Mathematics and Statistics for Data Science Session 7

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Linear Space

A set V is a linear space if it satisfies the following 3 categories of 10 axioms.

1. Closure Axioms

Axiom 1: Closure under addition

$$\forall\, x,\ y \in V \quad \exists\, z \in V \quad : \quad z = x + y$$

Axiom 2: Closure under multiplication by number

$$\forall\, x \in V \text{ and } \alpha \in \mathbb{R} \quad \exists\, z \in V \quad : \quad z = x\alpha = \alpha x$$

2. Axioms for addition

Axiom 3 : Commutative law

$$\forall\, x,\ y \in V \qquad x + y = y + x$$

Axiom 4 : Associative law

$$\forall\, x,\ y,\ z \in V \qquad (x + y) + z = x + (y + z) = y + (x + z)$$

Axiom 5: Existence of zero element

$$\exists\, \varnothing \in V, \qquad \forall\, x \in V : x + \varnothing = \varnothing + x = x$$

Axiom 6: Existence of negatives

$$\forall\, x \in V \qquad \exists - x = (-1)x : \quad x + (-x) = 0$$

3. Axioms for multiplication by number (scalar)

Axiom 7: Associative law

$$\forall\, x \in V \text{ and } \alpha,\ \beta \in \mathbb{R}: \quad \alpha(\beta x) = (\alpha\beta)x = \beta(\alpha x)$$

Axiom 8: Distributive law for addition in V

$$\forall\, x,\ y \in V \text{ and } \alpha \in \mathbb{R}: \quad \alpha(x+y) = \alpha x + \beta x$$

Axiom 9: Distributive law for addition of numbers

$$\forall\, x \in V \text{ and } \alpha,\ \beta \in \mathbb{R}: \quad (\alpha+\beta)x = \alpha x + \beta x$$

Axiom 10: Existence of Identity

$$\forall\, x \in V \quad 1 \cdot x = x$$

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Subspaces

Given a linear space V, a non-empty subset S of V is called subspace of V if S is also a linear space under the same operation as V.
Vector space is a specific form of linear space.

Example: $V = \{1, t, t^2, t^3\} \quad t \in \mathbb{R}$
Is function $f(t) = 3 + 2t^2 + \frac{1}{3}t$ linearly dependent in V?

$$f(t) = a \cdot 1 + b \cdot t + c \cdot t^2 + d \cdot t^3 \rightarrow a = 3,\ b = \frac{1}{3},\ c = 2,\ d = 0$$

Finite Basis

A finite set S of elements of a linear space V is called finite basis for V if S is independent and spans V.

Property

Let V be a finite dimensional linear space, then every finite basis for V has the same number of elements.

$$V_1 = \{1, t, 2t^2\} \rightarrow V_2 = \{1, 2t, t^2\}$$

They have the same elements, basis for polynomial order 2.

Example:

- Please verify that set $S = \{1, t, 2t\}$ is not a basis for the linear space of the polynomial of order "2".

$$t^2 = \alpha_1 1 + \alpha_2 t + 2\alpha_3 t \quad \nexists \alpha_1,\ \alpha_2,\ \alpha_3$$

- Is set S a basis for the linear space of polynomial of order "1"?

Recalls (Inner Products)

Example: Dot-product

$$xy = \sum_{i=1}^{n} x_i y_i \quad x = (x_1 ..., x_n), y = (y_1 ..., y_n)\ x \in V, y \in V$$

Definition: Let V be a real linear vector space, the inner product is a mapping $V \times V \to \mathbb{R}$ scalar.

Axiomatic properties:

- Symmetric: $(x, y) = (y, x)$
- Linear $(x, ay + bz) = a(x, y) + b(x, z)$ where $a, b \in \mathbb{R}$ (homogenity included)
- Non-negative: $(x, x) \geq 0$ with $(x, x) = 0$ iff $x = 0$

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Exam Exercise

$$V = \{1, t, t^2\}$$

$$(x(t), y(t)) = \int_a^b x(t)y(t)dt \quad a, b \in \mathbb{R},\ x, y \in V$$

Calculate a and b, if they exist, such that

1. $(1, t) = 0$
2. $(1, t^2) = 0$
3. $(t, t^2) = 0$

Remark

Two elements are orthogonal, iff their inner product equals 0.

1.)

$$(1,t)=\int_a^b 1t\,\mathrm{d}t=\frac{t^2}{2}\bigg|_b^a=\frac{b^2}{2}-\frac{a^2}{2}$$

$\Rightarrow$ if $a=-b\rightarrow(1,t)=0$, if $a=b\rightarrow(1,t)=0$ (trivial)

2.)

$$(1,t^2)=\int_a^b 1t^2\,\mathrm{d}t=\frac{t^3}{3}\bigg|_b^a=\frac{b^3}{3}-\frac{a^3}{3}$$

$\Rightarrow$ Just for $a=b\rightarrow(1,t^2)=0$ (trivial)

3.)

$$(t,t^2)=\int_a^b tt^2\,\mathrm{d}t=\frac{t^4}{4}\bigg|_b^a=\frac{b^4}{4}-\frac{a^4}{4}$$

$\Rightarrow$ if $a=-b\rightarrow(t,t^2)=0$, if $a=b\rightarrow(t,t^2)=0$ (trivial)

4.)
Calculate a and b ($a < b$), if they exist, such that the inner product defined as

$$(t, t^2 + t) = \int_a^b t(t^2 + t)\,\mathrm{d}t = 0 \to \int_a^b (t^3 + t^2)\,\mathrm{d}t = 0$$

$$\left(\frac{t^4}{4}+\frac{t^3}{3}\right)\Big|_b^a = \frac{b^4}{4}-\frac{a^4}{4}+\frac{b^3}{3}-\frac{a^3}{3}$$

$$\to 3b^4-3a^4+4b^3-4a^3=0$$

$$\to b^3(3b+4)-a^3(3a+4)=0$$

$$\to b^3(3b+4)=a^3(3a+4) \text{ if } a=0 \to b^3(3b+4)=0 \Leftrightarrow b=-\frac{4}{3}$$

$\Rightarrow$ With the two values $a=0,\ b=-\frac{4}{3}$ orthogonal.

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Mappings

Definition

A mapping T from a set V to another set W is denoted by
$T: \; v \to w$ by $x \to y = T(x)$
It associates exactly one element $y \in W$ for each $x \in V$
V is called domain, $T(x) \in W$ is called image of "x" under T.
Once can say: T maps x onto $T(x)$.

Linear Transformations

Let V and W be linear spaces over the same scalar field.
A mapping $T : v \to w$ is a linear transformation if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \quad \forall \, x, \; y \in V \text{ and } \alpha, \; \beta \in \; \mathbb{R}$$

Exam Exercises:

Exercise 1:

$$T : x \to 2x$$

Please verify if map T is a linear map.

$$T(x_1 + x_2) = T(x_1) + T(x_2) \text{ true if T is a linear map}$$

$$T(x_1 + x_2) = 2(x_1 + x_2)$$
$$T(x_1) = 2x_1; \quad T(x_2) = 2x_2$$
$$2(x_1 + x_2) = 2x_1 + 2x_2$$
$$\Rightarrow \text{Linear}$$

Exercise 2:

$$T : x \to 2x + 1$$

Please verify if map T is a linear map.

$$T(x_1 + x_2) = T(x_1) + T(x_2) \text{ true if T is a linear map}$$

$$T(x_1 + x_2) = 2(x_1 + x_2) + 1$$
$$T(x_1) = 2x_1 + 1; \quad T(x_2) = 2x_2 + 1$$
$$2(x_1 + x_2) + 1 \neq 2x_1 + 1 + 2x_2 + 1$$
$$\Rightarrow \text{Not linear}$$

Exercise 3:

$$T : x \to 2x^2$$

Please verify if map T is a linear map.

$$T(x_1 + x_2) = T(x_1) + T(x_2) \text{ true if T is a linear map}$$

$$T(x_1 + x_2) = 2(x_1 + x_2)^2 = 2(x_1^2 + 2x_2x_1 + x_2^2)$$
$$T(x_1) = 2x_1^2; \quad T(x_2) = 2x_2^2$$
$$2(x_1^2 + 2x_2x_1 + x_2^2) \neq 2x_1^2 + 2x_2^2$$
$$\Rightarrow \text{Not linear}$$

Exercise 4:

$$T : (x, y) \to x + y$$

Please verify if map T is a linear map.

$$\Rightarrow \text{Linear}$$

Exercise 5:

$$T : (x, y) \to x \cdot y$$

Please verify if map T is a linear map.

$$\Rightarrow \text{Not linear}$$

Nullity and Rank

Definition of Null Space:
Consider a linear transformation $T : v \to w$
The set $N(T) = \{x; x \in V, T(x) = 0\}$ is called the null space of T.

Property:
Every linear transformation maps" $\varnothing$" onto"$\varnothing$".

Dimensions
The dimension of the null space $N(T)$ is called the nullity of T.
The dimension of the range $T(V)$ is called rank of T.
Range is image (or collection) of domain.

Theorem (Nullity and Rank):
If dim V is finite, then dim $N(T) +$ dim $T(V) =$ dim V.