

## Using the Fourier Transform to Find

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$$

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June 5, 2023

We will derive the value of this sum for all  $x \in \mathbb{R}_{\geq 0}$  using fourier transforms.

*Proof.* First consider the convolution of an impulse train and a carefully chosen sinc wave, where  $k$  is some positive real number.

$$\begin{aligned} y(t) &= \sum_{n=-\infty}^{\infty} \delta(t - 2\pi n) * \frac{k}{\pi} \text{sinc}\left(\frac{k}{\pi}t\right) \\ &= \frac{k}{\pi} \sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{k}{\pi}(t - 2\pi n)\right) \\ &= \frac{k}{\pi} \sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{k}{\pi}t - 2kn\right) \end{aligned} \tag{1}$$

Note that a convolution in the time domain is a product in the frequency domain. So,

$$\begin{aligned} Y(\omega) &= \mathcal{F}\left(\sum_{n=-\infty}^{\infty} \delta(t - 2\pi n)\right) \cdot \mathcal{F}\left(\frac{k}{\pi} \text{sinc}\left(\frac{k}{\pi}t\right)\right) \\ &= \sum_{n=-\infty}^{\infty} \delta(\omega - n) \cdot \frac{k}{\pi} \frac{\pi^2}{k} \text{rect}\left(\frac{\pi\omega}{2k}\right) \\ &= \pi \sum_{n=-\infty}^{\infty} \delta(\omega - n) \cdot \text{rect}\left(\frac{\pi\omega}{2k}\right) \end{aligned}$$

Note that for all  $n$  such that  $|n| > \frac{k}{\pi}$ , the rect function effectively “zeroes out” the delta centered at  $w = n$ . Since these deltas are centered at discrete values of  $n$ ,  $|n|$  can be at most  $\left\lfloor \frac{k}{\pi} \right\rfloor$ . Thus,

$$Y(\omega) = \pi \sum_{n=-\lfloor k/\pi \rfloor}^{\lfloor k/\pi \rfloor} \delta(w - n)$$

For  $n \neq 0$ , this is just the fourier transform of a sum of cosine waves. We take the inverse fourier transform of  $Y(\omega)$  to find  $y(t)$ :

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}(Y(\omega)) \\ &= \mathcal{F}^{-1}(\pi\delta(\omega)) + \sum_{n=1}^{\lfloor k/\pi \rfloor} \cos(nt) \\ &= \frac{1}{2} + \sum_{n=1}^{\lfloor k/\pi \rfloor} \cos(nt) \end{aligned}$$

And this must be equal to the earlier result (1), when we simply took the convolution instead. So,

$$\frac{1}{2} + \sum_{n=1}^{\lfloor k/\pi \rfloor} \cos(nt) = \frac{k}{\pi} \sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{k}{\pi}t - 2kn\right) \quad (2)$$

Which is an intriguing result by itself.

To get the desired summation, simply plug in  $t = 0$  into equation (2) and use the fact that sinc is an even function.

$$\begin{aligned}
\frac{1}{2} + \sum_{n=1}^{\lfloor k/\pi \rfloor} 1 &= \frac{k}{\pi} \sum_{n=-\infty}^{\infty} \text{sinc}(-2kn) \\
\frac{1}{2} + \lfloor k/\pi \rfloor &= \frac{k}{\pi} \sum_{n=-\infty}^{\infty} \text{sinc}(2kn) \\
\frac{1}{2} + \lfloor k/\pi \rfloor &= \frac{k}{\pi} \left( \text{sinc}(0) + \sum_{n=1}^{\infty} \text{sinc}(2kn) + \text{sinc}(-2kn) \right) \\
\frac{\pi}{k} \left( \frac{1}{2} + \lfloor k/\pi \rfloor \right) &= 1 + 2 \sum_{n=1}^{\infty} \text{sinc}(2kn) \\
2 \sum_{n=1}^{\infty} \frac{\sin(2kn)}{2kn} &= \frac{\pi}{k} \left( \frac{1}{2} + \lfloor k/\pi \rfloor \right) - 1 \\
\sum_{n=1}^{\infty} \frac{\sin(2kn)}{n} &= \pi \left( \frac{1}{2} + \lfloor k/\pi \rfloor \right) - k
\end{aligned}$$

Since  $k$  is just some positive real number, we can let  $x = 2k$ . Then,

$$\sum_{n=1}^{\infty} \frac{\sin(xn)}{n} = \pi \left( \frac{1}{2} + \left\lfloor \frac{x}{2\pi} \right\rfloor \right) - \frac{x}{2}$$

□