Data simulation and calculation of betas

```
library(knitr)
library(kableExtra)
library(cowplot)
library(ggforce)
```

```
## Loading required package: ggplot2
```

```
library(latex2exp)
library(reshape2)
library(dplyr)
```

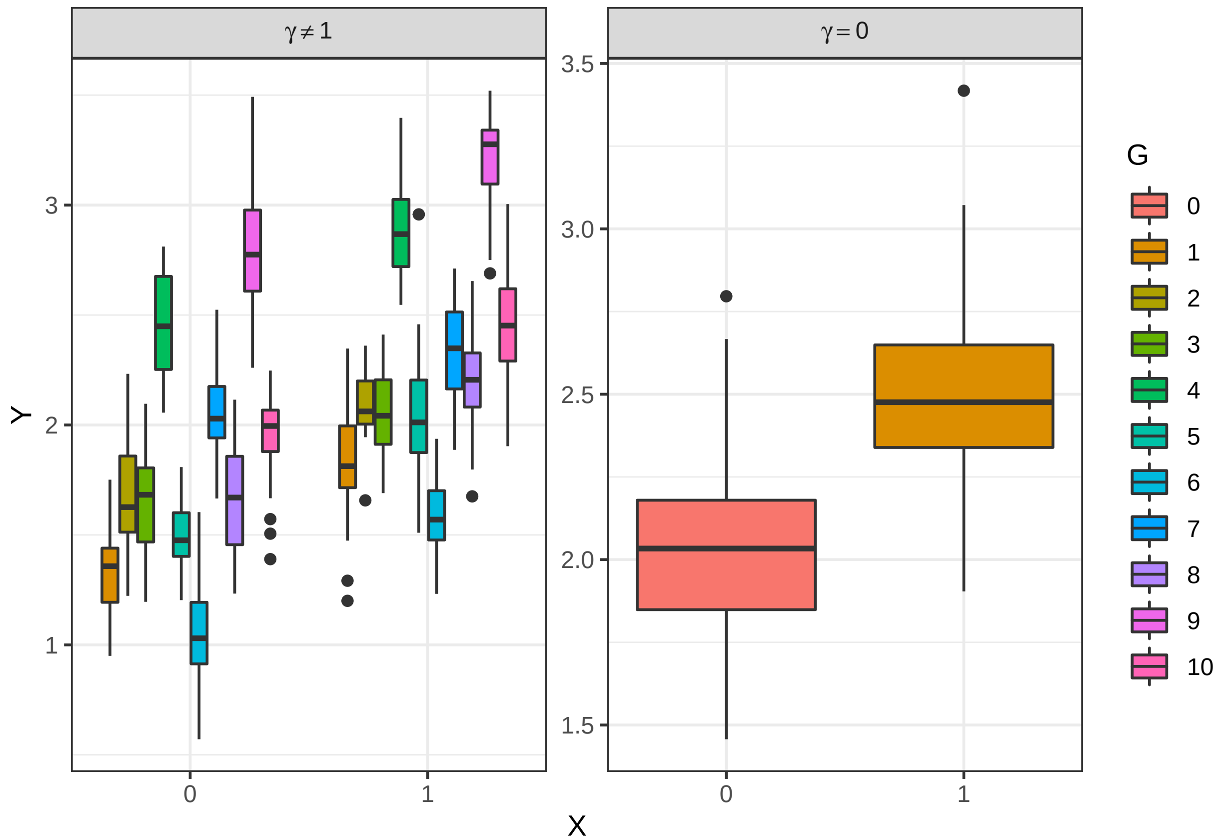
```
##
## Attaching package: 'dplyr'

## The following object is masked from 'package:kableExtra':
##
##     group_rows

## The following objects are masked from 'package:stats':
##
##     filter, lag

## The following objects are masked from 'package:base':
##
##     intersect, setdiff, setequal, union
```

```
knitr::opts_chunk$set(cache = FALSE, warning = FALSE, message = FALSE, cache.lazy = FALSE)
```

Simulation data

```
K <- 10 # Nombre de groupe
nK <- 50 # Nombre d'observations par groupe
N <- K * nK # Nombre total d'observation
G <- factor(rep(1:K, each = nK))
intercept <- 2
fixefEffect <- .5
aleaEffect <- rnorm(K, sd = .5)
bias <- rnorm(N, sd = .25)
X <- rbinom(N,size=1,prob = .5)
# X_gauss <- rnorm(N,1,2)
```

```
Y_withoutAlea <- intercept + fixefEffect * X + bias
# Y_G <- intercept + fixefEffect * X_gauss + aleaEffect[G]  + bias
Y <- intercept + fixefEffect * X + aleaEffect[G] + bias
dfWith <- data.frame(X, Y, G)
dfWithout <- data.frame(X, Y=Y_withoutAlea, G)
H0 <- formula(Y ~ X)
H1 <- formula(Y ~ X + (1|G))

dataPlot = cbind.data.frame(X=rep(factor(X),2), Y = c(Y, Y_withoutAlea),
                            G = factor(c(G,X),levels = 0:10),
                            Effect = factor(rep(c("gamma != 1", "gamma == 0"),
                                                each=length(X))))
ggplot(dataPlot) +
  geom_boxplot(aes(x=X, y=Y, fill = G)) +
  scale_x_discrete(expand = c(0, 0.5)) +
  theme_bw()+
  ggforce::facet_row(vars(Effect), scales = 'free', space = 'free',
                     labeller = "label_parsed")
```



Linear model with lm function

```
options(warn=-1)
X_prime <- cbind(1, X)
```

```r
# With Random Effect
lmModel <- lm(H0,data = dfWith)
Sigm <- sqrt(((summary(lmModel)$sigma)**2)*solve(t(X) %*% X))
betaLm <- lmModel$coefficients
SE <- (betaLm[-1] - confint(lmModel)[-1,][1])/1.96

A <- ggplot(dfWith, aes(x = X, y = Y, color = G) ) +
      geom_point() +
      geom_smooth(formula = as.formula(y ~ x), method = "lm", se=.3,aes(fill = G))+
  theme_bw()+theme(legend.position="none")+
  annotate(geom='text', label=TeX("$\\gamma \\neq 0$"), y=3.7, x=.5)


# Without Random Effect
lmModel <- lm(H0,data = dfWithout)
Sigm.1 <- sqrt(((summary(lmModel)$sigma)**2)*solve(t(X) %*% X))
betaLm.1 <- lmModel$coefficients
SE.1 <- (betaLm.1[-1] - confint(lmModel)[-1,][1])/1.96

B <- ggplot(dfWithout, aes(x = X, y = Y, color = G) ) +
      geom_point() +
      geom_smooth(formula = as.formula(y ~ x), method = "lm",se = .3, aes(fill = G))+
  theme_bw()+theme(axis.title.y = element_blank())+
  annotate(geom='text', label=TeX("$\\gamma=0$"), y=3, x=.5)
cowplot::plot_grid(A, B, labels = c('',''))
```

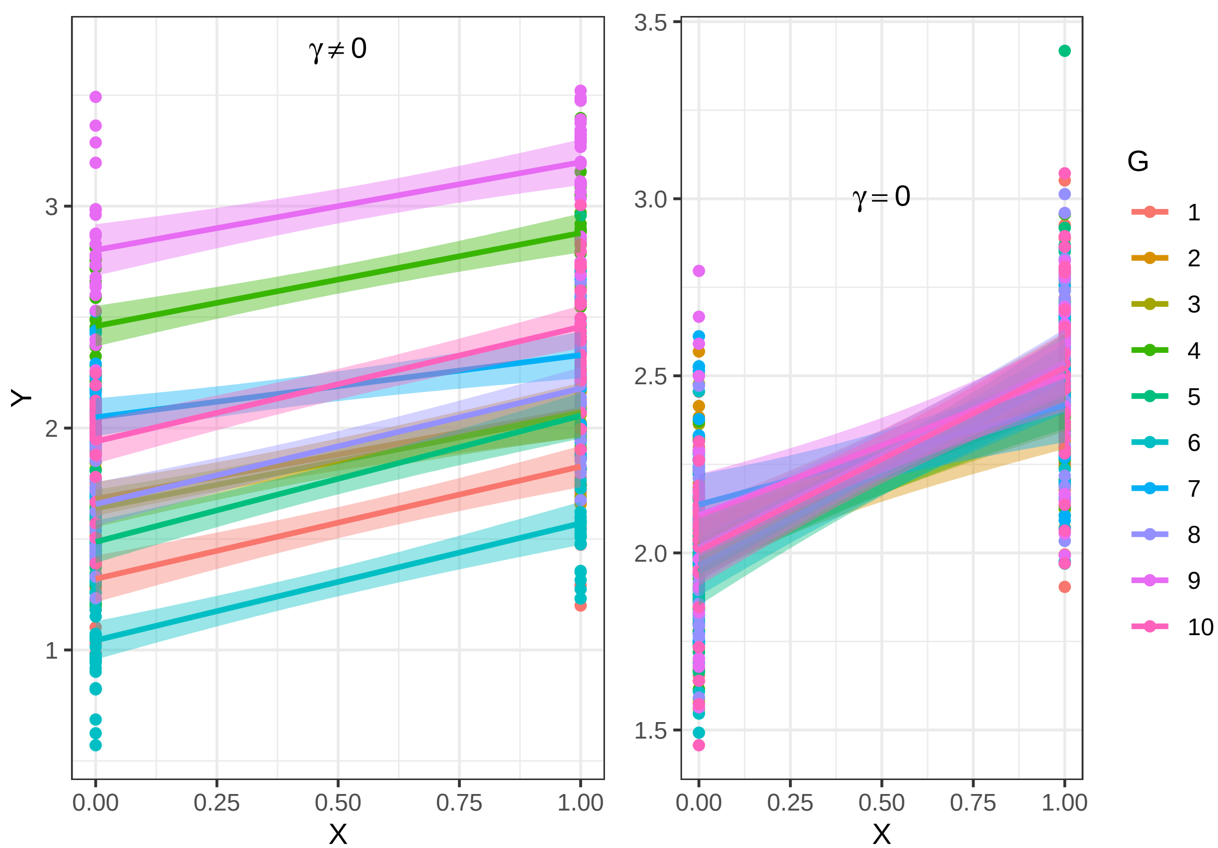


Table 1: Linear regression estimates with `lm()`

	$\hat{\beta}$	$SE(\hat{\beta})$	$\sigma(\hat{\beta})$
$\gamma \neq 0$	0.5111	0.0479	0.0349
$\gamma = 0$	0.4628	0.0216	0.0157

Mixed model with lmer function

```
options(warn=-1)
# With Random Effect
lmerModel <- lme4::lmer(H1, data=dfWith)
betaLmer <- lme4::fixef(lmerModel)
SE <- sqrt(diag(as.matrix(vcov(lmerModel))))[-1]
Sigm <- sqrt(stats::var(as.vector(betaLmer %*% t(X))))
A <- ggplot(dfWith, aes(x=X, y=Y, colour=G)) +
    geom_point(size=1.5) +
    geom_line(aes(y=predict(lmerModel), group=G), size=1.3) +
      theme_bw()+theme(legend.position="none")+
  annotate(geom='text', label=TeX("$\\gamma \\neq 0$"), y=3.7, x=.5)
# Without Random Effect
lmerModel.1 <- lme4::lmer(H1, data=dfWithout)
betaLmer.1 <- lme4::fixef(lmerModel.1)
SE.1 <- sqrt(diag(as.matrix(vcov(lmerModel.1))))[-1]
Sigm.1 <- sqrt(stats::var(as.vector(betaLmer.1 %*% t(X))))

B <- ggplot(dfWithout, aes(x=X, y=Y, colour=G)) +
    geom_point(size=1.5) +
    geom_line(aes(y=predict(lmerModel.1), group=G), size=1.3) +
  theme_bw()+theme(axis.title.y = element_blank())+
  annotate(geom='text', label=TeX("$\\gamma=0$"), y=3, x=.5)
cowplot::plot_grid(A, B, labels = c('',''))
```

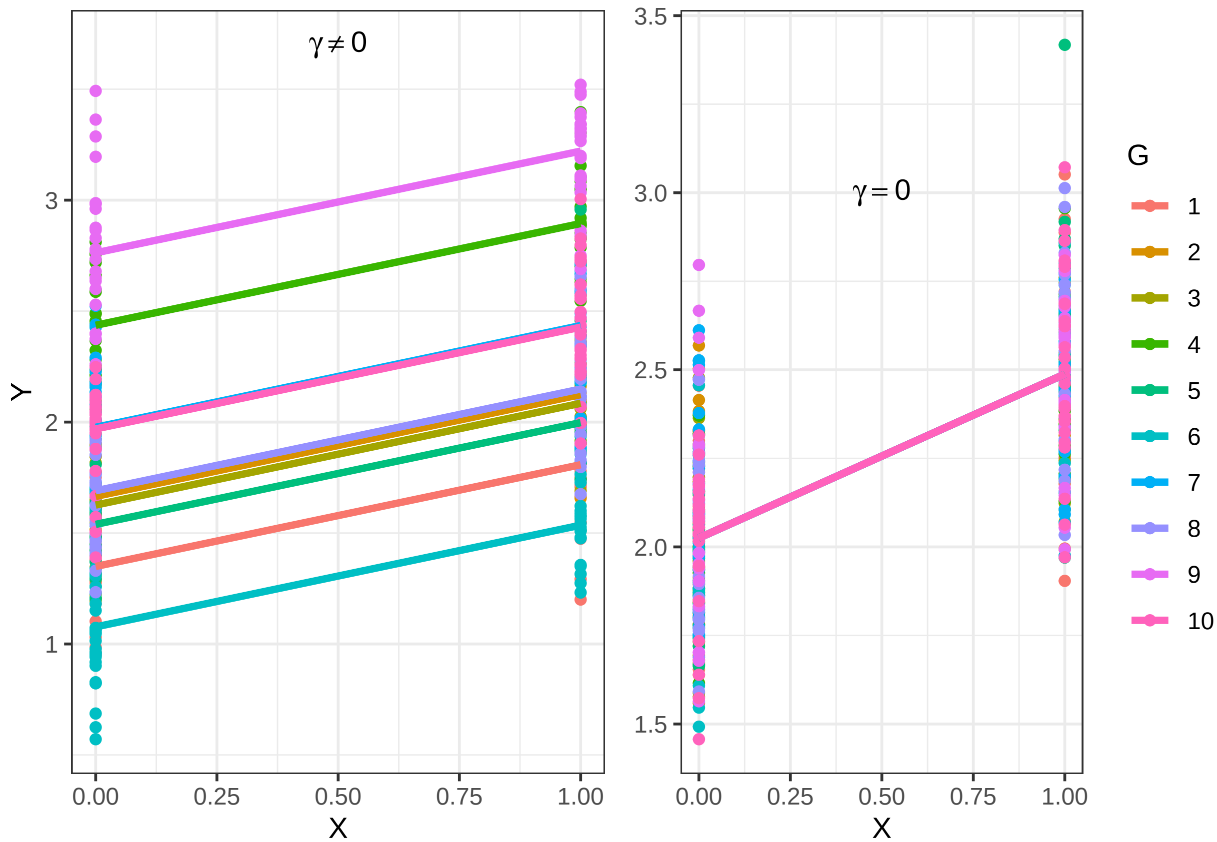


Table 2: Mixed Model estimates with `lmer()`

	$\hat{\beta}$	$SE(\hat{\beta})$	$\sigma(\hat{\beta})$
$\gamma \neq 0$	0.4585	0.0219	0.73
$\gamma = 0$	0.4628	0.0215	0.8186

OLS

```
OLS <- function(Y, X){
  modelmat <- model.matrix(~.,cbind.data.frame(X=X))
  indexes_X <- which(substring(colnames(modelmat), 1, 1) == "X")
  modX_OLS <- modelmat[, c(1, indexes_X), drop = FALSE]
  Y <- as.numeric(Y)
  betaOLS <- solve(crossprod(modX_OLS))%*%(t(modX_OLS)%*%Y)
  k <- ncol(modX_OLS)
  n <- nrow(modX_OLS)

  residuals <- as.matrix(Y - (betaOLS[1, , drop=FALSE]) - X * betaOLS[indexes_X, ,
                                                                   drop=FALSE])
  RSS <- as.numeric(t(residuals)%*%residuals)
  Sigma2 <- as.numeric(RSS/(n-k))
```

```
  Vb <- Sigma2*solve(t(X)%*% X)
  Sigm <- sqrt(Vb)
  OLSCOV <- 1/(n-k) * as.numeric(t(residuals)%*%residuals) * solve(t(modX_OLS)%*%modX_OLS)
  SE <- sqrt(diag(OLSCOV))[-1]
  return(list('betaOLS'=betaOLS[indexes_X, ,drop=FALSE], 'SE'=SE,'Sigm'=Sigm))
}
options(warn=-1)
# With Random Effect
res <- OLS(dfWith$Y,X)
# Without Random Effect
res.1 <- OLS(dfWithout$Y,X)
```

Table 3: OLS estimates

	$\hat{\beta}$	$SE(\hat{\beta})$	$\sigma(\hat{\beta})$
$\gamma \neq 0$	0.5111	0.0477	0.73
$\gamma = 0$	0.4628	0.0215	0.8186

Monte Carlo

Simulating data

The following represents a simulate a random intercepts model obtenaid 500 Monte-Carlo replicates. The method follows the following simulation framework from gaussian distributions with or without the presence of random effects:

$$Y = \beta_0 + \beta X + \gamma G + \epsilon \tag{1}$$

with $\beta_0 = 2$, $\beta \in \{0.5, 2, 5, 10\}$ the fixed effect of $X \sim \mathcal{B}\binom{n}{0.5}$ with $n = 500$ the number total of samples, $\epsilon \sim \mathcal{N}(0, 0.25)$ the bias associated, the random effect of group $\gamma \sim \mathcal{N}(0, \sigma_\gamma)$ if simulated with random effect with $\sigma_\gamma \in \{0.5, 5, 10, 20\}$) and $\gamma = 0$ if not random effect and the group $G \in \{1, \ldots, K\}$ with $K = 10$.

Running the Models

```
simOLS <- function (intercept=2, fixefEffects = c(.5,2,5,10), sds = c(.5,5,10,20),
                    K=10, nK=50, sims=5000){
  resAllOLS.1 <- resAllOLS.2 <- resAllMM.1 <- list()
  estimateAll <- matrix(0, length(fixefEffects)*length(sds)*sims, 12)
  yWithSimMean <- yWithoutSimMean <- matrix(0, 500, length(fixefEffects)*length(sds))
  n <- K * nK
  G <- factor(rep(1:K, each = nK))
  X <- rbinom(n,size=1,prob = .5)
  k = 1
  kk = 1
  for (fixefEffect in fixefEffects) {
    for (sdUnit in sds) {
        aleaEffect <- rnorm(K, sd = sdUnit)
```

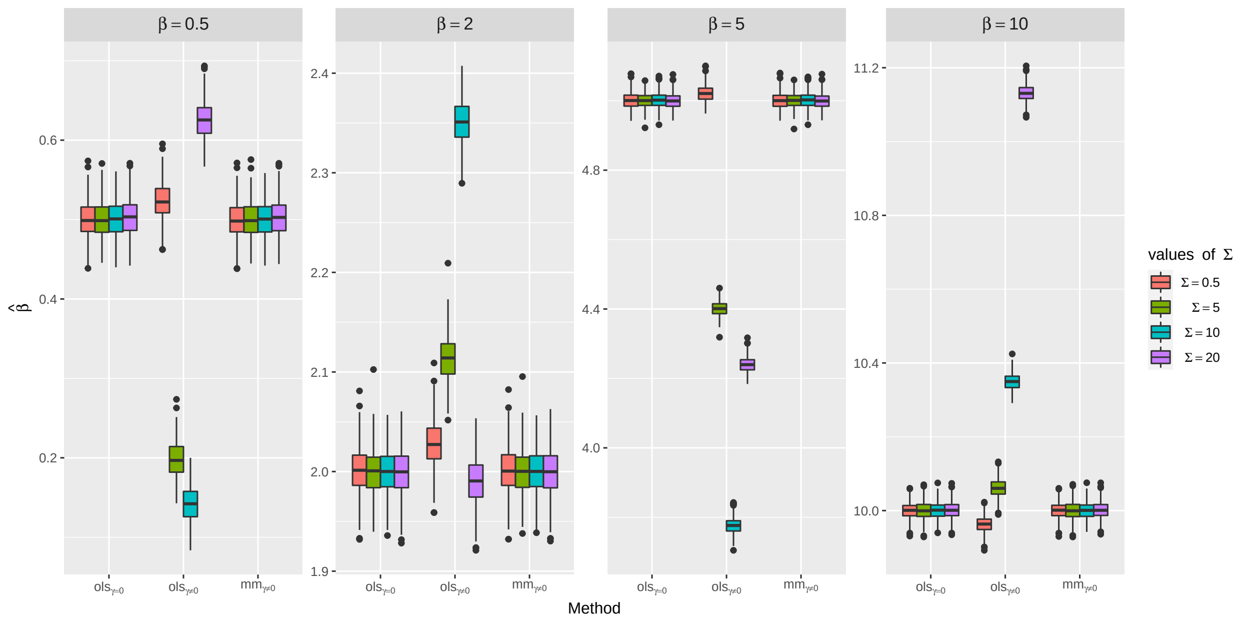
```
      resOLS.1 <- resOLS.2 <- resMM <- matrix(0, sims, 3)
      yOlsWith <- yOlsWithout <- matrix(0, 500, sims)
      for(i in 1:sims){
        options(warn=-1)
        bias <- rnorm(n, sd = .25)
        Y_with <- intercept + fixefEffect * X + aleaEffect[G] + bias
        Y_without <- intercept + fixefEffect * X + bias
        modOLS.1 <- OLS(Y_with,X)
        modOLS.2 <- OLS(Y_without,X)
        resOLS.1[i,] <- c(modOLS.1$betaOLS, modOLS.1$SE, modOLS.1$Sigm)
        resOLS.2[i,] <- c(modOLS.2$betaOLS, modOLS.2$SE, modOLS.2$Sigm)
        lmerModel.1 <- try({ lme4::lmer(Y_with ~ X + (1|G), REML = TRUE)}, silent = T)
        beta <- lme4::fixef(lmerModel.1)[-1]
        SE_b <- sqrt(diag(as.matrix(vcov(lmerModel.1))))[-1]
        Sigm <- sqrt(stats::var(as.vector(beta %*% t(X))))
        resMM[i,] <- c(beta, SE_b,Sigm)
        estimateAll[k,] <- c(fixefEffect, modOLS.1$betaOLS,modOLS.2$betaOLS,beta,
                         modOLS.1$SE,modOLS.2$SE,SE_b,
                         modOLS.1$Sigm,modOLS.2$Sigm,Sigm,sdUnit,i)
        yOlsWith[,i] <- Y_with; yOlsWithout[,i] <- Y_without
        k <- k+1
      }
      yWithSimMean[,kk] <- rowMeans(yOlsWith)
      yWithoutSimMean[,kk] <- rowMeans(yOlsWithout)
      kk <- kk + 1

      M <- apply(resOLS.1, 2, mean)
      S <- apply(resOLS.1, 2, sd)
      H_ols.1 <- matrix(c(M[1], S[1], M[2], S[2], M[3], S[3]), ncol = 2, byrow = TRUE)

      M <- apply(resOLS.2, 2, mean)
      S <- apply(resOLS.2, 2, sd)
      H_ols.2 <- matrix(c(M[1], S[1], M[2], S[2], M[3], S[3]), ncol = 2, byrow = TRUE)

      M <- apply(resMM, 2, mean)
      S <- apply(resMM, 2, sd)
      H_mm <- matrix(c(M[1], S[1], M[2], S[2], M[3], S[3]), ncol = 2, byrow = TRUE)

      dimnames(H_ols.1) <- dimnames(H_mm) <-
        dimnames(H_ols.2)  <- list( c('Beta','Standard Error', 'Sigm'), c('mean', 'se'))

      resAllOLS.1[[as.character(paste0(fixefEffect,"_",sdUnit))]] <- H_ols.1
      resAllOLS.2[[as.character(paste0(fixefEffect,"_",sdUnit))]] <- H_ols.2
      resAllMM.1[[as.character(paste0(fixefEffect,"_",sdUnit))]] <- H_mm
  }
}
estimateAll <- as.data.frame(estimateAll)
yWithSimMean <- as.data.frame(yWithSimMean)
yWithoutSimMean <- as.data.frame(yWithoutSimMean)
yWithSimMean <- cbind.data.frame(X,G,yWithSimMean)
yWithoutSimMean <- cbind.data.frame(X,G,yWithoutSimMean)
colnames(estimateAll) = c('fixefEffect','betaOLS.1','betaOLS.2','betaMM',
                          'seOLS.1','seOLS.2','seMM','sigmOLS.1',
```

```
                              'sigmOLS.2','sigmMM','sdUnit','simId')
  return (list('resAllOLS'=resAllOLS.1, 'resAllMM'=resAllMM.1,
               'resAllOLSWithoutAleaEffect'=resAllOLS.2,
               "estimateAll"=estimateAll,"yWithSimMean"=yWithSimMean,
               'yWithoutSimMean'=yWithoutSimMean))
}

resSim <- simOLS(sims = 500)

dataPLOT <- cbind.data.frame(betaEs = c(resSim$estimateAll[[2]],resSim$estimateAll[[3]],
                                        resSim$estimateAll[[4]]),
                             betas = factor(rep(resSim$estimateAll[[1]],3)),
                             Sigm = factor(rep(resSim$estimateAll[[11]],3)),
                             Method = as.factor(rep(c('OLS.1','OLS.2','MM'),
                                                    each=length(resSim$estimateAll[[1]]))))
```

Results

Plot estimation of β

```
dataPLOT$Bias = rep(resSim$estimateAll[[1]],3)-dataPLOT$betaEs

dataPLOT$betas <- gsub(0.5, "beta == 0.5",
           gsub(2, "beta == 2",
           gsub(10, "beta == 10",dataPLOT$betas)))
dataPLOT[dataPLOT$betas==5,'betas']="beta == 5"
dataPLOT$betas = factor(dataPLOT$betas,levels = c("beta == 0.5","beta == 2",
                                                  "beta == 5","beta == 10"))

dataPLOT$Sigm <- gsub(0.5, "Sigma == 0.5",
           gsub(10, "Sigma == 10",
           gsub(20, "Sigma == 20",dataPLOT$Sigm)))
dataPLOT[dataPLOT$Sigm==5,'Sigm']="Sigma == 5"
dataPLOT$Sigm = factor(dataPLOT$Sigm,levels = c("Sigma == 0.5","Sigma == 5",
                                                "Sigma == 10","Sigma == 20"))

ggplot(dataPLOT, aes(Method,betaEs))+
geom_boxplot(position="dodge",aes(fill=Sigm))+
facet_wrap(~betas, scales = "free_y",labeller = label_parsed,ncol = 4)+
  ylab(TeX("$\\hat{\\beta}$"))+
  scale_x_discrete(labels=c(TeX("$ols_{\\gamma=0}$"),TeX("$ols_{\\gamma\\neq 0}$"),
                            TeX("$mm_{\\gamma\\neq 0}$")))+
  theme(strip.text.x = element_text(size=12, face="bold"),
        strip.text.y = element_text(size=12, face="bold",)) +
  scale_fill_discrete(name=TeX("$values\\ of\\ \\Sigma$"),
                      labels=c(TeX("$\\Sigma=.5$"),TeX("$\\Sigma=5$"),
                         TeX("$\\Sigma=10$"),TeX("$\\Sigma=20$")))
```



Plot bias of etimates β

```
ggplot(data=dataPLOT, aes(x=Method, y=Bias))+
  geom_boxplot()+
  theme_bw()+
  facet_grid(betas~Sigm, scales = "free_y",labeller = label_parsed)+
  theme(strip.text.x = element_text(size=12, face="bold"),
        strip.text.y = element_text(size=12, face="bold",)) +
  ylab(TeX("$\\beta-\\hat{\\beta}$"))+
  scale_x_discrete(labels=c(TeX("$ols_{\\gamma=0}$"),TeX("$ols_{\\gamma\\neq 0}$"),
                            TeX("$mm_{\\gamma\\neq 0}$")))
```

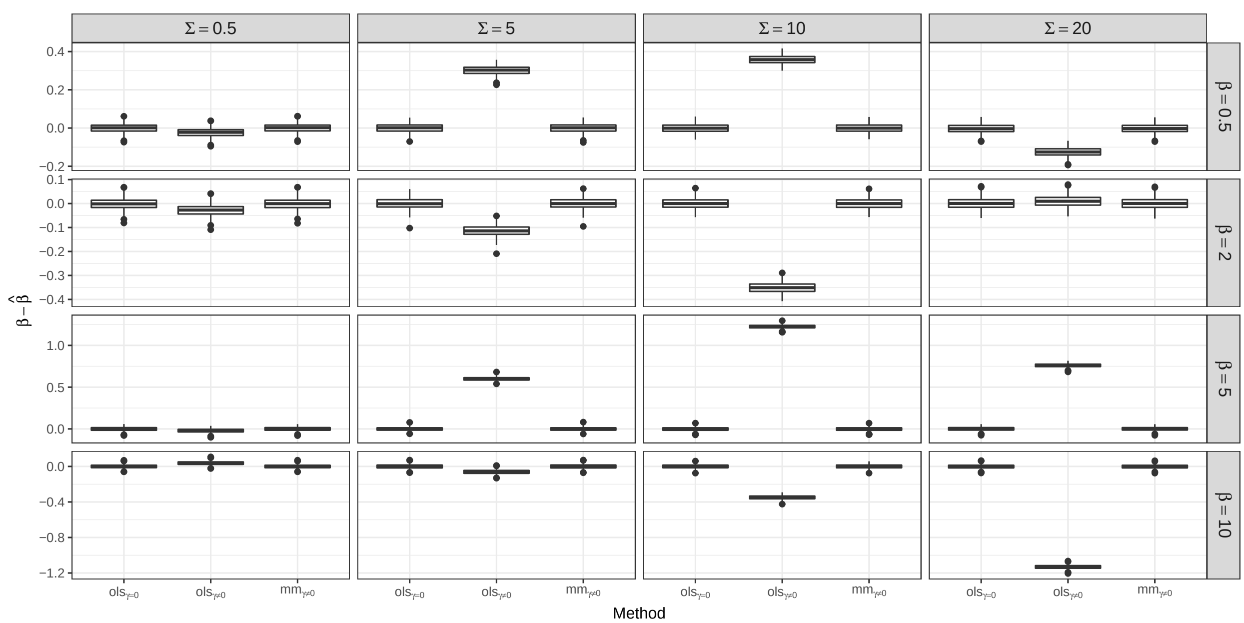


Table 4: OLS and Mixed Model estimates with 500 Monte-Carlo replicates

				$\mathbf{\Sigma = 0.5}$		$\mathbf{\Sigma = 5}$		$\mathbf{\Sigma = 10}$		$\mathbf{\Sigma = 20}$	
$\boldsymbol{\beta}$	Model	γ	Estimate	μ	SE	μ	SE	μ	SE	μ	SE
$\boldsymbol{\beta = 0.5}$	OLS	$\gamma \neq 0$	$\hat{\beta}$	0.5239	0.0225	0.198	0.0224	0.1422	0.0222	0.6252	0.0225
			$\boldsymbol{SE(\hat{\beta})}$	0.0463	0.001	0.4502	0.001	0.9211	0.001	1.9281	0.001
			$\boldsymbol{\sigma(\hat{\beta})}$	0.0334	7×10^{-4}	0.3247	7×10^{-4}	0.6642	7×10^{-4}	1.3903	7×10^{-4}
		$\gamma = 0$	$\hat{\beta}$	0.4998	0.0225	0.4999	0.0224	0.5007	0.0222	0.5025	0.0225
			$\boldsymbol{SE(\hat{\beta})}$	0.0224	7×10^{-4}	0.0223	7×10^{-4}	0.0224	7×10^{-4}	0.0224	7×10^{-4}
			$\boldsymbol{\sigma(\hat{\beta})}$	0.0161	5×10^{-4}	0.0161	5×10^{-4}	0.0161	5×10^{-4}	0.0161	5×10^{-4}
	MM	$\gamma \neq 0$	$\hat{\beta}$	0.5001	0.0225	0.4999	0.0223	0.5007	0.0224	0.5025	0.0227
			$\boldsymbol{SE(\hat{\beta})}$	0.0224	7×10^{-4}	0.0224	7×10^{-4}	0.0224	7×10^{-4}	0.0225	7×10^{-4}
			$\boldsymbol{\sigma(\hat{\beta})}$	0.2501	0.0113	0.25	0.0111	0.2504	0.0112	0.2513	0.0113
$\boldsymbol{\beta = 2}$	OLS	$\gamma \neq 0$	$\hat{\beta}$	2.0273	0.0228	2.1144	0.0216	2.351	0.0229	1.9903	0.0227
			$\boldsymbol{SE(\hat{\beta})}$	0.0318	8×10^{-4}	0.3584	0.0011	0.9513	0.001	1.1313	9×10^{-4}
			$\boldsymbol{\sigma(\hat{\beta})}$	0.0229	6×10^{-4}	0.2585	8×10^{-4}	0.686	7×10^{-4}	0.8158	7×10^{-4}
		$\gamma = 0$	$\hat{\beta}$	2.0006	0.0228	2.0005	0.0216	2.0001	0.0229	1.9996	0.0227
			$\boldsymbol{SE(\hat{\beta})}$	0.0225	7×10^{-4}	0.0224	7×10^{-4}	0.0224	7×10^{-4}	0.0224	7×10^{-4}
			$\boldsymbol{\sigma(\hat{\beta})}$	0.0162	5×10^{-4}	0.0161	5×10^{-4}	0.0161	5×10^{-4}	0.0161	5×10^{-4}
	MM	$\gamma \neq 0$	$\hat{\beta}$	2.001	0.0228	2.0004	0.0216	1.9999	0.0228	1.9996	0.0227
			$\boldsymbol{SE(\hat{\beta})}$	0.0225	7×10^{-4}	0.0225	7×10^{-4}	0.0225	7×10^{-4}	0.0225	7×10^{-4}
			$\boldsymbol{\sigma(\hat{\beta})}$	1.0007	0.0114	1.0004	0.0108	1.0001	0.0114	1	0.0114
$\boldsymbol{\beta = 5}$	OLS	$\gamma \neq 0$	$\hat{\beta}$	5.0209	0.0228	4.4006	0.0209	3.776	0.0225	4.24	0.022
			$\boldsymbol{SE(\hat{\beta})}$	0.0644	9×10^{-4}	0.7705	0.001	1.2872	0.001	1.8525	0.0011
			$\boldsymbol{\sigma(\hat{\beta})}$	0.0465	7×10^{-4}	0.5556	7×10^{-4}	0.9282	7×10^{-4}	1.3358	8×10^{-4}
		$\gamma = 0$	$\hat{\beta}$	5.0003	0.0228	5.0005	0.0209	5.002	0.0225	4.9995	0.022
			$\boldsymbol{SE(\hat{\beta})}$	0.0224	7×10^{-4}	0.0224	7×10^{-4}	0.0224	7×10^{-4}	0.0223	7×10^{-4}
			$\boldsymbol{\sigma(\hat{\beta})}$	0.0161	5×10^{-4}	0.0162	5×10^{-4}	0.0161	5×10^{-4}	0.0161	5×10^{-4}
	MM	$\gamma \neq 0$	$\hat{\beta}$	5.0003	0.0229	5.0003	0.0209	5.0019	0.0225	4.9994	0.0221
			$\boldsymbol{SE(\hat{\beta})}$	0.0225	7×10^{-4}	0.0225	8×10^{-4}	0.0224	7×10^{-4}	0.0224	7×10^{-4}
			$\boldsymbol{\sigma(\hat{\beta})}$	2.5006	0.0114	2.5006	0.0105	2.5015	0.0113	2.5002	0.011
$\boldsymbol{\beta = 10}$	OLS	$\gamma \neq 0$	$\hat{\beta}$	9.963	0.0217	10.0608	0.0235	10.3486	0.0219	11.1312	0.0225
			$\boldsymbol{SE(\hat{\beta})}$	0.0453	9×10^{-4}	0.4076	0.001	0.5963	0.001	1.9602	0.0011
			$\boldsymbol{\sigma(\hat{\beta})}$	0.0327	7×10^{-4}	0.2939	7×10^{-4}	0.43	7×10^{-4}	1.4136	8×10^{-4}
		$\gamma = 0$	$\hat{\beta}$	10.0006	0.0217	9.9998	0.0235	9.9996	0.0219	10.0013	0.0225
			$\boldsymbol{SE(\hat{\beta})}$	0.0224	7×10^{-4}	0.0224	7×10^{-4}	0.0224	7×10^{-4}	0.0224	7×10^{-4}
			$\boldsymbol{\sigma(\hat{\beta})}$	0.0161	5×10^{-4}	0.0161	5×10^{-4}	0.0161	5×10^{-4}	0.0162	5×10^{-4}
	MM	$\gamma \neq 0$	$\hat{\beta}$	10.0002	0.0218	9.9998	0.0235	9.9996	0.022	10.0012	0.0225
			$\boldsymbol{SE(\hat{\beta})}$	0.0224	7×10^{-4}	0.0225	7×10^{-4}	0.0224	7×10^{-4}	0.0225	7×10^{-4}
			$\boldsymbol{\sigma(\hat{\beta})}$	5.0011	0.0109	5.0009	0.0118	5.0008	0.011	5.0016	0.0113

Running models on the average of the simulations

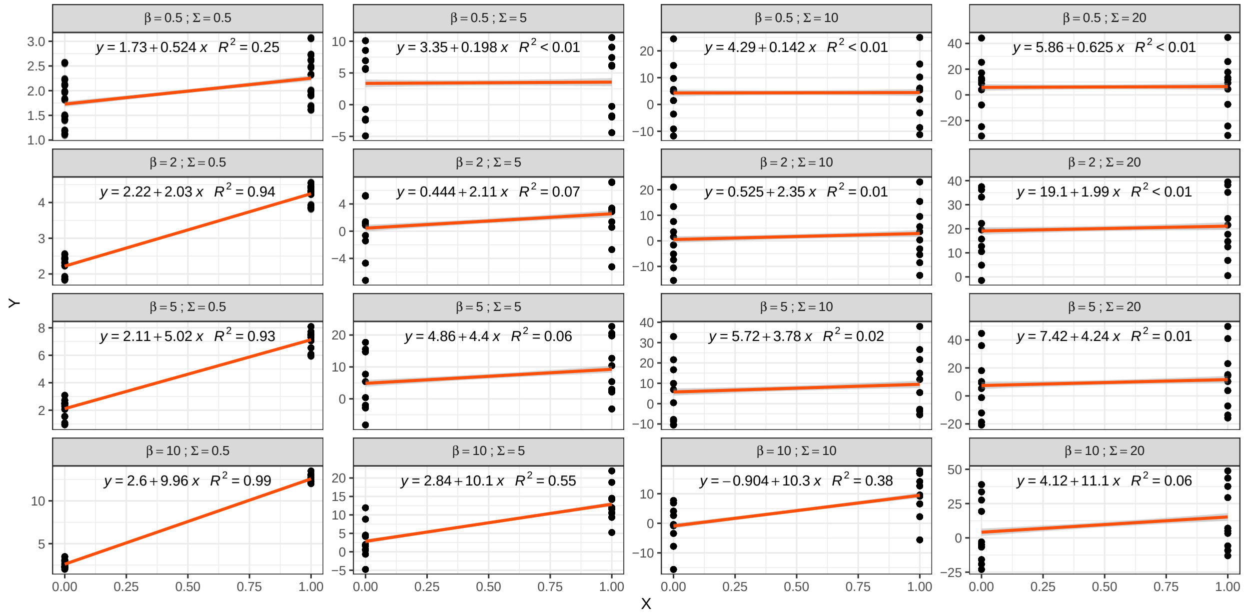
```
options(warn=-1)
colnames(resSim$yWithSimMean) <- colnames(resSim$yWithoutSimMean)<-c('X','G',
                                                                      names(resSim$resAllOLS))
resMeanSim <- matrix(0,nrow = length(colnames(resSim$yWithSimMean[,-c(1:2)])),9)
X <- resSim$yWithSimMean$X
G <- resSim$yWithSimMean$G
k=1
for (i in colnames(resSim$yWithSimMean[,-c(1:2)])) {
  lmerModel <- try({ lme4::lmer(H1,
                                data = cbind.data.frame(Y=resSim$yWithSimMean[,i],
                                                        X=X,
                                                        G=G),
                                REML = TRUE)}, silent = T)
  beta <- lme4::fixef(lmerModel)[-1]
  SE <- sqrt(diag(as.matrix(vcov(lmerModel))))[-1]
  Sigm <- sqrt(stats::var(as.vector(beta %*% t(X))))
  resMeanSim[k,] <- c(unlist(OLS(resSim$yWithSimMean[,i], X)),
                      unlist(OLS(resSim$yWithoutSimMean[,i], X)),
                      beta,SE,Sigm)
  k <- k+1
}
resMeanSim <- as.data.frame(resMeanSim)
```

Table 5: OLS and Mixed Model estimates with 500 Monte-Carlo replicates

		$\boldsymbol{OLS_{\gamma\neq 0}}$			$\boldsymbol{OLS_{\gamma=0}}$			$\boldsymbol{MM_{\gamma\neq 0}}$		
$\boldsymbol{\beta}$	$\boldsymbol{\Sigma}$	$\hat{\boldsymbol{\beta}}$	$\boldsymbol{SE}(\hat{\boldsymbol{\beta}})$	$\boldsymbol{\sigma}(\hat{\boldsymbol{\beta}})$	$\hat{\boldsymbol{\beta}}$	$\boldsymbol{SE}(\hat{\boldsymbol{\beta}})$	$\boldsymbol{\sigma}(\hat{\boldsymbol{\beta}})$	$\hat{\boldsymbol{\beta}}$	$\boldsymbol{SE}(\hat{\boldsymbol{\beta}})$	$\boldsymbol{\sigma}(\hat{\boldsymbol{\beta}})$
$\beta = 0.5$	$\Sigma = 0.5$	0.5239	0.0406	0.0293	0.4998	0.001	7×10^{-4}	0.4999	0.001	0.25
	$\Sigma = 5$	0.198	0.4497	0.3243	0.4999	0.001	7×10^{-4}	0.5	0.001	0.25
	$\Sigma = 10$	0.1422	0.9208	0.664	0.5007	0.001	7×10^{-4}	0.5007	0.001	0.2504
	$\Sigma = 20$	0.6252	1.9279	1.3902	0.5025	0.001	7×10^{-4}	0.5025	0.001	0.2513
$\beta = 2$	$\Sigma = 0.5$	2.0273	0.0225	0.0162	2.0006	0.0011	8×10^{-4}	2.0005	0.0011	1.0005
	$\Sigma = 5$	2.1144	0.3577	0.258	2.0005	0.001	7×10^{-4}	2.0004	0.001	1.0004
	$\Sigma = 10$	2.351	0.9511	0.6858	2.0001	9×10^{-4}	7×10^{-4}	1.9999	9×10^{-4}	1.0001
	$\Sigma = 20$	1.9903	1.1311	0.8156	1.9996	0.001	7×10^{-4}	1.9996	0.001	1
$\beta = 5$	$\Sigma = 0.5$	5.0209	0.0604	0.0436	5.0003	0.001	7×10^{-4}	5.0002	0.001	2.5006
	$\Sigma = 5$	4.4006	0.7701	0.5553	5.0005	0.001	7×10^{-4}	5.0003	0.001	2.5006
	$\Sigma = 10$	3.776	1.287	0.9281	5.002	0.001	7×10^{-4}	5.0019	0.001	2.5015
	$\Sigma = 20$	4.24	1.8524	1.3358	4.9995	9×10^{-4}	7×10^{-4}	4.9995	0.001	2.5002
$\beta = 10$	$\Sigma = 0.5$	9.963	0.0394	0.0284	10.0006	0.001	7×10^{-4}	10.0004	0.001	5.0012
	$\Sigma = 5$	10.0608	0.407	0.2935	9.9998	0.001	7×10^{-4}	9.9998	0.001	5.0009
	$\Sigma = 10$	10.3486	0.5959	0.4297	9.9996	0.001	7×10^{-4}	9.9996	0.001	5.0008
	$\Sigma = 20$	11.1312	1.9601	1.4135	10.0013	0.001	7×10^{-4}	10.0012	0.001	5.0016

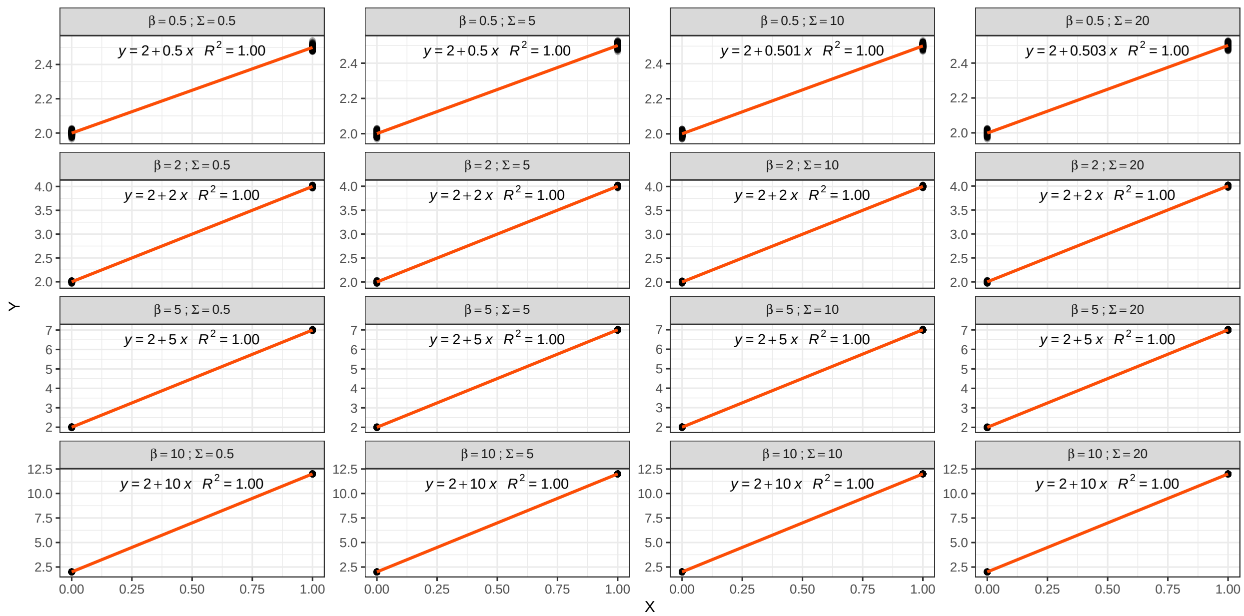
Visualization of the models

```
dataMod <- melt(resSim$yWithSimMean, id.vars=c("X","G"),value.name = "Y")
dataMod$variable = recode_factor(dataMod$variable,
              "0.5_0.5"="beta==0.5 ~ ';' ~ Sigma == 0.5",
              "0.5_5"="beta == 0.5 ~ ';' ~ Sigma == 5","0.5_10"="beta == 0.5 ~ ';' ~ Sigma == 10",
              "0.5_20"="beta == 0.5 ~ ';' ~ Sigma == 20",
              "2_0.5"="beta == 2 ~ ';' ~ Sigma == 0.5", "2_5"="beta == 2 ~ ';' ~ Sigma == 5",
              "2_10"="beta == 2 ~ ';' ~ Sigma == 10",
              "2_20"="beta == 2 ~ ';' ~ Sigma == 20","5_0.5"="beta == 5 ~ ';' ~ Sigma == 0.5",
              "5_5"="beta == 5 ~ ';' ~ Sigma == 5","5_10"="beta == 5 ~ ';' ~ Sigma == 10",
              "5_20"="beta == 5 ~ ';' ~ Sigma == 20",
              "10_0.5"="beta == 10 ~ ';' ~ Sigma == 0.5",
              "10_5"="beta == 10 ~ ';' ~ Sigma == 5","10_10"="beta == 10 ~ ';' ~ Sigma == 10",
              "10_20"="beta == 10 ~ ';' ~ Sigma == 20")

ggplot(data = dataMod, aes(x = X, y = Y)) +
   geom_point(aes(X, Y), alpha = 0.3)+
     geom_smooth(formula = as.formula(y~x),aes(x = X, y = Y),
                method = "lm", colour="#FC4E07", fullrange = TRUE, se = TRUE)+
  ggpmisc::stat_poly_eq(formula = as.formula(y~x),
             aes(label=paste(..eq.label.., ..rr.label.., sep = "~~~")),
             parse = TRUE, label.x.npc = "center", size = 3.45)+
  theme(strip.text.x = element_text(size=12, face="bold"),
strip.text.y = element_text(size=12, face="bold",))+theme_bw()+
  facet_wrap(~ variable, ncol=4,  scales = "free_y",labeller = label_parsed)
```

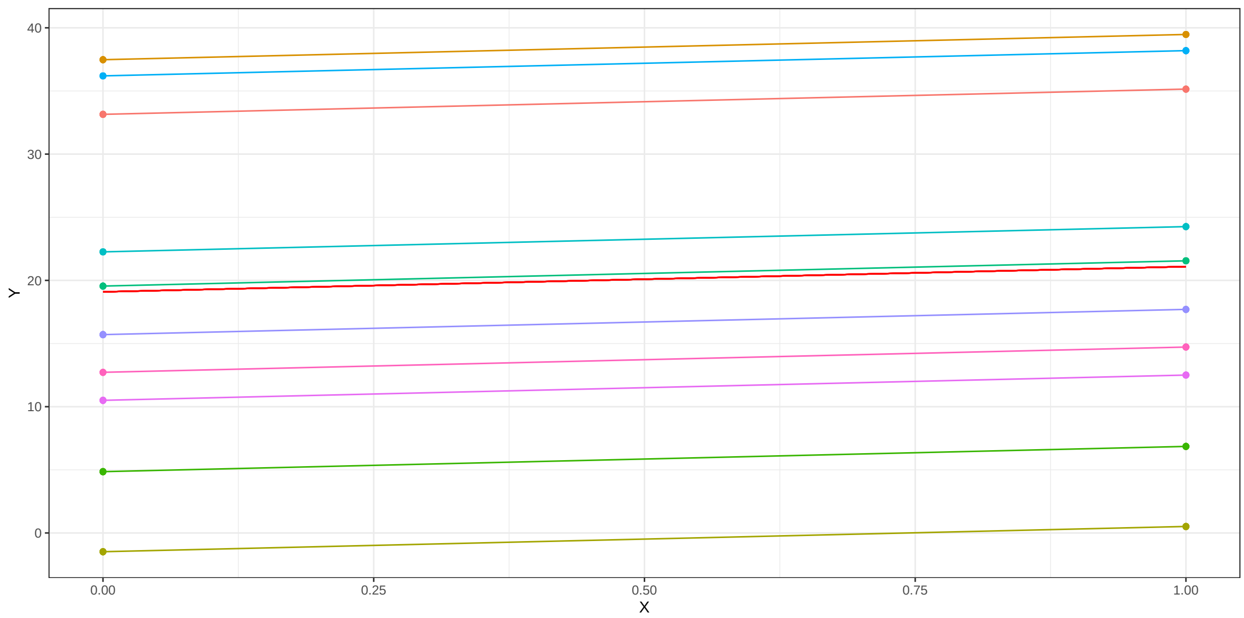



```
dataMod <- melt(resSim$yWithoutSimMean, id.vars=c("X","G"),value.name = "Y")
dataMod$variable = recode_factor(dataMod$variable,
              "0.5_0.5"="beta == 0.5 ~ ';' ~ Sigma == 0.5",
              "0.5_5"="beta == 0.5 ~ ';' ~ Sigma == 5","0.5_10"="beta == 0.5 ~ ';' ~ Sigma == 10",
              "0.5_20"="beta == 0.5 ~ ';' ~ Sigma == 20",
              "2_0.5"="beta == 2 ~ ';' ~ Sigma == 0.5", "2_5"="beta == 2 ~ ';' ~ Sigma == 5",
              "2_10"="beta == 2 ~ ';' ~ Sigma == 10",
              "2_20"="beta == 2 ~ ';' ~ Sigma == 20","5_0.5"="beta == 5 ~ ';' ~ Sigma == 0.5",
              "5_5"="beta == 5 ~ ';' ~ Sigma == 5","5_10"="beta == 5 ~ ';' ~ Sigma == 10",
              "5_20"="beta == 5 ~ ';' ~ Sigma == 20",
              "10_0.5"="beta == 10 ~ ';' ~ Sigma == 0.5",
              "10_5"="beta == 10 ~ ';' ~ Sigma == 5","10_10"="beta == 10 ~ ';' ~ Sigma == 10",
              "10_20"="beta == 10 ~ ';' ~ Sigma == 20")


ggplot(data = dataMod, aes(x = X, y = Y)) +
   geom_point(aes(X, Y), alpha = 0.3)+
     geom_smooth(formula = as.formula(y~x),aes(x = X, y = Y),
                 method = "lm", colour="#FC4E07", fullrange = TRUE, se = TRUE)+
  ggpmisc::stat_poly_eq(formula = as.formula(y~x),
             aes(label=paste(..eq.label.., ..rr.label.., sep = "~~~")),
```

```
              parse = TRUE, label.x.npc = "center", size = 3.45)+
  theme(strip.text.x = element_text(size=12, face="bold"),
strip.text.y = element_text(size=12, face="bold",))+theme_bw()+
  facet_wrap(~ variable, ncol=4,  scales = "free_y",labeller = label_parsed)
```

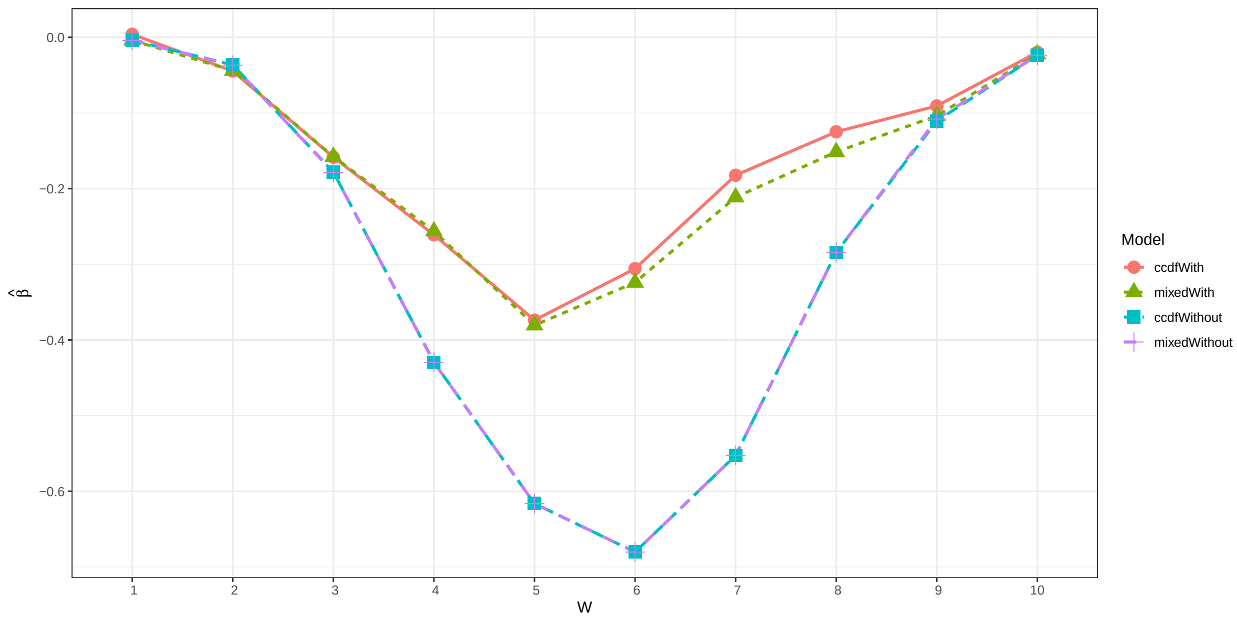



```
colnames(resSim$yWithSimMean)[10]="Y"
lmerModel <- lme4::lmer(H1, data=resSim$yWithSimMean)
ab_lines <- coef(lmerModel)[["G"]] %>%
  tibble::rownames_to_column("G") %>%
  rename(intercept = `(Intercept)`)
ab_lines$G <- factor(ab_lines$G, levels=1:K)
ggplot(resSim$yWithSimMean,aes(x = X, y = Y, colour=G)) +
  geom_point() +
    geom_line(aes(y=predict(lmerModel), group=G)) +
  geom_line(colour="red",aes(y=predict(lmerModel,re.form=NA),group=G))+
      theme_bw()+theme(legend.position="none")
```



CCDF application

```
ccdfRes <- data.frame(ccdfWith =c(0.004 ,-0.0446, -0.1584, -0.2611, -0.3738, -0.3058, -0.1823,-0.1249, -
                      mixedWith =c(-0.004 ,-0.044, -0.1577, -0.2563, -0.3806, -0.3241, -0.2114, -0.1515
                      ccdfWithout = c(-0.004, -0.03644, -0.1781, -0.4295, -0.6163, -0.6804, -0.5527, -0
                      mixedWithout = c(-0.0041, -0.0369, -0.1785, -0.4295, -0.6163,-0.6804, -0.5527, -0
                      indBeta = 1:10)


ccdfRes <- melt(ccdfRes, id.vars="indBeta",value.name = "Estimate",variable.name = "Model")



 ggplot(ccdfRes, aes(indBeta, Estimate, group=Model, colour=Model)) +
     geom_line(aes(linetype=Model), size=1) +
  geom_point(aes(shape=Model),size=4)+
   scale_x_discrete(limits=1:10)+
   labs(x = TeX("$W$"),y=TeX("$\\hat{\\beta}$"))+
  theme(strip.text.x = element_text(size=12, face="bold"),
        strip.text.y = element_text(size=12, face="bold"))+
   theme_bw()
```



0.0
−0.2
−0.4
−0.6
$\hat{\beta}$
1
2
3
4
5
6
7
8
9
10
W
Model
ccdfWith
mixedWith
ccdfWithout
mixedWithout