Mathematics and Statistics for Data Science Lecture 3 Exam Exercises

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Exam Exercises 1.)

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} ; \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} ; \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Please show if the vectors are linearly dependent or independent.

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = \alpha_1 \begin{bmatrix}1\\0\\1\end{bmatrix} + \alpha_2 \begin{bmatrix}1\\2\\1\end{bmatrix} + \alpha_3 \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\begin{aligned} 0 &= 2\alpha_2 \rightarrow \alpha_2 = 0 \\ 0 &= \alpha_1 + \alpha_2 \rightarrow \alpha_1 = 0 \\ 0 &= \alpha_1 + \alpha_2 + \alpha_3 \rightarrow \alpha_3 = 0 \end{aligned}$$

$\Rightarrow$ Vectors are independent.

2.)

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} ; \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} ; \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} ; \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \quad \subset \mathbb{R}^3$$

Please show if the vectors are linearly dependent or independent.

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = \alpha_1 \begin{bmatrix}1\\0\\1\end{bmatrix} + \alpha_2 \begin{bmatrix}1\\2\\1\end{bmatrix} + \alpha_3 \begin{bmatrix}0\\0\\1\end{bmatrix} + \alpha_4 \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\begin{cases} 0 = \alpha_1 + \alpha_2 + \alpha_4 \\ 0 = 2\alpha_2 + \alpha_4 \\ 0 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \end{cases} \rightarrow \alpha_4 = -2\alpha_2$$

$$\begin{cases} 0 = \alpha_1 + \alpha_2 - 2\alpha_2 \\ 0 = \alpha_1 + \alpha_2 + \alpha_3 - 2\alpha_2 \end{cases}$$

$$\begin{cases} 0 = \alpha_1 - \alpha_2 \\ 0 = \alpha_1 - \alpha_2 + \alpha_3 \end{cases} \rightarrow \alpha_1 = \alpha_2;\ \alpha_3 = 0$$

$$\begin{bmatrix}1\\1\\1\end{bmatrix} = \alpha_1 \begin{bmatrix}1\\0\\1\end{bmatrix} + \alpha_2 \begin{bmatrix}1\\2\\1\end{bmatrix} + \alpha_3 \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\begin{aligned} 1 &= \alpha_1 + \alpha_2 & \qquad (2) \to \alpha_1 &= \frac{1}{2} \\ 1 &= 2\alpha_2 & \qquad (1) \to \alpha_2 &= \frac{1}{2} \\ 1 &= \alpha_1 + \alpha_2 + \alpha_3 & \qquad (3) \to \alpha_3 &= 0 \end{aligned}$$

$\Rightarrow$ Vectors are dependent.

3.)

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} ; \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$$

Please show if the vectors are linearly dependent or independent.

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\begin{cases} 0 = \alpha_1 + \alpha_2 \\ 0 = 0 \end{cases} \rightarrow \alpha_1 = -2\alpha_2$$

$\Rightarrow$ Vectors are dependent.

4.)
Find a line passing through $P = (1, 1)$ and parallel to $A = (-1, -2)$.

$$\{P + tA\} = L(P, A)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$\rightarrow \quad \begin{aligned} x &= 1 - t \\ y &= 1 - 2t \end{aligned}$$

$$\begin{aligned} t &= 1 - x \\ y &= 1 - 2(1 - x) \\ y &= 1 - 2 + 2x \\ y &= 2x - 1 \end{aligned}$$

5.)

Find a line $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ passing through point $P = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ generated by vector $v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x = t + 1$$
$$y = t + 1$$
$$z = t + 1$$

$$\begin{cases} x = y & \rightarrow \textit{Plane} \\ x = z & \rightarrow \textit{Plane} \end{cases}$$

$$\rightarrow x = y = z$$

6.)

Find a line $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ passing through point $P = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ generated by vector $v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned} x &= t + 2 \\ y &= t \\ z &= t + 1 \end{aligned}$$

$$\begin{cases} x = y + 2 \\ z = y + 1 \end{cases}$$