Mathematics and Statistics for Data Science Session 9

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The Gauss-Jordan Method for the solution of linear systems

We can:

- Interchange two equations
- Multiply all terms in an equation by a non-zero scalar
- Add in algebraic sense two equations

to obtain a "triangular system (matrix)" or diagonal system (matrix).

$$a_{11}x_1 + a_{12}x_2 = c_1$$
$$a_{21}x_1 + a_{22}x_2 = c_2$$

A possible idea is to obtain the following diagonal system (matrix):

$$ax_1 + 0x_2 = b$$
$$0x_1 + cx_2 = d$$

For some $a, b, c, d \in \mathbb{R}$

$$x_2 = \frac{d}{c}, \quad x_1 = \frac{b}{a}$$

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Another possible idea is to obtain the following triangular system (matrix):

$$ex_1 + fx_2 = g$$
$$0x_1 + hx_2 = i$$
For some $e, f, g, h, i \in \mathbb{R}$

Example:

$$\begin{cases} x_1 + 3x_2 = 1 \\ x_1 + 2x_2 = 2 \end{cases}$$

$$\begin{cases} x_1 + 3x_2 = 1 \\ 0 + x_2 = -1 \end{cases} \rightarrow x_1 = 4$$

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Injective linear transformation

Considering a linear transformation $T : V \to W$ then T is injective if whenever $T(x) = T(y) \to x = y$ with $x, y \in V$ (which is logically equivalent to the contrapositive, $x \neq y \to T(x) \neq T(y)$)

Theorem
$T : V \to W$ is a linear transformation, then T is injective iff the kernel of $T = 0$ $N(T) = K(T) = 0$ (trivial kernel).

$$T : \mathbb{R}^2 \to \mathbb{R}, \quad T(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = 1,\ T(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = 1 \text{ non-injective}$$

$$T : (x, y) \to ax + by \ a, b \in \mathbb{R}$$

$$T : \mathbb{R}^2 \to \mathbb{R}, \quad T(\begin{bmatrix}1\\2\end{bmatrix}) = 1,\ T(\begin{bmatrix}1\\3\end{bmatrix}) = 1 \text{ non-injective}$$

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$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1 \to a \cdot 1 + b \cdot 2 = 1 \to 1 - 2b \to \begin{bmatrix} 1-2b & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\to K(T) = \begin{bmatrix} 1 \\ \frac{2b-1}{b} \end{bmatrix} x$$

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1, \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 2 \to b = 1 \to a = -1 \to \begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$\to N(A) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x \to A \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} x = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} x = -x + x = 0$$

$$\dim(K(T)) + \dim(T) = \dim(V)$$

$$T : \mathbb{R}^2 \to \mathbb{R}^2, \quad T(\begin{bmatrix}1\\4\end{bmatrix}) = \begin{bmatrix}1\\2\end{bmatrix},\ T(\begin{bmatrix}1\\3\end{bmatrix}) = \begin{bmatrix}1\\2\end{bmatrix}$$

$$\text{non-injective} \quad T : (x, y) \to (ax + by, cx + dy)$$

$$\begin{cases} a + 4b = 1 \\ c + 4d = 2 \\ a + 3b = 1 \\ c + 3d = 2 \end{cases} \to a = 1,\ b = 0,\ c = 2,\ d = 0$$

$$\Rightarrow \begin{bmatrix}1 & 0\\2 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \to N(A) = K(T) = \begin{bmatrix}0\\ *\end{bmatrix} y$$

$$T : \mathbb{R}^2 \to \mathbb{R}^3, \quad T \to A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 4 \end{bmatrix} \to T : (x, y) = (x+2y, 3x+4y, x+4y)$$

$$A \cdot N(A) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} x + 2y = 0 \\ 3x + 4y = 0 \\ x + 4y = 0 \end{cases} \to y = 0,\ x = 0 \text{ (injective, 0 dimension)}$$

$$T : \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad T \rightarrow A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \\ 1 & 2 \end{bmatrix} \rightarrow T : (x, y) = (x+2y, 3x+6y, x+2y)$$

two columns, which are dependent

$$A \cdot N(A) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 6 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} x + 2y = 0 \\ 3x + 6y = 0 \\ x + 2y = 0 \end{cases} \rightarrow y = 0,\ x = 0 \text{ (injective, 0 dimension)}$$

$$T : \mathbb{R}^2 \to \mathbb{R}, \quad T\begin{bmatrix}1\\2\end{bmatrix} = 2, T\begin{bmatrix}1\\2\end{bmatrix} = 1$$

$\Rightarrow$ not a transformation, not a function, does not work!
We can speak in the "same way" about Transformation, Matrices and linear systems. We say that there is an isomorphismus in between!
Example: Is the following transformation a injective one?

$$\begin{cases} x_1 + 2x_2 = 1 \\ 2x_1 + 4x_2 = 2 \end{cases}$$

$$\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$

1. step:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_{1H} \\ x_{2H} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow N(A) = \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_{2H}$$

The linear system does not represent an injective transformation.

Example: Is the following transformation a injective one?

$$T : \mathbb{R}^2 \to \mathbb{R}, \quad T \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2, T \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1$$

$\Rightarrow$ not a transformation, not a function, doesnt work!

$$T : (x, y) \to ax + by \quad a, b \in \mathbb{R}$$

$$a + 2b = 2$$

$$a + 2b = 1$$

$$\Rightarrow 2 = 1 \to \text{Contradiction!}$$

Theorem

Suppose that $T : V \to W$ is an injective linear transformation and a set $v = \{v_1, v_2, ..., v_t\}$ consists of the linear independent vector, then $R = \{T(v_1), T(v_2), ..., T(v_t)\}$ is a set of linear independent vectors.

Please look at the following example:

$$T : (x, y) = (2x, 2x + y)$$
$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \to \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$
$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} \to \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Theorem

Linear transformation and bases.
Suppose that $T : V \to W$ is a linear transformation and $B = \{v_1, v_2, ..., v_m\}$ a basis of V, then T is injective iff $R = \{T(v_1), T(v_2), ..., T(v_m)\}$ is an independent set of W.

Try to do the following example with the solution in the following slides:

$$T : (x, y) \to (ax + by, cx + dy) \quad T : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \to \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$a + 2b = 1 \rightarrow b = \frac{1}{2}$$

$$c + 2d = 2 \rightarrow d = -1$$

$$a = 2$$

$$c = 4$$

$$\Rightarrow$$

$$A = \begin{bmatrix} 2 & -\frac{1}{2} \\ 4 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -\frac{1}{2} \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x - \frac{y}{2} = 0 \rightarrow y = 4x$$

$$4x - y = 0 \rightarrow y = 4x$$

$$\Rightarrow N(A) = \begin{bmatrix} 1 \\ 4 \end{bmatrix} x \rightarrow \text{not injective}$$