Convex optimization HW3

KALLEL Mahdi

November 22, 2020

Exercise 1

1. From the previous homework we have :

$$\|u\|_1^* = \begin{cases} 0 & \text{if } \|u\|_\infty \leq 1 \\ +\infty & \text{otherwise} \end{cases}$$

(LASSO) can be written as :

$$\begin{aligned} \min_w \frac{1}{2}\|z\|_2^2 + \lambda\|w\|_1 \\ \text{s.t. } z = Xw - y \end{aligned} \quad [1.1]$$

The dual is :

$$\begin{aligned} g(v) &= \inf_{w,z} \mathcal{L}(w, z, v) \\ &= y^T v + \inf_z \left(\frac{1}{2}\|z\|_2^2 + v^T z\right) + \inf_w \left(\lambda\|w\|_1 - \left(X^T v\right)^T w\right) \end{aligned}$$

$h : z \mapsto \frac{1}{2}\|z\|_2^2 + v^T z$ is convex and differentiable.

$\nabla h(z) = z + v$ $\nabla h(z) = 0$ iff $z = -v$.

$\implies$ The minimum of h is $\frac{1}{2}\|v\|_2^2 - \|v\|_2^2 = -\frac{1}{2}\|v\|_2^2$.

The second term can be reformulated using $\|\cdot\|_1^*$:

$$\inf_w \lambda\|w\|_1 - \left(X^T v\right)^T w = \sup_w \left(\frac{1}{\lambda}X^T v\right)^T w - \|w\|_1 = \left\|\frac{1}{\lambda}X^T v\right\|_1^*$$

Thus,

$$g(v) = y^T v - \frac{1}{2}\|v\|_2^2 + \left\|\frac{1}{\lambda}X^T v\right\|_1^*$$

The dual can be re writte as :

$$\max_{v} y^T v - \frac{1}{2}\|v\|_2^2$$
$$\text{s.t. } \left\|\frac{1}{\lambda}X^T v\right\|_\infty \leq 1$$

The constraint can be reformulated

$$\begin{aligned}
\left\|\frac{1}{\lambda}X^T v\right\|\infty \leq 1 &\text{ iff } \forall i, -1 \leq \left[\frac{1}{\lambda}X^T v\right]_i \leq 1 \\
&\text{iff } \forall i, \left[\frac{1}{\lambda}X^T v\right]_i \leq 1 \text{ and } \left[-\frac{1}{\lambda}X^T v\right]_i \leq 1 \\
&\text{iff } Av \leq \lambda \mathbf{1_{2d}} \text{ where } A = \begin{pmatrix} X^T \\ -X^T \end{pmatrix}
\end{aligned}$$

The problem can be re written as :

$$\boxed{\begin{aligned} &\min_{v} v^T Q v + p^T v \quad \text{with } Q = \frac{1}{2}I_n,\ p = -y,\ b = \lambda \cdot \mathbf{1}_{2d} \\ &\text{s.t. } Av \leq b \end{aligned}}$$

2. For questions 2 and 3 please check the code