Probability and Statistics MAT 271E

PART 4

Exercise Questions

Assist. Prof. Dr. Ümit KARADOĞAN

Course originally developed by : Prof. Dr. Mehmetçik BAYAZIT, Prof. Dr. Beyhan YEĞEN

Example:

In a given business venture a man can make a profit of \$300 with probability 0.6 or take a loss of \$100 with probability 0.4.

Determine his expectation.

Solution:

$$\text{Expectation} = (\$300)(.6) + (-\$100)(.4) = \$180 - \$40 = \$140.$$

Example:

You take out a fire insurance policy on your home. The annual premium is \$300. In case of fire, the insurance company will pay you \$200,000. The probability of a house fire in your area is 0.0002.

a. What is the expected value?

b. What is the insurance company's expected value?

c. Suppose the insurance company sells 100,000 of these policies. What can the company expect to earn?

SOLUTION

a. $$\text{Expected value} = \underbrace{(0.0002)(\overbrace{199{,}700}^{200{,}000\,-\,300})}_{\text{Fire}} + \underbrace{(0.9998)(-300)}_{\text{No Fire}} = -\$260.00$$

The expected value over many years is $-\$260$ per year. Of course, your hope is that you will never have to collect on fire insurance for your home.

b. The expected value for the insurance company is the same, except the perspective is switched. Instead of $-\$260$ per year, it is $+\$260$ per year. Of this, the company must pay a large percent for salaries and overhead.

c. The insurance company can expect to gross \$30,000,000 in premiums on 100,000 such policies. With a probability of 0.0002 for fire, the company can expect to pay on about 20 fires. This leaves a gross profit of \$26,000,000.

$$100000 \times 300 - (0.0002 \times 100000 \times 200000) = 26{,}000{,}000 \text{ dollars profit}$$

Example:

Comparing Two Expected Values

A child asks his parents for some money. The parents make the following offers.

Father's offer: The child flips a coin. If the coin lands heads up, the father will give the child \$20. If the coin lands tails up, the father will give the child nothing.

Mother's offer: The child rolls a 6-sided die. The mother will give the child \$3 for each dot on the up side of the die.

Which offer has the greater expected value?

SOLUTION

Father's offer:

$$\text{Expected value} = \underbrace{\left(\frac{1}{2}\right)}_{\text{Probability of heads}}\underbrace{(20)}_{\text{Payoff for heads}} + \underbrace{\left(\frac{1}{2}\right)}_{\text{Probability of tails}}\underbrace{(0)}_{\text{Payoff for tails}} = \$10$$

Mother's offer: There are six possible outcomes.

	A	B	C	D
1	Number	Payoff	Probability	Expected Value
2	1	\$3.00	16.67%	\$0.50
3	2	\$6.00	16.67%	\$1.00
4	3	\$9.00	16.67%	\$1.50
5	4	\$12.00	16.67%	\$2.00
6	5	\$15.00	16.67%	\$2.50
7	6	\$18.00	16.67%	\$3.00
8	Total			**\$10.50**
9				

Example:

Your company is considering developing one of two cell phones. Your development and market research teams provide you with the following projections.

- **Cell phone A:**

  Cost of development: $2,500,000

  Projected sales: 50% chance of net sales of $5,000,000
  30% chance of net sales of $3,000,000
  20% chance of net sales of $1,500,000

- **Cell phone B:**

  Cost of development: $1,500,000

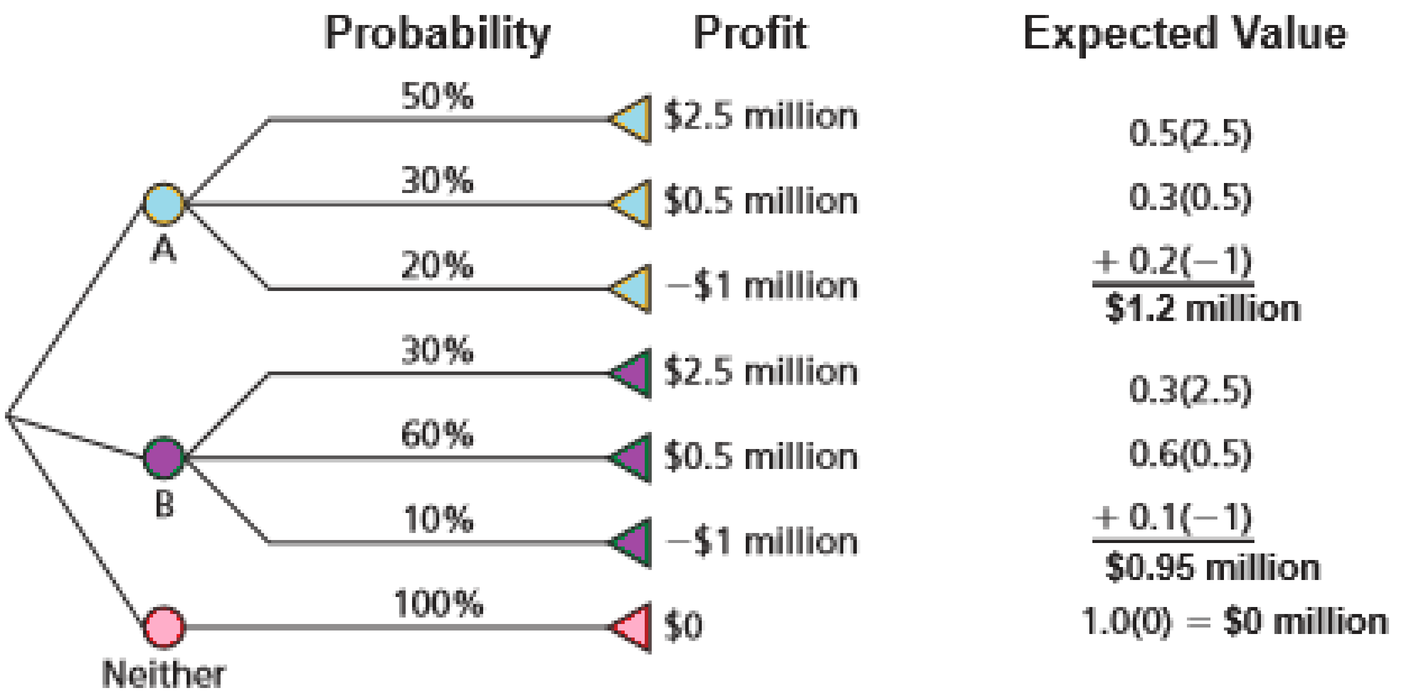
  Projected sales: 30% chance of net sales of $4,000,000
  60% chance of net sales of $2,000,000
  10% chance of net sales of $500,000

Which model should your company develop? Explain.

Example:

SOLUTION

A *decision tree* can help organize your thinking.



Although cell phone A has twice the risk of losing $1 million, it has the greater expected value. So, using expected value as a decision guideline, your company should develop cell phone A.

Example:

A *speculative investment* is one in which there is a high risk of loss. What is the expected value for each of the following for a $1000 investment?

a. Speculative investment

- Complete loss: 40% chance
- No gain or loss: 15% chance
- 100% gain: 15% chance
- 400% gain: 15% chance
- 900% gain: 15% chance

b. Conservative investment

- Complete loss: 1% chance
- No gain or loss: 35% chance
- 10% gain: 59% chance
- 20% gain: 5% chance

SOLUTION

a. Speculative investment

DATA

	A	B	C	D
1	Result	Payoff	Probability	Expected Value
2	Complete loss	-$1,000	40%	-$400
3	No gain or loss	$0	15%	$0
4	100% gain	$1,000	15%	$150
5	400% gain	$4,000	15%	$600
6	900% gain	$9,000	15%	$1,350
7	Total		100%	$1,700
8				

b. Conservative investment

	A	B	C	D
1	Result	Payoff	Probability	Expected Value
2	Complete loss	-$1,000	1%	-$10
3	No gain or loss	$0	35%	$0
4	10% gain	$100	59%	$59
5	20% gain	$200	5%	$10
6	Total		100%	$59
7				

This example points out the potential gain and the risk of investment. The speculative investment has an expected value of $1700, which is a high return on investment. If you had the opportunity to make 100 such investments, you would have a high likelihood of making a profit. But, when making only 1 such investment, you have a 40% chance of losing everything.

Example:

Find (a) $E(X)$, (b) $E(X^2)$, and (c) $E[(X-\bar{X})^2]$ for the following probability distribution.

X	8	12	16	20	24
$p(X)$	1/8	1/6	3/8	1/4	1/12

$$Var(X) = E((X-\mu_X)^2)$$

$$Var(X) = \sum_{x_i} (x_i - \mu_X)^2 p(x_i)$$

Solution:

(a) $E(X) = \Sigma X p(X) = (8)(1/8) + (12)(1/6) + (16)(3/8) + (20)(1/4) + (24)(1/12) = 16$

This represents the *mean* of the distribution.

(b) $E(X^2) = \Sigma X^2 p(X) = (8)^2(1/8) + (12)^2(1/6) + (16)^2(3/8) + (20)^2(1/4) + (24)^2(1/12) = 276$

This represents the *second moment* about the origin zero.

(c) $E[(X-\bar{X})^2] = \Sigma(X-\bar{X})^2 p(X)$

$= (8-16)^2(1/8) + (12-16)^2(1/6) + (16-16)^2(3/8) + (20-16)^2(1/4) + (24-16)^2(1/12) = 20$

This represents the *variance* of the distribution.

Example:

Suppose that X is a random variable that takes on values 0, 2 and 3 with probabilities 0.3, 0.1, 0.6 respectively.

Let $Y = 3(X - 1)^2$.

(a) What is the expectation of X?

(b) What is the variance of X?

(c) What is the expection of Y?

Example:

(a) We first make the probability tables

X	0	2	3
prob.	0.3	0.1	0.6
Y	3	3	12

$$\Rightarrow E(X) = 0 \cdot 0.3 + 2 \cdot 0.1 + 3 \cdot 0.6 = 2$$

(b) $E(X^2) = 0 \cdot 0.3 + 4 \cdot 0.1 + 9 \cdot 0.6 = 5.8$

$$\Rightarrow \mathrm{Var}(X) = E(X^2) - E(X)^2 = 5.8 - 4 = 1.8.$$

(c) $E(Y) = 3 \cdot 0.3 + 3 \cdot 0.1 + 12 \cdot 0.6 = 8.4.$

Example: Suppose that X takes values between 0 and 1 and has probability density function 2x. Compute Var(X) and Var(X^2).

We will make use of the formula $\mathrm{Var}(Y) = E(Y^2) - E(Y)^2$. First we compute

$$E[X] = \int_0^1 x \cdot 2x dx = \frac{2}{3}$$

$$E[X^2] = \int_0^1 x^2 \cdot 2x dx = \frac{1}{2}$$

$$E[X^4] = \int_0^1 x^4 \cdot 2x dx = \frac{1}{3}.$$

Example:

Thus,

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

and

$$\mathrm{Var}(X^2) = E[X^4] - (E[X^2])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$