In this class we are going to finish our discussions of probabilities, permutations and combinations.

Last time we covered probabilities, we discussed both independent and dependent probabilities using dice and marbles as examples.

Discuss with the class the results computed yesterday involving conditional probabilities on various values for the deck of cards.

Permutations and Combinations

A permutation of a set is an arrangement of the objects in the set in a particular order. Whereas a combination is an arrangement or selection where order is not important. To simplify a permutation is nothing more than an ordered combination.

Let's start by looking at permutations where we will allow for repeated values. Suppose that I have 10 items to choose from and because we are allowing repeated values each time I choose I will have 10 items to choose from. If I choose 4 of them there are: 10 * 10 * 10 * 10 permutations. Restated, I have 10 choices the first time, then I have 10 choices for the second time, and so on.

Mathematically we would say we had

x * x * x * … (n times)

An easier way to write that is by using an exponent

x * x * x * … (n times) = x^n

So for our example above we would have

10 * 10 * 10 * 10 = 10^4 = 10000 permutations

For a concrete example, think of a lock that has tumblers instead of a key. If there were 4 dials and each dial had 10 numbers (0…9) then we would have 10,000 permutations of the locks combination.

As a class example select another set of values *x* and a choice *n* and have the class compute the number of permutations.

Now let's add the restriction of not allowing for repeated values.

For an example you can choose a word like AUTO, and show the number of two-character permutations from the three letters in the word. This one is easy for the class to list.

AU AT AO UT UO TO OT OU TU OA TA UA

We can see if we use our formula above we would come up with the wrong value.

4^2 = 16 instead of 12.

Why is this?

This is because we have now restricted our permutations to not allowing for repetitions.

This is fairly easy when we are dealing with such small values but what happens when our number of possible values and choice of permutation is large?

It's much harder to list all the permutations. Refer back to the lock example above. If we restrict repeated values, how many permutations are there?

Have the class think about this and come up with ideas that might work.

To solve this problem without repetition we will select from the 10 possibilities for the first value, then from the remaining 9 for the second, 8 for the third and so on.

We would then get $10 * 9 * 8 * 7 = 5040$.

We can see that this is the first part of 10 factorial. 10 factorial (written 10!) is the results of $10 * 9 * 8 * 7 * 6 * 5 * 4 * 3 * 2 * 1$.

So to come up with our solution we need to take our 10 factorial and remove everything form 6 down, which is removing 6 factorial. How do we get 6 factorial? It is the difference of our total possibilities and the size of our permutations ($10 – 4 = 6$).

Our result can now be written as $10!/(10 – 4)! = 10!/6! = 5040$

We can now construct a formula as the following
$\text{Perm}(n,k) = n!/(n – k)!$

Going back to our AUTO example above we have
$\text{Perm}(4,2) = 4!/(4 – 2)! = 4!/2! = 12$

You can show the class this Excel as well by using the PREMUT function.
PERMUT(10,4)

As a class exercise, select other values and have the class compute the permutations. As a substitute, you can use the four different colored marbles from the probability section instead of the AUTO example or as an additional example to really hammer the point home. Mix difficulties so that the class can do some by hand and some that they will need Excel to find the result.

Next let's look at combinations without repetition.
As we defined above a combination is an arrangement or selection where order is not important.

Go back to the AUTO example and suppose we want to find the number of combination of size 2 without repetition. Since there are 4 possibilities and we want combination of size 2 we call this a 4 choose 2 combination. Note that since order is not important TO is the same as OT. Again with this small set we can write out the solution.

AU AT AO UT UO TO

Again, as with the permutations this is fairly easy when we are dealing with such small values but what happens when our number of possible values and choice of combination is large?

Let's return to our lock example but this time put the restiction that we can not use the same number more than once (no repetition).
This time we want to use our 10 numbers and find the number of combinations with 4 values or using our notation we want to find 10 choose 4.

Since combination are just permutation where order does not matter. Our high-level formula is just to take the number of permutations and reduce it by the number of ways the objects could be in order.

We discussed earlier how many ways n number of items can be put in order: $n!$

With this knowledge and our previous permutation formula, our formula for combinations would be:

$\mathrm{Comb}(n,k) = (n!/(n-k)!) * 1/k! = n!/(k!(n-k)!)$

By multiplying our permutation formula by $1/k!$, we are reducing it by the number of times k can be ordered.

Our lock example would now be:

Comb(10,4) = 10!/(4!(10-4)!) = 210

Going back to our AUTO example above we have
Comb(4,2) = 4!/(2!(4 – 2)!) = 24/4 = 6

You can show the class this Excel as well by using the COMBIN function.
COMBIN(10,4)

As a class exercise, select other values and have the class compute the combinations. Mix difficulties so that the class can do some by hand and some that they will need Excel to find the result.

There are as many examples that the class can do as you can come up with. Assign the class many of these permutation and combination problems and then go over them as a class, maybe having the students come up from and work them out.

As one last example of how they can apply this, let's look at the lottery.

What is the probability of winning the lottery?

What if you have to select 6 numbers that range from 1 to 49, how many possibilities are there?

The answer is how many permutations are there in 49 numbers when selecting 6.
Perm(49,6) = 49!/(49 – 6)! = 49!/43! = 10,068,347,520 ~ 10 billion

Since there is only one possible ordering of the numbers to win the lottery the probability of winning is approximately 1 in 10 billion.

Now let's change our definition of a winning lottery choice. Let's say the there are still 49 numbers and you select 6 but now order is not important. This time we are looking at a combination instead of a permutation.

Comb(49,6) = 49!/(6!(49 – 6)!) = 49!/(6! * 43!) = 13,983,816 ~ 14 million

Again there is only one combination of the 6 numbers that win the lottery, but now our probability of winning has increased to approximately 1 in 14 million.

As we can see, by relaxing our restriction of ordering we have greatly improved our chances on winning.

Even with this more favorable lottery our chances of winning are very small. You would have to buy approximately 140,000 tickets, each with different numbers, to give yourself a 1% chance of winning.

As an in class extended exercise or homework exercise, have the students compute the probability of winning a lottery where your select 7 numbers that range from 1 to 99.