Quantum Computing - Midterm (4 problems, Takehome)
Work individually on this. You may ask the instructor any questions you have of course.

1 Problems

1. Let $\{|0\rangle, |1\rangle\}$ be an orthonormal basis of $\mathbb{C}^2$ and define the states $|\pm\rangle = 1/\sqrt{2}(|0\rangle \pm |1\rangle)$. Consider the CNOT gate:

$$CNOT = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes \sigma_x$$

I.e., $CNOT\,|a,b\rangle = |a, a \oplus b\rangle$ for $a, b \in \{0,1\}$. Describe its action when given an input in the X basis (i.e., when $a, b \in \{+,-\}$). In particular, write out the action of CNOT on X basis input states, where the output is also written in the X basis. Then describe the action in words.

2. Let f be a classical function:

$$f : \{0,1\}^n \to \{0,1\}^m.$$

There are no guarantees about how the function behaves. Let U_f be the quantum oracle:

$$U_f |x, y\rangle = |x, f(x) \oplus y\rangle,$$

for $x \in \{0,1\}^n$ and $y \in \{0,1\}^m$. Consider the following process: First, initialize an n qubit register and an m qubit register in the state $|0 \cdots 0\rangle \otimes |0 \cdots 0\rangle$ (for a total of $n + m$ 0's). Next, apply the n-fold Hadamard to the first register leaving the second register alone. Apply the U_f gate to the resulting state. The state is now:

$$|\psi\rangle = U_f \cdot (H^{\otimes n} \otimes I_m) |0, 0\rangle.$$

(Here, I use I_m to mean the m-dimensional identity operator.) At this point, measure only the second m-qubit register but not the first n-qubit register. Assume your measurement result is $|y\rangle$ for some bit string $y \in \{0,1\}^m$. Prove using the measurement postulate that the resulting state collapses to:

$$\frac{1}{\sqrt{|J_y|}} \sum_{x \in J_y} |x\rangle \otimes |y\rangle,$$

where:

$$J_y = \{x : f(x) = y\}.$$

Hint: Write out $|\psi\rangle$ as a superposition in the computational basis. Next, your measurement operators should be $\{M_y\}_{y \in \{0,1\}^m}$ where $M_y = I_n \otimes |y\rangle\langle y|$. Use the measurement postulate to determine what the post measurement state is if the outcome is some particular y.

What is the probability of observing any particular y in terms of J_y?

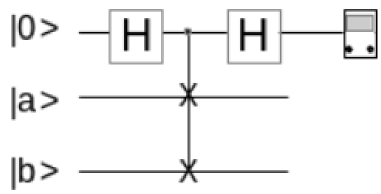

Figure 1: Input: A state of the form $|0, a, b\rangle$. The probability of observing a $|0\rangle$ after measuring allows you to determine $|\langle a|b\rangle|^2$ (after repeated uses of this algorithm with copies of the input).

3. A *controlled swap gate* $CSWAP$ is a gate acting on three systems: a control qubit (two dimensional), and two target systems (which, for this problem, may be of arbitrary, but equal, dimension). If the control qubit is the left-most system, then the CSWAP acts as follows:

$$
\begin{aligned}
CSWAP\,|0, a, b\rangle &= |0, a, b\rangle \\
CSWAP\,|1, a, b\rangle &= |1, b, a\rangle\,.
\end{aligned}
$$

   (a) Consider the algorithm shown in Figure 1. The CSWAP gate is depicted as a vertical line passing through three wires with a *solid circle* on the control wire (top most) and an "x" on each of the two target wires. This algorithm is meant to allow an experimenter to estimate $|\langle a|b\rangle|^2$ for arbitrary (normalized) $|a\rangle$ and $|b\rangle$. Prove that the probability of observing a $|0\rangle$ after the final measurement on the top (qubit) wire can give you this quantity. Namely, show that:

$$P(0) = \frac{1}{2} + \frac{1}{2}|\langle a|b\rangle|^2$$

   (Hint: Recall that $\langle x, y|z, w\rangle = \langle x|z\rangle \cdot \langle y|w\rangle$.
   Note that, if given a sufficient number of states prepared in the form $|a\rangle \otimes |b\rangle$, one could repeat this experiment multiple times to obtain an estimate of the desired inner-product.

4. **Quantum Random Walks (5 parts)**. Consider the following *classical* random walk: A *walker* sits on an integer line starting at position 0. At each time step, the walker chooses randomly to move left (with probability 1/2) or right (with probability 1/2). After choosing, the walker moves in that direction.

Problem 1: Fill in the chart below (Table 1) with the probability of finding the walker at position x (columns) at time t (rows, with time $t = 0$ being the start when the walker is at position 0 and hasn't moved yet). We filled in the first two rows.

	-4	-3	-2	-1	0	1	2	3	4
$t = 0$					1				
$t = 1$				$\frac{1}{2}$		$\frac{1}{2}$			
$t = 2$									
$t = 3$									

Table 1: Table for Problem 7.1. You may leave a cell empty to mean a probability of 0. The first two rows are filled in for you already.

Now, a *Quantum Random Walk* involves a quantum particle moving left, right, *or both* in a superposition of left and right! To model this, we use a joint Hilbert space:

$$\mathcal{H}_C \otimes \mathcal{H}_P,$$

where $\mathcal{H}_C$ is called the *coin space* and $\mathcal{H}_P$ is the *position space*. The coin space is spanned by orthonormal basis $\{|\mathtt{L}\rangle, |\mathtt{R}\rangle\}$ (i.e., it is a qubit), and the position space is spanned by orthonormal basis $\{\cdots, |-1\rangle, |0\rangle, |1\rangle, \cdots\}$ (don't worry that the space is not finite dimensional - since the number of time steps you are considering is finite, this won't be any different than what we learned in class!).

We begin, at time $t = 0$ with the walker (now a quantum particle) being initialized to the state:

$$|\psi_0\rangle = |\mathtt{R}\rangle \otimes |0\rangle .$$

At each subsequent time step, first a "coin flip" operator is applied to the coin space. This can be any unitary operator - let's use the Hadamard H where $H|\mathtt{L}\rangle = \frac{1}{\sqrt{2}}(|\mathtt{L}\rangle + |\mathtt{R}\rangle)$ and $H|\mathtt{R}\rangle = \frac{1}{\sqrt{2}}(|\mathtt{L}\rangle - |\mathtt{R}\rangle)$.

Next, a "Shift" operator S is applied which moves the position of the walker based on the state of the coin space. Namely S maps basis states as follows:

$$S\left|\mathtt{L}\right\rangle \otimes \left|x\right\rangle = \left|\mathtt{L}\right\rangle \otimes \left|x-1\right\rangle$$
$$S\left|\mathtt{R}\right\rangle \otimes \left|x\right\rangle = \left|\mathtt{R}\right\rangle \otimes \left|x+1\right\rangle .$$

We may write this in the following way:

$$S = \left|\mathtt{L}\right\rangle\left\langle\mathtt{L}\right| \otimes \sum_x \left|x-1\right\rangle\left\langle x\right| + \left|\mathtt{R}\right\rangle\left\langle\mathtt{R}\right| \otimes \sum_x \left|x+1\right\rangle\left\langle x\right| .$$

It can be shown that S is unitary.

Putting it all together, at each time step, the walker, which is in state $\left|\psi_{t-1}\right\rangle$ evolves to the state $\left|\psi_t\right\rangle = S(H \otimes I_P)\left|\psi_{t-1}\right\rangle$, where I_P is the identity operator on the position space $\mathcal{H}_P$. After T time steps, one measures the position space to determine the location of the walker. You will show that this distribution is very different from the classical case!

Problem 2: Write out the state of the walker, *before measurement*, at time steps $t = 1, 2$ and 3. Assume that at time $t = 0$, the walker is in position $\left|\psi_0\right\rangle = \left|\mathtt{R}\right\rangle \otimes \left|0\right\rangle$.

$$\left|\psi_0\right\rangle = \left|\mathtt{R}\right\rangle \otimes \left|0\right\rangle$$
$$\left|\psi_1\right\rangle =$$
$$\left|\psi_2\right\rangle =$$
$$\left|\psi_3\right\rangle =$$

Problem 3: For each of the time steps you did above, now assume a measurement is made of the position space only. Fill in Table 2 for your answer.

	-4	-3	-2	-1	0	1	2	3	4
$t = 0$					1				
$t = 1$									
$t = 2$									
$t = 3$									

Table 2: Table for Problem 7.3. You may leave a cell empty to mean a probability of 0. The first row is filled in.

Problem 4: While for $t = 0, 1, 2$ this is the same distribution, for larger t this is a very different distribution than the classical case (and if you have more time - or a computer - you can compute more time steps and realize the distribution continues to differ drastically). Comment on why this is. After all, shouldn't the Hadamard coin flip mean the quantum walker is choosing 50/50 to go left or right (same as the classical case)?

Problem 5: Building off your answer to part 4, can you think of a simple way to turn the quantum walk into a classical one?