 \end{bmatrix}\mathbf{v}$$

36

Analysis of Nonlinear Resistive Circuits

Linear approximation of the nonlinear element at the operating point Q can be obtained using Taylor series expansion

$$v_N(t) = f(i_N) = f(I_Q) + \frac{df(i)}{di_N}(i - I_Q)\Big|_Q + \text{h.o.t}$$

- The first term $V_Q = f(I_Q)$ is obtained from DC analysis. The solutions to a circuit with dc input are called operating points. The term dc analysis refers to the determination of operating points.
- The second term $v(t) = \frac{df(i)}{di_N}(i - I_Q)\Big|_Q$ is obtain form ac analysis (small signal analysis). We assume that the applied signal (which are ac signal) has a sufficiently small voltage or current (in magnitude).

The solution

From the superposition

$$v_N(t) = V_Q + R_Q i(t)$$

37

37

DC Analysis

How to solve the nonlinear equation which is obtained from DC Analysis ?

- Analytic approach :

$$aV_Q^2 + bV_Q + c = 0$$

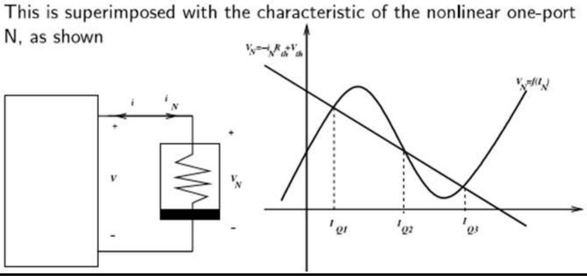
- Numerical method : The numerical method is very useful in solving nonlinear equations. The Newton-Raphson method is the most commonly used numerical method for finding dc operating points.
- Graphic Method (load line): Using Equivalent Circuit of the one-port, we have

$$V = iR_{th} + V_{th}$$

from KCL: $i = -i_N$ and KVL: $V = V_N$ we will have

$$V_N = -i_N R_{th} + V_{th}$$

This is superimposed with the characteristic of the nonlinear one-port N, as shown

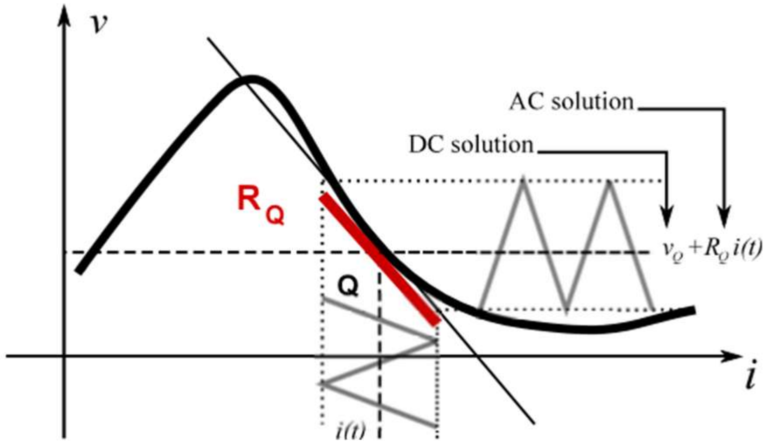


38

38

AC Analysis

An operating point specifies a region in the $v - i$ plane in the neighborhood of which the actual voltage and current in the circuit vary as a function of time.



AC Analysis

Amplitude of the AC signal is small compare to the operating point. to replacing the nonlinear characteristic by its linear approximation about the operating point Q.

$$v(t) = \frac{df(i)}{di_N}(i - i_Q)\bigg|_Q$$

The term $\frac{df(i)}{di_N}\Big|_Q$ is the slope of the nonlinear characteristic at the operating point Q.

$$R_Q = \frac{df(i)}{di_N}\bigg|_Q$$

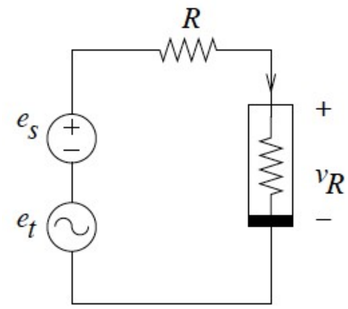
is called the "small-signal" resistance of the nonlinear element at the operating point Q.
Using R_Q in the circuit *small-signal equivalent circuit is obtained about operating point Q.*

Example

For the following circuit, R = 3.5Ω, e_s = 9V, e(t) = 0.1sin(10t).

The nonlinear resistor is characterized by:

$V_R = i_R^3 - 6i_R^2 + 9i_R$



DC Analysis:

$e = i_R R + V_R$

$e = i_R R + i_R^3 - 6i_R^2 + 9i_R$ -> I_R = 2A, V_R = 2V

AC Analysis:

Linearize the nonlinear resistor around I_R=2A

Resistance around the operating point (Q) is:

$R_Q = dV_R/di_R \mid_Q$ (derivative value at the operating point

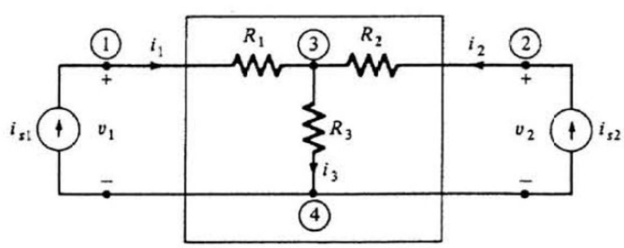
$R_Q = 3i_R^2 - 12i_R + 9 \mid_{iR=2} = -3\ \Omega$

$v_R = R_Q e(t) / (R_Q + R) = -0.6\sin(10t)$ for the AC source

Complete solution (superposition)

$V_R = 2 - 0.6\sin(10t)$

Linear Resistive Two Port (Current-Driven)



KCL

is1 = i1

is2 = i2

i3 = i1 + i2

KVL

v1 = (R1+R3)i1 + R3i2

v2 = R3i1 + (R2+R3)i2

Current controlled representation

Resistance matrix:

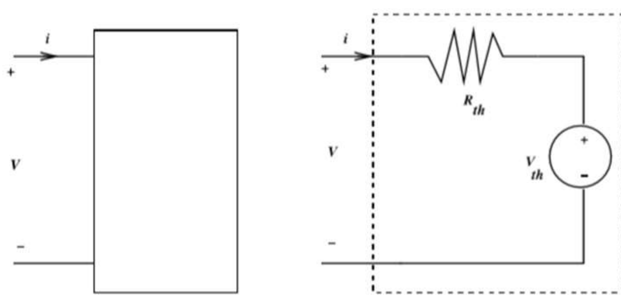
$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{Ri} = \begin{bmatrix} R_1 + R_3 & R_3 \\ R_3 & R_2 + R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

Linear Resistive Two Port

6 representations of a two-port

Representations	Scalar equations	Vector equations
Current-controlled	$v_1 = r_{11}i_1 + r_{12}i_2$ $v_2 = r_{21}i_1 + r_{22}i_2$	$\mathbf{v} = \mathbf{R}\mathbf{i}$
Voltage-controlled	$i_1 = g_{11}v_1 + g_{12}v_2$ $i_2 = g_{21}v_1 + g_{22}v_2$	$\mathbf{i} = \mathbf{G}\mathbf{v}$
Hybrid 1	$v_1 = h_{11}i_1 + h_{12}v_2$ $i_2 = h_{21}i_1 + h_{22}v_2$	$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \mathbf{H}\begin{bmatrix} i_1 \\ v_2 \end{bmatrix}$
Hybrid 2	$i_1 = h'_{11}v_1 + h'_{12}i_2$ $v_2 = h'_{21}v_1 + h'_{22}i_2$	$\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = \mathbf{H}'\begin{bmatrix} v_1 \\ i_2 \end{bmatrix}$
Transmission 1†	$v_1 = t_{11}v_2 - t_{12}i_2$ $i_1 = t_{21}v_2 - t_{22}i_2$	$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \mathbf{T}\begin{bmatrix} v_2 \\ -i_2 \end{bmatrix}$
Transmission 2†	$v_2 = t'_{11}v_1 + t'_{12}i_1$ $-i_2 = t'_{21}v_1 + t'_{22}i_1$	$\begin{bmatrix} v_2 \\ -i_2 \end{bmatrix} = \mathbf{T}'\begin{bmatrix} v_1 \\ i_1 \end{bmatrix}$

Thevenin Equivalent Circuit



I. Method

- One-port N is driven by an ideal current source.
- Find the terminal voltage in terms of the internal energy sources inside the network and the external current source.

Then the terminal voltage is obtained such as

$$V = R_{th}i + V_{th}$$