Mathematics and Statistics for Data Science Session 10

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The linear span of finite set of vectors

Let $S = \{A_i : i = 1, ..., k\}$ be a set of k vectors in $\mathbb{R}^n$.
A vector $x \in \mathbb{R}^n$ is said to be spanned by S if we can write

$$x = \sum_{i=1}^{k} \alpha_i A_i \text{ for some } \alpha_i \in \mathbb{R}$$

The set of all vectors by S is called the linear span $L(S)$ of S.

Example

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}; \quad S_2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$L(S_1) \subseteq L(S_2)$$

$$L(S_1) = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \quad \forall \alpha \in \mathbb{R}$$

$$L(S_2) = \left\{ \begin{bmatrix} \alpha \\ 0 \end{bmatrix}; \begin{bmatrix} 0 \\ \beta \end{bmatrix} \right\} \ \forall \alpha, \beta \in \mathbb{R}$$

or

$$L(S_2) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \forall \alpha, \beta \in \mathbb{R}$$

$\Rightarrow L(S_1) \subseteq L(S_2)$

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Definition (Basis)

A basis for $\mathbb{R}^n$ is a finite set S of vectors that is linear independent and spans $\mathbb{R}^n$.
If S is also orthogonal it is called an orthogonal basis.

Properties

- Every basis for $\mathbb{R}^n$ contains exactly "n" elements.
- Any linear independent set of vectors in $\mathbb{R}^n$ is a subset of a basis for $\mathbb{R}^n$
- Any set of n linear independent vectors in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$

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Example

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} ; \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\} \quad \text{is a basis for } \mathbb{R}^2$$

$$L(S) = \left\{ \begin{bmatrix} \alpha \\ 0 \end{bmatrix} ; \begin{bmatrix} 0 \\ \beta \end{bmatrix} \right\} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \quad \forall \alpha, \beta$$

$$L(S) = \mathbb{R}^2$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$0 = \alpha_1 \rightarrow \alpha_1 = 0$$
$$0 = 3\alpha_2 \rightarrow \alpha_2 = 0$$

$$L(S) = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} ; \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\} = L(S_2) = \left\{ \begin{bmatrix} 1 \\ \beta \end{bmatrix} ; \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\} \quad \forall \beta$$

$$L(S_3) = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} ; \begin{bmatrix} \beta \\ 0 \end{bmatrix} \right\} = \mathbb{R}_x$$

$$L(S_4) = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \mathbb{R}_y$$

$$\begin{bmatrix} 1 \\ \beta \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$1 - \alpha_1 + \alpha_2 \cdot 0 \to \alpha_1 = 1$$

$$\beta = \alpha_1 \cdot 0 + 3 \cdot \alpha_2 \to \alpha_2 = \frac{\beta}{3} \quad \forall \beta$$

Write $v = (2, -5, 3)$ as a linear combination of $u_1 = (1, -3, -2)$; $u_2 = (2, -4, 2)$; $u_3 = (1, -5, 7)$

1.)
Are $u_1,\ u_2,\ u_3$ linear dependent?

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix}$$

$$\begin{aligned} 0 &= \alpha_1 + 2\alpha_2 + \alpha_3 \qquad \rightarrow \alpha_1 = -2\alpha_2 - \alpha_3 \\ 0 &= -3\alpha_1 - 4\alpha_2 - 5\alpha_3 \\ 0 &= -2\alpha_1 - \alpha_2 + 7\alpha_3 \end{aligned}$$

$$\begin{cases} 0 = 6\alpha_2 + 3\alpha_3 - 4\alpha_2 - 5\alpha_3 \\ 0 = 4\alpha_2 + 2\alpha_3 - \alpha_2 + 7\alpha_3 \end{cases}$$

$$\rightarrow \alpha_1 = 0,\ \alpha_2 = 0,\ \alpha_3 = 0$$

$\Rightarrow$ independent