Finding The Volume of Lego Bricks Using Double Integration

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Objective

This project aims to give insight and a practical application into how double integration in a three-dimensional space can be used to find the volume of structures. The project also reinforces key concepts learned in MATH-251, such as equations containing three variables and XYZ bounds in a 3D space.

Introduction/Theory

The goal of integration in a two-dimensional space is to determine the area underneath a function/curve; however, we live in a three-dimensional world. When converting the concept of integration to a 3D space, we use double integrals. Instead of finding the area underneath a curve, double integrals can be utilized to find the volume underneath a plane. When calculating the volume of an object without calculus, we can multiply the length times the width times the height. Since the length times height is equal to the area of a plane, the formula for finding volume can also be thought of as the area of one of the surfaces times the width of the object. Similarly, when taking the integral with respect to y or x, we find the area underneath the plane. We then multiply it by the width: the value of the second integral.

In this project, we will calculate and determine the area of a Lego structure in three different ways: by counting the bricks, by integration of 1x1 pillars, and by integration of the plane function of the lamina. To begin, we construct a total of 125 1x1x1 unit Lego blocks as shown in Figure 1. We then stack these blocks to create pillars of different heights as shown in Figure 2. Finally, we build a 5x5x5 unit 3D structure as shown in Figure 3, Figure 4, and Figure 5. We can imagine this structure as living in the first octant of a three-dimensional Cartesian coordinate system with the tallest pillar closest to the origin. After building the Lego structure, we label each pillar as shown in Figure 6 and use double integration to determine the volume of that pillar. utilize our knowledge gained in MATH-251 to determine the correct equation of the plane covering the structure and use an online 3D calculator to verify the equation.

Figure 1

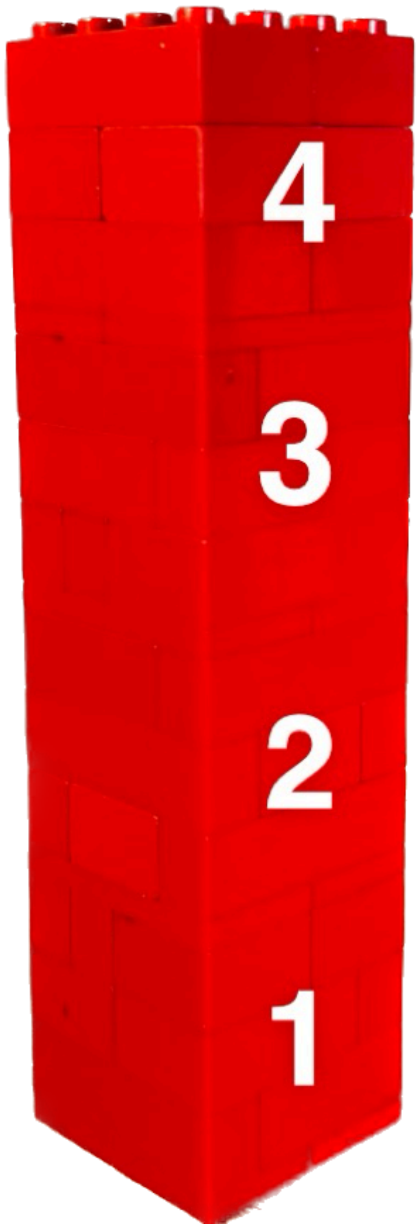


Figure 2

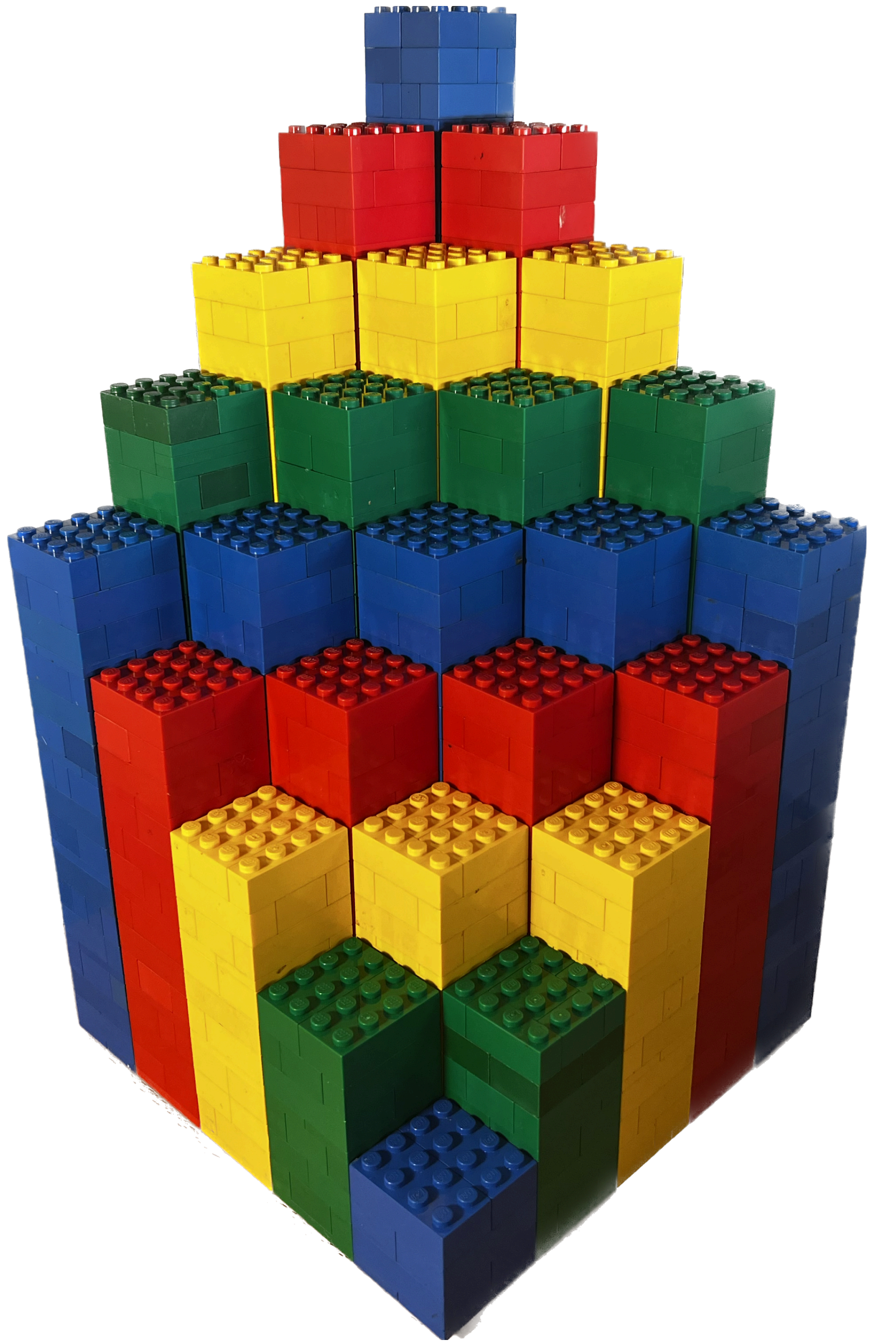


Figure 3

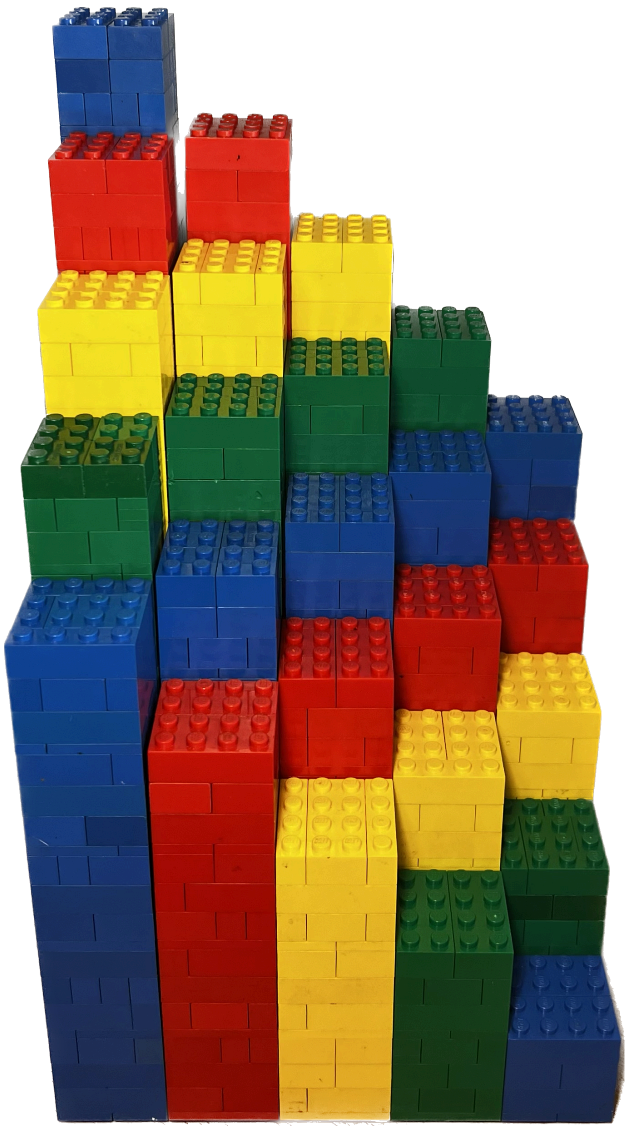
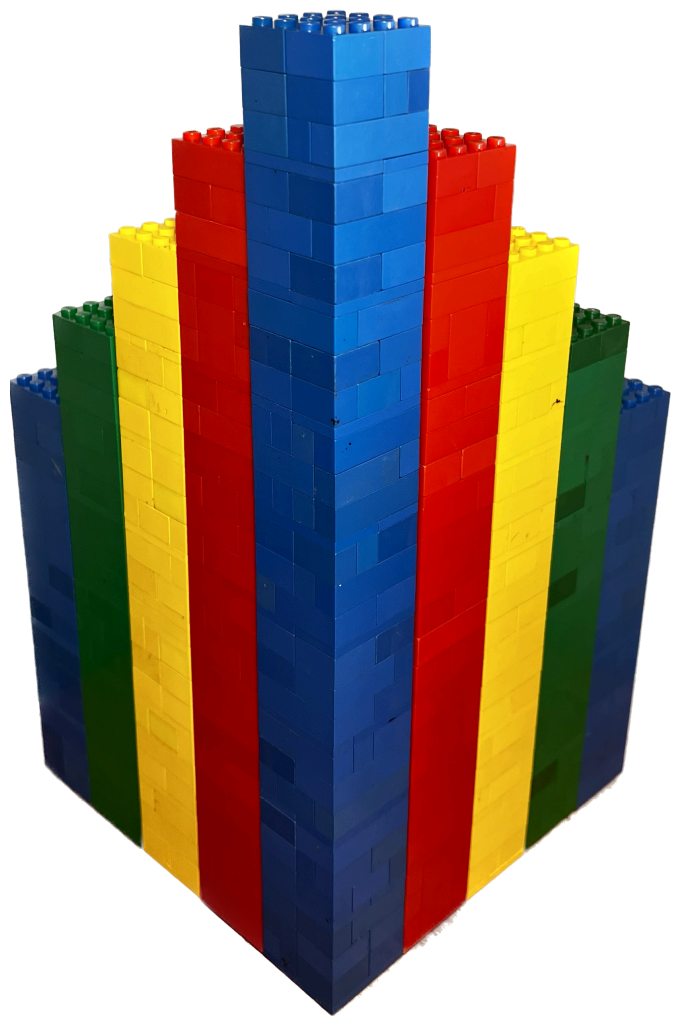


Figure 4

Figure 5



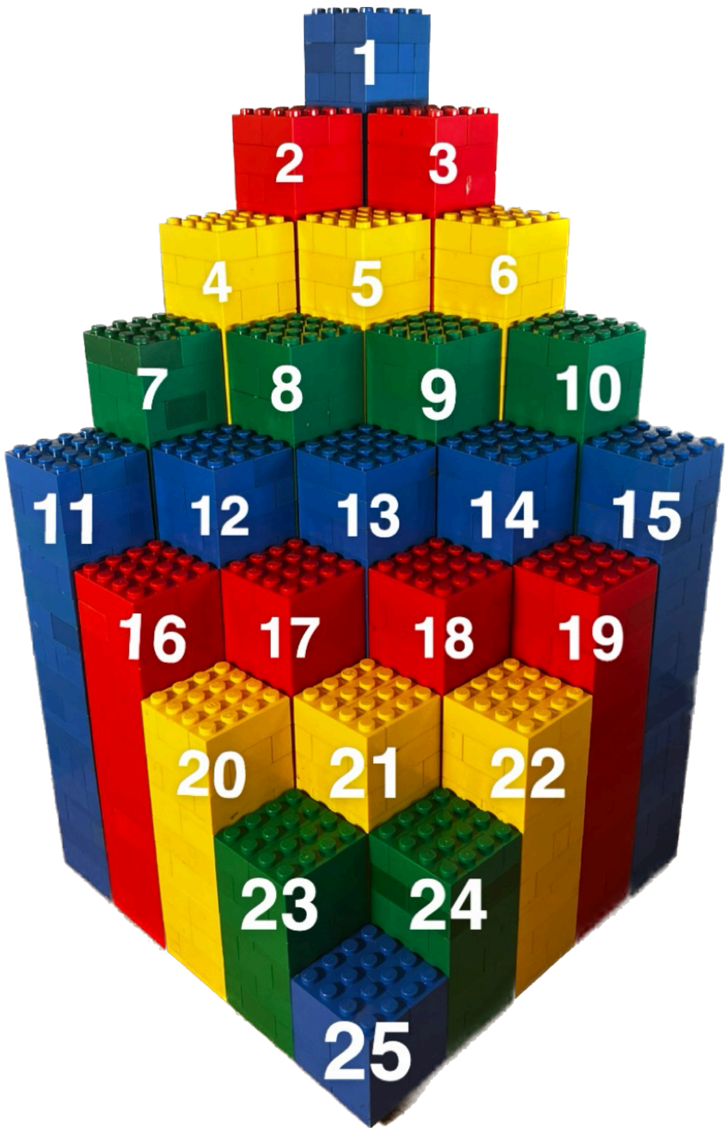


Figure 6 Each pillar is numbered to keep track of its volume and corresponding integral.

<u>Results</u>

Pillar Integrals

1. $\int_{0}^{1}\int_{0}^{1} 9 \, dy \, dx = 9\int_{0}^{1} y \, dx = 9\int_{0}^{1} (1 - 0) \, dx = 9\int_{0}^{1} 1dx = 9x = 9(1 - 0) = \mathbf{9 \ units^3}$
2. $\int_{1}^{2}\int_{0}^{1} 8 \, dy \, dx = 8\int_{1}^{2} y \, dx = 8\int_{1}^{2} (1 - 0) \, dx = 9\int_{1}^{2} 1dx = 8x = 8(2 - 1) = \mathbf{8 \ units^3}$

3. $\int_0^1\int_1^2 8\,dy\,dx = 8\int_0^1 y\,dx = 8\int_0^1 (2-1)\,dx = 8\int_0^1 1dx = 8x = 8(1-0) = \mathbf{8\ units^3}$

4. $\int_2^3\int_0^1 7\,dy\,dx = 7\int_2^3 y\,dx = 7\int_2^3 (1-0)\,dx = 7\int_2^3 1dx = 7x = 7(3-2) = \mathbf{7\ units^3}$

5. $\int_1^2\int_1^2 7\,dy\,dx = 7\int_1^2 y\,dx = 7\int_1^2 (2-1)\,dx = 7\int_1^2 1dx = 7x = 7(2-1) = \mathbf{7\ units^3}$

6. $\int_0^1\int_2^3 7\,dy\,dx = 7\int_0^1 y\,dx = 7\int_0^1 (3-2)\,dx = 7\int_0^1 1dx = 7x = 7(1-0) = \mathbf{7\ units^3}$

7. $\int_3^4\int_0^1 6\,dy\,dx = 6\int_3^4 y\,dx = 6\int_3^4 (1-0)\,dx = 6\int_3^4 1dx = 6x = 6(4-3) = \mathbf{6\ units^3}$

8. $\int_2^3\int_1^2 6\,dy\,dx = 6\int_2^3 y\,dx = 6\int_2^3 (2-1)\,dx = 6\int_2^3 1dx = 6x = 6(3-2) = \mathbf{6\ units^3}$

9. $\int_1^2\int_2^3 6\,dy\,dx = 6\int_1^2 y\,dx = 6\int_1^2 (3-2)\,dx = 6\int_1^2 1dx = 6x = 6(2-1) = \mathbf{6\ units^3}$

10. $\int_0^1\int_3^4 6\,dy\,dx = 6\int_0^1 y\,dx = 6\int_0^1 (4-3)\,dx = 6\int_0^1 1dx = 6x = 6(1-0) = \mathbf{6\ units^3}$

11. $\int_4^5\int_0^1 5\,dy\,dx = 5\int_4^5 y\,dx = 5\int_4^5 (1-0)\,dx = 5\int_4^5 1dx = 5x = 5(5-4) = \mathbf{5\ units^3}$

12. $\int_3^4\int_1^2 5\,dy\,dx = 5\int_3^4 y\,dx = 5\int_3^4 (2-1)\,dx = 5\int_3^4 1dx = 5x = 5(4-3) = \mathbf{5\ units^3}$

13. $\int_2^3\int_2^3 5\,dy\,dx = 5\int_2^3 y\,dx = 5\int_2^3 (3-2)\,dx = 5\int_2^3 1dx = 5x = 5(3-2) = \mathbf{5\ units^3}$

14. $\int_1^2\int_3^4 5\,dy\,dx = 5\int_1^2 y\,dx = 5\int_1^2 (4-3)\,dx = 5\int_1^2 1dx = 5x = 5(2-1) = \mathbf{5\ units^3}$

15. $\int_0^1\int_4^5 5\,dy\,dx = 5\int_0^1 y\,dx = 5\int_0^1 (5-4)\,dx = 5\int_0^1 1dx = 5x = 5(1-0) = \mathbf{5\ units^3}$

16. $\int_4^5\int_1^2 4\,dy\,dx = 4\int_4^5 y\,dx = 4\int_4^5 (2-1)\,dx = 4\int_4^5 1dx = 4x = 4(5-4) = \mathbf{4\ units^3}$

17. $\int_3^4\int_2^3 4\,dy\,dx = 4\int_3^4 y\,dx = 4\int_3^4 (3-2)\,dx = 4\int_3^4 1dx = 4x = 4(4-3) = \mathbf{4\ units^3}$

18. $\int_2^3 \int_3^4 4 \, dy \, dx = 4\int_2^3 y \, dx = 4 \int_2^3 (4 - 3) \, dx = 4 \int_2^3 1dx = 4x = 4(3 - 2) = \mathbf{4 \text{ units}^3}$

19. $\int_1^2 \int_4^5 4 \, dy \, dx = 4\int_1^2 y \, dx = 4 \int_1^2 (5 - 4) \, dx = 5 \int_1^2 1dx = 4x = 4(2 - 1) = \mathbf{4 \text{ units}^3}$

20. $\int_4^5 \int_2^3 3 \, dy \, dx = 3\int_4^5 y \, dx = 3 \int_4^5 (3 - 2) \, dx = 3 \int_4^5 1dx = 3x = 3(5 - 4) = \mathbf{3 \text{ units}^3}$

21. $\int_3^4 \int_3^4 3 \, dy \, dx = 3\int_3^4 y \, dx = 3 \int_3^4 (4 - 3) \, dx = 3 \int_3^4 1dx = 3x = 3(4 - 3) = \mathbf{3 \text{ units}^3}$

22. $\int_2^3 \int_4^5 3 \, dy \, dx = 3\int_2^3 y \, dx = 3 \int_2^3 (5 - 4) \, dx = 3 \int_2^3 1dx = 3x = 3(3 - 2) = \mathbf{3 \text{ units}^3}$

23. $\int_4^5 \int_3^4 2 \, dy \, dx = 2\int_4^5 y \, dx = 2 \int_4^5 (4 - 3) \, dx = 2 \int_4^5 1dx = 2x = 2(5 - 4) = \mathbf{2 \text{ units}^3}$

24. $\int_3^4 \int_4^5 2 \, dy \, dx = 2\int_3^4 y \, dx = 2 \int_3^4 (5 - 4) \, dx = 2 \int_3^4 1dx = 2x = 2(4 - 3) = \mathbf{2 \text{ units}^3}$

25. $\int_4^5 \int_4^5 1 \, dy \, dx = 1\int_4^5 y \, dx = 1 \int_4^5 (4 - 3) \, dx = 1 \int_4^5 1dx = 1x = 1(5 - 4) = \mathbf{1 \text{ unit}^3}$

After adding all the pillars' volumes/integral values, we get a total of **125 units^3**

Plane Equation Integral:

1. $\int_0^5 \int_0^5 (10 - x - y) \, dy \, dx = \int_0^5 (10y - xy - \frac{y^2}{2}) \, dx = \int_0^5 (37.5 - 5x) \, dx = 37.5x - \frac{5x^2}{2}$

$= 37.5(5) - 5(25)/2 = \mathbf{125 \text{ units}^3}$

The area determined by counting each 1x1x1 Lego block is **125 units^3**.

Discussion and Conclusions

After determining the volume by counting the unit blocks, solving 25 double integrals, and solving an integral that represents the entire structure, it is evident that all methods are accurate. However, when one cannot count the unit blocks below a function, using double integrals is a prominent method. Breaking apart the volume between a function and plane into smaller pillars and continuing that process for each pillar encompasses the embodiment of integral calculus: the summation of infinitely small elements to define a whole function.

This project successfully illustrated how double integration can be employed to determine the volume of three-dimensional structures in real-world applications. On top of functioning as an engaging and fun exploration of Calculus III concepts, the project was also a great thought experiment regarding how adding infinitely tiny segments is the basis of integration and calculus.