Mathematics and Statistics for Data Science Lecture 2 Some Geometric Concepts

Prof. Dr.-Ing. Paolo Mercorelli

October 22, 2020

LEUPHANA
UNIVERSITY OF LÜNEBURG

Orthogonality of Vectors

Definition

Two nonzero vectors A, $B \in \mathbb{R}^n$ are called perpendicular, or orthogonal if $A \cdot B = 0$.
$\|A + B\|^2 = \|A\|^2 + \|B\|^2 + 2A \cdot B$
If $A \cdot B = 0 \rightarrow \|A + B\|^2 = \|A\|^2 + \|B\|^2$ (Phytagoras)

Projections: Angle Between Vectors in n-Space

Consider 2 vectors A, $B \in \mathbb{R}^n$
$A = C + \alpha B$ with $C \cdot B = 0$

$$\rightarrow B \cdot A = B \cdot C + \alpha B \cdot B$$

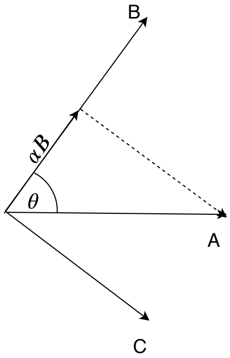
$$\rightarrow B \cdot A = \alpha \left\|B\right\|^2$$

$$\Rightarrow \alpha = \frac{B \cdot A}{\left\|B\right\|^2}$$

We have $cos\theta = \frac{\|\alpha B\|}{\|A\|}$

So that $\quad cos\theta = \frac{|\alpha|\|B\|}{\|A\|} = \frac{|A \cdot B|}{\|A\|\|B\|}$

$$\Rightarrow \|A\| \, \|B\| \, cos\theta = A \cdot B$$



Linear Combination

Definition 1

X is called a linear combination of vectors (set of vectors) $\{A_i\}$ $i = 1, ..., n$ if

$$X = \sum_i x_i A_i$$

x_i are scalars

Definition 2

We say that a set of vectors $\{A_i\}$ $i = 1, ..., n$ are linear independent if

$$\sum_{i=1}^{n} \alpha_i A_i = 0 \rightarrow \alpha_i = 0 \qquad \forall i$$

This means that set $\{A_i\}$ generates the vector 0 only in a trivial way.

Remark

If we have a set of vectors belonging to $\mathbb{R}^n$ consisting of "m" elements
if $m > n \rightarrow$ the set is in a set of depending elements
if $m \leq n \rightarrow$ the set must be checked

$$S = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix} ; \begin{bmatrix} 0 \\ 3 \end{bmatrix} ; \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\} \rightarrow \textit{this set is a depending set } \forall \alpha, \beta \in \mathbb{R}$$

Exam Exercise

1.) Given the following vectors
$V_1 = (1, 2, 3)$; $V_2 = (-1, 3, -1)$; $V_3 = (0, 0, 1)$
Are these vectors linearly dependent or independent?
Please justify your answer.

$$\sum_{i=1}^{n} \alpha_i A_i = 0, \sum_{i=1}^{3} \alpha_i A_i = 0$$

$$\rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$$\{A_i\} = \{\ v_1, v_2, v_3\}$$

$$\alpha_1 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \cdot \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} + \alpha_3 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\alpha_1 - \alpha_2 + \alpha_3 \cdot 0 = 0$$

$$2\alpha_1 + 3\alpha_2 + \alpha_3 \cdot 0 = 0$$

$$3\alpha_1 - \alpha_2 + \alpha_3 = 0$$

$\alpha_1 = \alpha_2$
$2\alpha_1 + 3\alpha_2 = 0 \rightarrow \alpha_1 = 0 \rightarrow \alpha_2 = 0 \rightarrow \alpha_3 = 0$
This means that the given set of vector represents a linearly independent set.

If one of these three vectors is dependent, the set of vectors is called dependent, even though two are independent. If in a n-space vector there are n+1 (or more) vectors, they are dependent.
e.g. dependent:
$V_1 = (1, 0)$; $V_2 = (1, 1)$; $V_3 = (2, -1)$
$V_1 = (1, 0)$; $V_2 = (1, 1)$; $V_3 = (2, 2)$;

$V_1 = (1, 2, 3)$, $V_2 = (2, 0, 0)$ are independent (all elements zero when solving).
$V_1 = (1, 0, 0)$, $V_2 = (2, 0, 0)$ are dependent.

Definition: Point

A point is a vector (n-tuple) in $\mathbb{R}^n$

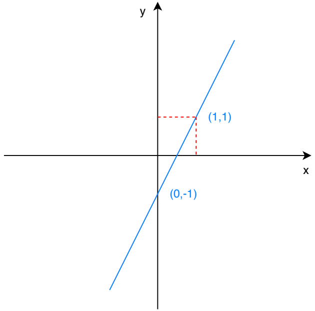
Definition: Line

Let P be a point and A a non-zero vector.
A line through P and parallel to A is the set of points
$L(P; A) = \{P + tA : t \in \mathbb{R}^n\} = \{P + tA\}$.
A point Q is on the line $L(P; A)$ if $Q = P + tA$ for some t

LEUPHANA
UNIVERSITY OF LÜNEBURG

Examples:

Find a line passing through point $P = (1, 1)$ with direction $v = (-1, -2)$.



$$L(P, v) = P + t \cdot v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$\begin{aligned} x &= 1 - t \rightarrow t = 1 - x \\ y &= 1 - 2t \\ y &= 1 - 2(1 - x) = 2x - 1 \end{aligned}$$