Probability and Statistics
MAT 271E

PART 3

Distribution of Random Variables

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DISTRIBUTIONS OF RANDOM VARIABLES

Random Variable

A **random variable** is defined mathematically as a real-valued function defined on a sample space (represented with a letter such as X or Y).

We can express the **probabilities** of various **random events** belonging to a **random variable** by a **distribution function** as a whole.

The expression of this function differs depending on whether the type of the variable is **discrete** or **continuous**.

DISTRIBUTIONS OF RANDOM VARIABLES

It is therefore necessary to divide random variables into two types (in terms of defining the probabilities): **discrete** or **continuous**

Discrete: A random variable is **discrete** if it has a **finite number of values** (we can assume a countable number of values),
i.e. number of elements (simple events) in the sample space is **finite**.

Discrete random variables usually arise from an experiment that involves counting.

Some examples...
Number of cars arriving at a crossroad in one minute
Number of rainy days in a year
Number of individuals who get a certain type of flu in a country in a year

DISTRIBUTIONS OF RANDOM VARIABLES

Continuous: A random variable is **continuous** if it is capable of assuming all the values in an interval,
i.e. the number of elements (simple events) in the sample space is **infinite**.

In experiments, because of the limited accuracy of the measuring devices, no random variables can be measured to be truly continuous in reality. However, we can sometimes abstractly take them to be continuous.

Some examples...
Discharge of a river
Wind velocity at a point
Weight loss during a dietary routine
Blood pressure

Discrete Random Variables

ProbabilityMass Function

A discrete random variable assumes each of its values with a certain probability.

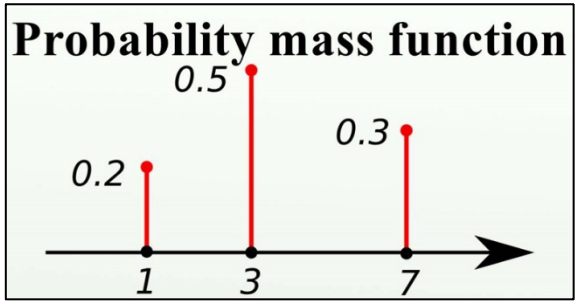
The probability mass function **(p.m.f. or PMF)** of a ***discrete random variable*** can be obtained by demonstrating the probabilities of various simple random events belonging to this discreterandom variableas **vertical lines** at $x_i$abscissas.

$$p(x_i)=P(X=x_i)$$

This function lets us see the probabilities of various simple events at one look (see the figure).

The total length of these vertical lines (the probabilities) is **always equal to 1**.

$$\sum_{x_i} p(xi) = 1$$



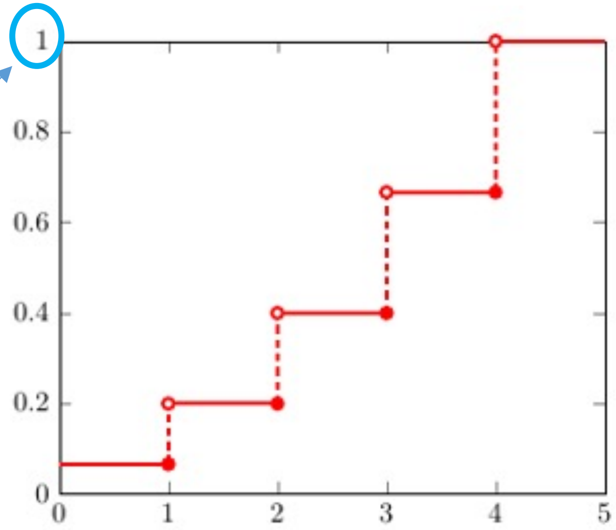
Discrete Random Variables

Cumulative Distribution Function

Another way of demonstration is to express the probability that the **random variable is equal to or smaller** than a certain value.

This function, which is quite significant in practice is called the ***cumulative distribution function*****(c.d.f. or CDF)**.

$$F(x_i)=P(\mathbf{X \leq x_i})$$



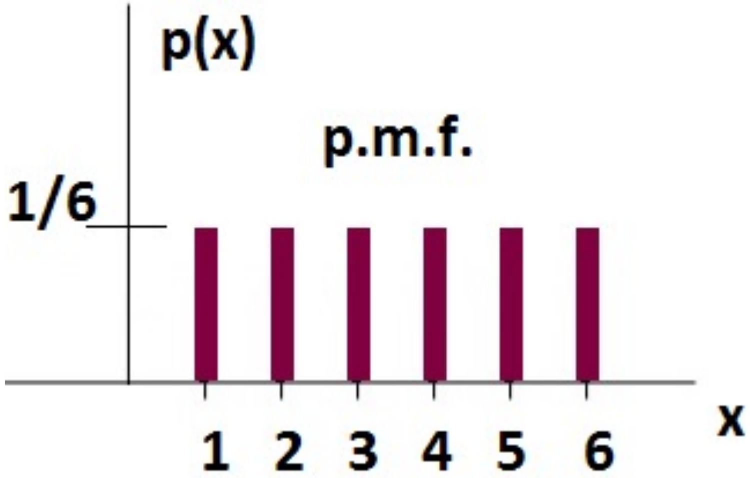
It can be seen from the definition of the functionthat **F(x)**function is a **stepwise increasing function from 0 to 1.**

$$F(x_i)=\sum_{x_j \leq x_i} p(xj)$$

It is quite easily understood that ***probability mass function*** *and* ***cumulative distribution function*** *contain the same information and once one of them is known the other can easilybe obtained.*

Discrete Random Variables

Example: Roll of a die **(probability mass function)**



$p(x_i)=P(X=x_i)$

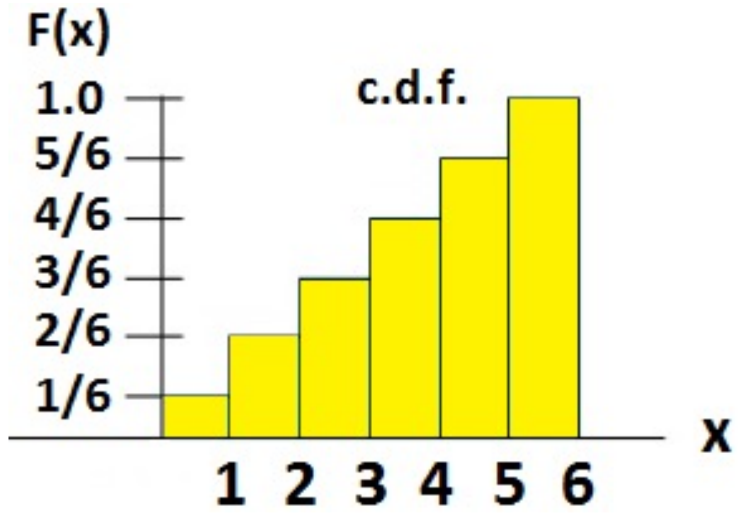
probability mass function

p(1)=**1/6**, p(2)= **1/6**, p(3)= **1/6**, p(4)= **1/6**, p(5)= **1/6**, p(6)= **1/6**

$$\sum_{x_i} p(xi) = 1/6 + 1/6 + 1/6 + 1/6 + 1/6 + 1/6 = \mathbf{1}$$

Discrete Random Variables

Example: Roll of a die **(cumulative distribution function, CDF)**



x_i	$F(X \leq x_i)$
1	$F(X \leq 1) = 1/6$
2	$F(X \leq 2) = 2/6$
3	$F(X \leq 3) = 3/6$
4	$F(X \leq 4) = 4/6$
5	$F(X \leq 5) = 5/6$
6	$F(X \leq 6) = \mathbf{1}$

cumulative distribution function

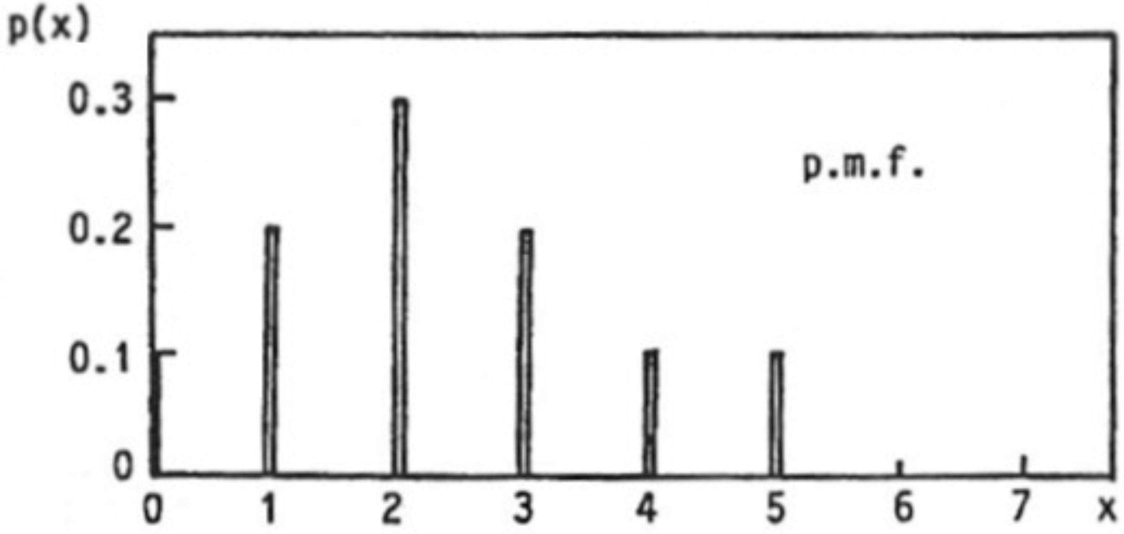
$$F(x_i) = P(X \leq x_i)$$

$$F(x_i) = \sum_{x_j \leq x_i} p(xj)$$

Discrete Random Variables

Example: The number of cars stopping at a traffic light

$p(x_i)=P(X=x_i)$



(probability mass function)

P(0)=0.1, p(1)=0.2, p(2)=0.3, p(3)=0.2, p(4)=0.1, p(5)=0.1, p(6)=0, p(7)=0

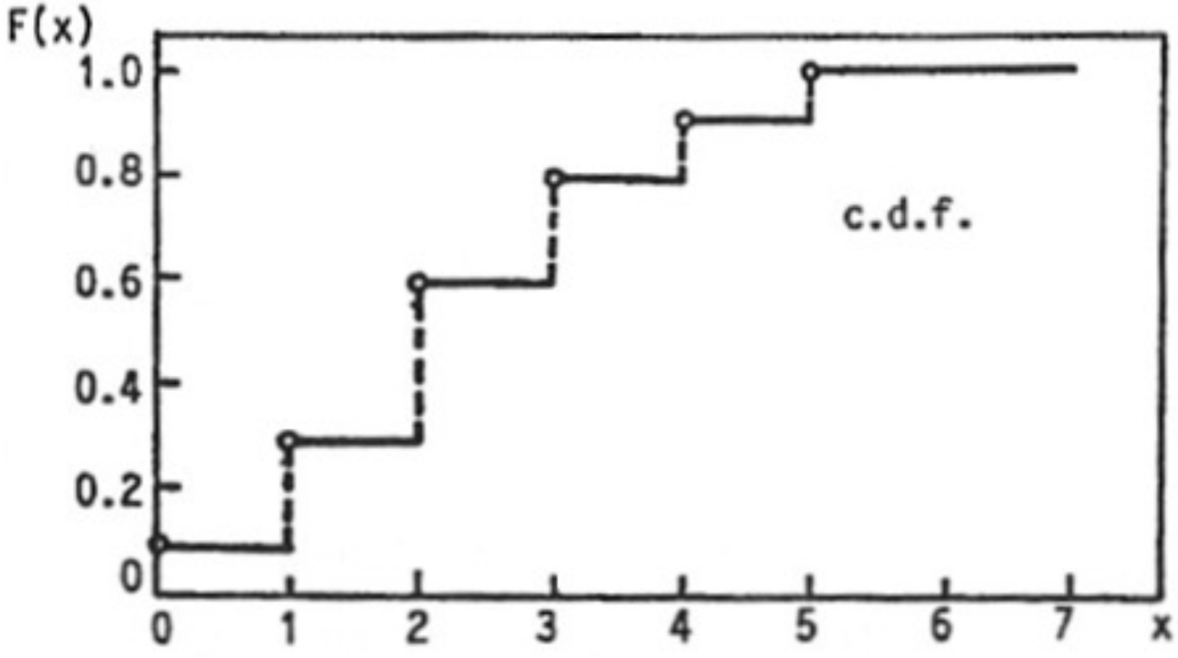
$$\sum_{x_i} p(xi) = 0.1 + 0.2 + 0.3 + 0.2 + 0.1 + 0.1 + 0 + 0 = \mathbf{1}$$

Discrete Random Variables

Example: The number of cars stopping at a traffic light

$F(x_i)=P(X\leq x_i)$

$F(x_i)=\sum_{x_j\leq x_i} p(xj)$



x_i	$F(X\leq x_i)$
0	$F(X\leq 0)=0.1$
1	$F(X\leq 1)=0.3$
2	$F(X\leq 2)=0.6$
3	$F(X\leq 3)=0.8$
4	$F(X\leq 4)=0.9$
5	$F(X\leq 5)=1$

(cumulative distribution function)

$P(0)=0.1$, $p(1)=0.2$, $p(2)=0.3$, $p(3)=0.2$, $p(4)=0.1$, $p(5)=0.1$, $p(6)=0$, $p(7)=0$

Continuous Random Variables

A **continuous** random variable can take **infinite** number of values (simple events in the sample space).

In other words, such a variable can take all real numbered values (or all the values within an interval).

Many of the random variables which are met in engineering problems are **continuous**, such as precipitation depth, flow volume, strength of a material, age of a material, ambient temperature, moisture...

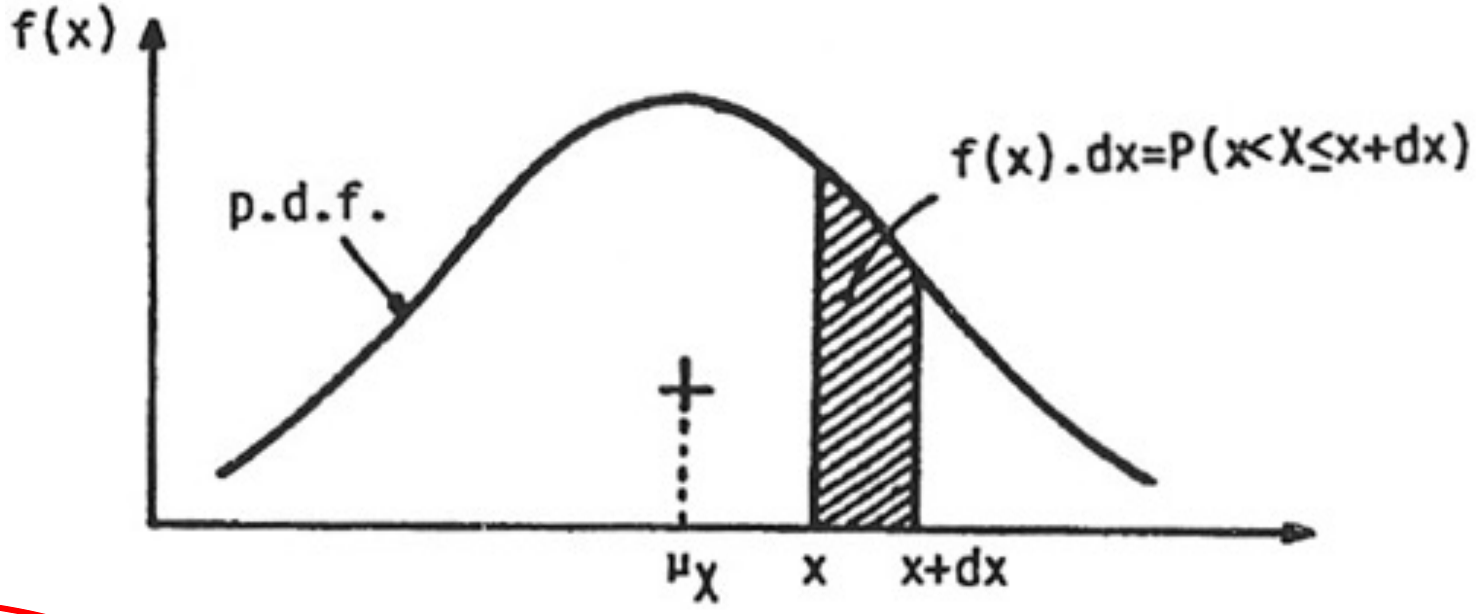
Since the number of values a continuous random variable can take is infinite and sum of the probabilities of taking these values is equal to **1**, the probabilities of **simple events such as $X=x_i$ will approach zero**.

This is the reason for defining the probability of a **compound event** such as the variable remaining **in an interval x and $x+dx$** and giving up considering probabilities of simple events.

Continuous Random Variables

Probability Density Function

Thus the ***probability density function*** **(p.d.f. or PDF)** **f(x)** can be defined as follows



f(x)dx=P(x≤X≤x+dx)

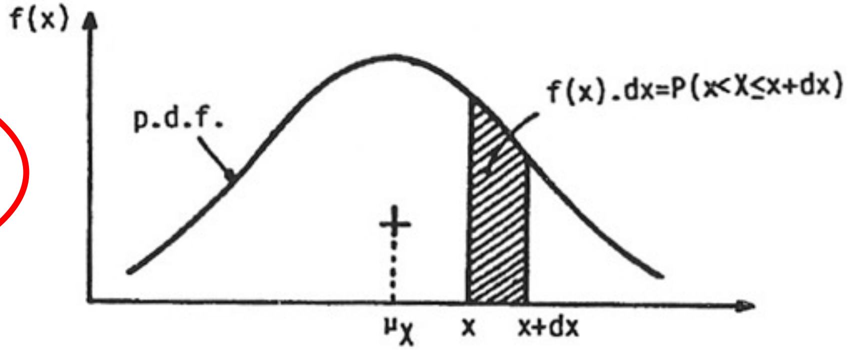
Here, **the** **area** bounded by the **curve** ***f(x)***, the **x-axis** and the **verticals** drawn from points ***x*** *and* ***x+dx*** shows the **probability** of the variable having a value within the interval ***(x, x+dx)***.

Continuous Random Variables

Probability Density Function

The probability of the variable taking a value within a finite **interval** can be calculated by dividing this interval into small pieces and summing up the probabilities of the variables being in these small intervals.

$$P(x_1 < X \leq x_2) = \int_{x_1}^{x_2} f(x)dx$$



The probability that the variable takes a value in the interval (-∞, ∞) is **always 1**.

$$\int_{-\infty}^{\infty} \mathbf{f(x)dx = 1}$$

Continuous Random Variables

Cumulative Distribution Function

The definition of ***cumulative distribution function*** **(c.d.f. or CDF)**does not change in case of continuous variables.

F(x)=P(X≤x)

The following relation exists between **f(x)** *and* **F(x)***:*

$$\boldsymbol{f(x)} = \frac{d}{dx}\left(\int_{-\infty}^{x} f(u)du\right) = \frac{\boldsymbol{dF(x)}}{\boldsymbol{dx}}$$

Cumulative distribution function:

$$F(x) = P(-\infty < X \le \boldsymbol{x}) = \int_{-\infty}^{x} f(u)du$$

Continuous Random Variables

Cumulative Distribution Function

The cumulative distribution function will always satisfy the following conditions:

$0 \leq F(x) \leq 1$

$F(-\infty) = 0$

$F(\infty) = 1$

$F(x+\varepsilon) \geq F(x) \quad \text{for } \varepsilon > 0$

$F(x_2) - F(x_1) = P(x_1 < X \leq x_2)$

Continuous Random Variables

Example

On a 50 km long highway, the traffic accidents are uniformly distributed (f(x)=C).

What is the probability of having an accident between the 20. and 30. kilometers of the highway?

The value of constant C can be calculated as follows:

$\int_0^{50} C dx = 50C = 1$ C=0.02

The *cumulative distribution function* of the **random variable X** showing the **distance of the point of the traffic accident to the origin of the road** can be found as follows:

$$F(x) = \int_0^x f(x)dx = \int_0^x 0.02dx = 0.02x$$

F(0)=0,   F(50)=1

Continuous Random Variables

Example

The **probability** of having an accident between the 20. and 30. kilometers of the highway can be computed as:

$$P(20 < X < 30) = F(30) - F(20)$$

$$= 0.02*30 - 0.02*20 = 0.20$$

Continuous Random Variables

Example

The maximum wind velocity, which will be used in the design of a tall building, can be shown with the following ***probability density function***:

$$f(x) = ke^{-\lambda x} \qquad x > 0$$

It is also given that the probability of maximum wind velocity remaining smaller than 70 km/hour is 0.9.

Using $\int_{-\infty}^{\infty} f(x)dx = 1$

the following equality can be written:

$$\int_0^{\infty} k\, e^{-\lambda x} dx = \frac{-k}{\lambda} e^{-\lambda x} \Big|_0^{\infty} = \frac{k}{\lambda} = 1$$

Continuous Random Variables

Example

Therefore, k=λ

The p.d.f (probability density function) becomes:

$f(x) = \lambda e^{-\lambda x}$ x≥0

The *cumulative distribution function*

$$\boldsymbol{F(x)} = \int_{-\infty}^{x} f(u)\,du = \int_{0}^{x} \lambda\, e^{-\lambda u} du = -e^{-\lambda u} \Big|_{0}^{x}$$

$$= 1 - e^{-\lambda x} \quad x \geq 0$$

Parameter λ can be calculated as follows under the assumption that the probability of maximum wind velocity remaining smaller than 70 km/hour is 0.9.

$$F(70) = 1 - e^{-70\lambda} = 0.9 \quad \lambda=0.033$$

Continuous Random Variables

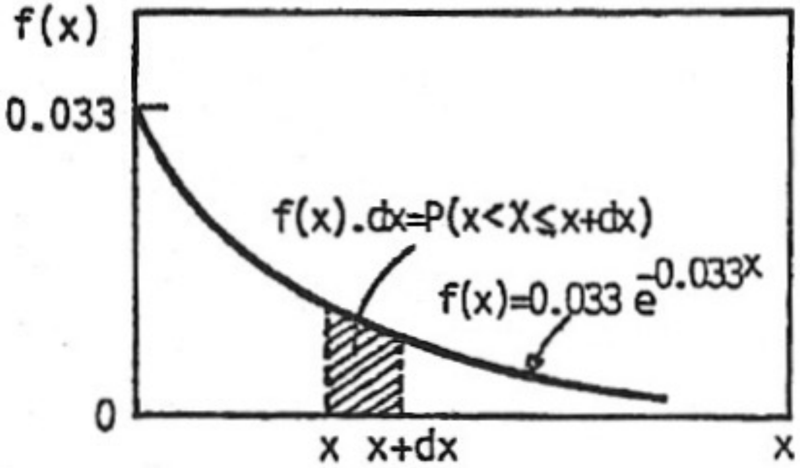
Example

By substituting into the corresponding equations, the following expressions are obtained:

λ=0.033

probability density function (p.d.f.)

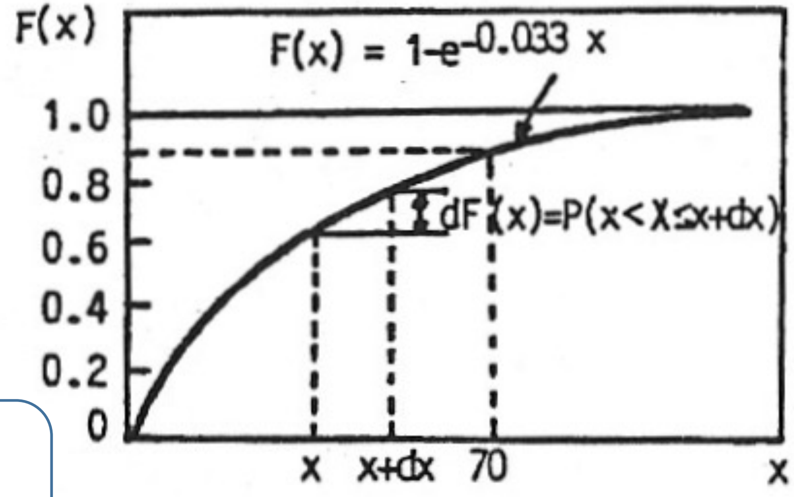
f(x) = $0.033\ \mathrm{e}^{-0.033\mathrm{x}}$



(p.d.f.)

cummulative distribution function (c.d.f.)

$\mathbf{F}(\boldsymbol{x}) = 1 - \mathrm{e}^{-0.033\mathrm{x}}$x≥0



(c.d.f.)

For example, the probability of wind velocity **exceeding** 80 km/hour can be calculated as:

1 - F(80) = 1 - $(1\text{-}\mathrm{e}^{-0.033*80}) = 0.07$

$$\boldsymbol{F}(\boldsymbol{x}) = \int_{-\infty}^{x} f(u)\,du$$

non-exceedance

Multivariable Distributions

A **multivariate** probability distribution is one that contains **two or more random variables**.

These random variables **might or might not be correlated**.

The probabilities of events belonging to the variables are expressed by:

joint probability functions

For discrete variables → **joint probability mass function**

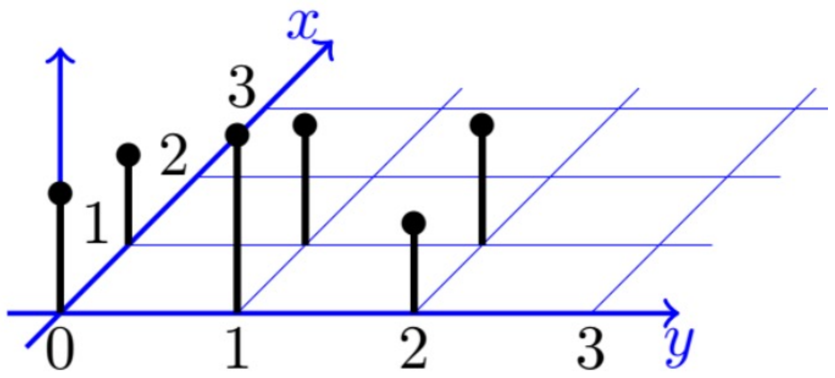
For continuous variables → **joint probability density function**

For both discrete and continuous variables → **joint cumulative distribution functions**

Multivariable Distributions

Joint Probability Mass Function

(Discrete variables)



for **discrete** X and Y variables:

$P(x_i,y_i) = P((X = x_i) \cap (Y = y_j))$

Showing the probability that **in the same observation**:

X takes a value of $\mathbf{x_i}$

and Y takes a value of $\mathbf{y_j}$

Multivariable Distributions

Joint Cumulative Distribution Function

(Discrete variables)

For these two **discrete** variables (X and Y), it is defined as:

$$F(x,y) = P\big((X \leq x) \cap (Y \leq y)\big) = \sum_{x_i \leq x} \sum_{y_j \leq y} p(x_i, y_j)$$

Multivariable Distributions

Marginal Probability Mass Function

(Discrete variables)

If the distribution of **only one of the variables** is taken into consideration **in a two-variable distribution**, the ***marginal probability mass function*** of the variables is obtained, for variable X:

$$p(x) = P(X = x) = \sum_{y_j} p(x, y_j)$$

Multivariable Distributions

Marginal Cumulative Distribution Function

(Discrete variables)

Marginal cumulative distribution function is similarly defined as:

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} p(x_i) = \sum_{x_i \leq x} \sum_{\boldsymbol{y_j}} p(x_i, y_j) = F(x, \infty)$$

If the value of Y approaches infinity in the ***joint*** *cumulative distribution function*, the ***marginal*** *cumulative distribution function* of X is obtained.

Multivariable Distributions

Conditional Probability Mass Function

(Discrete variables)

In a two-variable distribution, **under the condition** that one of the variables takes a certain value, the distribution of the other variable is called **conditional probability mass function**.

$$\mathrm{p}(x_i|y_j) = P(X = x_i|\boldsymbol{Y = y_j}) = \frac{P((X=x_i)\cap(Y=y_j))}{P(Y=y_j)} = \frac{p(x_i,y_j)}{p(y_j)}$$

In this equation, the expression to the right of the sign| shows a **condition** and not a random event.

$$\mathrm{p}(x_i, y_j) = p(x_i|y_j)p(y_j) = p(y_j|x_i)p(x_i)$$

Marginal Probability Mass Function
vs.
Conditional Probability Mass Function

The **marginal p.m.f** and the **conditional p.m.f.** of a random variable do not certainly have to be the same.
The two functions being different shows that the condition for one of the variables affects the probabilities belonging to the other variable.

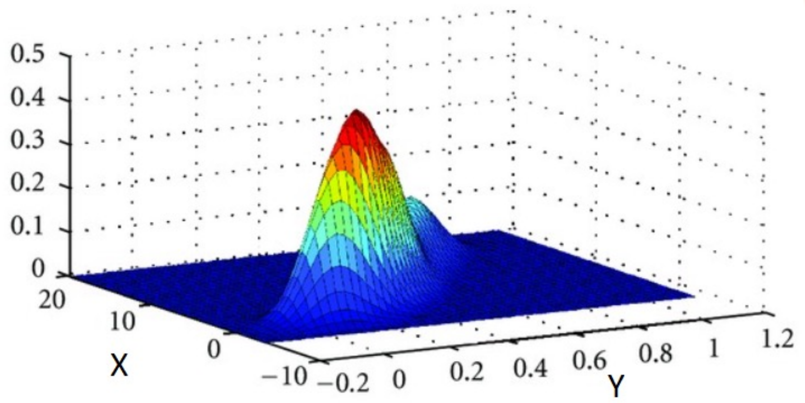
This shows that there exists a relation between the two variables (the two variables are **not independent**).

In such a case, if the value of one of the variables in an observation is known, the probabilities of events belonging to the other variable can be estimated more correctly.

Multivariable Distributions

Joint Probability Density Function

(Continuous variables)



For two **continuous** random variables the ***joint probability density function*** is defined similarly.

f(x,y) shows the probability of:

X being in the interval (x,x+dx) and Y being in the interval (y,y+dy).

The function always satisfies the following condition:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{f}(\mathrm{x},\mathrm{y})\mathrm{dxdy} = 1$$

Multivariable Distributions

Joint Cumulative Distribution Function

(Continuous variables)

For the two variables (X and Y), it is defined as:

$$F(x, y) = P\big((X \leq x) \cap (Y \leq y)\big)$$

$$= P\big((-\infty < X \leq x) \cap (-\infty < Y \leq y)\big)$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v) du dv$$

$$\mathrm{F(x,y)} = \int_{-\infty}^{\mathrm{x}} \int_{-\infty}^{\mathrm{y}} \mathrm{f(u,v)dudv}$$

$$\mathrm{f(x,y)} = \frac{\partial^2 \mathrm{F(x,y)}}{\partial \mathrm{x} \partial \mathrm{y}}$$

Marginal Probability Density Function

(Continuous variables)

Marginal probability density function of variable **X**:

$$f(x) = \int_{-\infty}^{\infty} f(x,y)dy$$

Multivariable Distributions

Marginal Cumulative Distribution Function

(Continuous variables)

Marginal cumulative distribution function of variable **X**:

$$F(x) = P(X \leq x) = F(x, \infty)$$

ConditionalProbabilityDensity Function

Conditional probability density function of **X** for a **given value of Y**:

$$f(x|y) = \frac{f(x, y)}{f(y)}$$

Conditional Cumulative Distribution Function

Conditional cumulative distribution function of **X** for a **given value of Y**:

$$F(x|y) = P(X \leq x|Y = y) = \int_{-\infty}^{x} f(u|y)du$$

As a special case:

If the events related to the variable X are independent of events related to the variable Y: they are independent.
In case X and Y variables are ***independent***, the following equations can be written:

$f(x|y)=f(x)$
$f(y|x)=f(y)$
$f(x,y)=f(x)f(y)$
$F(x,y)=F(x)F(y)$
$F(x|y)=F(x)$
$F(y|x)=F(y)$

The equations above given for two random variables can be also be derived for the joint distributions of more than two random variables.

Example (M. Bayazıt, B. Oğuz, Example 2.13, pg 32)

(Discrete variables)

X is the number of vehicles passing through a road in 1 minute.

Y is the number of vehicles a counting device counts during the same 1 minute.

Since the vehicle counting device does not work with exact correctness, these two random variables differ from one another.

As a result of the observations made, the following values are obtained for the $p(x_i,y_j)$ (**joint**) **probability mass function** of the variables **X** and **Y**.

X	Y				
	0	1	2	3	4
0	0.25	0	0	0	0
1	0.04	0.36	0	0	0
2	0.01	0.03	0.16	0	0
3	0	0.01	0.02	0.07	0
4	0	0	0.0025	0.015	0.0325

Example (M. Bayazıt, B. Oğuz, Example 2.13, pg 32)

The fact that the probabilities corresponding to the same values of X and Y (X=Y=0, X=Y=1, X=Y=2, X=Y=3, X=Y=4) are high shows that the device works correctly for most of the time. Using these values, the probability of the device working correctly can be found.

$$P(Y = X) = \sum_{x_i} P[(X = x_i) \cap (Y = x_i)] = \sum_{x_i} p(x_i, x_i)$$

P(Y=X)=p(0,0)+p(1,1))+p(2,2))+p(3,3))+p(4,4)

=0.25+0.36+0.16+0.07+0.0325=0.8725

According to this, the probability that the device makes an error is

1-0.8725=0.1275

Example (M. Bayazıt, B. Oğuz, Example 2.13, pg 32)

Using the given values, the ***marginal probability mass functions*** of X and Y can be found.

$$p(x) = P(X = x) = \sum_{y_j} p(x, y_j)$$

X	Y 0	1	2	3	4
0	0.25	0	0	0	0
1	0.04	0.36	0	0	0
2	0.01	0.03	0.16	0	0
3	0	0.01	0.02	0.07	0
4	0	0	0.0025	0.015	0.0325

x_i	0	1	2	3	4
$p(x_i)$	0.25	0.40	0.20	0.10	0.05

$$p(x_i) = \sum_{y_j=0}^{4} p(x_i, y_j)$$

X	Y 0	1	2	3	4
0	0.25	0	0	0	0
1	0.04	0.36	0	0	0
2	0.01	0.03	0.16	0	0
3	0	0.01	0.02	0.07	0
4	0	0	0.0025	0.015	0.0325

y_j	0	1	2	3	4
$p(y_j)$	0.30	0.40	0.1825	0.085	0.0325

$$p(y_j) = \sum_{x_i=0}^{4} p(x_i, y_j)$$

Example (M. Bayazıt, B. Oğuz, Example 2.13, pg 32)

The ***conditional probability mass function*** of *X* when the measuring device counts *Y=1*.

$$\mathrm{p}(x_i|y_j) = P(X = x_i|Y = y_j) = \frac{P((X=x_i)\cap(Y=y_j))}{P(Y=y_j)} = \frac{p(x_i,y_j)}{p(y_j)}$$

x_i	0	1	2	3	4
$p(x_i\|y=1)$	0	0.90	0.075	0.025	0
	0/0.40	0.36/0.40	0.03/0.40	0.01/0.40	0/0.40

$$p(x_i|y=1) = \frac{p(x_i,1)}{0.40}$$

X	Y = 0	1	2	3	4
0	0.25	0	0	0	0
1	0.04	0.36	0	0	0
2	0.01	0.03	0.16	0	0
3	0	0.01	0.02	0.07	0
4	0	0	0.0025	0.015	0.0325

Example (M. Bayazıt, B. Oğuz, Example 2.13, pg 32)

It is seen that ***conditional probability mass function*** is different from the ***marginal probability mass function*** of *X*, showing that *X* and Y are **dependent** (see slide 27).

There is a strong relationship between X and Y.

The probability that the correct number of vehicle is X=1 when the measuring device counts Y=1 is 0.90, a value very close to 1.

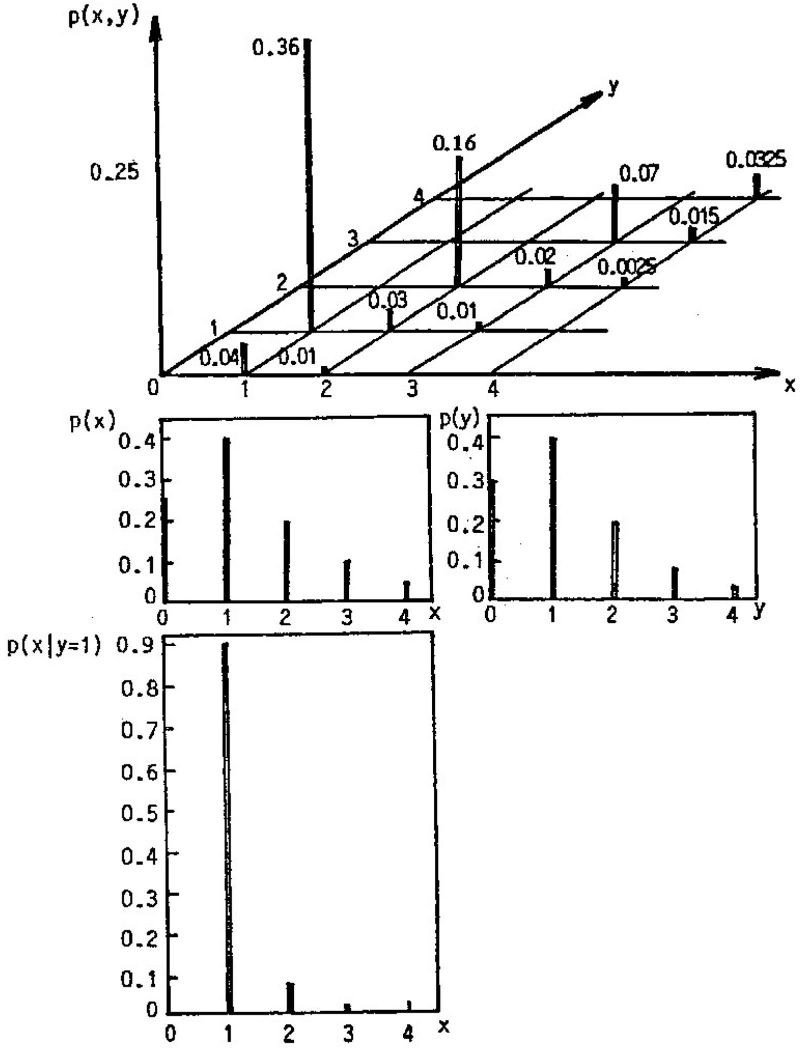
(If the measuring device worked with exact correctness, X and Y would be the same variable with p(x)=p(y)).

Example (M. Bayazıt, B. Oğuz, Example 2.13, pg 32)

$p(x_i, y_j)$ **joint probability mass function**

$p(x_i)$ and $p(y_j)$ **marginal probability mass functions**

$p(x_i|y=1)$ **conditional probability mass function**



Example (M. Bayazıt, B. Oğuz, Example 2.14, pg 33)

(Continuous variables)

In a building, the **joint probability <u>density</u> function**of ***X***(the **cost of labour**, billion TL) and ***Y***(the **cost of material**, billion TL) is:

$$f(x,y) = 2y\,exp[-y(x+2)] \qquad\qquad x,y \geq 0$$

The **marginal p.d.f. of X**:

$$f(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{0}^{\infty} 2y\,exp[-y(x+2)]dy = 2/(x+2)^2$$

Example (M. Bayazıt, B. Oğuz, Example 2.14, pg 33)

(Continuous variables)

The **marginal p.d.f. of Y**:

$$f(y) = \int_{-\infty}^{\infty} f(x, y)dx = \int_{0}^{\infty} 2y\, exp[-y(x+2)]dx = 2\exp(-2y)$$

Example (M. Bayazıt, B. Oğuz, Example 2.14, pg 33)

(Continuous variables)

The **conditional p.d.f.** of **X**:

$$f(x|y) = \frac{\mathrm{f(x,y)}}{\mathrm{f(y)}} = \mathrm{y\ exp(-xy)} \neq f(x)$$

The **conditional p.d.f.** of **Y**:

$$f(y|x) = \frac{\mathrm{f(x,y)}}{\mathrm{f(x)}} = \mathrm{y}\ (x+2)^2\ \mathrm{exp}\big(-\mathrm{y(x+2)}\big) \neq f(y)$$

It can be concluded that the variables X and Y are **not independent**.

Example (M. Bayazıt, B. Oğuz, Example 2.14, pg 33)

(Continuous variables)

The **c.d.f.** of variable **X**:

Marginal cumulative distribution function:

$$F(x) = P(-\infty < X \leq x) = \int_{-\infty}^{x} f(u)du$$

$$F(x) = \int_{0}^{x} f(u)du = \int_{0}^{x} [2/(u+2)^2]du = 1 - [2/(x+2)]$$

Example (M. Bayazıt, B. Oğuz, Example 2.14, pg 33)

(Continuous variables)

The probability that the **cost of labour** exceeds 1 billion TL:

Marginal cumulative distribution function:

$$F(x) = \int_0^x f(u)du = \int_0^x [2/(u+2)^2]du = 1 - [2/(x+2)]$$

P(**X>1**)=1-F(1)

F(1)=1-$[2/(x+2)] = 1/3$

P(X>1)=0.67

Example (M. Bayazıt, B. Oğuz, Example 2.14, pg 33)

(Continuous variables)

The **c.d.f.** for the **conditional** distribution of **X**:

Conditional cumulative distribution function:

$$F(x|y) = P(X \leq x|Y = y) = \int_{-\infty}^{x} f(u|y)du$$

$$F(x|y) = \int_{0}^{x} f(u|y)du = \int_{0}^{x} y\exp(-uy)\,du = 1 - \exp(-xy)$$

Example (M. Bayazıt, B. Oğuz, Example 2.14, pg 33)

(Continuous variables)

The probability that the **cost of labour** exceeds 1 billion TL **in case** the cost of the material **Y=1**:

Conditional cumulative distribution function of X for a **given value of Y**:

$$F(x|y) = P(X \leq x|Y = y) = \int_{-\infty}^{x} f(u|y)du$$

$$\mathrm{F(x|y)} = \int_{0}^{\mathrm{x}} \mathrm{f(u|y)du} = \int_{0}^{\mathrm{x}} \mathrm{y\,exp(-uy)\,du} = 1 - \mathrm{exp(-xy)}$$

$\mathrm{P(X > 1|Y = 1)}$=1-$\mathrm{F(1|1)}$

$\mathrm{F(1|1)} = 1 - \exp(-1)$

$\mathrm{P(X > 1|Y = 1)}$=1-1+ $\exp(-1) = 0.37$

It is seen that the cost of the material affects the cost of labour.

Example (M. Bayazıt, B. Oğuz, Example 2.15, pg 35)

(Continuous variables)

The **joint probability density function** (p.d.f.) of the continuous variables X and Y is given as:

$$f(x,y)=\begin{cases}\dfrac{x(1+3y^2)}{4} & 0<x<2,0<y<1\\[2ex] 0 & elsewhere\end{cases}$$

Checking that the given equation is in fact a p.d.f.

It is seen that at all points *f(x,y)* $\geq$ *0 and:*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$$

$$\int\int_{-\infty}^{\infty} f(x,y)dx\,dy=\int_0^1\int_0^2\frac{x(1+3y^2)}{4}dx\,dy$$

$$=\int_0^1\left(\frac{x^2}{8}+\frac{3x^2y^2}{8}\Big|_0^2\right)dy=\int_0^1\left(\frac{1}{2}+\frac{3y^2}{2}\right)dy=\frac{y}{2}+\frac{y^3}{2}\Big|_0^1=1$$

Example (M. Bayazıt, B. Oğuz, Example 2.15, pg 35)

(Continuous variables)

Let us find the probability of the event that the random variables remain in region

$0 < x < 1$, **$1/4 < y < 1/2$** :

Joint cumulative distribution function:

$$p = \int_{1/4}^{1/2} \int_{0}^{1} \frac{x(1+3y^2)}{4} dx\, dy = \int_{1/4}^{1/2} \left(\frac{x^2}{8} + \frac{3x^2y^2}{8} \Big|_0^1 \right) dy$$

$$= \int_{1/4}^{1/2} \left(\frac{1}{8} + \frac{3y^2}{8} \right) dy = \frac{y}{8} + \frac{y^3}{8} \Big|_{1/4}^{1/2} = 23/512$$

Example (M. Bayazıt, B. Oğuz, Example 2.15, pg 35)

(Continuous variables)

Marginal probability density function of variable **X**:

$$f(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^1 \frac{x(1+3y^2)}{4}dy = \frac{xy}{4} + \frac{xy^3}{4}\Big|_0^1 = \frac{x}{2} \qquad 0<x<2$$

Marginal probability density function of variable **Y**:

$$f(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^2 \frac{x(1+3y^2)}{4}dx = \frac{x^2}{8} + \frac{3x^2y^2}{8}\Big|_0^2 = \frac{1+3y^2}{2} \qquad 0<y<1$$

Example (M. Bayazıt, B. Oğuz, Example 2.15, pg 35)

(Continuous variables)

Conditional probability density function of **X** for a **given value of Y**.

$$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{x(1+3y^2)/4}{(1+3y^2)/2} = \frac{x}{2} \qquad 0 < x < 2$$

The probability that **X** remains between 1/4 and 1/2 given that Y=1/3.

Conditional cumulative distribution functionof **X** for a **given value of Y**:

$$F(x|y) = P(X \leq x|Y = y) = \int_{-\infty}^{x} f(u|y)du$$

$$P(1/4 < X \leq 1/2|Y = 1/3) = \int_{1/4}^{1/2} \frac{x}{2} dx = 3/64$$

Example (M. Bayazıt, B. Oğuz, Example 2.15, pg 35)

(Continuous variables)

Let us check whether variables **X** and **Y** are dependent or not:

$$f(x,y)=\frac{x(1+3y^2)}{4}=\frac{x}{2}\frac{1+3y^2}{2}=f(x)f(y)$$

This equation shows that the variables X and Y are **independent**.

In that case, a condition given for Y **does not affect** the distribution of X.

$$f(x|y)=f(x)=\frac{x}{2}$$

Therefore, the **conditional probability density function** and **marginal probability density function** of X are the same.