Mathematics and Statistics for Data Science Session 5 Preliminaries on Matrixes

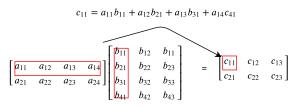
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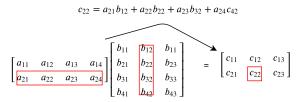
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Matrix multiplication









Definition: Transpose of a Matrix Transpose of a matrix A (left) then the transpose is A^T (right):

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

Properties:

$$(A \cdot B)^T = B^T A^T$$

$$(A \cdot C \cdot B)^T = B^T C^T A^T$$

$$(A \cdot B \cdot C \cdot D)^T = D^T C^T B^T A^T$$



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Definition: Inversion of a Matrix

We say a matrix B is the inversion of a matrix A, if $B \cdot A = I$ (identity matrix). A is a squared matrix. $B \cdot A \neq A \cdot B$

Remark

An identity matrix is a matrix with following structure

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

Find a matrix B such that $B \cdot A = I$



$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$a = 1$$
$$2a + b = 0$$
$$c = 0$$
$$2c + d = 1$$
$$B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \text{ (inversion of matrix A)}$$



$$A \cdot B = I \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$a + 2c = 1$$
$$b + 2d = 0$$
$$c = 0$$
$$d = 1$$

$$B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$
 (inversion of matrix A)

$$B \cdot A = A \cdot B, I = I$$
 (true if A is an inversion of B)
 $B \cdot A \neq A \cdot B$ (in general)



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Definition: Orthogonal Matrix

We say that A is an orthogonal matrix if $A^T = A^{-1}$. $(A^{-1}$ is the inverse of A)

Remark

$$A^T \cdot A = I = A \cdot A^T$$
 (A is a squared matrix)



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Definition: Determinant

Given a squared matrix A and fixed i, then

$$\det A = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det A_{ij}$$

Where A_{ij} is called a minor ij.

Remark

Minor i, j is the matrix excluding the i row and j column

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Properties:

- 1. det $A = \det A^T$
- 2. If A has row(s)/column(s) of zeros, then det A = 0
- 3. If A has two identical rows or columns, then the det A=0
- 4. If A has rows/columns linear depending, then det A = 0
- 5. If two rows/columns are interchanging, then det $A = \det B$
- If a row/column is multiplied by a scalar k, then det A = k det B



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- 7. Adding rows/columns, then det $A = \det B$
- 8. $\det(A \cdot B) = \det(A) \cdot \det(B)$
- 9. If A is invertible, then det $A \neq 0$
- 10. If det $A \neq 0$ then A is invertible
- 11. If A is invertible, then $A \cdot x = 0$ has only zero solutions
- 12. $A \cdot x = 0$ with only zero solution if det $A \neq 0$

Remark

 $A \cdot x = 0$ states the homogenous part of a linear system.



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Example for property 7:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix} \rightarrow \det(A) = -2, \det(B) = 6 - 8 = -2$$

$$|A|=$$
 determinant $A o$ operator o scalar (positive or negative).

$$|A| = \det A \rightarrow \operatorname{scalar}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \to \det(A) = a_{11}a_{22} - a_{12}a_{21}$$



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Example:

$$\det\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} = 4 - 6 = -2$$

$$\det A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$



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Definition: Eigenvalue, Eigenvector of a Matrix

Given a squared matrix $A_{n\times n}$, we define eigenvalue of this matrix the scalar λ , such that $A\cdot v=\lambda v$, where v is called eigenvector.

Remark

A is a squared matrix $(n \times n)$ and v is its vector $n \times 1$.

$$A \cdot v = \lambda v \to (A - \lambda I) \cdot v = 0 \to v \in N(A - \lambda I)$$
 where N is the "Null" or "Kernel" of $(A - \lambda I)$.
$$\det(A - \lambda I) = 0 \to \text{ Characteristic polynomial of the matrix } A.$$



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Exam Exercises

1.)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Please calculate the eigenvalues and eigenvectors.



$$\begin{split} \det \left(A - \lambda I \right) &= 0 \\ \det \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) &= 0 \\ \Leftrightarrow \det \begin{bmatrix} 1 - \lambda & 2 \\ 3 & 4 - \lambda \end{bmatrix} &= 0 \\ \Leftrightarrow \left(1 - \lambda \right) \cdot \left(4 - \lambda \right) - 6 &= 0 \\ \Leftrightarrow 4 - 5\lambda + \lambda^2 - 6 &= 0 \\ \Leftrightarrow \lambda^2 - 5\lambda - 2 &= 0 \\ \Leftrightarrow \lambda &= \frac{5 \pm \sqrt{33}}{2} \Rightarrow \lambda_1 &= \frac{5 - \sqrt{33}}{2}, \ \lambda_2 &= \frac{5 + \sqrt{33}}{2} \end{split}$$



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We say that a vector v belongs to the "Kernel" of $(A-\lambda I)$ if

$$(A - \lambda I) \cdot v = 0$$

$$\begin{pmatrix}
\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix} \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{pmatrix}
\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \frac{5-\sqrt{33}}{2} & 0 \\ 0 & \frac{5-\sqrt{33}}{2} \end{bmatrix} \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} a \\ b \end{bmatrix} \\
\begin{pmatrix}
\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \frac{5+\sqrt{33}}{2} & 0 \\ 0 & \frac{5+\sqrt{33}}{2} \end{bmatrix} \end{pmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} c \\ d \end{bmatrix}$$



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2.)

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$$

Please calculate the eigenvalues and eigenvectors.



$$\begin{aligned} &\det \left(A - \lambda I \right) = 0 \\ &\det \left(\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0 \\ &\Leftrightarrow \det \begin{bmatrix} 1 - \lambda & 2 \\ 0 & 4 - \lambda \end{bmatrix} = 0 \\ &\Leftrightarrow (1 - \lambda)(4 - \lambda) = 0 \\ &\Rightarrow \lambda_1 = 1 \to \nu_1 \\ &\Rightarrow \lambda_2 = 4 \to \nu_2 \end{aligned}$$



$$(A - \lambda I) \cdot v = 0$$

$$\left(\begin{bmatrix}1 & 2\\0 & 4\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \rightarrow \begin{bmatrix}0 & 2\\0 & 3\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow 2b = 0, 3b = 0 \rightarrow b = 0, \forall a \rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} a$$

$$\left(\begin{bmatrix}1 & 2\\0 & 4\end{bmatrix} - \begin{bmatrix}4 & 0\\0 & 4\end{bmatrix}\right)\begin{bmatrix}c\\d\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \rightarrow \begin{bmatrix}-3 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}c\\d\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow$$
 $-3c + 2d = 0 \rightarrow d = \frac{3c}{2}, 0c + 0d = 0 \rightarrow v_2 = \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix} c$