Probability and Statistics
MAT 271E

PART 2

Elements of Probability Theory

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ELEMENTS OF PROBABILITY THEORY

Random Variable

Probability theory studies **variables that have uncertain behavior**.

Such variables, whose values cannot be predicted with certainty, are called **random variables**.

Uncertainties of random variables is due to the effects of **a large number of unknown properties involved**.

Therefore:

Random variables cannot be studied by a **deterministic approach**.

A **probabilistic approach** is needed. (See course slides **Part 1**).

Random Event

A **random event** is defined as: the **occurrence** of a particular value for a **random variable** in an observation.

It is possible to only **estimate** the chance of an event to take place.

Example:

When you flip a coin

There are 2 random events with equal chances: heads or tails

When you roll a die:

There are 6 random events (integers 1-6) with equal chances.

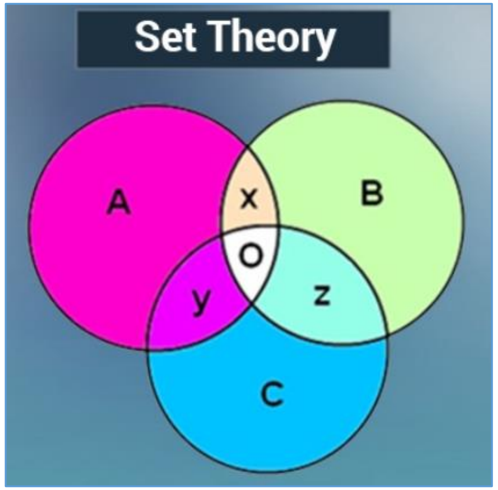
ELEMENTS OF SET THEORY

Probability theory uses the concepts of **set theory**.

In this theory, a **set** consists of exactly defined elements.

Name of the set is expressed with **capital letters**.

If its **element** is a letter, it should be a **small letter**.



$F=\{d\}$

$D=\{1,2,3,4,5,6\}$

Order of elements is **not significant**.

If we write:

$d \in F,\ x \notin F$

It means d is an element of the set F, but x is not.

$\varnothing=\{\ \}$

An **empty set** has no elements.

ELEMENTS OF SET THEORY

Elements of a **subset** are also the elements of the set.

For example: $D=\{1,2,3,4,5,6\}$

Set E, which consists of even numbers (when you throw a die), is a **subset** of D.

$E=\{2,4,6\}\subset D$

If for example, set X is not a subset of D it is expressed as: $X\not\subset D$

If $E\subset D$, this means that the region corresponding to E is also a part of the region corresponding to D.

If two sets have no common elements, their intersection is the **empty set**.

The set of even numbers E (when throwing a die) and the set of odd numbers O are **disjoint sets** (they do **not intersect**).

The set consisting of elements of two sets is their **union**.

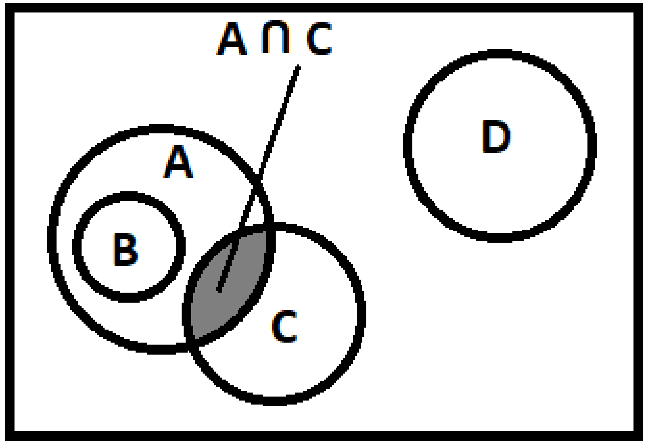
For example: when throwing a die, the union of E and O is set D.

$E \cup O=D$

Venn diagram

A **Venn diagram** shows the relations between the sets and subsets.

In a Venn diagram, each set is represented by the region inside a closed shape.



From the Venn diagram above, we can write:

$B \subset A \quad C \not\subset A \quad D \not\subset A$

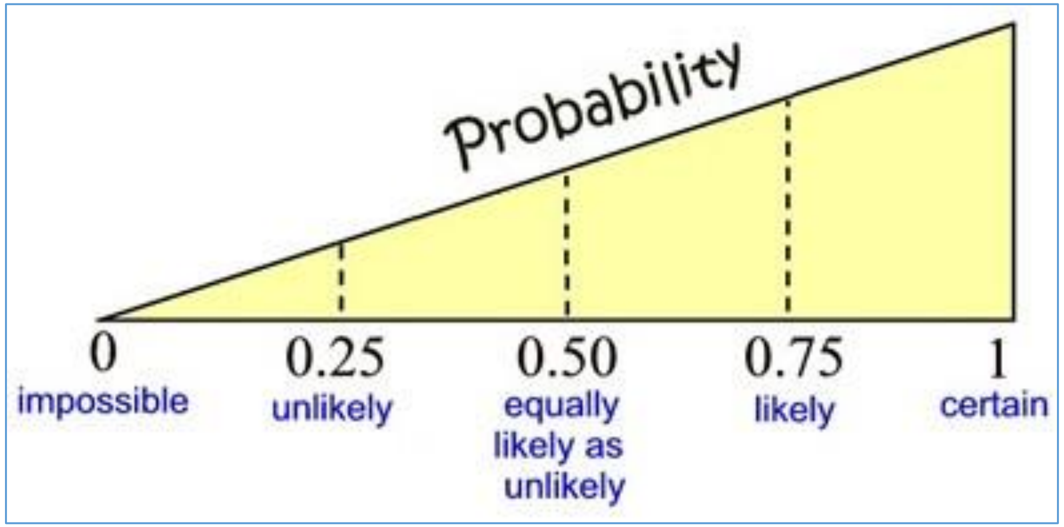
The set consisting of common elements of two sets is their **intersection**.

The shaded region (intersection of A and C) is expressed as:

$A \cap C$ or **AC**

Probability Concept

In probability theory, the **chance of the occurrence of a random event** is defined as its **probability**.



Each random event has a certain probability that varies in the range of 0 to 1.

Probability Concept

In a probability expression:

$$\boxed{P(X=x_i)=p_i}$$

The **random variable** is denoted by a **capital** letter (X)

Its **value** in an observation is expressed as the corresponding **small** letter (x)

$X=x_i$ is a **random event**.

P is the **symbol for the probability** of the event

p_i is the **probability** (a number between 0 and 1) of the event $X=x_i$.

p_i =**0** means that the event $X=x_i$ will **never** occur.

p_i =**1** means that the event $X=x_i$ will **always** occur in all observations.

Probability Concept

How can we estimate **the probabilities of random events**?

In case of simple variables, the probabilites can be determined by logic.

For example:

There are six random events in the throw of a die with equal probabilites.

P(X=1)=P(X=2)=P(X=3)=P(X=4)=P(X=5)=P(X=6)

$$\sum_{i=1}^{6} P(X = i) = 1$$

Therefore

P(X=1)=P(X=2)=P(X=3)=P(X=4)=P(X=5)=P(X=6)=1/6

Probability Concept

Example:

If two dice are rolled, what is the probability that the sum of the two will be equal to 7.

6*6=36 possible outcomes.

Sum of two dice=7

(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)   six possible outcomes

P(sum of two dice=7)=6/36=1/6

Probability Concept

Example:

From a total of 212 members in a sports club, 36 members play tennis, 28 play squash and 18 play basketball.

22 of the members play both tennis and squash

12 of the members play both tennis and basketball

9 of the members play both squash and basketball

4 of the members play all three sports

How many members of this club play at least one of three sports?

N: the number of members of the club

T: set of members that play tennis

S: set of members that play squash

B: set of members that play basketball

Probability Concept

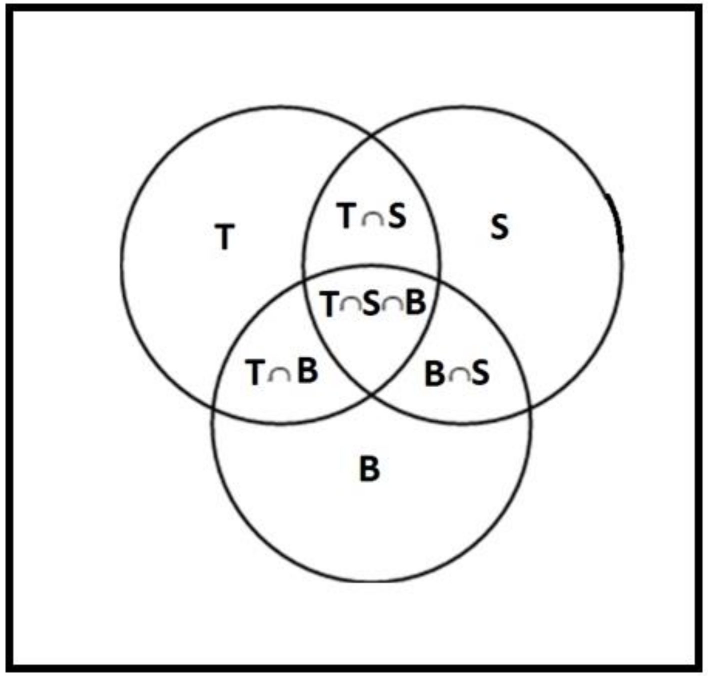
Example:

$$P(T \cup S \cup B) = P(T) + P(S) + P(B) - P(TS) - P(TB) - P(SB) + P(TSB)$$

$$= \frac{36 + 28 + 18 - 22 - 12 - 9 + 4}{N}$$

$$= \frac{43}{N}$$

$=43/212=0.20$



Probability Concept

Suppose the die is somehow loaded so that the probability of an even number is **two** times the probability of an odd number.

P(X=2)=P(X=4)=P(X=6)=**2**P(X=1)=**2**P(X=3)=**2**P(X=5)

$$\sum_{i=1}^{6} P(X=i)=1$$

Thererefore;

P(X=2)=P(X=4)=P(X=6)=**2/9**

P(X=1)=P(X=3)=P(X=5)=**1/9**

In engineering problems, it may be very difficult to estimate the **probabilities** as above so we usually base our estimates on **frequencies**.

Frequency

Frequency of a random event is the ratio of **the number of time it occurs** to **the total number of observations**.

If the event $\mathbf{X=x_i}$ is observed $\mathbf{n_i}$ times during **N** experiments, its frequency is:

$$f_i = n_i / N$$

The **probability** of a random event is defined as the limit of its **frequency** as the **number of observations approaches infinity**.

$$p_i = \lim_{N \to \infty} \left(\frac{n_i}{N}\right)$$

Although we can never make an infinite number of observations, it can be assumed that **$\mathbf{f_i}$ approaches $\mathbf{p_i}$ quite rapidly as N increases**.

Frequency

For example, if no precipitation has been observed for n_i=900 days along a period of N=1500 days, then the probability of no precipitation can be estimated to be:

$$P(X=0)=900/1500=0.60$$

As the observation period increases, the estimated frequency will be a <u>better estimate</u> of the true probability.

Probabilities of Simple and Compound Random Events

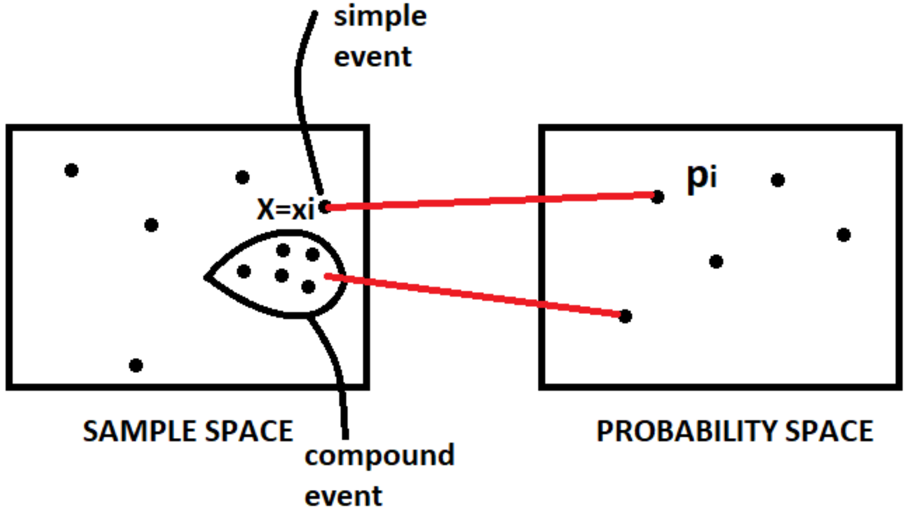
Sample space of a random variable consists of **all the values** that it can take.

Each point (element) of this space is a sample point that corresponds to a **random event**.

SAMPLE SPACE
{Head, Tail}

SAMPLE SPACE
{1, 2, 3, 4, 5, 6}

Compound random event is the union of a number of (**simple** or **compound**) random events.



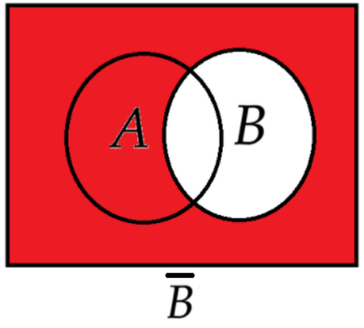
Probabilities of Simple and Compound Random Events

Each *compound random event* is a subset of the *sample space*.

A certain point (probability) in the **probability space** corresponds to each point (for simple random events) or region (compound random event) in the **sample space**.

P=0 corresponds to the empty set that has no points in the sample space.

P=1 corresponds to the set consisting of all the points in the sample space.



$\overline{B}$ $\Rightarrow$ *complement of B*

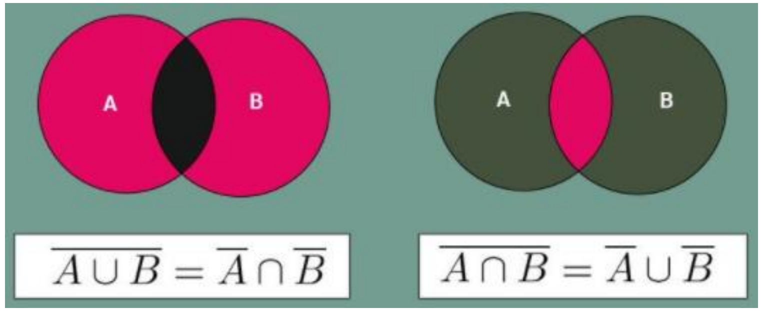
Points that are not in a region B of the sample space are in the region $\bar{B}$, called the **complementary event**. (Insomebooks, B'orB^{c}areused instead of $\bar{B}$.)

$$\boxed{P(B)+P(\bar{B})=1}$$

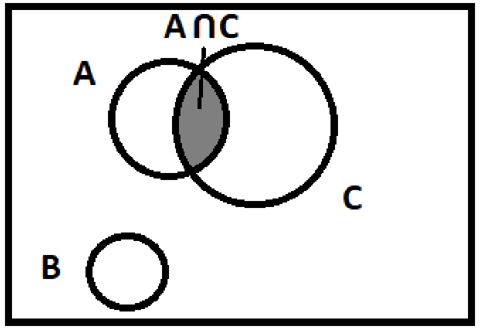
For example, when you throw a die, the event that an **odd number (1,3,5)** shows **(O)** is the complimetary event **($\bar{E}$)** of the case of an **even number (2,4,6)** :

P(O)=1-P(E)=P($\bar{E}$)

Probabilities of Simple and Compound Random Events



If two events have no common points, they are called **disjoint events**. Their intersection is an empty set.



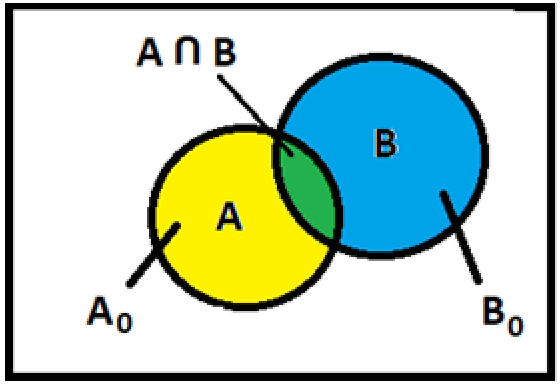
Events A and B are **disjoint** events. $A \cap B = \varnothing$.

The probability of a compound event consisting of two **disjoint** events is the sum of their probabilities.

$P(A \cup B) = P(A)+P(B) \qquad \text{if } A \cap B = \varnothing.$

Probabilities of Simple and Compound Random Events

The probability of the union of two events that are not disjoint:



$$P(A)=P(A \cap B)+P(A_0)$$

$$P(B)=P(A \cap B)+P(B_0)$$

$$P(A \cup B) = P(A)+P(B)- P(A \cap B)$$

$$P(A \cup B) = P(A_0)+P(B_0)+ P(A \cap B)$$

Example:

$P(E)=P\{2,4,6\}=1/2$

$P(S)=P\{1,2\}=1/3$

The intersection $E \cap S$ has only one element, 2, with a probability of 1/6.

$$P(E \cup S) = P\{1,2,4,6\}=P(E)+P(S)- P(E \cap S)$$

$$=1/2+1/3-1/6=2/3$$

Example (M. Bayazıt, B. Oğuz, Example 2.1, pg 15)

The number of vehicles waiting for a left turn at a cross-section is observed to vary between 0 and 6, with the following probabilities:

P(X=0)=4/60, P(X=1)=16/60, P(X=2)=20/60 , P(X=3)=14/60, P(X=4)=3/60, P(X=5)=2/60, P(X=6)=1/60

What is the probability (of the event) that **more than 3 vehicles** are waiting for a left turn?

What is the probability that **less than or equal to 3 vehicles** are waiting for a left turn?

There are **7 simple events** in the sample space of the random variable **X (*the number of vehicles waiting for a left turn*)**. **The sum of their probabilities is 1**.

The probability that **more than 3 vehicles** are waiting is calculated taking three simple disjoint events.

$P(X>3)=P(X=4)+P(X=5)+P(X=6)=3/60+2/60+1/60=6/60=1/10$

The event **less than or equal to 3 vehicles** waiting is the complementary of the above event.

$P(X\leq 3)=1 - P(X>3)=1-1/10=9/10$

Example (M. Bayazıt, B. Oğuz, Example 2.2, pg 15)

There are 3 bulldozers at a construction site, each having a probability of **no failure** during the total period of construction equal to 0.50.

Let us consider the **random variable X** as «**the number of bulldozers in operation throughout the construction period**».

There are 4 events in the sample space of X, as X can be equal to 0,1,2 or 3. Let us compute their probabilities:

A bulldozer **in operation** is denoted by **S**, bulldozer **not in operation** by **F**.

8 occurences are possible for the **3 bulldozers**: FFF, FFS, FSF, SFF, FSS, SFS, SSF, SSS

These occurences have equal probabilities because the probability of failure (F) 50% is assumed to be equal to the probability of no failure (S) 50%.

Since the sum of the probabilities is 1, each of the above occurences has a probability of **1/8(=1/2*1/2*1/2)**.

Let us check the probabilites of the random events.

Example (M. Bayazıt, B. Oğuz, Example 2.2, pg 15)

Let us check the probabilites of the random events of the **variable X**.

P(X=0)=P(FFF)=1/8

P(X=1)=P(FFS)+P(FSF)+P(SFF)=1/8+1/8+1/8=3/8 (1 **bulldozer <u>in operation</u> throughout the construction period)**

P(X=2)=P(FSS)+P(SFS)+P(SSF)=1/8+1/8+1/8=3/8

P(X=3)=P(SSS)=1/8

Example (M. Bayazıt, B. Oğuz, Example 2.3, pg 16)

It is known that the probability that a job is completed

in 2-4 days is 0.5,

in 4-6 days is 0.25,

in 2-6 days is 0.55

What is the probability that the job is completed in **4** days?

Let us define the following events

2-4 days: $A=\{(X=2) \cup (X=3) \cup (X=4)\}$ $P(A)=0.50$

4-6 days: $B=\{(X=4) \cup (X=5) \cup (X=6)\}$ $P(B)=0.25$

The union and intersection of the events are:

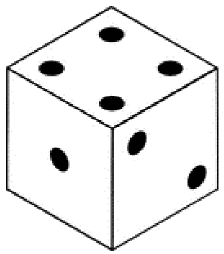
2-6 days: $A \cup B = \{ (X=2) \cup (X=3) \cup (X=4) \cup (X=5) \cup (X=6)\}$, $P(A \cup B)=0.55$

$A \cap B = \{X=4\}$

The probability of intersection (event taking place in 4 days)

$P(A \cap B) = P(X=4)=P(A)+P(B)- P(A \cup B)=0.50+0.25-0.55=0.20$

Example:



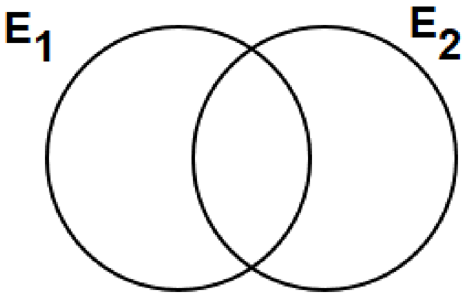
Find the probability of a **4** turning up **<u>at least once</u>** in **two** tosses of a fair die.

Let $\mathbf{E_1}$=event «4» on the first toss

$\mathbf{E_2}$= event «4» on the second toss

$\mathbf{E_1+E_2}$= event that 4 turns up **at least once** = event «4» on the first toss or «4» on the second toss or both.

The total number or equally likely ways in which both dice can fall is 6 x 6 =36



Number of ways in which (**E_1 occurs, but not E_2**) = 5

Number of ways in which (**E_2 occurs, but not E_1**) = 5

Number of ways in which (**both E_1 and E_2**) occurs = 1

Thus, the number of ways in which **at least one of the events E_1 or E_2** occurs is

5+5+1=11, and thus $P(E_1+E_2)=11/36$

Factorial

A factorial is represented by the sign (!).

n factorial ⇨ n!

Factorial is the product of all the whole numbers between 1 and n, where n must always be positive.

For example:

$$1! = 1 = 1$$

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

$0! = 1$ (special case)

Permutation

A **permutation** is an arrangement of things in **a certain order**.

PERMUTATIONS

The number of permutations of n things taken r at a time is

$$_nP_r = \frac{n!}{(n-r)!}.$$

Jane has **6 different** pencils.

Find the number of different orders in which **all 6 pencils** can be arranged.

The number of pencils arranged: 6 out of 6

$$_6P_6 = \frac{6!}{(6-6)!} = \frac{6!}{0!} = \frac{6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1}{1} = 720$$

Permutation

Jane has **6 different** pencils**.**

Find which **3 pencils** arranged at a time <u>when the order is also important.</u>

The number of pencils is 6.

$$ {}_6P_3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1}{3 \bullet 2 \bullet 1} = 6 \bullet 5 \bullet 4 $$

The pencils are arranged 3 at a time.

$$ = 120 $$

There are 120 permutations i.e, if 3 pencils are taken from 6 pencils each time, where their order is also important, a total of 120 different outcomes is possible.

Combination

A **combination** is a selection of things in any order.

COMBINATIONS

The number of combinations of n things taken r at a time is

$$_{n}C_{r} = \frac{_{n}P_{r}}{r!} = \frac{n!}{r!(n-r)!}.$$

Jane has **6 different** pencils.

Find their combination (i.e. order is not important).

$$_{6}C_{6} = \frac{6!}{6!(6-6)!} = 1$$

Combination

Jane has **6 different** pencils**.**

Find the number of ways you can take **3 pencils at a time if the order is not important**.

The number of pencils is 6.

$$_{6}C_{3} = \frac{6!}{3!(6-3)!} = \frac{6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1}{(3 \bullet 2 \bullet 1)(3 \bullet 2 \bullet 1)} = \frac{6 \bullet 5 \bullet 4}{3 \bullet 2 \bullet 1} = 20$$

The pencils are taken 3 at a time.

There are 20 combinations for taking 3 out of the 6 pencils (<u>order is not important</u>.).

Permutations and Combinations

Example:

If no letter is used more than once, there is only **1 combination** of the **first 3 letters of the alphabet**.

Meanwhile, there are 6 **permutations** of the **first 3 letters of the alphabet (i.e. order is important in counting the permutations)**

6 Permutations: ABC, ACB, BAC, BCA, CAB, and CBA

$${}_3P_3 = \frac{3!}{(3-3)!} = \frac{3!}{0!} = \mathbf{6} \qquad {}_3C_3 = \frac{3!}{3!} = \mathbf{1}$$

(ABC, ACB, BAC, BCA, CAB, and CBA are considered to be the same **combination** of A, B, and C because the **order does not matter**.)

Permutations and Combinations

Example:

If no letter is used more than once, there are 60 permutations and 10 combinations of the first **5** letters of the alphabet (A,B,C,D,E) when taken **3** at a time. To see this, look at the list of permutations (60 in total).

ABC	ABD	ABE	ACD	ACE	ADE	BCD	BCE	BDE	CDE
ACB	ADB	AEB	ADC	AEC	AED	BDC	BEC	BED	CED
BAC	BAD	BAE	CAD	CAE	DAE	CBD	CBE	DBE	DCE
BCA	BDA	BEA	CDA	CEA	DEA	DBC	CEB	DEB	DEC
CAB	DAB	EAB	DAC	EAC	EAD	DCB	EBC	EBD	ECD
CBA	DBA	EBA	DCA	ECA	EDA	DBC	ECB	EDB	EDC

$_5P_3 = 60$

$_5C_3 = 10$

(For example ABC, ACB, BAC, BCA, CAB, and CBA are considered to be the same **combination** of A, B, and C because the **order does not matter**, but they are different permutations.)

Example:

- How many words can be formed by using **3** letters from the word "TDZSR"?

(Hint: for example RTZ, RZT, TRZ, TZR, ZRT, ZTR are all different words and should be counted separately, i.e. the order of the letters is important.)

Solution :

The word "TDZSR" contains 5 different letters (T,D,Z,S,R).

Therefore, required number of letters $\mathbf{= n! / (n - r)!}$ $= 5! / (5 - 3)! = 5! / 2! = 120 / 2 = 60$

- How many words can be formed by using **5** letters from the word "TDZSR"?

(Hint: the order of the letters is important.)

Solution :

The required number of letters $\mathbf{= n! / (n - r)!}$ $= 5! / (5 - 5)! = 5! = 120$

Example:

- In how many ways, can we select a team of 4 students from a given choice of 15 ?

(Hint: in a team, the order of the students is not important.)

Solution :

Number of possible ways of selection **= n! / (r!** x **(n – r)!)** = 15 ! / [(4 !) x (11 !)]

=> Number of possible ways of selection = (15 x 14 x 13 x 12) / (4 x 3 x 2 x 1) = 1365

- In how many ways can a group of 5 members be formed by selecting 3 boys out of 6 and 2 girls out of 5 ?

(Hint: the order of the members is not important.)

Solution :

Number of ways 3 boys can be selected out of 6 = 6 ! / [(3 !) x (3 !)] = (6 x 5 x 4) / (3 x 2 x 1) = 20

Number of ways 2 girls can be selected out of 5 = 5 ! / [(2 !) x (3 !)] = (5 x 4) / (2 x 1) = 10

Therefore, total number of ways of forming the group = 20 x 10 = 200

Two-Dimesional and Conditional Sampling Spaces

So far we have seen events with a single random variable.

Let us see the case where there are **two random variables** X and Y.

In this case, we can talk of **two-dimensional sample spaces** with points (x_i, y_j)

where

$\mathbf{X=x_i}$ is the **random event** of the **variable X**

$\mathbf{Y=y_j}$ is the **random event** of the **variable Y**

A certain **probability $\mathbf{p_{ij}}$** corresponds to each **point $\mathbf{(X=x_i, Y=y_j)}$** in the **probability space**.

$$\boxed{P(X=x_i\,,Y=y_j)= p_{ij}}$$

Two-Dimesional and Conditional Sampling Spaces

Conditional sample space

One-dimensional space consisting of points **$(x_i|y_j)$** where **$Y=y_j$** denotes a **condition**.

Random event **$(X=x_i|Y=y_j)$** corresponds to:

the **occurrence of $X=x_i$**

with the **condition that $Y=y_j$**

The event at the right of the vertical line is considered **not as a random event**, but as a **condition**.

Conditional probability is defined as:

$$\boxed{P(X=x_i\,|Y=y_j)=\,p_{i|j}}$$

Two-Dimesional and Conditional Sampling Spaces

Example:

Let us consider the test where concrete beams are loaded.

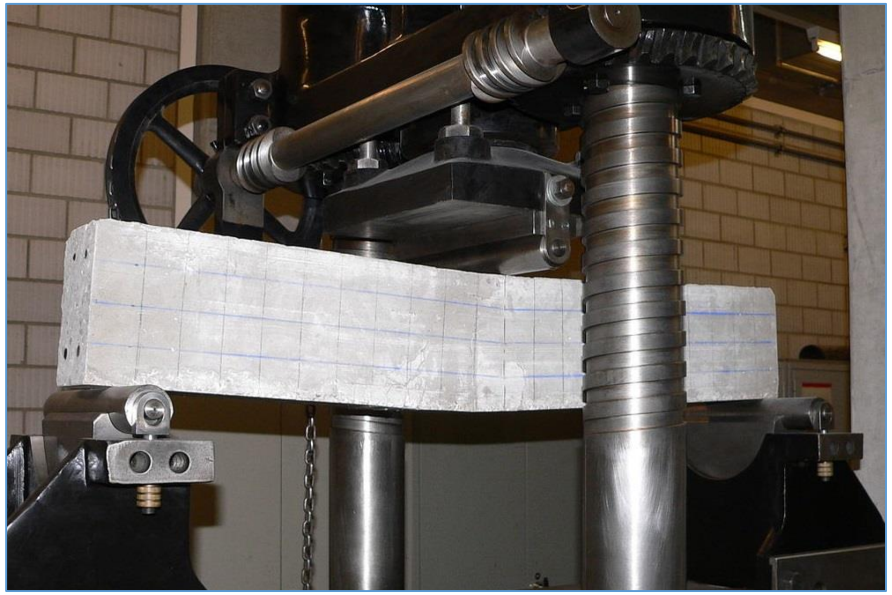
Y : the crack load of a beam

X : the load at which the beam fails

We can define:

the **two-dimensional** sample space as **$(X=x_i, Y=y_j)$**

the **conditional** sample space as **$(X=x_i \mid Y=y_j)$**



In this case,

$(X=10 \mid Y=6)$ is

the **event** that the failure load is **X=10 kN**

on the **condition** that the first crack occurred at **Y=6 kN.**

Two-Dimesional and Conditional Sampling Spaces

The **probability of a conditional event** can be computed as follows:

Writing A=$\{X=x_i\}$ and B=$\{Y=y_j\}$

The **probability theory** provides us with the following expression:

$$\mathrm{P}(\mathrm{A}|B) = \frac{P(A \cap B)}{P(B)} \qquad P(B) \neq 0$$

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

$$\boxed{\mathrm{P}(\mathrm{A} \cap \mathrm{B}) = \mathrm{P}(\mathrm{A}|\mathrm{B})\mathrm{P}(\mathrm{B}) = \mathrm{P}(\mathrm{B} \cap \mathrm{A}) = \mathrm{P}(\mathrm{B}|\mathrm{A})\mathrm{P}(\mathrm{A})}$$

(Intersection and union of sets are commutative.)

Two-Dimesional and Conditional Sampling Spaces

Example:

Let us compute the probability that **2** occurs in the throw of a die, knowing that an **even number** has occurred.

P(A)=P(2)= 1/6

P(B)=1/2 (probability of an even number)

P(A∩B)=P(2)=1/6 (probability of the event that an even number has occurred and also 2 has occurred= probability that 2 has occurred)

$$\mathrm{P}(\mathrm{A}|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{1/2} = 1/3$$

The conditional probability of 2 in a throw with the outcome of an even number (**<u>when</u>** it is an even number or **<u>under the condition that</u>** it is an even number) =1/3

Two-Dimesional and Conditional Sampling Spaces

Example:

The probability that there is <u>at least 10 mm of rain on a rainy day</u>

Suppose the probability of a rainy day is P(B)=0.20

The probability of a day with a rain of at least 10 mm is P(A)=0.03.

$P(A \cap B)=P(A)=0.03$ (probability of the event that a day is rainy and has at least 10 mm of rain = probability of at least 10 mm of rain)

Then, the probability that there is at least 10 mm of rain **on a rainy day** is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.03}{0.20} = 0.15$$

Two-Dimesional and Conditional Sampling Spaces

Example:

A coin is flipped **twice** to see whether it lands on heads **h** or tails **t**.

What is the **conditional probability** that **both flips land on heads**, given that:

a) the **first flip lands on heads**?

b) **at least one flip lands on heads**?

Two-Dimesional and Conditional Sampling Spaces

Example:

Let B=$\{$(h,h)$\}$ be the event that **both flips land on heads**

Let F=$\{$(h,h), (h,t)$\}$ be the event that the **first flip lands on heads**

Let A=$\{$(h,h), (h,t), (t,h)$\}$ be the event that **at least one flip lands on heads**

Sample space S=$\{$(h,h), (h,t), (t,h), (t,t)$\}$ *(all four are equally likely)*

The probability of the event that **both flips land on heads** under the **condition** that (a) **the first flip lands on heads** can be obtained from $\mathrm{P}(\mathrm{B} \cap \mathrm{F})$= $\mathrm{P}(\{(h,h)\})$

$$\mathrm{P}(\mathrm{B}|\mathrm{F}) = \frac{\mathrm{P}(\mathrm{B} \cap \mathrm{F})}{\mathrm{P}(\mathrm{F})} = \frac{\mathrm{P}(\{(h,h)\})}{\mathrm{P}(\{(h,h),(h,t)\})} = \frac{1/4}{2/4} = 1/2$$

Two-Dimesional and Conditional Sampling Spaces

Example:

Let B={(h,h)} be the event that **both flips land on heads**

Let F={(h,h), (h,t)} be the event that the **first flip lands on heads**

Let A={(h,h), (h,t), (t,h)} be the event that **at least one flip lands on heads**

The probability of the event that **both flips land on heads** under the **condition** that (b) **at least one flip lands on heads** can be obtained from $\mathrm{P}(\mathrm{B} \cap \mathrm{A})= \mathrm{P}(\{(h,h)\})$

$$\mathrm{P}(\mathrm{B}|\mathrm{A}) = \frac{\mathrm{P}(\mathrm{B} \cap \mathrm{A})}{\mathrm{P}(\mathrm{A})} = \frac{\mathrm{P}(\{(h,h)\})}{\mathrm{P}(\{(h,h),(h,t),(\mathrm{t,h})\})} = \frac{1/4}{3/4} = 1/3$$

Two-Dimesional and Conditional Sampling Spaces

Example:

The conditional probability that both flips land on heads given that the first one does is **1/2**.

The conditional probability that both flips land on heads given that at least one does is only **1/3**.

This second result may be seem to be surprising. Initially it may sound like: if at least one flip lands on heads, there are two possible results: Either they both land on heads or only one does, which is not correct.

Initially there are 4 equally likely outcomes S={(h,h), ***(h,t), (t,h)****, (t,t)}.*

Because the information that ***at least one flip lands*** *on heads is equivalent to the information that the outcome is not (t,t).*

Therefore, we are left with 3 equally likely outcomes (h,h), (h,t), (t,h), only one of which results in both flips landing on heads which means that the probability is 1/3.

Two-Dimesional and Conditional Sampling Spaces

Example:

Ahmet is 80 percent certain that his missing key is in one of the two pockets of his hanging jacket, being 40 percent certain it is in the left-hand pocket and 40 percent certain it is in the right-hand pocket.
If a search of the left-hand pocket does not find the key, what is the conditional probability that it is in the other pocket?

If we let **L** be the event that the key is in the left-hand pocket of the jacket, and **R** be the event that the it is in the right-hand pocket, then the desired probability $P(R|\bar{L})$ can be obtained as follows:

$$P(R|\bar{L}) = \frac{P(R\cap\bar{L})}{P(\bar{L})}$$

$$= \frac{P(R)}{1 - P(L)} = 0.40/(1-0.40) = 0.40/0.60$$

$$= 2/3$$

Two-Dimesional and Conditional Sampling Spaces

Example:

Suppose that a box contains **8 red balls** and **4 white balls**.

We draw **2 balls** from the box **without replacement**.

a) What is the probability that **both balls drawn are red**?

Let **R1** and **R2** denote, respectively, the events that the first and second balls are red.

Now, given that the first ball selected is red, there are 7 remaining red balls and 4 white balls, so $P(R_2 | R_1)=7/11$.

As $P(R_1)$ is clearly 8/12=2/3, the desired probability is:

$$P(R_1 \cap R_2) = P(R_2 | R_1)P(R_1)$$

$$= \left(\frac{7}{11}\right)\left(\frac{2}{3}\right)=\left(\frac{14}{33}\right)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad P(B) \neq 0$$

$$\boxed{P(A \cap B) = P(A|B)P(B) = P(B \cap A) = P(B|A)P(A)}$$

(Intersection andunion of setsarecommutative.)

Let us elaborate on the above equations this time for **more than two events**:

For the intersection of **3 events**(A, B and C):

$$\boxed{\begin{aligned} P(A \cap B \cap C) &= P(A|B \cap C)P(B \cap C) \\ &= P(A|B \cap C)P(B|C)P(C) \end{aligned}}$$

(Intersection andunion of setsarealsoassociative.)

In the examples above we found $\mathbf{P}(\mathbf{A}|\boldsymbol{B})$**≠P(A)** which means that **the condition <u>changes</u> the probability of the event**.

On the other hand, if the two events *A* and *B* are **not interrelated**,one of the events will have<u>**no effect**</u> on the other:

$$\boxed{\mathrm{P}(\mathrm{A}|\mathrm{B}) = \mathrm{P}(\mathrm{A}), \quad \mathrm{P}(\mathrm{B}|A) = \mathrm{P}(\mathrm{B})}$$

In this case, the events *A* and *B* are called <u>**probabilistically independent.**</u>

For **independent events:**

The equation $\mathrm{P}(\mathrm{A} \cap \mathrm{B}) = \mathrm{P}(\mathrm{A}|\mathrm{B})\mathrm{P}(\mathrm{B}) = \mathrm{P}(\mathrm{B}|\mathrm{A})\mathrm{P}(\mathrm{A})$

becomes: $\mathrm{P}(\mathrm{A} \cap \mathrm{B}) = \mathrm{P}(\mathrm{A})\mathrm{P}(\mathrm{B})$ **(independent events)**

In the case of **three independent events**:

The equation $P(A \cap B \cap C) = P(A|B \cap C)P(B \cap C) = P(A|B \cap C)P(B|C)P(C)$

becomes: $\boxed{P(A \cap B \cap C) = P(A)\,P(B)P(C)}$ **(independent events)**

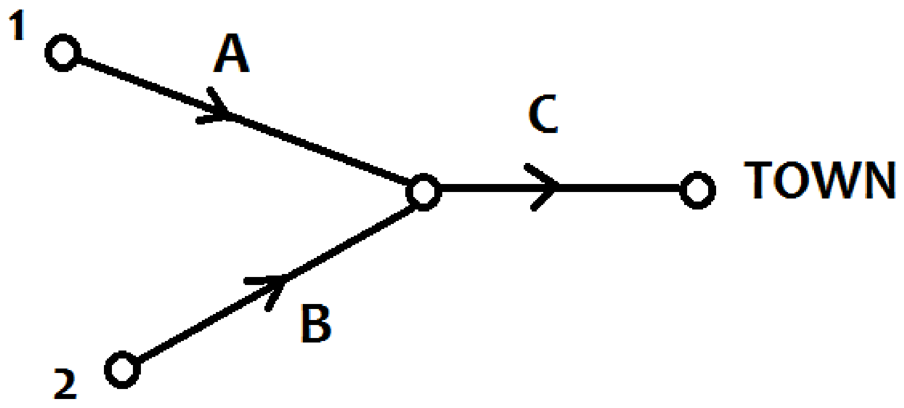
Therefore, the probability of two or more **mutually independent** events occuring together equals the **product of their individual probabilities**.

Example (M. Bayazıt, B. Oğuz, Example 2.4, pg 18)

Water is supplied to a town from the sources 1 and 2 by the pipes A, B and C (see the picture).

The **probabilies of failure** in these pipes are P(A)=0.15, P(B)=0.10 and P(C)=0.02.

Let us determine the probability that the town gets no water, if the events are**independent**.



No water can be supplied when both pipe *A* **and** pipe *B* fail, **or** pipe C fails.

Thus wemust compute the probability of the **union of *A*∩ *B* and C(A∩B) U C**

Example (M. Bayazıt, B. Oğuz, Example 2.4, pg 18)

Adaptingequation $P(A \cup B) = P(A)+P(B) - P(A \cap B)$ on slide19 given for two events to the situation of three events as:

$$P((A \cap B) \cup C) = P(A \cap B)+P(C)-P((A \cap B)\cap C)$$

Assuming that the failure events are **independent**, the probabilities of intersections can be determined as:

$$P(A \cap B)=P(A)P(B)=0.15*0.10=0.015$$

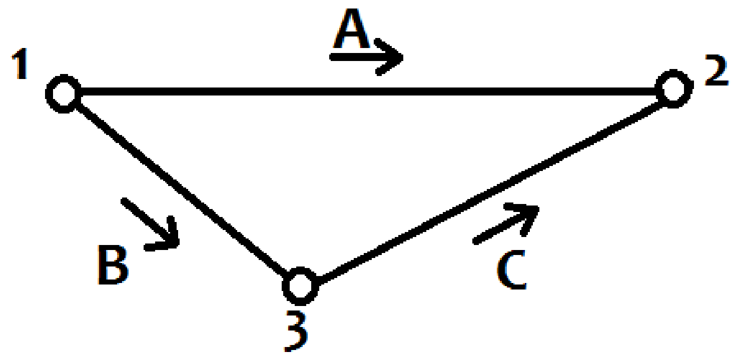
$$P((A \cap B) \cap C) = P(A \cap B)P(C)=0.015*0.02=0.0003$$

Substituting these into the equation for the probability of union:

$$P((A \cap B) \cup C) = 0.015+0.02-0.0003=0.0347$$

Example (M. Bayazıt, B. Oğuz, Example 2.5, pg 18)

One can get to town 2 from town 1 either by route*A* or by routes *B* and C through town 3.



In winter, the probabilities of the routes being **open** are:

P(A)=0.40, P(B)=0.75, P(C)=0.67.

These events are **not independent**:

The probability of route C to be open when (under the condition that) B is open is given as $P(C|B)=0.80$

The probability of route Ato be open when (under the condition that) both B **and** C are open is

$P(A|B\cap C)=0.50$.

Example (M. Bayazıt, B. Oğuz, Example 2.5, pg 18)

Let us determine the probability that one can get **from 1 to 2** in winter:

The travel between the points 1 and 2 is possible if: route ***A* is open** or

both *B* and C are open.

Let us find the probability that both *B* and C are open.

$P(C \cap B)=P(C|B)P(B)=0.80*0.75=0.60$

Now the probability of travel can be determinedby adaptingthe equation

on slide19 given for two events to the situation of **three events** as:

$\mathbf{P(A \cup (B \cap C)} = P(A)+P(B\cap C)-P(A \cap (B\cap C))$

where the probability $P(A \cap (B\cap C))$can be computed using the (conditional probability) equation.

$P(\mathbf{A}\cap (\mathbf{B}\cap \mathbf{C}))=P(\mathbf{A}|(\mathbf{B} \cap \mathbf{C}))\ P(\mathbf{B}\cap \mathbf{C})=0.50*0.60=0.30$

Substituting into the equation for the probability of union:

$P(A \cup (B \cap C) = P(A)+P(B\cap C)-P(A \cap (B\cap C))= 0.40+ 0.60- 0.30= 0.70$

Example (M. Bayazıt, B. Oğuz, Example 2.6, pg 19)

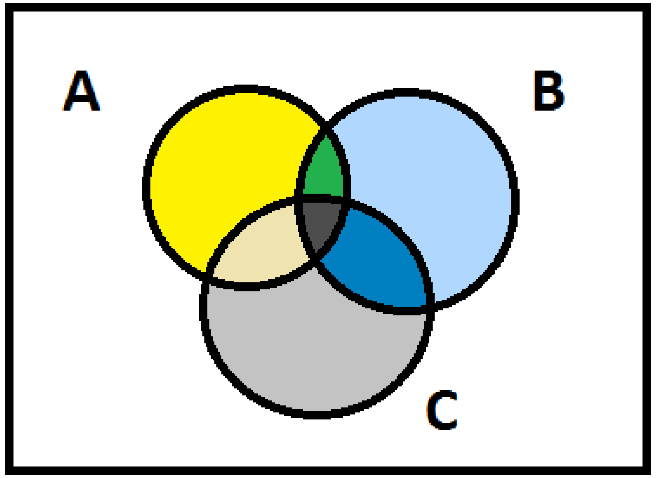
A structural frame consists of 3 elements with the following **probabilities of failure**:

$P(A)=0.05$, $P(B)=0.04$, $P(C)=0.03$.

The frame fails when **at least one of its elements** fails.

The events of failure of the elements are assumed to be **independent**.

What is the probability of **failure** for the frame?



Venn diagram for the events A, B and C

Example (M. Bayazıt, B. Oğuz, Example 2.6, pg 19)

We must determine the probability of the **union P(A U B U C)**.

The probability of the union of three events can be computed using the Venn diagram.

From the diagram it can be seen that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

The probabilities of intersection can be computed as the **products** of the probabilities of individual events because the events are assumed to be **independent**:

$$P(A \cap B) = P(A)P(B) = 0.05 \times 0.04 = 0.002$$

$$P(A \cap C) = P(A)P(C) = 0.05 \times 0.03 = 0.0015$$

$$P(B \cap C) = P(B)P(C) = 0.04 \times 0.03 = 0.0012$$

$$P(A \cap B \cap C) = P(A)P(B)P(C) = 0.05 \times 0.04 \times 0.03 = 0.00006$$

Substituting into the first equation we get the **probability of failure of the frame**:

$$P(A \cup B \cup C) = 0.05 + 0.04 + 0.03 - 0.002 - 0.0015 - 0.0012 + 0.00006 = 0.11536 .$$

Example (M. Bayazıt, B. Oğuz, Example 2.6, pg 19)

We could also solve the problem by computing the probability that **no element of the framefails**since the events of **failure and no failure** are **complementary**.

The separate probabilities of **no failure**:

P($\bar{A}$)= 1-P(A) = 0.95

P($\bar{B}$)= 1-P(B) = 0.96

P($\bar{C}$)= 1-P(C) = 0.97

Because of the assumption of **independence**,the probability of **no failure :**

P($\bar{A} \cap \bar{B} \cap \bar{C}$)=P($\bar{A}$)P($\bar{B}$)P($\bar{C}$) = 0.95 x 0.96 x 0.97= 0.88464

The probability of the failure of the frameis:

P($A \cap B \cap C$)= 1 - P($\bar{A} \cap \bar{B} \cap \bar{C}$)= 1- 0.88464 = 0.11536

Example (M. Bayazıt, B. Oğuz, Example 2.7, pg 20)

A structural frame rests on two footings with probabilities of settlingequal to 0.1 for each footing. Settling of theframehappenswhen**at leastone**of thefootings fail.

The probability that **one of the footings settles when the otherdoes** is 0.8.

Let us determine the **probability of settling of theframe**.

We are looking for the probability **P(A U B)**where A and B denote the events of settling of the footings.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

where the probability of intersection is:

$P(A \cap B) = P(A|B)P(B) = \mathbf{0.8} * 0.1 = 0.08$

Substituting into the first equation:

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.1 + 0.1 - 0.08 = 0.12$

The two events were **not independent.**

Example (M. Bayazıt, B. Oğuz, Example 2.7, pg 20)

If the events A and B were **independent**:

$P(A \cap B) = P(A)P(B) = \mathbf{0.1} * 0.1 = 0.01$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.1 + 0.1 - 0.01 = \mathbf{0.19}$

If the events A and B were **functionally dependent** (one event uniquely identifies another event), i.e. $P(A|B)=1$:

$P(A \cap B) = P(A|B)P(B) = \mathbf{1} * 0.1 = 0.1$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.1 + 0.1 - 0.1 = \mathbf{0.10}$

The probability of settling of the frame may vary in the range of **0.10** and **0.19** as a function of the **dependence** of the events *A* and *B.*

Example: A ball is randomly drawn from a box containing **6 red balls**, **4 white balls**, and **5 blue balls**. Three balls are drawn successively from the box. Find the probability that they are drawn in the order **red**, **white**, and **blue** if each ball is (a) **replaced** (put back into the box) and (b) **not replaced**.

Let R = event "red" on the first draw, W = event "white" on the second draw, and B = event "blue" on the third draw. We require $P\{RWB\}$.

(*a*) If each ball is replaced, then R, W, and B are independent events and

$$P\{RWB\} = P\{R\}P\{W\}P\{B\} = \left(\frac{6}{6+4+5}\right)\left(\frac{4}{6+4+5}\right)\left(\frac{5}{6+4+5}\right) = \left(\frac{6}{15}\right)\left(\frac{4}{15}\right)\left(\frac{5}{15}\right) = \frac{8}{225}$$

(*b*) If each ball is not replaced, then R, W, and B are dependent events and

$$P\{RWB\} = P\{R\}\ P\{W|R\}\ P\{B|WR\} = \left(\frac{6}{6+4+5}\right)\left(\frac{4}{5+4+5}\right)\left(\frac{5}{5+3+5}\right)$$

$$= \left(\frac{6}{15}\right)\left(\frac{4}{14}\right)\left(\frac{5}{13}\right) = \frac{4}{91}$$

Where $P\{B|WR\}$ is the conditional probability of getting a blue ball if a white and red ball have already been chosen.

Example:

An insurance company believes that people can be divided into two types: those who are accident prone and those who are not.

The company's statistics show that an **accident-prone** person will have an accident at some time withinfixed 1-year period with a probability of 0.40,

whereas this probability decreases to 0.20 for a person who is **not accident prone**.

If we assume that 30 percent of the population is accident prone, what is the probability that a new policyholder will have an accident within a year of purchasing a policy?

We shall obtain the desired probability by first <u>conditioning upon whether or not the policyholder is accident prone</u>.

Let $\mathbf{A_1}$ denote the event that the policyholder will have an accident within a year and let **C** denote the event that the policyholder is accident-prone.

Hence, the desired probability is given by:

$$P(A_1) = P(A_1 \cap \mathbf{C}) + P(A_1 \cap \bar{\mathbf{C}}) = P(A_1 | \mathrm{C})P(\mathrm{C}) + P(A_1 | \bar{\mathrm{C}})P(\bar{\mathrm{C}})$$

$$= (0.4)(0.3) + (0.2)(0.7) = 0.26$$

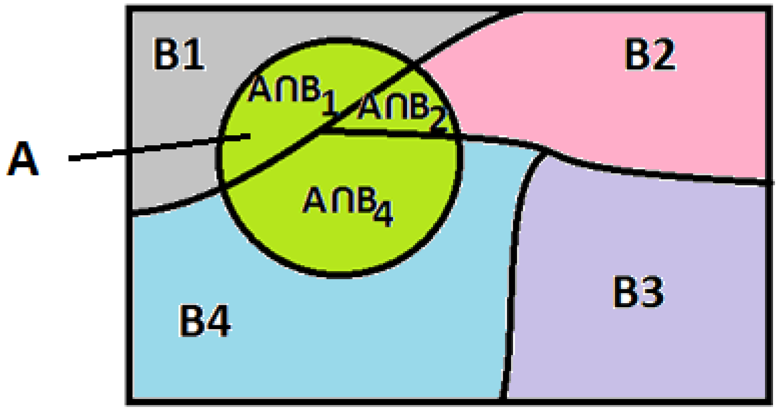
Suppose that a new policyholder has an accident within a year of purchasing a policy. What is the probability that he or she is accident prone?

$$P\left(\mathrm{C} \mid \mathrm{A}_1\right) = \frac{P(\mathrm{C} \cap \mathrm{A}_1)}{P(\mathrm{A}_1)} = \frac{P(\mathrm{C})P(\mathrm{A}_1 \mid \mathrm{C})}{P(\mathrm{A}_1)} = \frac{(0.3)(0.4)}{(0.26)} = \frac{6}{13}$$

Total Probability Theorem and Bayes Theorem

Let the sample space be composed of **disjoint events** $B_1, B_2, ..., B_n$.

$$A = (A \cap B_1) \cup (A \cap B_2) \cup ... \cup (A \cap B_n)$$



Sample space for deriving the Total Probability Theorem

Using the expression for the probability of the union of disjoint events

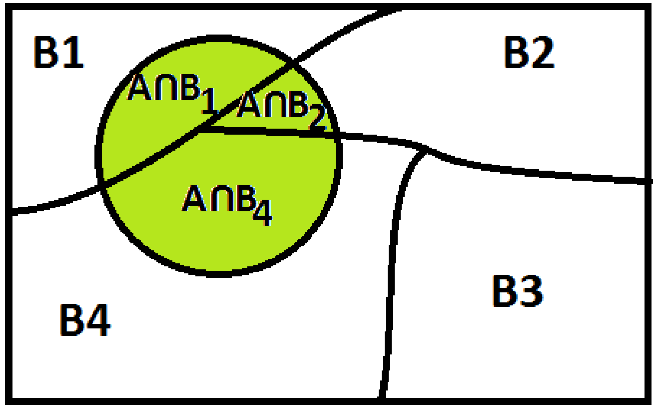
$$\mathbf{P(A) = P(A \cap B_1) \cup P(A \cap B_2) \cup ... \cup P(A \cap B_n)}$$

because the events $A \cap B_1, A \cap B_2, ... , A \cap B_n$ are disjoint.

Total Probability Theorem and Bayes Theorem

Probability of $A \cap B_i$ can be written in the following form

$$\boxed{P(A \cap B_i) = P(A|B_i)P(B_i)}$$



Therefore,

$$\boxed{P(A)=P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+...+P(A|B_n)P(B_n)=\sum_{i=1}^{n} P(A|Bi)P(Bi)}$$

This is called **Total Probability Theorem**.

Total Probability Theorem and Bayes Theorem

For any k(k=1,2, ... ,n) the conditional probability of $P(B_k|A)$ can be expressed as:

$$P(B_k|A) = \frac{P(B_k \cap A)}{P(A)}$$

$$P(B_k|A) = \frac{P(A|B_k)P(B_k)}{P(A)}$$

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

Using the **total probability theorem** for P(A):

$P(A)=P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+...+P(A|B_n)P(B_n)=\sum_{i=1}^{n} P(A|Bi)P(Bi)$

$$P(B_k|A) = \frac{P(A|B_k)P(B_k)}{\sum_{i=1}^{n} P(A|Bi)P(Bi)}$$

This expression, known as the **Bayes Theorem** makes it possible to compute probabilities like **$P(B_k|A)$** when the probabilities **$P(A|B_k)$** **(k=1,2, ... ,n)** are known.

It is used to modify the probabilities based on previous experience using the results of later experiments.

Example (M. Bayazıt, B. Oğuz, Example 2.8, pg 22)

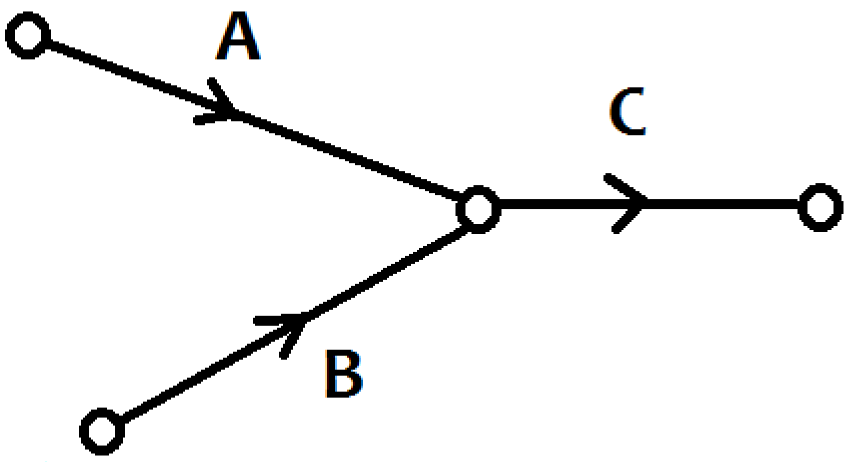
The probabilities of choking for routes **A** and **B** are P(A)=0.10 and P(B)=0.20.

The events of choking are **dependent** so that P(A|B)=0.50 and P(B|A)=1.

Route **C** is choked when **one of the routes A** and **B** is choked.

Route **C** is choked with the probability of 0.20 when **both A** and **B** are open.

What is the probability of choking of route **C**?



Example (M. Bayazıt, B. Oğuz, Example 2.8, pg 22)

This problem can be solved by the application of the **total probability theorem**.

4 events are possible when the routes **A**and **B**are considered together.

Their probabilities are (they add up to be 1):

$P(A \cap B) = P(A|B)\, P(B) = 0.50 * 0.20 = 0.10$ ***(both A and B choked)***

$P(\overline{A} \cap B) = P(\overline{A}|B)\, P(B) = (1-0.50) * 0.20 = 0.10$ ***(A open and B choked)***

$P(A \cap \overline{B}) = P(\overline{B}|A)\, P(A) = (1-1) * 0.10 = 0$ ***(A choked and B open)***

$P(\overline{A} \cap \overline{B}) = 1 - (0.10 + 0.10 + 0) = 0.80$ ***(both A and B open)***

(These are the probabilities of both A and B choked, A open and B choked, A choked and B open, both A and B open, respectively. The last probability is determined as the probability of the complementary event.)

Example (M. Bayazıt, B. Oğuz, Example 2.8, pg 22)

Substituting all into the **total probability theorem**equation on slide 63:

$$\boxed{P(A) = P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+...+P(A|B_n)P(B_n)=\sum_{i=1}^{n} P(A|Bi)P(Bi)}$$

P(C) =P(C|**A**∩ **B**)P(**A**∩ **B**)+P(C|$\bar{\mathbf{A}}$ ∩ **B**)P($\bar{\mathbf{A}}$ ∩ **B**)+P(C|**A**∩ $\bar{\mathbf{B}}$)P(**A** ∩ $\bar{\mathbf{B}}$)+P(C|$\bar{\mathbf{A}}$ ∩ $\bar{\mathbf{B}}$)P($\bar{\mathbf{A}}$ ∩ $\bar{\mathbf{B}}$)

P(C|A∩ B) = P(C|$\bar{A}$ ∩ B) = P(C|A∩ $\bar{B}$) = 1 as C is choked when **either A and B or both of them are choked**.

P(C|$\bar{A}$ ∩ $\bar{B}$) = 0.2 as C is choked with the probability of 0.2 when **both A and B are open**.

P(C) =P(C|**A**∩ **B**)P(**A**∩ **B**)+P(C|$\bar{\mathbf{A}}$ ∩ **B**)P($\bar{\mathbf{A}}$ ∩ **B**)+P(C|**A**∩ $\bar{\mathbf{B}}$)P(**A** ∩ $\bar{\mathbf{B}}$)+P(C|$\bar{\mathbf{A}}$ ∩ $\bar{\mathbf{B}}$)P($\bar{\mathbf{A}}$ ∩ $\bar{\mathbf{B}}$)

= 1 x0.1+ 1 x0.1+1 x 0+0.2 x 0.8= **0.36**

The probability of choking of route C is determined as 0.36 using the **total probability theorem**.

Example (M. Bayazıt, B. Oğuz, Example 2.9, pg 23)

A material is estimated to be of **good quality ($\mathbf{G}$)** with probability 0.70, and **poor quality ($\overline{\mathbf{G}}$)** with probability 0.30.

To approve these estimates, a quality test is performed on the material.

This test however, is not perfectly reliable.

It is known that **good quality** material **passes the test** with the probability of 0.80; **poor quality** material passes it with the probability of 0.10.

Let us determine an **improved estimate** of the material being of good or poor quality.

A: The event that the material **passes the test**

Example (M. Bayazıt, B. Oğuz, Example 2.9, pg 23)

A: The event that the material passes the test

$P(A \mid G) = 0.80$

$P(A \mid \overline{G}) = 0.10$

We can determine an improved estimate for the **quality of being good when the material passes the test,** $P(G \mid A)$, by *Bayes theorem*:

$$P(B_k|A) = \frac{P(B_k)P(A \mid B_k)}{\sum_{i=1}^{n} P(A \mid Bi)P(Bi)}$$

$$P(G|A) = \frac{P(A|G).P(G)}{P(A|G).P(G) + P(A|\overline{G}).P(\overline{G})}$$

$$= \frac{0.80 \times 0.70}{0.80 \times 0.70 + 0.10 \times 0.30} = 0.95$$

Example (M. Bayazıt, B. Oğuz, Example 2.10, pg 23)

The concrete at a construction site is classified (before doing tests) based on previous experience as : (B_1=20 MPa, B_2=30 MPa, B_3=40 MPa)

C20 (=B_1) with the probability 0.3;

C30 (=B_2) with the probability 0.6;

C40 (=B_3) with the probability 0.1;

To imrove these estimates, a **test** is performed on a sample.

The **test (A)** is known to behave as follows:

Actual value

Test result

$P(A \mid B_k)$			
	B_k		
A	B_1	B_2	B_3
B_1	0.75	0.15	0
B_2	0.25	0.70	0.25
B_3	0	0.15	0.75

Example (M. Bayazıt, B. Oğuz, Example 2.10, pg 23)

From this table we can see that a sample of:

quality C30 (B_2) will appear in the test

as C30 (B_2) with the probability 0.70;

as C20 (B_1) with the probability 0.15;

as C40 (B_3) with the probability 0.15;

$$P(B_k|A) = \frac{P(B_k)P(A \mid B_k)}{\sum_{i=1}^{n} P(A \mid B_i)P(B_i)}$$

Let us assume that the test gives the result B_2 i.e $A=B_2$.

We can compute the new estimates $P(B_k/B_2)$using the test results and the *Bayes theorem*:

$$P(B_1|\underbrace{B_2}_{A=B_2}) = \frac{P(B_1)P(\underbrace{B_2}_{A=B_2}|B_1)}{P(B_1)P(B_2|\ B_1)+P(B_2)P(B_2|\ B_2)+P(B_3)P(B_2|B_3)}$$

$$= \frac{0.3 \times 0.25}{0.3 \times 0.25 + 0.6 \times 0.7 + 0.1 \times 0.25} = 0.14$$

Example (M. Bayazıt, B. Oğuz, Example 2.10, pg 23)

$A=B_2$ $A=B_2$

$$P(B_2|B_2)=\frac{P(B_2)P(B_2|B_2)}{P(B_1)P(B_2|B_1)+P(B_2)P(B_2|B_2)+P(B_3)P(B_2|B_3)}$$

$$=\frac{0.6\times 0.7}{0.3\times 0.25+0.6\times 0.7+0.1\times 0.25}=0.81$$

$A=B_2$ $A=B_2$

$$P(B_3|B_2)=\frac{P(B_3)P(B_2|B_3)}{P(B_1)P(B_2|B_1)+P(B_2)P(B_2|B_2)+P(B_3)P(B_2|B_3)}$$

$$=\frac{0.1\times 0.25}{0.3\times 0.25+0.6\times 0.7+0.1\times 0.25}=0.05$$

Example (M. Bayazıt, B. Oğuz, Example 2.10, pg 23)

Before the test, the probabilities of the quality B_1 , B_2 and B_3 were **0.3, 0.6** and **0.1**, respetively.

After one test with the result of $A=B_2$ now they are modified to **0.14, 0.81** and **0.05**.

The probability of the concrete being B_2 is now found to be higher because the test confirmed this assumption.

If we carry out further tests and if for example $A=B_2$ again, the probabilities of B_1, B_2 and B_3 will be changed to 0.06, 0.92 and 0.02, respectively.

(If we again do the same calculations using Bayes theorem.)

Example

A laboratory blood test is 95 percent effective in detecting a certain disease when it is, in fact, present. However, the test also gives «a false positive» result for 1 percent of the healthy persons tested.

(That is if a healthy person is tested, the test result will imply that he or she has the disease with the probability of 0.01,.)

If 0.5 % of the population actually has the disease, what is the probability that **a person has the disease given that the test result is positive**?

D: the event that the person tested **has the disease**

E: the event that the **test result is positive**

Then the desired probability is:

$$P(D|E) = \frac{P(DE)}{P(E)}$$

Example

$$P(B_k|A) = \frac{P(B_k)P(A \mid B_k)}{\sum_{i=1}^{n} P(A \mid B_i)P(B_i)}$$

$$\begin{aligned} P(D|E) &= \frac{P(DE)}{P(E)} \\ &= \frac{P(E|D)P(D)}{P(E|D)P(D) + P(E|\bar{D})P(\bar{D})} \\ &= \frac{(.95)(.005)}{(.95)(.005) + (.01)(.995)} \\ &= \frac{95}{294} \approx .323 \end{aligned}$$

Thus, only 32% of those people, whose tests are positive, actually have the disease.

Sounds surprising ?

Example

Check:

Since 0.5% of the population actually has the disease, it means that 1 person out of 200 tested will have it.

The test correctly confirms that this person has the disease with 0.95 probability.

Thus, out of every 200 people tested, the test will correctly confirm that 0.95 person has the disease.

On the other hand, out of the 199 healthy people, the test will **incorrectly** state that 199*0.01=1.99 people have the disease.

Hence, for every 0.95 diseased person, that the test **correctly** states ill, there are 1.99 people, that the test **incorrectly** states ill.

0.95/(0.95+1.99)=0.323 = 32.3 %