Mathematics and Statistics for Data Science Session 8

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5.) Exam Exercise
Calculate the linear map $T: \mathbb{R}^3 \to \mathbb{R}^2$ such that

$$T(\begin{bmatrix}1\\1\\2\end{bmatrix}) \to \begin{bmatrix}1\\3\end{bmatrix}, \quad T(\begin{bmatrix}1\\2\\1\end{bmatrix}) \to \begin{bmatrix}0\\0\end{bmatrix} \text{ and } T(\begin{bmatrix}1\\0\\0\end{bmatrix}) \to \begin{bmatrix}1\\1\end{bmatrix}$$

$$T: (x, y, z) \in \mathbb{R}^3 \to (ax + by + cz, dx + ey + fz) \in \mathbb{R}^2$$

$$
\begin{aligned}
a + b + 2c &= 1 \\
d + e + 2f &= 3
\end{aligned}
$$

$$
\begin{aligned}
a + 2b + c &= 0 \\
d + 2e + f &= 0
\end{aligned}
$$

$$
\begin{aligned}
a &= 1 \\
d &= 1
\end{aligned}
$$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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Exam Exercise

1.)
Given $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ please calculate null space (kernel) of it, if it exists.

$$\mathrm{Null}(T) = \mathrm{Null}(A) = \{x : Ax = 0\} \ T : \mathbb{R}^3 \to \mathbb{R}^2$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \forall a, b, c \in \mathbb{R}$$

$$\begin{aligned} a + 3b = 0 &\Rightarrow a = -3b \\ 2a + b + c = 0 &\Rightarrow -6b + b + c = 0 \\ &\Rightarrow c = 5b \end{aligned}$$

$$\Rightarrow x = \begin{bmatrix} -3b \\ b \\ 5b \end{bmatrix} = b \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix}$$

2.)
Given $A = \begin{bmatrix} 1 & 3 & 0 \end{bmatrix}$ please calculate null space (kernel) of it, if it exists.

$$\text{Null}(T) = \text{Null}(A) = \{x : Ax = 0\} \; T : \mathbb{R}^3 \to \mathbb{R}^1$$

$$\begin{bmatrix} 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$a + 3b = 0 \Rightarrow a = -3b \quad \forall c$$

$$\Rightarrow N(A) = \begin{bmatrix} -3b \\ b \\ c \end{bmatrix} = b \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow \text{Dimensionality } = 2$$

3.)

Given $A = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ please calculate null space (kernel) of it, if it exists.

$$\mathrm{Null}(T) = \mathrm{Null}(A) = \{x : Ax = 0\} \quad T : \mathbb{R}^3 \to \mathbb{R}^3$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} a + 3b &= 0 \\ a + c &= 0 \\ a &= 0 \end{aligned}$$

$$\Rightarrow N(A) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \text{ Dimensionality } = 0$$

4.)

Given $A = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 0 & 1 \\ 2 & 6 & 0 \end{bmatrix}$ please calculate null space (kernel) of it, if it exists.

$$\mathrm{Null}(T) = \mathrm{Null}(A) = \{x : Ax = 0\} \; T : \mathbb{R}^3 \to \mathbb{R}^3$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 1 & 0 & 1 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$a + 3b = 0 \Rightarrow a + 3b = 0$$
$$a + c = 0 \Rightarrow a = -c$$
$$2a + 6b = 0 \Rightarrow 2a + 6b = 0$$

$$\Rightarrow N(A) = b \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix} \rightarrow \text{Dimensionality } = 1$$

Matrices associated to the the transformation

For all $S, T : v \to w$ and scalar c we have
$m(S+T) = m(S) + m(T)$ and $m(cT) = cm(T)$.
Moreover, if $m(S) = m(T) \to S = T$.

Properties

- Associative: $A(BC) = (AB)C = ABC$
- Right Distributive law: $(A+B)C = AC + BC$
- Left Distributive law: $C(A+B) = CA + CB$

  Remark: $AB \neq BA$ (not commutative)

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Application to linear systems

Theorem
For each linear system $\sum_{k=1}^{n} a_{ik}x_k = c_i$ for $i = 1, 2, ..., m$ there is (x_k are unknown) associated another system $\sum_{k=1}^{n} a_{ik}x_k = 0$ for $i = 1, 2, ..., m$ (homogenous system)

The general solution of the linear system consists of the sum of all independent solutions of the homogeneous system plus a particular solution of the non-homogeneous one.

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Remark

For $i = 1, 2$

$$a_{11}x_1 + a_{12}x_2 = c_1$$
$$a_{21}x_1 + a_{22}x_2 = c_2$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

1.)

$$\begin{cases} x_1 + 2x_2 = 1 \\ 2x_1 + 3x_2 = 3 \end{cases}$$

Homogeneous:

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 = 0, \quad x_2 = 0$

Non-homogeneous:

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 3 \end{bmatrix} x_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$\Rightarrow x_1 = 3, \quad x_2 = -1$

2.)

$$\begin{cases} x_1 + 2x_2 + x_3 = 1 \\ x_1 + 3x_2 + 2x_3 = 2 \end{cases}$$

3.)

$$\begin{cases} x_1 + 2x_2 = 1 \\ 2x_1 + 4x_2 = 2 \end{cases}$$

Homogeneous:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 = -2x_2, \rightarrow N(A) = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}, N(A) = x_1 \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} \text{ (inverse)}$$

Non-homogeneous:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow x_1 = 1 - 2x_2, \quad 2 = 2 \;\; \forall x_2 \to v = \begin{bmatrix} 1 - 2x_2 \\ x_2 \end{bmatrix}$$

Considering for instance a particular solution of this $(x_2 = 1)$ we have:

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{1}$$

$$v^* = v_1 + N(A)$$

$$v^* = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$\forall \; x_2$.