Probability and Statistics
MAT 271E

PART 3

Exercise Questions

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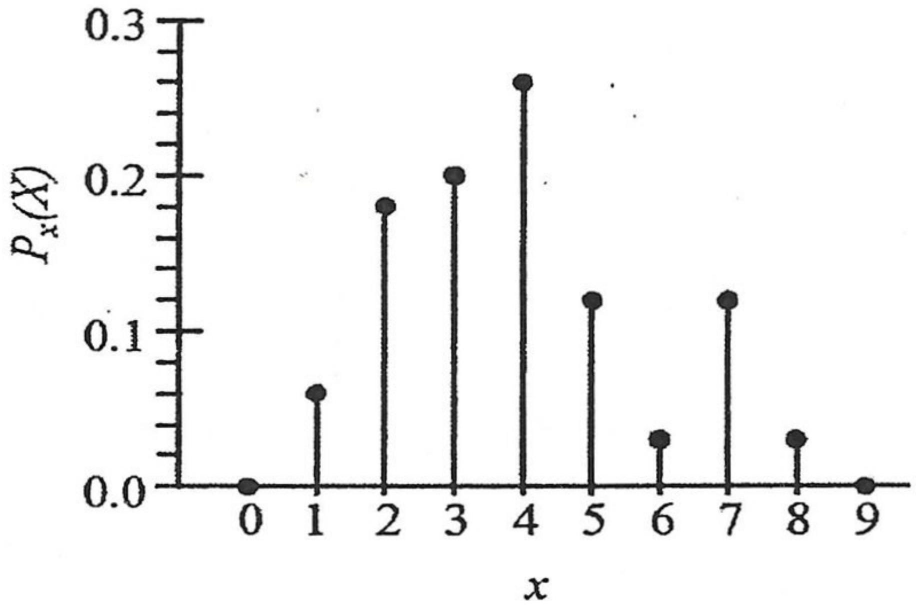
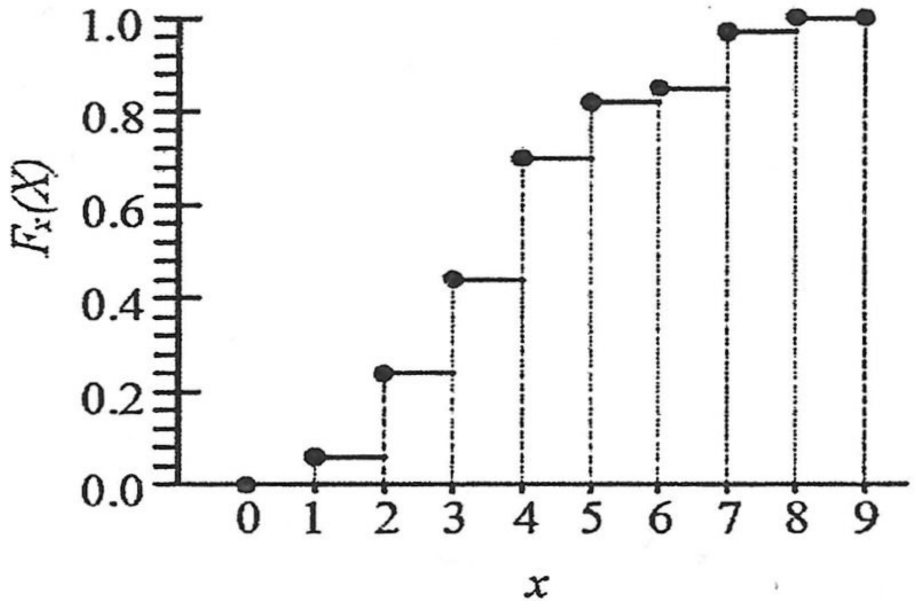
Example: The probability of a river having floods in a period of time is presented in a table. Plot the probability mass function and the cumulative distribution function for flood.

N. of floods	0	1	2	3	4	5	6	7	8	>8
p(x)	0.00	0.06	0.18	0.20	0.26	0.12	0.03	0.12	0.03	0
F(x)	0.00	0.06	0.24	0.44	0.70	0.82	0.85	0.97	1.00	1.00





Example:

The time (msec) needed for a chemical reaction to complete can be defined with a cumulative distribution function (cdf):

$F(x) = 0 \qquad (x < 0)$

$F(x) = 1\text{-}e^{-0.01x} \qquad (0 \leq x)$

a) Find the probability density function.

$f(x) = F(x)/dx$

$f(x) = 0.01e^{-0.01x} \qquad (0 \leq x)$

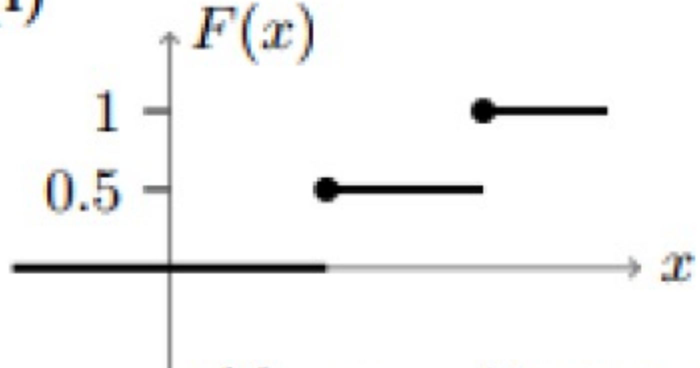
b) What is the probability that the reaction is completed in equal to or less than 200 msec.

$P[x \leq 200] = F(200) = 1 - e^{-0.01*200} = 1 - e^{-2} = 0.865$

Example

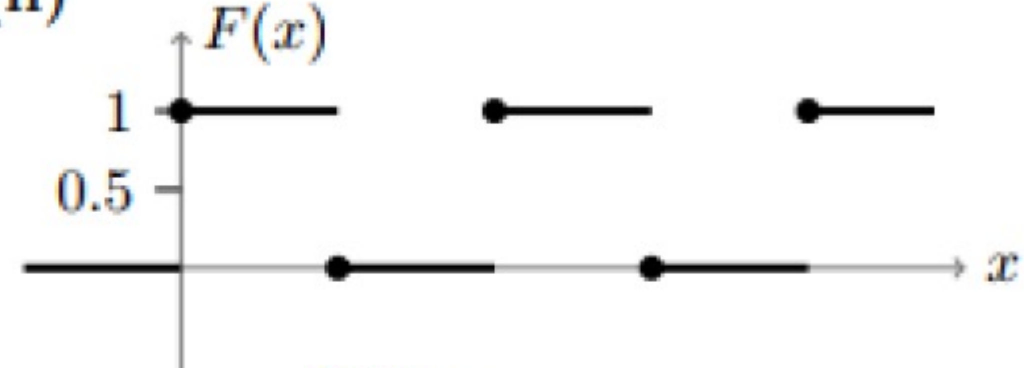
For each of the following say whether it can be the graph of a cdf. If it can be, say whether the variable is discrete or continuous. (Cumulative Distribution Function)

(i)



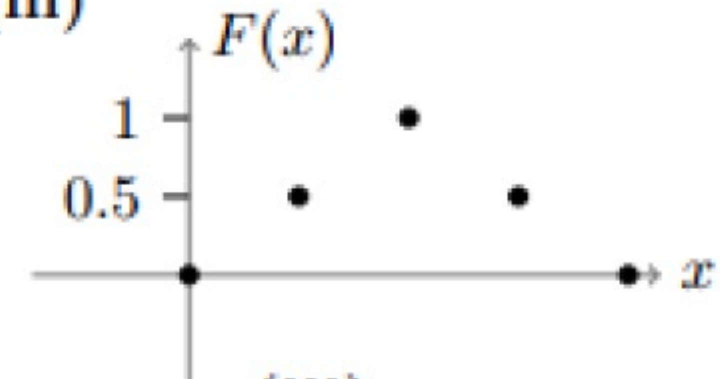
(i) yes, discrete,

(ii)



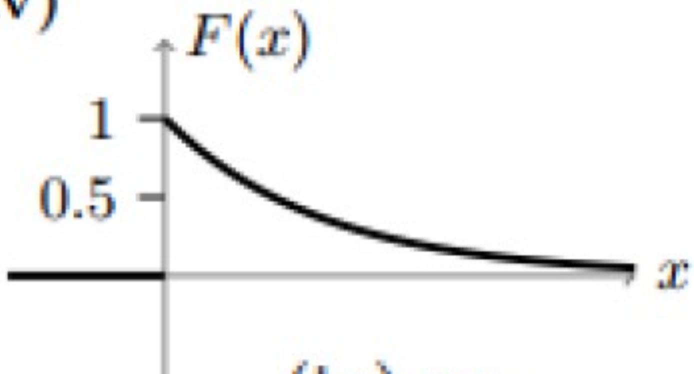
(ii) no,

(iii)



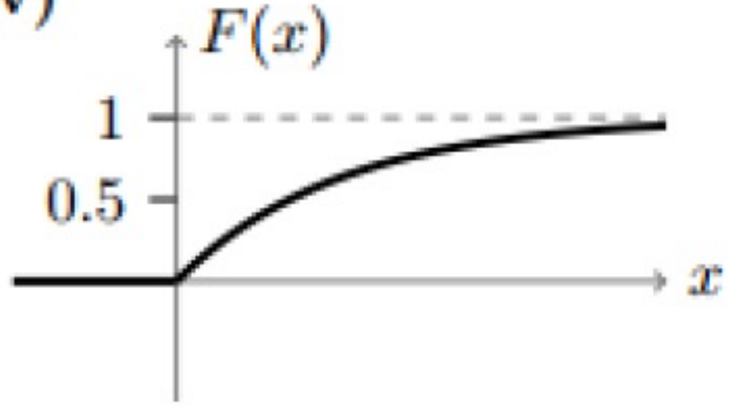
(iii) no,

(iv)



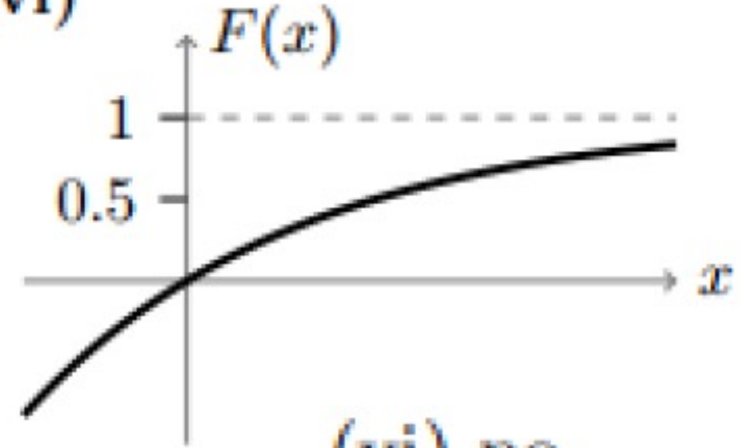
(iv) no,

(v)



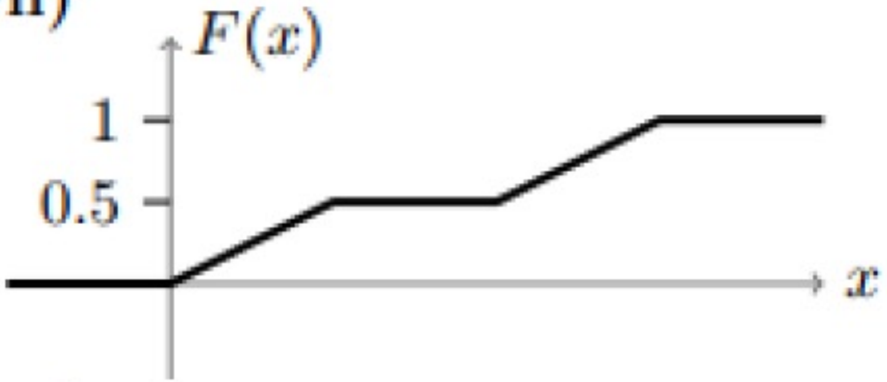
(v) yes, continuous

(vi)



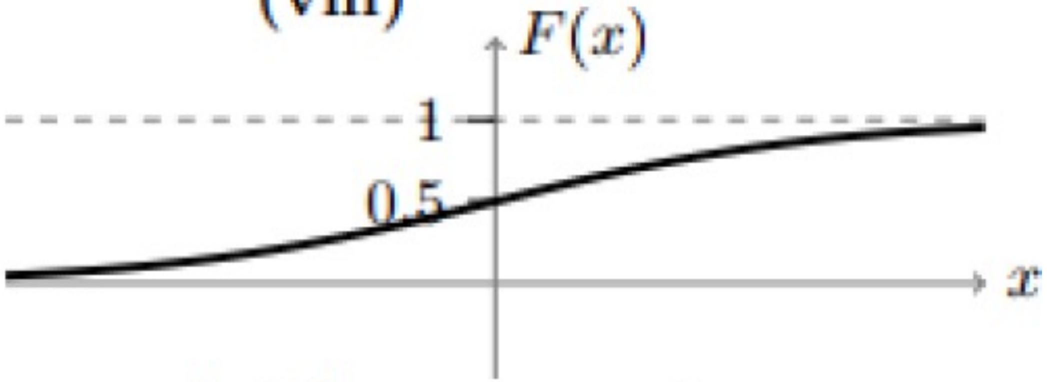
(vi) no

(vii)



(vii) yes, continuous,

(viii)



(viii) yes, continuous.

Example: A dam may be collapsed under two conditions. One is that the capacity of the dam is exceeded (event A), the other is earthquake (event B). In a given year, the probability of these events are P[A] = 0.02, P[B] = 0.01. Events A and B are independent.

a) Find the probability of the dam to collapse in any year.

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

$$P[A \cap B] = P[A]P[B] \quad \rightarrow \quad \text{A and B are independent}$$

$$P[A \cup B] = 0.02 + 0.01 - 0.02 \times 0.01 = 0.0298$$

Example:

b) Find the probability of the dam not to collapse in the second year under the condition that it did not collapse the first year.

The sample space:

$$\Omega \equiv [(A \cap B), (A \cap B^c), (A^c \cap B), (A^c \cap B^c)]$$

The event of **collapsing**:

$$A + B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$$

The probability of the dam **not to collapse** in a given year:

$$P[A^c \cap B^c] = 1 - P[A \cup B] = 1 - 0.0298 = 0.9702$$

The probability of the dam not to collapse in the second year under the condition that it did not collapse the first year:

$$P[((A^c \cap B^c)_2 | (A^c \cap B^c)_1)] = \frac{P[(A^c \cap B^c)_1 \cap (A^c \cap B^c)_2]}{P[(A^c \cap B^c)_1]}$$

Example: Since the events A and B are independent.

$$P[((A^c \cap B^c)_2 | (A^c \cap B^c)_1)] = P((A^c \cap B^c)_2) = 0.9702$$

The probability of the dam not to collapse in the first two years:

$$P[(A^c \cap B^c)_1 \cap (A^c \cap B^c)_2] = P[(A^c \cap B^c)_2]^2 = 0.9702^2$$

c) The probability of the dam not to collapse for 50 years

The probability of the dam not to collapse for m years (because the events are independent):

$$P[(A^c \cap B^c)_1 \cap (A^c \cap B^c)_2 \cap \ldots \cap (A^c \cap B^c)_m] = P[A^c \cap B^c]^m$$

The probability of the dam not to collapse for 50 years:

$$P[A^c \cap B^c]^{50} = 0.9702^{50} = 0.2203$$

Example: What should **a** be, for the equation below to be defined as a probability density function?

$$f(x) = ax^2 \quad \rightarrow \quad (0 \le x \le 5) \quad \text{(zero elsewhere)}$$

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{-\infty}^{0} f(x)dx + \int_{0}^{5} f(x)dx + \int_{5}^{\infty} f(x)dx = 1$$

The first and third terms are 0:

$$\int_{0}^{5} ax^2 dx = \left| \frac{ax^3}{3} \right|_0^5 = 1$$

$$a = \frac{3}{125} \quad \rightarrow \quad f(x) = \frac{3x^2}{125}$$

Example: After finding the probability density function in the previous question, calculate the probabilities for the following values of x:

a) Less than or equal to 2;
b) Between 1 and 3;
c) Greater than 4;
d) Greater than or equal to 4;
e) Greater than 6.

Ans:

$$f(x) = \frac{dF(x)}{dx}$$

$$F(x) = \frac{x^3}{125} \quad \rightarrow \quad (0 \leq x \leq 5)$$

a) $P[x \leq 2] = F(2) = 8/125$

Example:

b) Between 1 and 3;

$$P[1 \le x \le 3] = F(3) - F(1) = \frac{3^3}{125} - \frac{1^3}{125} = \frac{26}{125}$$

c) Greater than 4;

$$P[x > 4] = 1 - F(4) = 1 - \frac{4^3}{125} = \frac{63}{125}$$

d) Greater than or equal to 4;

$$P[x \ge 4] = \frac{63}{125}$$

(Because the function is continuous) $P\,[x = 4] = 0$

Example:

e) Greater than 6;

$$P[x > 6] = 1 - F(6)$$

$$F(x) = \int_{-\infty}^{0} 0dx + \int_{0}^{5} \frac{3x^2}{125} + \int_{5}^{x} 0dx$$

$$F(6) = 0 + 1 + 0 = 1$$

$$P[x > 6] = 1 - F(6) = 1 - 1 = 0$$

Example: In a wind farm, there are two wind speed measurement devices (measuring X and Y). To statistically evaluate X and Y, the joint probability mass function is given below. (The last column and the last row show the marginal probability mass functions.)

	Y=0 (1)	Y=1 (2)	Y=2 (3)	Y=3 (4)	P(x)* (5)
X=0	0.2910	0.0600	0.0000	0.0000	0.3510
X=1	0.0400	0.3580	0.0100	0.0000	0.4080
X=2	0.0100	0.0250	0.1135	0.0300	0.1785
X=3	0.0005	0.0015	0.0100	0.0505	0.0625
P(y)=*	0.3415	0.4445	0.1335	0.0805	$\sum$=1.00

a) What is the probability that X and Y measure the same value:

$$P[0,0] + P[1,1] + P[2,2] + P[3,3]$$
$$= 0.2910 + 0.3580 + 0.1135 + 0.0505 = 0.813$$

Example:

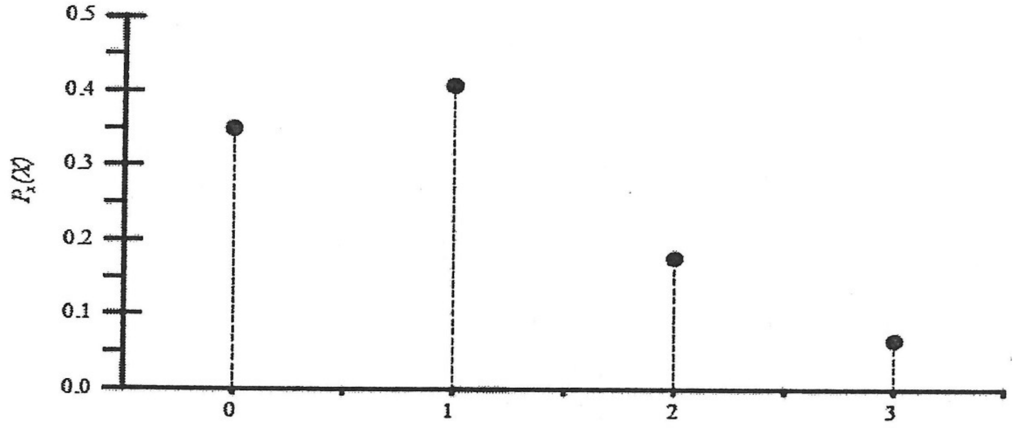
b) Find the marginal probability mass function of X.

$$P[x=0]=\sum_{y=0}^{3}P[0,y]=0.2910+0.0600+0.00+0.00=0.3510$$

$$P[x=1]=\sum_{y=0}^{3}P[1,y]=0.0400+0.3580+0.0100+0.00$$
$$=0.4080$$

$$P[x=2]=\sum_{y=0}^{3}P[2,y]=0.0100+0.0250+0.1135+0.0300$$
$$=0.1785$$

$$P[x=3]=\sum_{y=0}^{3}P[3,y]=0.0005+0.0015+0.0100+0.0505$$
$$=0.0625$$



Example:

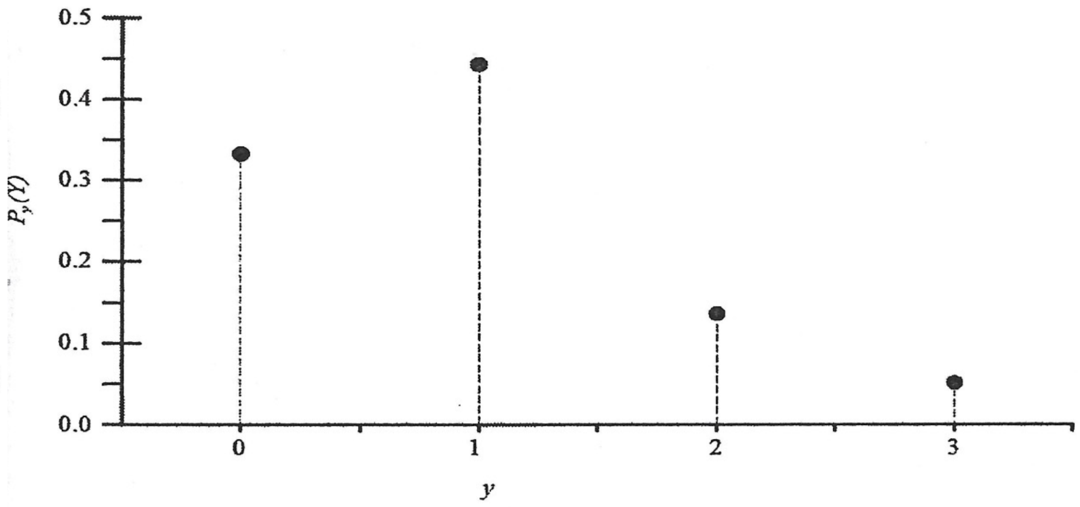
c) Find the marginal probability mass function of Y.

$$P[y=0] = \sum_{x=0}^{3} P[x,0] = 0.2910 + 0.0400 + 0.0100 + 0.0005$$
$$= 0.3415$$

$$P[y=1] = \sum_{x=0}^{3} P[x,1] = 0.0600 + 0.3580 + 0.0250 + 0.0015$$
$$= 0.4445$$

$$P[y=2] = \sum_{x=0}^{3} P[x,2] = 0.0000 + 0.0100 + 0.1135 + 0.0100$$
$$= 0.1335$$

$$P[y=3] = \sum_{x=0}^{3} P[x,3] = 0.0000 + 0.0000 + 0.0300 + 0.0505$$
$$= 0.0805$$



Example:

d) What is the conditional probability mass function of *X* *for Y=1*.

$$P[X=0|Y=1]=\frac{P[0,1]}{P[1]}=\frac{0.0600}{0.4445}=0.1350$$

$$P[X=1|Y=1]=\frac{P[1,1]}{P[1]}=\frac{0.3580}{0.4445}=0.8054$$

$$P[X=2|Y=1]=\frac{P[2,1]}{P[1]}=\frac{0.0250}{0.4445}=0.00562$$

$$P[X=3|Y=1]=\frac{P[3,1]}{P[1]}=\frac{0.0015}{0.4445}=0.003$$

$$\sum_{x_i|y=1} P[x_i|1]=0.1350+0.8054+0.0562+0.0034=1.00$$

Example:

e) What is the conditional probability mass function of X *for* $Y \geq 1$.

$$\sum_{y_j \geq 1} P[0, y_j] = 0.0600 + 0.00 + 0.00 = 0.0600$$

$$\sum_{y_j \geq 1} P[1, y_j] = 0.3580 + 0.0100 + 0.00 \quad = 0.3680$$

$$\sum_{y_j \geq 1} P[2, y_j] = 0.0250 + 0.1135 + 0.0300 \quad = 0.1685$$

$$\sum_{y_j \geq 1} P[3, y_j] = 0.0015 + 0.0100 + 0.0505 \quad = 0.0620$$

$$\sum_{y_j \geq 1} P[y_j] = 0.4445 + 0.1335 + 0.0805 = 0.6585$$

Example:

e)

$$\sum_{y_j \geq 1} P[y_j] = 0.4445 + 0.1335 + 0.0805 = 0.6585$$

for $Y \geq 1$

$$P[X = 0|Y \geq y] = \frac{0.0600}{0.6585} = 0.0911$$

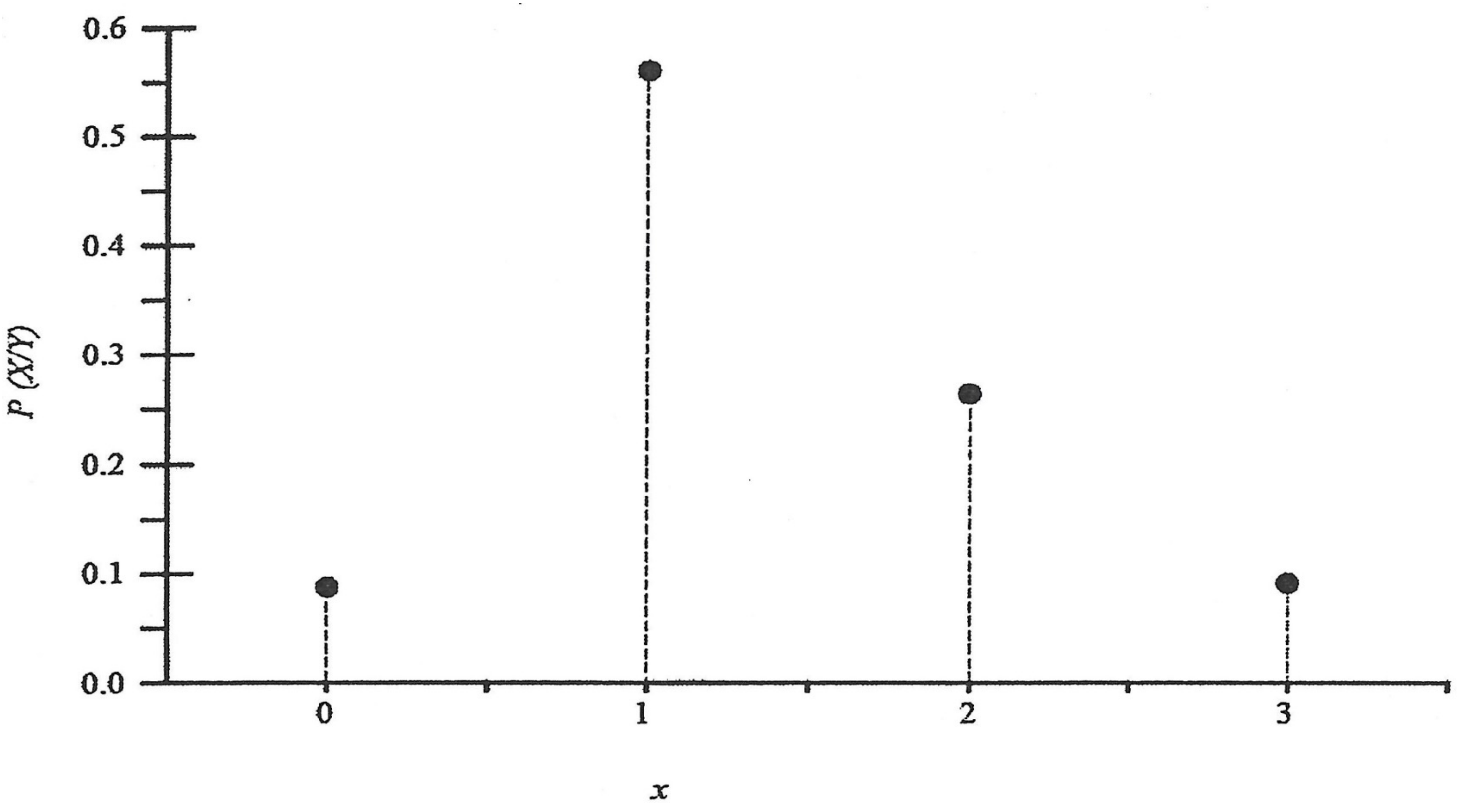
$$P[X = 1|Y \geq y] = \frac{0.3680}{0.6585} = 0.5588$$

$$P[X = 2|Y \geq y] = \frac{0.1685}{0.6585} = 0.2559$$

$$P[X = 3|Y \geq y] = \frac{0.0620}{0.6585} = 0.0942$$

Example:

e)



Example

Let X and Y be two continuous random variables with joint pdf

$$f(x,y) = cx^2y(1+y) \quad \text{for } 0 \leq x \leq 3 \text{ and } 0 \leq y \leq 3,$$

and $f(x,y) = 0$ otherwise.

(a) Find the value of c.

(b) Find the probability $P(1 \leq X \leq 2, 0 \leq Y \leq 1)$.

(c) Find the marginal pdf $f(x)$ directly from $f(x,y)$

Ans: (a) Total probability must be 1, so

$$1 = \int_0^3 \int_0^3 f(x, y)\, dy\, dx$$

$$= \int_0^3 \int_0^3 c(x^2 y + x^2 y^2)\, dy\, dx$$

$$= c \cdot \frac{243}{2}$$

Therefore, $c = \dfrac{2}{243}$.

(b) $$P(1 \le X \le 2, 0 \le Y \le 1) = \int_1^2 \int_0^1 f(x,y)\,dy\,dx$$
$$= \int_1^2 \int_0^1 c(x^2 y + x^2 y^2)\,dy\,dx$$
$$= c \cdot \frac{35}{18}$$
$$= \frac{70}{4374} \approx 0.016$$

(c) by integrating over the entire range for y,

$$f(x) = \int_0^3 f(x,y)\,dy = cx^2\left(\frac{3^2}{2} + \frac{3^3}{3}\right) = c\frac{27}{2}x^2 = \frac{1}{9}x^2$$