

# Some Concepts in Category Theory

- CMIT Lecture

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## Let's Recall

- CAT, FUN, NAT
- Mono, epi & isomorphisms
- Initial, terminal & zero objects
- Yoneda Lemma

## ◦ Universal Property of Kernel

Let  $f: G \rightarrow H$  be a gp homomorphism. Then by defn.  
 $\text{Ker}(f) := \{g \in G \mid f(g) = 0\}$

Note that  $\text{Ker}(f) \subset G_1$ , so we have the inclusion

$$\text{Ker}(f) \xhookrightarrow{i} G_1 \xrightarrow{f} H \quad \text{&} \quad f \circ i = 0.$$

Now assume  $X$ , a group &  $\omega: X \rightarrow G$  such that  
 $f \circ \omega = 0$

$$\Rightarrow \forall x \in X, f \circ \omega(x) = 0 \Rightarrow \omega(x) \in \text{Ker}(f)$$

so we have a unique map  $\Theta: X \rightarrow \text{Ker}(f)$   
 $x \mapsto \omega(x)$

$$\begin{array}{ccccc}
 & \text{Ker}(f) & \xrightarrow{i} & G_1 & \xrightarrow{f} H \\
 & \swarrow \exists! \theta & \uparrow \omega & \nearrow & \\
 X & & & &
 \end{array}
 \quad \begin{array}{l}
 x \in X, \\
 i \circ \theta(x) = \theta(i) \circ \omega(x)
 \end{array}$$

Is Kernel, the only gp with this property ??

Assume,  $\exists$  another gp  $K \xrightarrow{j} G_1 \xrightarrow{f} H$  with  $f \circ j = 0$   
 & for any  $w: X \rightarrow G_1$

$$\begin{array}{ccccc}
 K & \xrightarrow{j} & G_1 & \xrightarrow{f} & H \\
 \swarrow \exists! \alpha & \uparrow \omega & \nearrow & & \\
 X & & & &
 \end{array}$$

since  $f \circ j = 0$ ,  $\exists! \alpha: K \rightarrow \text{Ker}(f)$

also  $f \circ i = 0 \Rightarrow \exists! \beta: \text{Ker}(f) \rightarrow K$

Why?

$$\begin{array}{ccccc}
 \text{Ker}(f) & \xrightarrow{i} & G_1 & \xrightarrow{f} & H, \text{ so } \alpha \circ \beta : \text{Ker}(f) \rightarrow \text{Ker}(f) \\
 \downarrow \beta & \uparrow \omega & \nearrow & & \\
 f \circ \alpha & \xrightarrow{j} & G_1 & \xrightarrow{f} & H \\
 \downarrow \alpha & \uparrow \omega & \nearrow & & \\
 \text{Ker}(f) & & & &
 \end{array}$$

$\exists i \circ (\alpha \circ \beta) = i$   
 But we have  $i \circ 1_H = i$   
 $\Rightarrow \alpha \circ \beta = 1_{\text{Ker}(f)}$  hence  $f \circ \alpha \cong \text{Ker}(f)$

### \* Universal property:

Given a group homomorphism  $f: G_1 \rightarrow H$ ,  
 $(\text{Ker}(f), i)$  is the unique group with  $\text{Ker}(f) \xrightarrow{i} G_1 \xrightarrow{f} H$   
 $f \circ i = 0$  such that for any other  $(X, \omega)$   $\omega: X \rightarrow G_1$   
 with  $f \circ \omega = 0$ ,  $\exists! \theta: X \rightarrow \text{Ker}(f)$   $\exists \omega = i \circ \theta$

## • Cokernel

Let  $f: V \rightarrow W$  be a linear transformation between vector spaces  $V, W$ .

$$\text{Coker}(f) := W / \text{Im}(f).$$

We have

$$V \xrightarrow{f} W \xrightarrow{p} \text{Coker}(f) \quad p \circ f = 0$$

with the property if  $\exists \bar{p}: W \rightarrow Y$ ,  $Y$  being a v.s with  $\bar{p} \circ f = 0$ , then  $\exists! \theta: \text{Coker}(f) \rightarrow Y$

$$\begin{array}{ccccc} V & \xrightarrow{f} & W & \xrightarrow{p} & \text{Coker}(f) \\ & & \searrow \bar{p} & & \swarrow \exists! \theta \\ & & X & & \end{array}$$

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XERCISE : State the Universal Property of Cokernel

- What might be Cokernel of a map  $f: G \rightarrow H$  in category of groups?

## • Equalizer

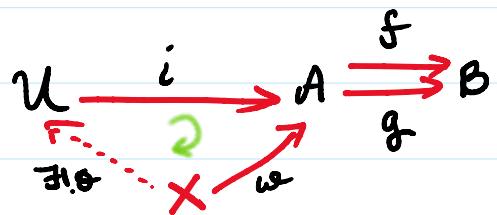
Let  $f, g: A \rightarrow B$  be two set maps.

Define  $U = \{a \in A \mid f(a) = g(a)\}$ , then we have

$$U \xrightarrow{i} A \xrightarrow{\begin{matrix} f \\ g \end{matrix}} B \quad \text{with } f \circ i = g \circ i$$

If  $\exists x \xrightarrow{\omega} A$ , with  $f \circ \omega = g \circ \omega$ . Then  $\forall n \in X$   
 $f \circ \omega(n) = g \circ \omega(n) \Rightarrow \omega(n) \in U$

so we have a unique map  $\Theta: X \rightarrow U$



The pair  $(U, i)$  is called the equalizer for  $f, g: A \rightarrow B$

## Exercise: Is it Unique?

State the Universal property of Equalizer.

In Set, equalizer  $\Leftrightarrow$  monomorphism  $\Leftrightarrow$  injective

### Remark

- Kernel is a special case of an equalizer  
 $\text{Ker}(G \xrightarrow{f} H) = \text{Equalizer } (G \xrightarrow{f-g} H)$ .

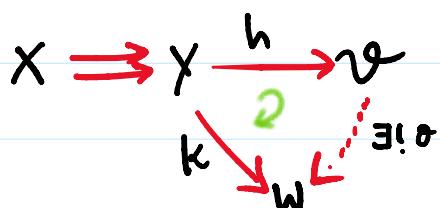
- In category of groups/vector space etc  
 $\text{Equalizer } (G \xrightarrow{f-g} H) = \text{Ker } (f-g)$ .

### Coequalizer

Given morphisms  $X \xrightarrow{f} Y$ , the coequalizer is the universal object

$$X \xrightarrow{f} Y \xrightarrow{h} V \quad h \circ f = h \circ g$$

with the property, if  $\exists Y \xrightarrow{k} W$  with  $k \circ f = k \circ g$



# Exercise: What is a coequalizer in Set? Grp?

- Products and Coproducts of two objects.

Recall that if  $A, B$  are sets,  $A \times B = \{(a, b) \mid a \in A, b \in B\}$ .

Do we have a universal property?

Of course. As soon as we define  $A \times B$ , we note that we get two maps (projections)

$$P_1 : A \times B \longrightarrow A \\ (a, b) \longmapsto a$$

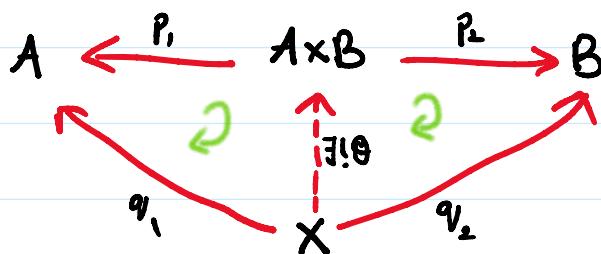
$$P_2 : A \times B \longrightarrow B \\ (a, b) \longmapsto b$$

Suppose given a set  $X$  & two maps  $\alpha_1, \alpha_2$

$$\alpha_1 : X \longrightarrow A \\ \alpha_2 : X \longrightarrow B$$

then, we can define a map (unique?)

$$\theta : X \longrightarrow A \times B \\ x \longmapsto (\alpha_1(x), \alpha_2(x))$$

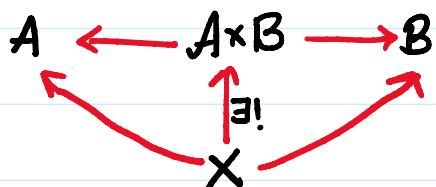


Voila!!!

if it exists

Given two objects  $A, B$  in a category, the product of  $A \& B$  denoted as  $A \times B$  is the unique object with maps

Given two objects  $A$  and  $B$ , the product of  $A$  and  $B$  denoted as  $A \times B$  is the unique object with maps  $A \xleftarrow{P_1} A \times B \xrightarrow{P_2} B$  such that given any other  $X \xleftarrow{\alpha_1} A \times B \xrightarrow{\alpha_2} B$ ,  $\exists! \theta : X \rightarrow A \times B$



Exercise : Prove that product of  $A$  &  $B$  is unique if it exists

- In category of groups, direct product is the product
- In category of topological spaces, the product is the cartesian product with product topology.

### Interesting example

Let  $(A, \leq)$  be a partially ordered set. That is

- $\leq$  is reflexive
- $\leq$  is antisymmetric
- $\leq$  is transitive

we can consider  $(A, \leq)$  as a category

$$\text{Obj}(A) = A$$

$$A \rightarrow B \text{ exists iff } A \leq B.$$

What is the product of  $A$  &  $B$ .

What about in a category  $\mathbb{Z}$ , where

$$\text{Obj}(\mathbb{Z}) = \mathbb{Z}^+$$

$$m \rightarrow n \text{ exists iff } m | n.$$

$\text{obj}(\mathcal{C}) \leftarrow$

$m \rightarrow n$  exists iff  $m|n$ .

## Coproducts

$$\begin{array}{ccccc} & & A & \xrightarrow{\quad} & A \sqcup B \\ & & \swarrow & & \downarrow & \swarrow \\ & & X & & B & \xleftarrow{\quad} \end{array}$$

(2)      (3)      (2)

- State the universal property.
- Coproduct of two sets  $A, B$  is their disjoint union.
- coproduct of two vector spaces is the direct sum.
- Coproduct in  $(A, \leq)$ , the poset ?

Convince yourself

## Pull backs

Given

$$\begin{array}{ccc} & B & \\ & \downarrow & \\ A & \longrightarrow & Z \end{array}$$

, pullback is

$$\begin{array}{ccc} A \times_Z B & \longrightarrow & B \\ \downarrow & \text{(2)} & \downarrow \\ A & \longrightarrow & Z \end{array}$$

st.

$\nexists \exists x : x \rightarrow B, x \rightarrow A$  with

## Push forwards

Given

$$\begin{array}{ccc} Z & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & & \end{array}$$

$$\begin{array}{ccc} Z & \longrightarrow & B \\ \downarrow & \text{(2)} & \downarrow \\ A & \longrightarrow & A \sqcup_Z B \end{array}$$

st

$\exists$

$A \rightarrow x, B \rightarrow x$  commuting

$$\begin{array}{ccccc} & & x & \xrightarrow{\quad} & A \times_Z B \\ & & \swarrow & \text{(3!) } & \downarrow & \swarrow \\ & & A & \xrightarrow{\quad} & B \\ & & \downarrow & \text{(2)} & \downarrow \\ & & A & \longrightarrow & Z \end{array}$$

$$\begin{array}{ccccc} & & z & \xrightarrow{\quad} & B \\ & & \downarrow & \text{(2)} & \downarrow \\ & & A & \xrightarrow{\quad} & A \sqcup_Z B \\ & & \downarrow & \text{(2)} & \downarrow \\ & & A & \xrightarrow{\quad} & x \\ & & \downarrow & \text{(3!) } & \downarrow \\ & & x & & \end{array}$$

## UNIVERSAL ARROWS

Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a functor, given  $A \in \text{Obj}(\mathcal{D})$

Define a category  $\mathcal{B}_A$

$$\text{Obj}(\mathcal{B}_A) = \left\{ (c, f) \mid F(c) \xrightarrow{f} A \right\}$$

$$\text{Hom}\left((c, f), (d, g)\right) = \left\{ c \xrightarrow{\alpha} d \mid \begin{array}{c} F(c) \xrightarrow{F(\alpha)} F(d) \\ f \searrow \text{?} \quad g \swarrow \\ A \end{array} \right\}$$

A **Universal Arrow** from  $F$  to  $A$  is a terminal object in  $\mathcal{B}_A$

that is, it is a pair  $(v, \theta) \in \mathcal{B}_A$   
 with the property, if  $\exists f: F(c) \rightarrow A$  for any  $c \in \mathcal{C}$   
 then  $\exists!$  map

$$\begin{array}{ccc} F(c) & \xrightarrow{\quad} & F(v) \\ f \searrow \text{?} & & \downarrow \theta \\ & & A \end{array}$$

We can also consider  $\mathcal{B}^A$

$$\text{Obj}(\mathcal{B}^A) = \left\{ (c, f) \mid A \xrightarrow{f} F(c) \right\}$$

$$\text{Hom}\left((c, f), (d, g)\right) = \left\{ c \xrightarrow{\alpha} d \mid \begin{array}{c} c \xrightarrow{f} F(c) \\ F(c) \xrightarrow{\alpha} F(d) \\ \downarrow \quad \searrow \text{?} \quad \swarrow \\ F(c) \xrightarrow{F(g)} F(d) \end{array} \right\}$$

A **Universal Arrow** from  $A$  to  $F$  is an initial object in  $\mathcal{B}^A$  (work out what)

initial object in  $\mathcal{C}$  (work out what that means)

# LIMITS & COLIMITS

Let  $J$  be a small category. Let  $F: J \rightarrow \mathcal{C}$ .  
 $F$  belongs to the functor category  $\text{Fun}[J, \mathcal{C}]$ .  
We have the functor

$$\Delta: \mathcal{C} \rightarrow \text{Fun}[J, \mathcal{C}]$$

$$A \mapsto \Delta_A$$

$$\Delta_A(j \rightarrow j') = A \xrightarrow{\quad 1_A \quad} A$$

then a limit of shape  $J$  in  $\mathcal{C}$  is a universal arrow from  $\Delta$  to  $F$ .

We use  $\varprojlim F$  to denote the limit of  $F$ .

A colimit of  $F$  of shape  $J$  is an universal arrow from  $F$  to  $\Delta$  (denoted  $\varinjlim F$ )

Okay!!!. But what does it mean???

Universal arrow from  $\Delta$  to  $F \Rightarrow$  terminal object in  $\text{Fun}[J, \mathcal{C}]_F$

which is a pair  $(\varprojlim F, \eta)$   $\Delta(\varprojlim F) \xrightarrow{\eta} F$

So given  $(i \xrightarrow{\alpha} j)$  in  $J$

$$\begin{array}{ccc}
 & \Delta(\varprojlim F)(\alpha) & \\
 \Delta(\varprojlim F)(i) & \xrightarrow{\quad} & \Delta(\varprojlim F)(j) \\
 \downarrow \eta_i & \text{2} & \downarrow \eta_j \\
 F(i) & \xrightarrow{\quad} & F(j) \\
 & F(\alpha) &
 \end{array}$$

$$\begin{array}{ccc}
 & \varprojlim F & \\
 \eta_i & \text{2} & \eta_j \\
 F(i) & \xrightarrow{\quad} & F(j) \\
 & F(\alpha) &
 \end{array}
 \quad \text{Cone}$$

Given any other  $(D, \epsilon) \in \text{Fun}[\mathcal{I}, \mathcal{C}]_F$   
ie a cone

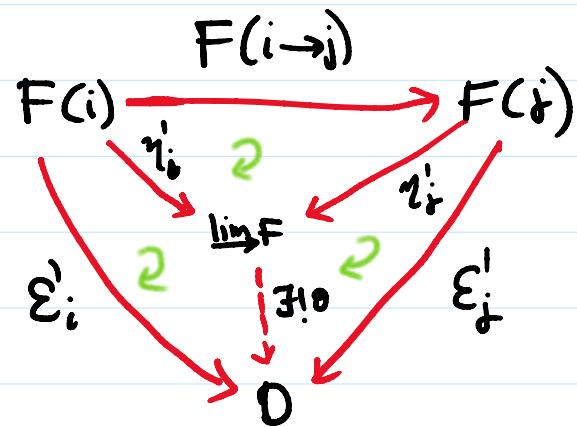
$$\begin{array}{ccc}
 D & & \\
 \xi_i & \text{2} & \xi_j \\
 F(i) & \xrightarrow{\quad} & F(j)
 \end{array}$$

then  $\exists!$  map  $\Theta: D \rightarrow \varprojlim F$

$$\begin{array}{ccc}
 D & & \\
 \xi_i & \text{2} & \xi_j \\
 F(i) & \xrightarrow{\quad} & F(j) \\
 & \varprojlim F & \\
 & \eta_i & \eta_j \\
 & \text{2} & \text{2} \\
 & F(i \rightarrow j) &
 \end{array}$$

Colimit

## Colimit



THANKS

Oh! Wait. There's more (provided we have time)

We have seen that given a cone  $(D, \varepsilon)$  with base  $F$ , there exist a unique map  $D \xrightarrow{\theta} \limleftarrow F$

Conversely if given an arrow  $D \xrightarrow{\theta} \limleftarrow F$ ,  $(D, \eta \circ \theta)$  forms a cone. Thus we have the bijection

$$\text{Cone}(-, F) \cong \text{Hom}(-, \limleftarrow F)$$

This gives another characterization of a limit.

$\limleftarrow F$  is the object representing the functor  $\text{Cone}(-, F)$

$$\text{Cone}(-, F) : \mathcal{C}^{\text{op}} \rightarrow \text{Set}$$