Mathematics and Statistics for Data Science Session 11

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Similarity

Two squared matrices A and B are similar if there exists a matrix P such that $B = P^{-1}AP$

Properties

- A matrix A is diagonalizable iff there exists a matrix P such that $I = P^{-1}AP$.
- A matrix A can be represented by diagonal matrix D iff there exists a basis S of V consisting of eigenvectors of A. The diagonal elements of D are the eigenvalues of A.
- If two matrices A and B are similar, then they have the same eigenvalue.
- Matrix A is diagonalizable iff it has n linearly independent eigenvectors and $D = P^{-1}AP$, where $D = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$ and P is the matrix of the eigenvectors.

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Exercise:
Given $A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$,
verify that matrix A and B are similar.

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Given $A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$ please calculate matrix P, such that $I = P^{-1}AP$.

$$\det(A - \lambda \cdot I) = 0$$

$$\det\left(\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0 \to \det\left(\begin{bmatrix} 4-\lambda & 2 \\ 3 & -\lambda - 1 \end{bmatrix}\right) = 0$$

$$-(4-\lambda)(1+\lambda) - 6 = 0 \to -\lambda^2 + 3\lambda + 10 = 0 \to \lambda^2 - 3\lambda - 10 = 0$$

$$\Rightarrow \lambda_1 = -2$$

$$\Rightarrow \lambda_2 = 5$$

$$\lambda_1 : \left(\begin{bmatrix}4 & 2\\3 & -1\end{bmatrix} + 2\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \rightarrow \begin{bmatrix}6 & 2\\3 & 1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\rightarrow 6a + 2b = 0.\ \ 3a + b = 0 \rightarrow b = -2a\ \ \forall a$$

$$\Rightarrow v_1 = \begin{bmatrix}1\\-3\end{bmatrix} a\ \forall a.$$

$$\lambda_2 : \left(\begin{bmatrix}4 & 2\\3 & -1\end{bmatrix} - 5\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \rightarrow (\begin{bmatrix}-1 & 2\\3 & -6\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\rightarrow -a + 2b = 0 \rightarrow a = 2b\ \ \forall b, 3a - 6b = 0$$

$$\Rightarrow v_2 = \begin{bmatrix}2\\1\end{bmatrix} b\ \forall b.$$

$$P = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$$

$$P^{-1} \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \; P^{-1}P = I \to \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\to a - 3b = 1, \; 2a + b = 0, \; c - 3d = 0, \; 2c + d = 1$$

$$\to a = \frac{1}{7}, \; b = -\frac{2}{7}, \; c = \frac{3}{7}, \; d = \frac{1}{7}$$

$$P^{-1} = \begin{bmatrix} \frac{1}{7} & -\frac{2}{7} \\ \frac{3}{7} & \frac{1}{7} \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & -\frac{2}{7} \\ \frac{3}{7} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} -2 & 4 \\ 15 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} -14 & 0 \\ 0 & 35 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix} = B$$