Mathematics and Statistics for Data Science Session 5 Preliminaries on Matrixes

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November 10, 2020

Matrix multiplication

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + a_{14}c_{41}$$

$$\begin{bmatrix} \boxed{a_{11} \quad a_{12} \quad a_{13} \quad a_{14}} \\ a_{21} \quad a_{22} \quad a_{23} \quad a_{24} \end{bmatrix} \begin{bmatrix} \boxed{b_{11}} & b_{12} & b_{11} \\ \boxed{b_{21}} & b_{22} & b_{23} \\ \boxed{b_{31}} & b_{32} & b_{33} \\ \boxed{b_{41}} & b_{42} & b_{43} \end{bmatrix} = \begin{bmatrix} \boxed{c_{11}} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

$$c_{22} = a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} + a_{24}c_{42}$$

$$\begin{bmatrix} a_{11} \quad a_{12} \quad a_{13} \quad a_{14} \\ \boxed{a_{21} \quad a_{22} \quad a_{23} \quad a_{24}} \end{bmatrix} \begin{bmatrix} b_{11} & \boxed{b_{12}} & b_{11} \\ b_{21} & \boxed{b_{22}} & b_{23} \\ b_{31} & \boxed{b_{32}} & b_{33} \\ b_{41} & \boxed{b_{42}} & b_{43} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & \boxed{c_{22}} & c_{23} \end{bmatrix}$$

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Definition: Transpose of a Matrix
Transpose of a matrix A (left) then the transpose is A^T (right):

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

Properties:

- $(A \cdot B)^T = B^T A^T$
- $(A \cdot C \cdot B)^T = B^T C^T A^T$
- $(A \cdot B \cdot C \cdot D)^T = D^T C^T B^T A^T$

Definition: Inversion of a Matrix

We say a matrix B is the inversion of a matrix A, if $B \cdot A = I$ (identity matrix). A is a squared matrix.
$B \cdot A \neq A \cdot B$

Remark

An identity matrix is a matrix with following structure

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

Find a matrix B such that $B \cdot A = I$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$a = 1$$

$$2a + b = 0$$

$$c = 0$$

$$2c + d = 1$$

$$B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \text{(inversion of matrix A)}$$

$$A \cdot B = I \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$a + 2c = 1$$

$$b + 2d = 0$$

$$c = 0$$

$$d = 1$$

$$B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \text{(inversion of matrix A)}$$

$$B \cdot A = A \cdot B, I = I \text{ (true if } A \text{ is an inversion of } B\text{)}$$

$$B \cdot A \neq A \cdot B \text{ (in general)}$$

Definition: Orthogonal Matrix

We say that A is an orthogonal matrix if $A^T = A^{-1}$.
(A^{-1} is the inverse of A)

Remark

$A^T \cdot A = I = A \cdot A^T$ *(A is a squared matrix)*

Definition: Determinant

Given a squared matrix A and fixed i, then

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

Where A_{ij} is called a minor ij.

Remark
Minor i, j is the matrix excluding the i row and j column

Properties:

1. $\det A = \det A^T$
2. If A has row(s)/column(s) of zeros, then $\det A = 0$
3. If A has two identical rows or columns, then the $\det A = 0$
4. If A has rows/columns linear depending, then $\det A = 0$
5. If two rows/columns are interchanging, then $\det A =$ - $\det B$
6. If a row/column is multiplied by a scalar k, then $\det A = k$ $\det B$

7. Adding rows/columns, then $\det A = \det B$
8. $\det (A \cdot B) = \det (A) \cdot \det (B)$
9. If A is invertible, then $\det A \neq 0$
10. If $\det A \neq 0$ then A is invertible
11. If A is invertible, then $A \cdot x = 0$ has only zero solutions
12. $A \cdot x = 0$ with only zero solution if $\det A \neq 0$

Remark
$A \cdot x = 0$ *states the homogenous part of a linear system.*

Example for property 7:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},\ B = \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix} \rightarrow \det(A) = -2,\ \det(B) = 6 - 8 = -2$$

$|A|$ = determinant $A \rightarrow$ operator $\rightarrow$ scalar (positive or negative).
$|A| = \det A \rightarrow$ scalar

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \rightarrow \det(A) = a_{11}a_{22} - a_{12}a_{21}$$

Example:

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 4 - 6 = -2$$

$$\det A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Definition: Eigenvalue, Eigenvector of a Matrix

Given a squared matrix $A_{n\times n}$, we define eigenvalue of this matrix the scalar λ, such that $A \cdot v = \lambda v$, where v is called eigenvector.

Remark

A is a squared matrix $(n \times n)$ and v is its vector $n \times 1$.

$$A \cdot v = \lambda v \rightarrow (A - \lambda I) \cdot v = 0 \rightarrow v \in N(A - \lambda I)$$

$$\textit{where} \quad N \quad \textit{is} \quad \textit{the} \quad \text{"}\textit{Null}\text{"} \quad \textit{or} \quad \text{"}\textit{Kernel}\text{"} \quad \textit{of} \quad (A - \lambda I).$$

$$\det(A - \lambda I) = 0 \rightarrow \text{ Characteristic polynomial of the matrix } A.$$

Exam Exercises

1.)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Please calculate the eigenvalues and eigenvectors.

$$
\begin{aligned}
&\det\,(A - \lambda I) = 0\\
&\det\left(\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0\\
&\Leftrightarrow \det\begin{bmatrix}1-\lambda & 2\\ 3 & 4-\lambda\end{bmatrix} = 0\\
&\Leftrightarrow (1-\lambda)\cdot(4-\lambda) - 6 = 0\\
&\Leftrightarrow 4 - 5\lambda + \lambda^2 - 6 = 0\\
&\Leftrightarrow \lambda^2 - 5\lambda - 2 = 0\\
&\Leftrightarrow \lambda = \frac{5 \pm \sqrt{33}}{2} \Rightarrow \lambda_1 = \frac{5-\sqrt{33}}{2},\ \lambda_2 = \frac{5+\sqrt{33}}{2}
\end{aligned}
$$

We say that a vector v belongs to the "Kernel" of $(A - \lambda I)$ if

$$(A - \lambda I) \cdot v = 0$$

$$\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix} \right) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \frac{5-\sqrt{33}}{2} & 0 \\ 0 & \frac{5-\sqrt{33}}{2} \end{bmatrix} \right) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \frac{5+\sqrt{33}}{2} & 0 \\ 0 & \frac{5+\sqrt{33}}{2} \end{bmatrix} \right) \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} c \\ d \end{bmatrix}$$

2.)

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$$

Please calculate the eigenvalues and eigenvectors.

$$
\begin{aligned}
&\det\left(A - \lambda I\right) = 0 \\
&\det\left(\begin{bmatrix}1 & 2\\ 0 & 4\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0 \\
&\Leftrightarrow \det\begin{bmatrix}1-\lambda & 2\\ 0 & 4-\lambda\end{bmatrix} = 0 \\
&\Leftrightarrow (1-\lambda)(4-\lambda) = 0 \\
&\Rightarrow \lambda_1 = 1 \rightarrow v_1 \\
&\Rightarrow \lambda_2 = 4 \rightarrow v_2
\end{aligned}
$$

$$(A - \lambda I) \cdot v = 0$$

$$\left(\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2b = 0, 3b = 0 \rightarrow b = 0, \forall a \rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} a$$

$$\left(\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right) \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} -3 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -3c + 2d = 0 \rightarrow d = \frac{3c}{2}, 0c + 0d = 0 \rightarrow v_2 = \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix} c$$