Homework 10
CS 3385
Due March 17 at the end of class

1. Show that equation (15.4) follows from equation (15.3) and the initial condition $T(0) = 1$. This is most easily done using induction.

2. Show, by means of a counterexample, that the following "greedy" strategy does not always determine an optimal way to cut rods. Define the density of a rod of length i to be p_i/i, that is, its value per inch. The greedy strategy for a rod of length n cuts off a first piece of length i, where $1 \le i \le n$, having maximum density. It then continues by applying the greedy strategy to the remaining piece of length $n - i$. Use $n = 4$ for your counterexample. Show which cuts are made and how much revenue they result in.

3. Consider a modification of the rod-cutting problem in which, in addition to a price p_i for each rod, each cut incurs a fixed cost of c. The revenue associated with a solution is now the sum of the prices of the pieces minus the costs of making the cuts. Give a dynamic-programming algorithm to solve this modified problem.

4. Draw the recursion tree for the `mergeSort` procedure on an array of 16 elements. Explain why memoization fails to speed up a good divide-and-conquer algorithm such as `mergeSort`.

5. Suppose that in the rod-cutting problem we also have limit l_i on the number of pieces of length i that we are allowed to produce, for $i = 1, 2, \ldots, n$. Show that the optimal-substructure property no longer holds. You can show this by example.

6. Imagine that you wish to exchange one currency for another. You realize that instead of directly exchanging one currency for another, you might be better off making a series of trades through other currencies, winding up with the currency you want. Suppose that you can trade n different currencies, numbered $1, 2, \ldots, n$, where you start with currency 1 and wish to wind up with currency n. You are given, for each pair of currencies i and j, an exchange rate r_{ij}, meaning that if you start with d units of currency i, you can trade for dr_{ij} units of currency j. A sequence of trades may entail a commission, which depends on the number of trades you make. Let c_k be the commission that you are charged when you make k trades. Show that, if $c_k = 0$ for all $k = 1, 2, \ldots, n$, then the problem of finding the best sequence of exchanges from currency 1 to currency n exhibits optimal substructure. Then show that if commissions c_k are arbitrary values, then the problem of finding the best sequence of exchanges from currency 1 to currency n does not necessarily exhibit optimal substructure.

7. Consider the following two DNA sequences:

   ```
   ATCC
   ACGC
   ```

   An optimal alignment will look something like this (this is an example of the format; it is not an optimal alignment):

   ```
   A TC C
   AC GC
   ```

   Give the optimal alignment and total cost for the two sequences given the following costs:

   (a) gap insertion = 2; mismatch = 1
   (b) gap insertion = 1; mismatch = 2

8. Give an $O(n^2)$-time algorithm to find the longest monotonically (not strictly monotonically) increasing subsequence of a sequence of n numbers.