

Probability and Combinatorics

The science of uncertainty... and gambling

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#MathForDevs

Table of Contents

- Probability
- Combinatorics
- Algebra of events
- Conditional probability
- Applications
- Statistical distributions
- Central limit theorem

Probability

Definition and principles

Some Definitions

- The scientific method relies on **experiments**
 - Initial conditions → outcome
 - Usually we control the initial conditions to isolate the outcome
- **Random event**
 - A set of outcomes of an experiment
 - Each outcome happens with a certain **probability**
- **Random variable**
 - An expression whose value is the outcome of the experiment
 - Usually denoted with $X, Y, Z...$ (capital letters)
- **It is not possible to predict the next outcome of a random event!**
 - But we can perform the same experiment **many times** (trials)
 - The patterns and laws become more apparent with more trials

Frequency

- Let's perform the same experiment many times
 - Under the same conditions
 - ... such as throwing a dice
- Assign a frequency to each number $i = \{1, 2, \dots, 6\}$ that the dice shows
 - m – number of trials we got i , n – all trials
- As n increases, f_i "stabilizes" around some number
- We cannot perform infinitely many experiments
 - But we can "extend" the trials mathematically
- $$p(A) = \lim_{n \rightarrow \infty} \frac{m}{n}$$
- We call this the probability of outcome **A**
 - **Statistical definition** of probability

$$f_i = \frac{m_i}{n}$$

Examples

- Rolling a dice
 - Possible outcomes: $\{1, 2, 3, 4, 5, 6\}$
 - Fair dice – all outcomes are equally likely
$$p(1) = p(2) = \dots = p(6) = 1/6$$
- Tossing a fair coin
 - Possible outcomes: $\{0 = \textit{heads}, 1 = \textit{tails}\}$
$$p(0) = p(1) = 1/2$$
- Tossing an unfair coin
$$p(0) = 0,3; \quad p(1) = 0,7$$
- Note that
 - The probability $p \in [0; 1]$
 - It can also be expressed as percentage: $p \in [0\%; 100\%]$
 - The sum of all probabilities is equal to 1

Countable and Uncountable Outcomes

- In some cases, the number of outcomes is finite
- But some random variables have **infinitely many** outcomes
- Example: intervals
 - What is the probability that a real number $A \in [0; 100]$ chosen at random, is in the interval $[10; 20]$?
 - Answer: it depends only on the lengths of the intervals

$$p = \frac{20 - 10}{100 - 0} = 0.1 = 10\%$$

- The number of outcomes is infinite, but we are still able to compute probabilities
- **Probability density** – the probability of the result being in a tiny interval dx

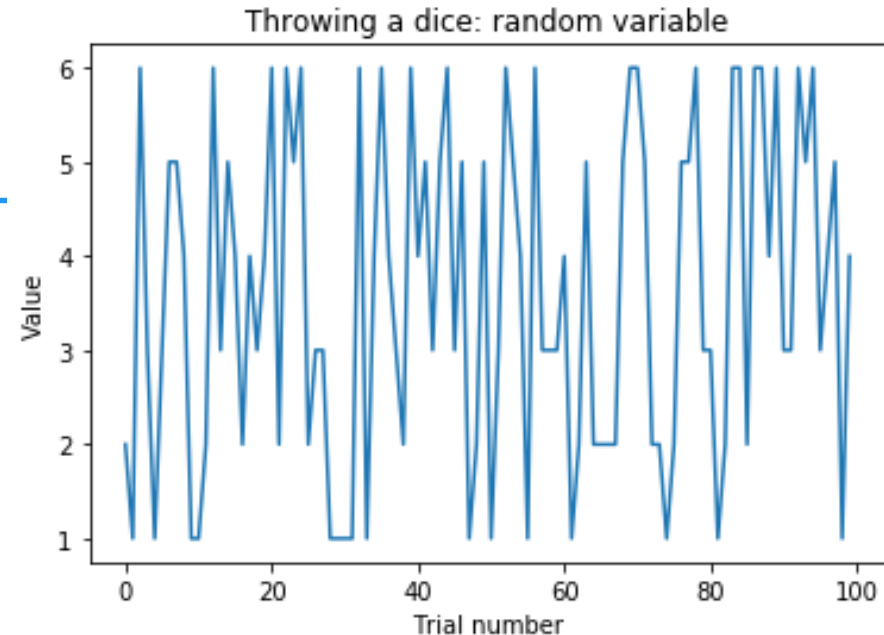
$$dp = \frac{dx}{b - a}$$

- a, b – both ends of the interval $[0; 100]$

Visualizing Random Variables

- To visualize a random variable, we plot the value as a function of the trial number
 - We can generate random values using **numpy**
 - Example: throwing a dice

```
def throw_dice():  
    return np.random.randint(1, 7) # from 1 to 6  
  
x = [throw_dice() for i in range(100)]  
plt.plot(x)  
plt.show()
```

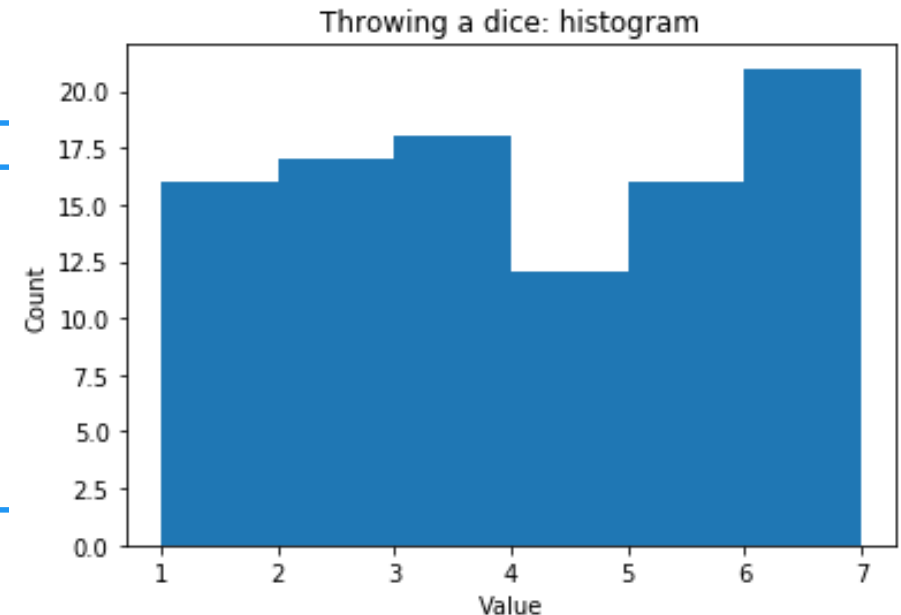


Visualizing Random Variables (2)

- The function we got is not very informative
 - Better way: show the frequency of each output
 - For each possible value of the random variable, count how many times we got that value
- This is called a **histogram**

```
# Counting all values
from collections import Counter
counts = Counter(x)
for number, count in counts.items():
    print(str(number) + ": " + str(count))
```

```
# Plotting a histogram
plt.title("Throwing a dice: histogram")
plt.hist(x, bins = range(1, 8))
plt.ylabel("Count")
plt.show()
```



Combinatorics

How to count things

Combinatorics

- Combinatorics deals with **counting** objects and groups of objects
- Prerequisites
 - Finite (countable) number of outcomes
 - All outcomes have equal probability
- Examples: gambling games
 - Roulette – all segments are equally likely
 - Card games – all card backs are the same
- Counting rules
 - Rules for computing a **combinatorial probability**
 - Show how many "desired" outcomes exist

Combinatorics (2)

- Notation
 - All outcomes: n
 - All experiment outcomes: k
 - Usually n is fixed and k depends on the experiment
- Types of **samples**
 - with repetition / without repetition
 - ordered / unordered
- Example: taking numbered balls out of a box
 - Take a ball, then return it to the box
 - Take a ball without returning it to the box (in this case $k \leq n$)
 - Take balls in a specific order (e.g. if they are numbered or colored)
 - Take balls in no specific order

Counting Rules

- **Rule of sum**

- m choices for one action, n choices for another action
- The two actions **cannot be done at the same time**
 \Rightarrow There are $m + n$ ways to choose one of these actions

- **Example**

- A woman will shop at **one** store in town today
 - North part of town – mall, furniture, jewellery (3 stores)
 - South part of town – clothing, shoes (2 stores)
- In how many ways she could visit one shop?
- Answer: $3 + 2 = 5$ ways

Counting Rules (2)

- **Rule of product**

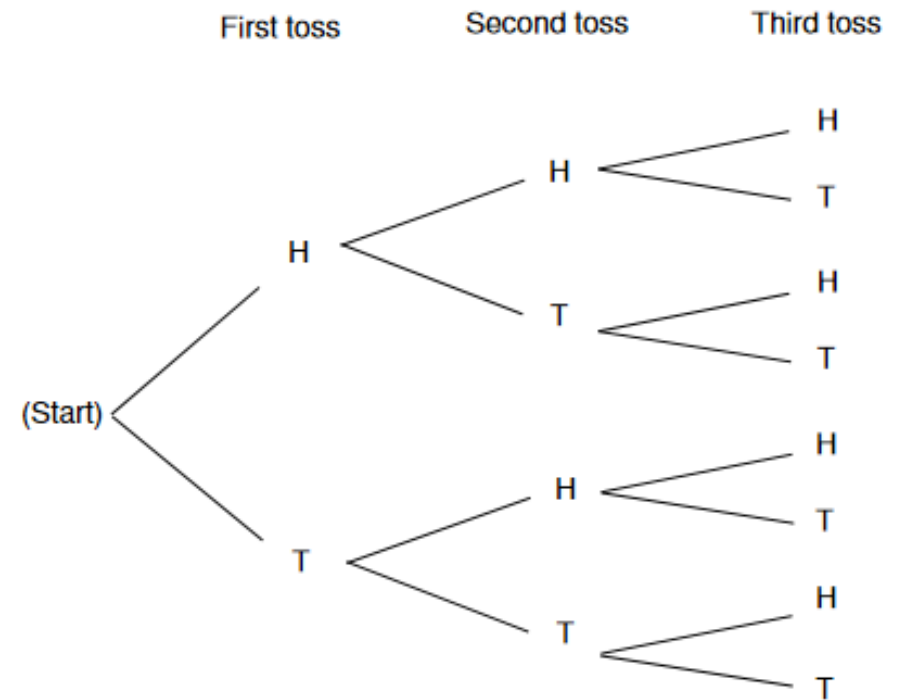
- m choices for one action, n choices for another action
- The two actions are performed **one after the other**
 \Rightarrow There are $m \cdot n$ ways to do both actions

- **Example**

- You have to decide what to wear
 - Shirts – red, blue, purple (3 colors)
 - Pants – black, white (2 colors)
- In how many ways can you create one outfit (shirt and pants)?
- Answer: $3 \cdot 2 = 6$ ways
 - For each choice of shirt, you can choose one color of pants
 - These are **independent**

Example: Three Coin Tosses

- Let's explore a graphic method of solving combinatorial problems called a **tree diagram**
 - Draw all intermediate results and the links between them
 - A "path" through the tree represents an outcome
 - Useful when the outcomes are relatively few
- What's the probability of getting 3 tails out of three coin tosses?
 - Answer: $1/8$
- What's the probability that both of these are true?
 - The first outcome is a head
 - The second outcome is a tail
 - Answer: $1/4$



Example 2: Eating at a Restaurant

- A restaurant offers
 - 5 choices of appetizer
 - 10 choices of main course
 - 4 choices of dessert
- You can choose one course, two **different** courses, or all three
- How many possible meals can you make?
 - One course: either appetizer, main course, or dessert: $5 + 10 + 4 = 19$
 - Two courses: total 110
 - Appetizer + main course: $5 \cdot 10 = 50$
 - Main course + dessert: $10 \cdot 4 = 40$
 - Appetizer + dessert: $5 \cdot 4 = 20$
 - Three courses: $5 \cdot 10 \cdot 4 = 200$
 - Total: $19 + 110 + 200 = 329$ possible meals

Permutations

- A permutation (without repetition) of a set A is any shuffling of all elements in A
 - The order matters
 - Notation: P_n
- Example:
 - If $A = \{1, 2, 3, 4\}$, some permutations are $\{1, 2, 3, 4\}$; $\{1, 4, 3, 2\}$; $\{2, 3, 4, 1\}$
- Number of permutations of n elements
 - n choices for the first element
 - $n - 1$ for the second one
 - Because the first one is already taken
 - $n - 2$ for the third one
 - 1 for the last one
 - Total: $n! = 1.2.3. \cdots .n$

Variations

- A variation is an **ordered subset** of k elements from A
- Notation: V_n^k
 - We read this as "Variations of n elements, k^{th} class"
- Example:
 - If $A = \{1, 2, 3, 4\}$, $k = 2$, some variations are $\{1, 2\}$; $\{4, 3\}$; $\{3, 1\}$; $\{4, 1\}$
- Number of variations
 - Same technique as in permutations
 - n choices for the first element
 - $n - 1$ for the second one
 - $(n - k + 1)$ for the last one

$$V_n^k = n.(n - 1).\cdots.(n - k + 1) = \frac{n!}{(n - k)!}$$

Combinations

- A combination is an **unordered subset** of k elements from A
- Notation: C_n^k
- Example:
 - If $A = \{1, 2, 3, 4\}$, $k = 2$, some combinations are $\{1, 2\}; \{3, 4\}; \{3, 1\}; \{4, 1\}$
- Number of combinations of n elements
 - Using a similar (but more involved) logic, we can prove that

$$C_n^k = \frac{n!}{(n-k)!k!}$$

- This is also known as "**n choose k**" (we choose k elements from n)

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Example Usages

- Shuffle a deck of cards
 - The same as generating a random permutation of 52 (or 54) elements
- Crack a password
 - How many 3-letter passwords are there (26 + 26 letters total)? V_{52}^3
- Generate all anagrams of a given word
 - Anagram: a different word using the same letters
 - Example: emits → items, mites, smite, times
 - Method:
 - Generate all permutations of the letters
 - For each permutation, find whether it's a valid word (check with a dictionary)
 - Return all valid words
- Make a fruit salad
 - Generate combinations of fruits (the order doesn't matter)
 - Possibly, combinations with repetition (if I love bananas, I'll take a double serving)

Probability Algebra

**Sets and probabilities,
geometry intuition**

Events

- **Event** – a result from the experiment
- **Elementary event**
 - One particular outcome
 - Example: outcomes of two coin flips: $\{HH\}, \{HT\}, \{TH\}, \{TT\}$
- Compound event
 - Consists of many elementary events
 - Example: getting an odd number from a dice
 - Consists of the elementary events 1, 3, 5
- **Event space** – the set Ω of all possible events
- The algebra of events is the same as the algebra of sets
 - ... and we already know these :)

Algebra of Events

- If event A happens with event B , A is a **consequence** of B : $A \subset B$
- If $A \subset B$ and $B \subset A$, then $A = B$
- **Complementary event**: \bar{A} happens iff A does **not** happen
- **Impossible event**: contains no elementary events: \emptyset
- **Product of events**: happens iff **A and B** happen: $C = A \cap B$
 - **Incompatible events**: $A \cap B = \emptyset$
- **Sum of events**: happens if **A, B or both** happen: $C = A \cup B$
 - If A and B are incompatible, $C = A + B$
- Observe that
 - Logical relations are the same as set operations (and event operations)
 - **AND**: intersection
 - **OR**: union
 - **NOT**: complement

Conditional Probability

- Additional information about the experiment outcome can change the probabilities
- Example:
 - "Hidden dice": someone rolls a dice and doesn't tell us the result
 - Probabilities: $1/6$ for every number
 - These are also called "a priori" probabilities
 - Now we know the number is even
 - This changes all outcome probabilities: $\{1 \rightarrow 0; 2 \rightarrow \frac{1}{3}; 3 \rightarrow 0; 4 \rightarrow \frac{1}{3}; 5 \rightarrow 0; 6 \rightarrow \frac{1}{3}\}$
 - These are called "a posteriori" probabilities
- Conditional probability
 - Probability of event A given event B
 - Math notation: $P(A|B)$

Conditional Probability (2)

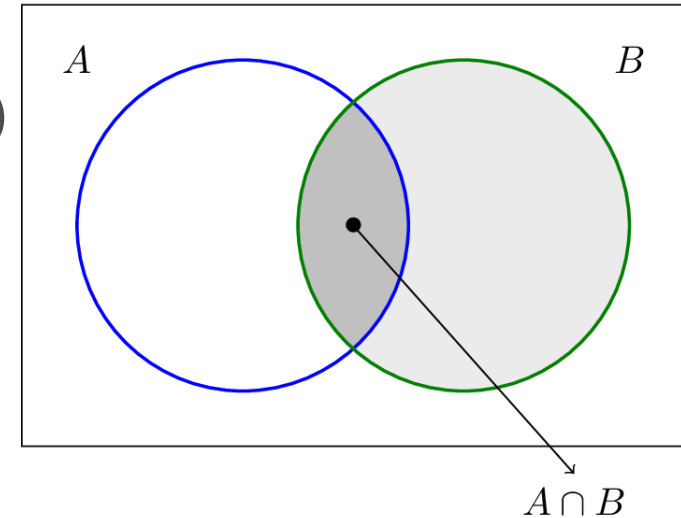
- More formally

- If we know B happened, the probability $P(A|B)$ corresponds to the part of the set B which is shared between A and B

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- In our example

- Event A : number on a fair dice
 - $A = \{1, 2, 3, 4, 5, 6\}$
- Event B : the number is even
 - $B = \{2, 4, 6\}$
- $A \cap B = \{2, 4, 6\}$
- $P(1|\text{even}) = 0; P(2|\text{even}) = \frac{1}{3}; \dots$



Event Independence

- Sometimes, an event doesn't influence another event
 - They are called independent events
- If two events are independent, knowledge of one **does not tell us anything** about the other
- More formally, $P(A \cap B) = P(A) \cdot P(B)$
 - If $P(B) \neq 0$, $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \cdot P(B)}{P(B)} = P(A)$
 - The same can be applied to A if $P(A) \neq 0$
- Example
 - 99% of all people who died of cancer, have consumed pickles
 - 99,8% of all soldiers have eaten pickles
 - <http://www.pleacher.com/mp/mhumor/pickles.html>
 - <http://www.dhmo.org/facts.html>

Bayes' Theorem

- The theorem tells us how to update the probabilities when we know some evidence

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(A|B)P(B)$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} \Rightarrow P(B \cap A) = P(B|A)P(A)$$

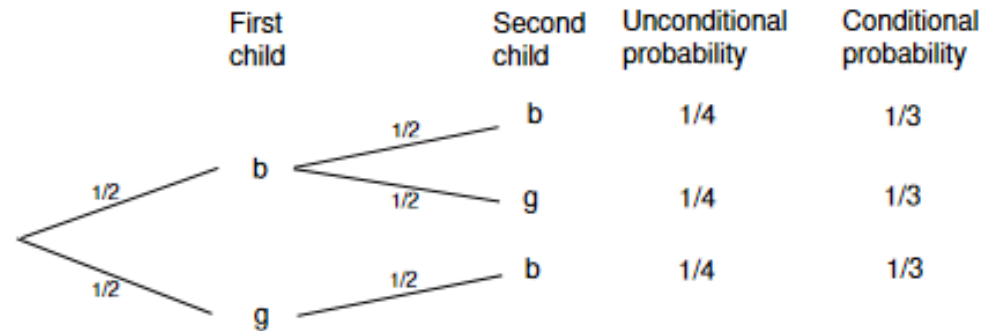
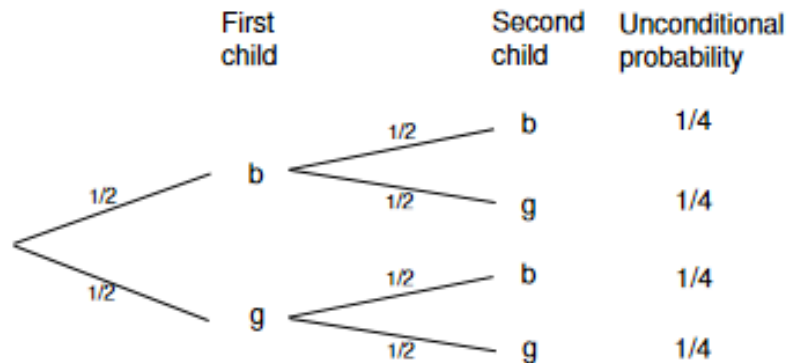
$$A \cap B = B \cap A \Rightarrow P(A|B)P(B) = P(B|A)P(A)$$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

- Example usage: spam detection
 - Consider each word w ; compute number of emails which contain it
 - m spam emails containing w ; n total emails containing w :
 - "Spamminess" of word: frequency $P(\text{word}|\text{spam}) = m/n$
 - "Spamminess" of email: $P(\text{spam}|\text{all words})$

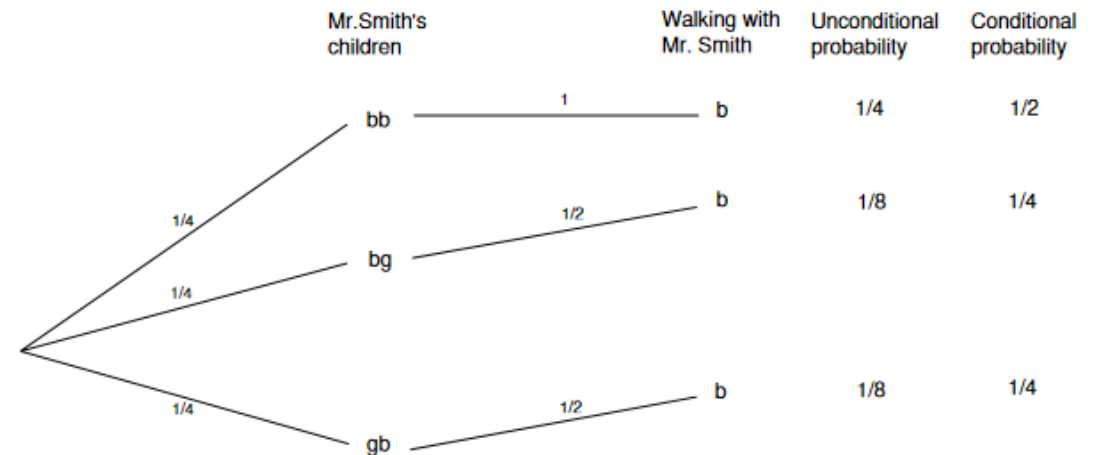
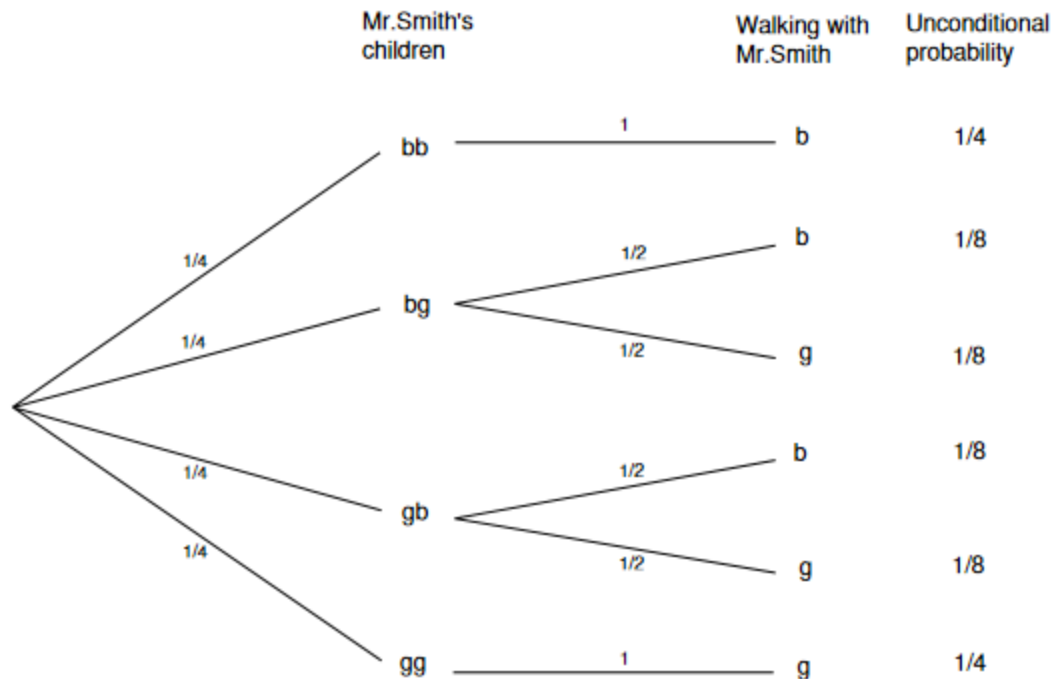
Example: Family Paradox 1

- A family has two children
 - One of them is a boy
 - What is the probability that both children are boys?
 - A child has a 0,5 chance of being a boy or a girl
- Intuitive answer: 0,25
 - But wait... let's exhaust all possibilities



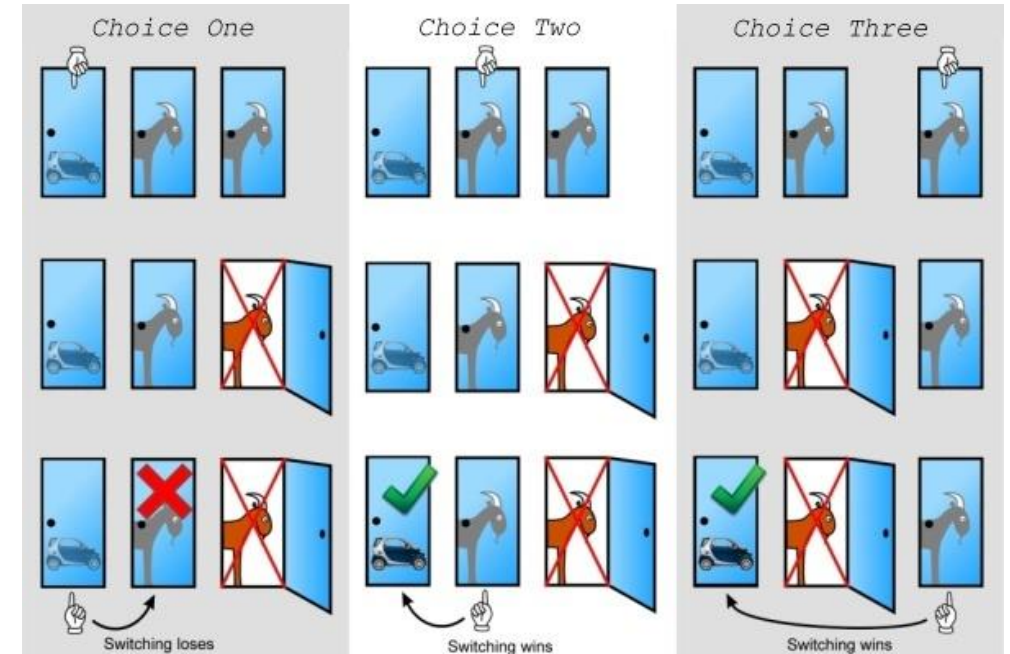
Example: Family Paradox 2

- Mr. Smith is the father of two children
 - When we meet him on the street, he introduces one as his son
 - What's the probability that the other child is a boy?
- Assumption
 - He is equally likely to take any child to a walk



Example: Monty Hall Problem

- In a game show, you have to choose between three doors
 - Behind one is a car, behind the other two – goats
- You pick a door
- The host reveals one of the two other doors – it's always a goat
- You have the option to keep your choice or switch doors
 - Which is the winning strategy?
- It turns out that the winning strategy is to always switch
 - This gives you $2/3$ chance of winning the car
- More details: [Quora](#)



Statistical Distributions

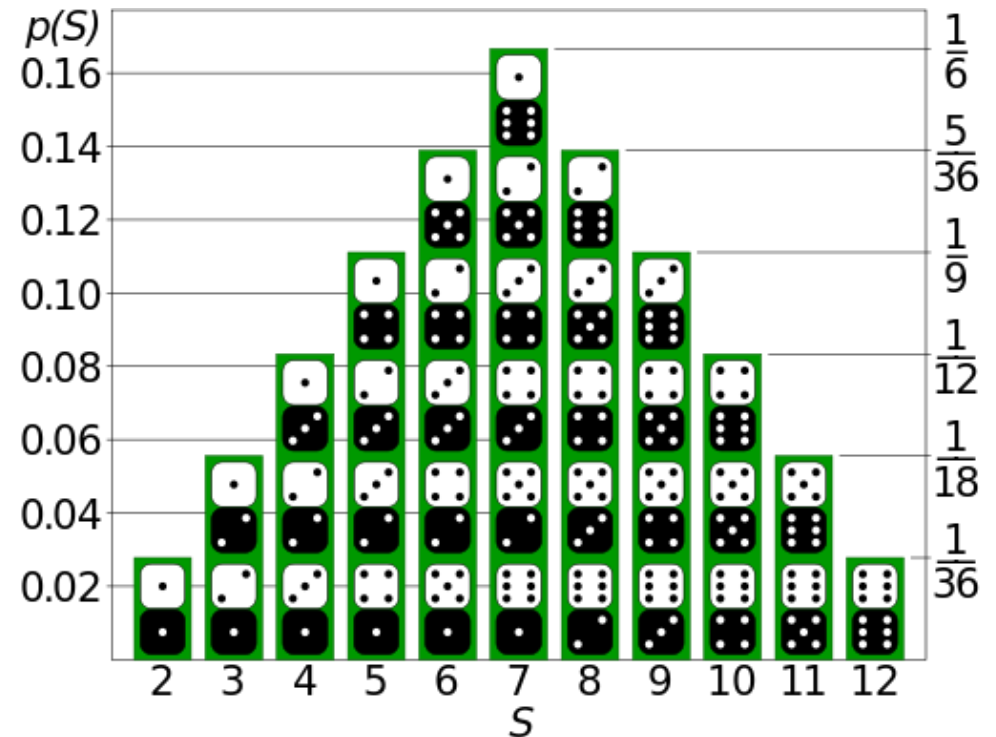
Seeing the results of our
experiments

Distributions

- We saw that random variables can be treated as functions
 - But they look funky
 - Don't have derivatives at most points
 - Difficult to work with
- We can instead take functions of these functions
 - Like we counted each outcome
 - Instead of graphing the real function, we made a histogram of counts
 - This gives us a much better idea what the random variable looks like
- These functions of functions are called **distributions**
 - In our example, we looked at the **frequency distribution**

Discrete Distribution

- Probability distribution function
 - A table which maps each outcome of a random variable to a probability: $p_X(x_i) = P(X = x_i)$
 - Also called **probability mass function** (pmf)
- Example: two die rolls
 - Random variable: sum of numbers
 - Outcomes: $\{2, 3, \dots, 12\}$
 - Probabilities:
 $P(2) = P(\{1, 1\}) = 1/36$
 $P(3) = P(\{1, 2\}) + P(\{2, 1\}) = 2/36$
 \vdots



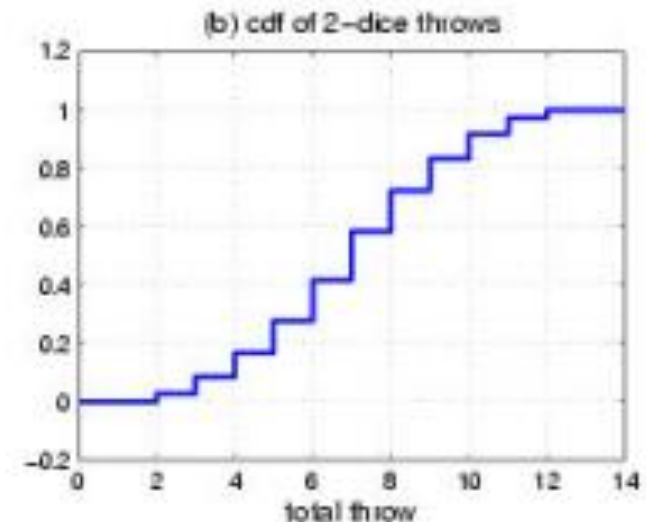
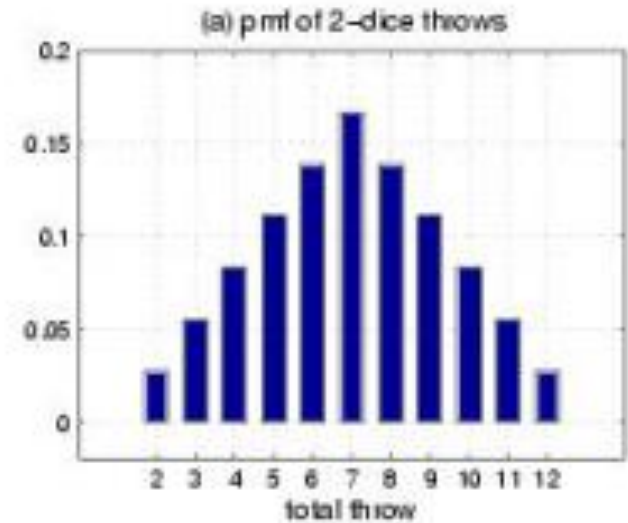
Discrete Distribution (2)

■ Cumulative distribution function

- A table which maps each outcome of a random variable to the probability of its value being less than or equal to a given number

$$F_X(x_i) = P(X \leq x_i)$$

- Also called **cumulative mass function** (cmf) or **cumulative density function** (cdf)
- Every cmf is non-decreasing
 - Usually starts at 0
 - Always ends at 1



Continuous Distribution

- **Cumulative density function** (cdf)

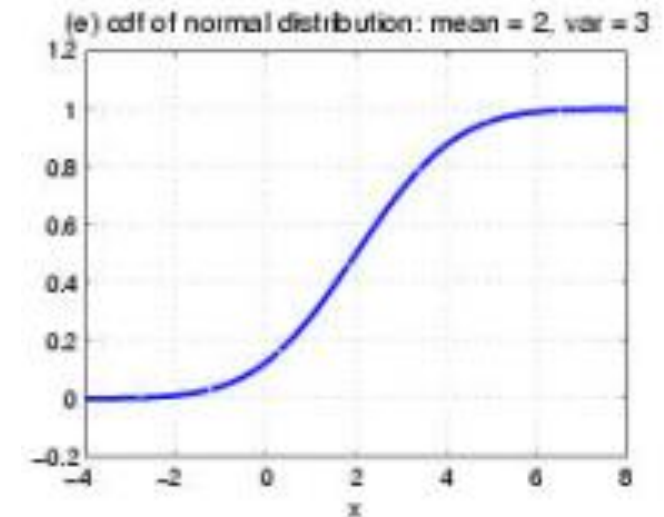
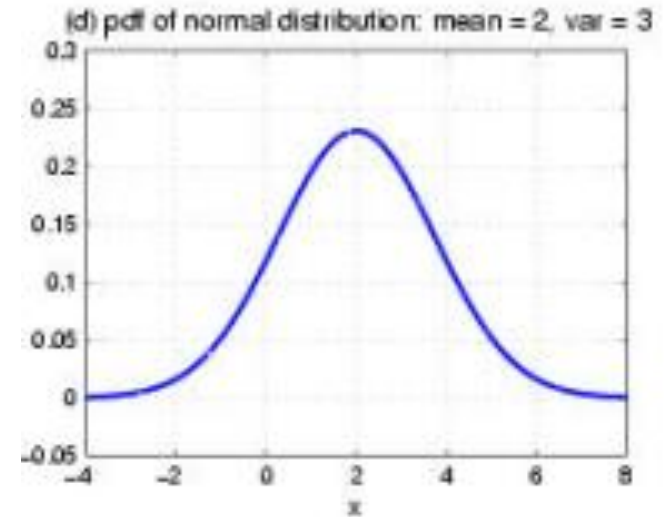
- Defined in the same way as the cmf:
$$F(x) = P(X \leq x)$$

- **Probability density function**

- Derivative of the cdf:

$$f(x) = \frac{dF(x)}{dx}$$

- Meaning: the probability of the function taking values in an infinitely small interval **around** x
- **The probability of observing any single value a is exactly 0**
 - The number of outcomes is ∞
 - $p(a) = \left[\frac{\# \text{ of values } a}{\infty} \right] = 0$



Common Distributions

**Probability and Statistics
Playing Together**

Bernoulli and Uniform Distributions

- Bernoulli distribution
 - The simplest distribution of a random variable
 - Value 0 with probability p
 - Value 1 with probability $q = 1 - p$
 - The two events are incompatible (mutually exclusive)
 - Example: coin flip (fair coin: $p = 0,5$)
 - ... Not so interesting on its own
 - But takes part in other distributions
- Uniform distribution
 - All values in some range $[a; b]$ are equally likely
 - Example: number on a fair dice
 - Also generalizes to continuous variables

Binomial Distribution

- n Bernoulli trials
 - Each trial has a "success" probability p
 - $n = 1 \Rightarrow$ Bernoulli distribution
 - Discrete distribution
 - Notation: $X \sim B(n, p)$
 - "X follows the binomial distribution with parameters n and p "
 - Probability mass function
- Cumulative function

$$f(k; n, p) = P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

$$F(k; n, p) = P(X \leq k) = \sum_{i=0}^{\lfloor k \rfloor} \binom{n}{i} p^i (1 - p)^{n-i}$$

Normal Distribution ❤️

- Origin: random errors in measurements
 - When we perform an experiment, there are many sources of error
- Example: throwing a dart at the origin of the (x, y) -plane
 - We aim at the origin
 - Random errors prevent us from hitting it every time
 - Sources of error
 - Hand shaking, air currents, distribution of mass inside the arrow, different viewing angles... and many more, some of which we can't even know
- Assumptions
 - The errors don't depend on the orientation of the coordinate system
 - The errors in x and y directions are independent: one doesn't influence the other
 - Large errors are less likely than small errors

Normal Distribution (2)

- We can derive the distribution of errors

- Distances from the origin

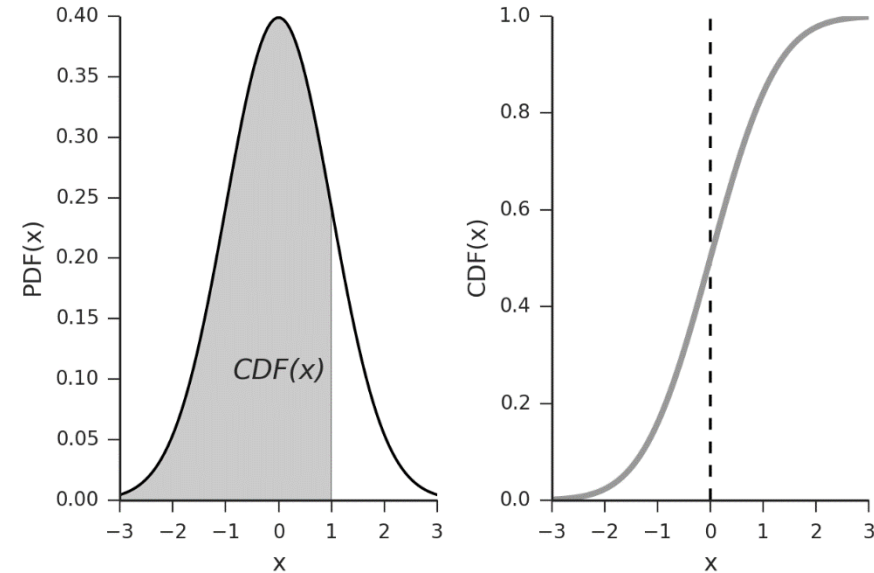
- Normal (Gaussian) distribution

- pdf: $p(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$
 - μ, σ – parameters
 - We'll see their real meaning next time
 - cdf: doesn't exist as a function, we can integrate numerically

- Complete derivation of the formula: [here](#)

- **Standard normal distribution:** $\mu = 0, \sigma = 1$

- Mainly for convenience



Central Limit Theorem

- The sum of many independent random variables tends to a normal distribution even if the original random variables are not normally distributed
 - In other words: The sampling distribution of the mean of any independent random variable will be normal or nearly normal if the sample is large enough
 - Large enough?
 - $n \in [30; 40]$ for most statisticians, but more is better
- Example: customers in a shop
 - Every customer has their own behavior, reasons, money, etc.
 - We can treat them as random variables with unknown distributions
 - The shop's earnings are approximately normally distributed
 - If there are many customers
 - We **don't even care** about the many sources of error: they will produce a **normal distribution** anyway

Summary

- Probability
- Combinatorics
- Algebra of events
- Conditional probability
- Applications
- Statistical distributions
- Central limit theorem

The image features a white background with two blue decorative bars. The top bar is a solid blue strip. The bottom bar is a gradient blue strip that transitions from a lighter blue on the left to a darker blue on the right. The word "Questions?" is centered in a blue, sans-serif font.

Questions?