NTNU
Norwegian University of Science and Technology

TBA4236 Theoretical Geomatics

Assignment 2

Geoid Height Modelling and National Height Systems

Peder Undheim, Tobias Andresen, Knut A. Lilleaas & Morten S. Edvardsen

March 23, 2024

Contents

1 Introduction

In this assignment we are going to produce a local geoid model fitted to a selection of provided measured points scattered around the Trondheim area. This will be done in 2 by utilizing the Least Squares Method to find a polynomial that represents the geoid surface. The heights of the points were measured with two different techniques, levelling and total station, and to account for the difference in accuracy between the two, we will assign different weights based on the measurement method. Furthermore, in 3, we will briefly discuss the national height systems in Norway.

2 Part A - Geoid height modeling

The calculations for this assignment were mainly done in Python. The source code can be found at: Geoid-Height-Modelling-and-National-Height-Systems.

2.1 Estimate the coefficients of the geoid model

To start off this assignment, we used the point HAVSTEIN as a local origin by setting its northing and easting to 0. This point was selected due to its beneficial central placement within the study area. Subsequently, we then adjusted the coordinates of the remaining points relative to the local origin established at HAVSTEIN. In this initial phase, our objective is to estimate the coefficients of the following polynomial for geoid height:

$$N = AX^2 + BXY + CY^2 + DX + EY + F \tag{1}$$

To solve this equation, we are using LSM.

$$\mathbf{X} = (\mathbf{A}^T\mathbf{P}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{P}\mathbf{F} \tag{2}$$

In which $\mathbf{A}$ is the design matrix for the geoid polynomial, $\mathbf{P}$ is the weight matrix, and $\mathbf{F}$ is the calculated geoid height matrix. We build the weight matrix $\mathbf{P}$ by assigning the weight w_i for each observation, where w_i is

$$w_i = \begin{cases} 4 & \text{if method is levelling} \\ 1 & \text{if method is total station} \end{cases} \tag{3}$$

And the calculated geoid heights are

$$N_i = h_{\text{ellipsoid},i} - h_{\text{orthometric},i} \tag{4}$$

This ultimately gives us the coefficient matrix $\mathbf{X}$, with the following resulting values:

$$\mathbf{X} = \begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} -5.9942 \times 10^{-10} \\ -8.1184 \times 10^{-10} \\ 8.3976 \times 10^{-11} \\ -3.9194 \times 10^{-5} \\ -3.0776 \times 10^{-5} \\ 39.6374 \times 10^{0} \end{pmatrix} \tag{5}$$

2.2 Compute the standard deviations of the coefficients

Now we are interested in finding the standard deviations of our estimated coefficients in the polynomial geoid model. To to this we want to compute the variance-covariance matrix, given by

$$\mathbf{C_{xx}} = \sigma^2(\mathbf{A}^T\mathbf{P}\mathbf{A})^{-1} \tag{6}$$

Where $\boldsymbol{\sigma}^2$, the variance of the residuals obtained from the least squares adjustment from above, is

$$\sigma^2 = \frac{\mathbf{V}^T\mathbf{P}\mathbf{V}}{n-e} \tag{7}$$

In the equation above, $\boldsymbol{n}$ and $\boldsymbol{e}$ represent the number of observations and the number of parameters (Coefficients), while $\boldsymbol{V}$ represents verbesserung given by the following equation:

$$\mathbf{V} = \mathbf{AX} - \mathbf{F} \tag{8}$$

From the output of the variance-covariance matrix C_{xx} derived in 6, we calculated the standard deviations for each coefficient by taking the square root of the diagonal, yielding:

$$\begin{pmatrix} \sigma_A \\ \sigma_B \\ \sigma_C \\ \sigma_D \\ \sigma_E \\ \sigma_F \end{pmatrix} = \begin{pmatrix} 2.357831377283972 \times 10^{-10} \\ 2.1523119255361066 \times 10^{-10} \\ 9.514435163334293 \times 10^{-11} \\ 1.3866869288409758 \times 10^{-6} \\ 7.948973492673901 \times 10^{-7} \\ 0.005479856463201371 \times 10^{1} \end{pmatrix} \tag{9}$$

2.3 Significance of the coefficients

Since the residuals of the regression model are assumed to be normally distributed, the test statistic for evaluating the significance of each regression coefficient is distributed according to Student's t-distribution. As we have 21 observations and 6 different coefficients we have a total of $\nu = 21 - 6 = 15$ degrees of freedom. To determine the significance of each coefficient with a certainty of 0.95 ($\alpha = 0.05$) we use the following condition:

$$\left| \frac{X_i}{\mathrm{SD}_i} \right| > t_{\frac{\alpha}{2},\nu} = t_{0.025,15} = 2.131 \tag{10}$$

Note that the standard deviations for each coefficient is taking the square root of the diagonal of C_{xx}. Calculating for each coefficient i, we get:

$$\begin{aligned} A:&\quad \text{Significant with value } 2.5423, \\ B:&\quad \text{Significant with value } 3.7720, \\ C:&\quad \text{NOT significant with value } 0.8826, \\ D:&\quad \text{Significant with value } 28.2642, \\ E:&\quad \text{Significant with value } 38.7172, \\ F:&\quad \text{Significant with value } 7233.2987. \end{aligned}$$

As C is deemed not significant, we reevaluate our model without the C parameter.

$$\mathbf{X} = \begin{pmatrix} A \\ B \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} -6.0149 \times 10^{-10} \\ -7.1133 \times 10^{-10} \\ -3.8741 \times 10^{-5} \\ -3.1114 \times 10^{-5} \\ 3.9639 \times 10^{1} \end{pmatrix} \tag{11}$$

Which ultimately gives us the polynomial for estimated geoid height:

$$N = -6.0149 \times 10^{-10} X^2 - 7.1133 \times 10^{-10} XY - 3.8741 \times 10^{-5} X - 3.1114 \times 10^{-5} Y + 39.6389 \tag{12}$$

2.4 Deflection of the vertical

To compute the pair of deflections of the vertical at two of the points in the Appendix 1, we calculated the derivatives of the model polynomial for both x and y. For these, we used our new model (N) without the C-coefficient, that we in the previous task found to be insignificant.

$$N = AX^2 + BXY + DX + EY + F \tag{13}$$

$$\frac{\partial N}{\partial X} = 2AX + BY + D \tag{14}$$

$$\frac{\partial N}{\partial Y} = BX + E \tag{15}$$

Furthermore, we used the formulas for the vertical deflection, and calculated the deflection in north- and east-direction for our two points. We chose to use the coordinates of the point MOHOLT and SJET.

$$\xi = -\frac{\partial N}{\partial X} \qquad \eta = -\frac{\partial N}{\partial Y} \tag{16}$$

	ξ	η
MOHOLT	2.61080348	1.99417952
SJET	1.96348488	1.68130489

Table 1: Deflections of verticals [milligon]

2.5 Calculation of heights with model and finding the standard deviations

First we estimated the geoid heights for all the points in the appendix 1 by inserting all the coordinates in our new local geoid model. Next, we used this formula to find the difference between our estimated heights from the model and the corresponding geoid heights from the appendix.

$$\mathbf{V} = \mathbf{AX} - \mathbf{F} \tag{17}$$

Then we calculated the standard deviation of levelled heights by using this formula:

$$\sigma_0 = \sqrt{\frac{\mathbf{V}^T\mathbf{PV}}{n-e}} = 0.0241 \tag{18}$$

Standard deviation of unit weight: 0.024094578767253957 Standard deviation of levelled heights: 0.012720104244239227

2.6 Computation of geoid heights using SkTrans

We used SkTrans to transform every coordinate height using the height reference model HREF2008a.bin. We then found the geoid height of the model by calculating the differences from the ellipsoidal height. Here is a table of all the transformed geoid heights:

N-coordinate	E-coordinate	HREF2008a	Our model	Difference
7031656.237	568780.646	39.664	39.639	0.025
7032450.406	569796.193	39.590	39.576	0.014
7032454.240	569784.222	39.590	39.576	0.014
7031952.892	571469.041	39.562	39.543	0.019
7034487.402	571578.304	39.447	39.432	0.015
7030500.761	570574.097	39.643	39.629	0.014
7032141.126	566821.215	39.727	39.682	0.045
7033941.628	568890.318	39.567	39.544	0.023
7033251.070	568449.389	39.615	39.586	0.029
7032874.708	570073.423	39.564	39.549	0.015
7033210.648	570149.359	39.548	39.533	0.015
7032505.663	570480.827	39.567	39.552	0.015
7029005.339	558064.206	40.018	40.051	-0.033
7023177.607	560908.590	40.139	40.122	0.017
7035989.149	559459.379	39.732	39.779	-0.047
7025043.846	568858.934	39.882	39.867	0.015
7034498.404	575639.545	39.327	39.297	0.030
7036210.065	571151.403	39.378	39.369	0.009
7036049.635	566740.556	39.546	39.527	0.019
7035057.295	579323.493	39.201	39.147	0.054
7032598.606	570218.125	39.571	39.556	0.015

Table 2: Coordinates, corresponding transformed geoid heights with rounded values, and differences.

2.7 Comparison of the two sets of computed geoid heights

We generally see that the geoid heights from HREF2008a is a little big bigger than the heights calculated from our model. When comparing the results with the levelled heights, we see that the heights calculated from our model are closer the heights from the HREF-model. This is likely, due to the quite small area we used for making our model. Our points are all located in Trondheim, which makes our model quite good on points in Trondheim. The HREF-model is likely better at points all over Norway, but not as specialized at points in this region. Therefore, making a local geoid height model is useful when working on a project where good accuracy is required.

3 Part B - National height systems

Norway employs the **NN2000 height system** on land, which supersedes the older NN1954 height system. The NN2000 system is anchored to the "Norway Normal Null 2000" datum and serves a broad spectrum of applications, including geodesy and engineering surveys throughout Norway. This vertical coordinate system is measured in meters and is oriented upwards, facilitating accurate height determination and mapping within the country (1). N2000 has replaced the old datum NN1954 in all Norwegian municipalities, with the difference between the two datums ranging from -15 to 35 centimeters, depending on geographic location (2). Reasons for phasing out the old NN1954 in favor of the new NN2000 include land uplift, ongoing sea-level rise, suitability for modern satellite-based measurement methods, and being tied to the European vertical reference system EVRF2000 (3).

In addition to the NN2000 height system on land, Norway employs a distinct vertical reference system for maritime maps, known as "sjøkartnull"(4). It serves as the baseline for measuring depths in nautical charts and navigation. Sjøkartnull is typically set at the lowest astronomical tide (LAT), a level rarely surpassed by water levels(5).

Moreover, the **Baltic Sea Chart Datum 2000 (BSCD2000)** has implications for height and depth measurements on nautical charts and related applications in the Baltic Sea region, encompassing Norway. The BSCD2000 introduces a unified reference level, aiming to standardize height systems across national boundaries, thereby enhancing navigation safety and hydrographic mapping efficiency. This system is particularly beneficial for applications such as offshore engineering and environmental studies, promoting consistent measurements across the Baltic Sea countries (6).

References

[1] NN2000 height - EPSG:5941. *epsg.io.* Available at `https://epsg.io/5941`.

[2] Kartverket - Slik bruker du NN2000 - . Available at `https://www.kartverket.no/til-lands/posisjon/ta-i-bruk-nn2000`

[3] Store Norske Leksikon - NN2000 (Lars Mæhlum). Available at `https://snl.no/NN2000`

[4] Kartverket - Hva er et referansenivå? Available at `https://www.kartverket.no/til-sjos/se-havniva/referanseniva/hva-er-et-referanseniva`

[5] Kartverket - Vannstandsnivå. Available at `https://www.kartverket.no/til-sjos/se-havniva/referanseniva/vannstandsniva`

[6] THE BALTIC SEA CHART DATUM 2000 (BSCD2000) - Implementation of a common reference level in the Baltic Sea. *International Hydrographic Review (IHR).* Available at `https://ihr.iho.int`.