Convex optim dm 2 :

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Ex 1 :

1) $\min_{x} c^T x$ $= (P)$

$Ax = b$

$x \geq 0$

$\mathcal{L}(x, \mu, \lambda) = \inf_{x} \{ c^T x - \mu^T x + \lambda^T (Ax - b) \}$

$\frac{\partial \mathcal{L}}{\partial x} = c - \mu + A^T \lambda$

If $c - \mu + A^T \lambda = 0$:

$g(\mu, \lambda) = - b\lambda^T$

If $c - \mu + A^T \lambda = V \neq 0$

$V = (v_i)_{i \neq 0}$ take $x = \begin{pmatrix} 0 \\ \frac{s}{v_1} \\ 0 \end{pmatrix}$

$\mathcal{L}(x, \mu, V) = x^T V - b^T \lambda = s - b^T \lambda$

$\lim_{s \to -\infty} \mathcal{L}(x_s, \mu, V) = -\infty$

$\Rightarrow \mathrm{dom}\, g_P = \{ \lambda, \mu \mid c - \mu + A^T \lambda = 0 \}$

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$$g_p(\lambda, \mu) = \begin{cases} -b\lambda^T & \lambda, \mu \in dom\, g \\ -\infty & else \end{cases}$$

The dual is P': $\lambda' = -\lambda$

$$\max\ -\lambda^T b \quad \Longleftrightarrow \quad \max\ \lambda'^T b$$
$$A^T\lambda + C = \mu \qquad\qquad A^T\lambda' \leq C$$
$$\mu \geq 0$$

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2) (D) $\min\ -b^T y$ $\Rightarrow$ $\mathcal{L}(y, v) = \inf_y \{-b^T y + v^T(A^T y - c)\}$
$A^T y \leq C$

$$\frac{\partial \mathcal{L}}{\partial y} = -b + Av$$

If $Av = b$:

$$g(v) = -v^T C$$

If $Av - b = x \neq 0$

$x = \begin{pmatrix} | \\ x_i \\ | \end{pmatrix} \neq 0$ take $y = \begin{pmatrix} 0 \\ \frac{s}{x_i} \\ 0 \end{pmatrix} \Rightarrow \mathcal{L}(y, v) = -v^T C + s$

$$\lim_{s \to -\infty} \mathcal{L}(y, v) = 0$$

$$\Rightarrow g = \begin{cases} -v^T C & \text{if } Av^T = b \\ -\infty & else \end{cases}$$

$$(D') = \max\ -v^T C$$
$$Av = b$$
$$v \geq 0$$

3) Self dual :

$$\min_{x,y} c^T x - b^T y \iff \min_{x} c^T x + \max_{y} b^T y$$

$Ax = b \qquad Ax = b \qquad A^T y$

$x \geq 0 \qquad x \geq 0$

$A^T y \leq c$

we are dealing with linear programs so $D \iff D'$

$P \iff P'$

$$\Rightarrow S-D \iff P' + D' = \max_{\lambda} \lambda^T b + \max_{v} -v^T C$$

$A^T \lambda \leq C \qquad Av = b, \; v \geq 0$

$$= \max_{v,\lambda} \{ \lambda^T b \underbrace{- v^T C}_{\in \mathbb{R}} \} = \max \{ b^T \lambda - C^T \lambda \}$$

$Av = b$

$A^T \lambda \leq C$

$v \geq 0$

$$= \max_{x,y} \{ b^T y - C^T x \} = -\min_{x,y} \{ C^T x - b^T y \}$$

$Ax = b$

$A^T y \leq 0$

$x \geq 0$

if x, y solve $-\min_{x,y} \{..\}$ they solve $\min \{ \}$

$\Rightarrow$ (S-D) is self dual.

4) I) we have:

$$g_{S-D}(\lambda,\mu,\vartheta) = g_P(\lambda,\mu) + g_D(\vartheta)$$

$$\max_{\lambda,\mu,\vartheta \in \mathrm{dom}\, g_{S-D}} g_{S-D}(\lambda,\mu,\vartheta) = \max_{\lambda,\mu \in \mathrm{dom}\, g_{S-D}} g_D(\lambda,\mu) + \max_{\vartheta \in \mathrm{dom}\, g_{S-D}} g_P(\vartheta)$$

$$= \max_{\lambda,\mu \in \mathrm{dom}\, \underline{\underline{g_D}}} g_D(\lambda,\mu) + \max_{\vartheta \in \mathrm{dom}\, \underline{\underline{g_P}}} g_P(\vartheta)$$

Let x^*, y^* be solution to P), (D)

strong duality $\Rightarrow$

$$c^T x^* = \max_{\lambda,\mu \in \mathrm{dom}\, g_D} g_D(\lambda,\mu)$$

$$-b^T y^* = \max_{\vartheta \in \mathrm{dom}\, g_P} g_P(\vartheta)$$

$\forall x, y \in \mathrm{dom}(S\text{-}D)$ $c^T x - b^T y \geq g_{S-D}(\lambda,\mu,\vartheta)$

$\forall x, y \ldots$ $c^T x - b^T y \geq \max_{\lambda,\mu,\vartheta \in \mathrm{dom}\, g_{S-D}} g_{S-D}(\mu,\lambda,\vartheta) = c^T x^* - c^T y^*$

for $x = x^*$, $y = y^*$, we get equality between primal and dual

$\Rightarrow$ x^*, y^* Is a solution of the primal (S-D)

4) II) from 3) we have

$$\min_{\substack{x,y \\ \in dom(S-D)}} f(x,y) \Leftrightarrow -\min_{\substack{x,y \\ \in dom(S-D)}} f(x,y)$$

If x^*, y^* are solution of the primal

$$f(x^*,y^*) = \min_{x,y \in \ldots} f(x,y)$$

$$f(x^*,y^*) = -\min_{x,y} f(x,y)$$

$$\Rightarrow f(x^*,y^*) = 0$$

Exercice 2:

1) $\|y\|_1^* = \sup_{\|x\|_1 \leq 1} x^T y$

$x^T y = \sum x_i y_i \leq \sum x_i \|y\|_\infty \leq \|y\|_\infty \sum x_i \leq \|y\|_\infty \sum |x_i|$

$\sup_{\|x\|_1 \leq 1} x^T y \leq \|y\|_\infty \|x\|_1 \leq \|y\|_\infty$

for $x_\alpha = \begin{pmatrix} 0 \\ 0 \\ sign(y_{max}) \\ 0 \\ 0 \end{pmatrix} \leftarrow \arg\max_k y_k$

We have $x_\alpha^T y = \|y\|_\infty \Rightarrow \|y\|_1^* = \|y\|_\infty$

$f^*(y) = \sup_x (y^T x - \|x\|_1) = \begin{cases} 0 & \text{if } \|y\|_\infty \leq 1 \\ +\infty & \|y\|_\infty > 1 \end{cases}$

2) $(R2S) = \min_x \|Ax - b\|_2^2 + \|x\|_1$

$\Leftrightarrow \min_{y = Ax - b} \|y\|_2^2 + \|x\|_1$

$L(x, y, v) = \inf_{x,y} \{ \|y\|_2^2 + \|x\|_1 + v^T(y + b - Ax) \}$

$= \underbrace{\inf_x \{ \|x\|_1 - v^T A x + v^T b \}}_{a} + \inf_y \{ \|y\|_2^2 + v^T y \}$

$b \quad (\inf_y \{ \|y\|_2^2 - v^T y \}$

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(a) from 1) we have $a = \begin{cases} \nu^T b & \text{if } \|\nu^T A\| \leq 1 \\ -\infty & \|\nu^T A\|_\infty > 1 \end{cases}$

(b) $\inf_y \{ \|y\|_2^2 - \nu^T y \}$

$$\frac{\partial}{\partial y} = 2y - \nu = 0 \Rightarrow \frac{\nu}{2} = y$$

our function is convex so this is a minimum

$$\inf_y \{ \|y\|_2^2 - \nu^T y \} = -\frac{1}{4} \|\nu\|_2^2$$

$$(a) + (b)\ L(x, y, \nu) = \begin{cases} \nu^T b - \frac{1}{4} \|\nu\|_2^2 & \text{if } \|\nu^T A\|_\infty \leq 1 \\ -\infty & \text{otherwise} \end{cases}$$

$$\text{Dual RLS} = \max_{\|\nu^T A\|_\infty \leq 1} \nu^T b - \frac{1}{4} \|\nu\|_2^2$$

Exercice 3:

1) Let $y : \mathbb{R}^{n \times n} \to \mathbb{R}^n$

$$w \to \begin{pmatrix} z(w, x_i, y_i) \\ | \end{pmatrix} = y(w)$$

Let $g : \mathbb{R}^n \times \mathbb{R}^{n^2} \to \mathbb{R}$

$$z, w \to \frac{1}{nZ} \mathbb{1}^T z + \frac{1}{2} \|W\|_2^2 = g(w, z)$$

$$(\text{Sep1}) = \min_{z = y(w)} g(w, z)$$

And we have $\begin{cases} z_i \geq 1 - y_i w^T x_i \\ z_i \geq 0 \end{cases} \Leftrightarrow z_i \geq \max(0, 1 - y_i w^T x_i)$

$$(\text{Sep1}) \Leftrightarrow \min g(w, y(w)) \Leftrightarrow \min_{w} \; g(w, z) \quad z = y(w)$$

$$(\text{Sep2}) \Leftrightarrow \min g(w, z) \quad z \geq y(w), \; z \geq 0$$

Let w, x be feasible points of (Sep 2)

by construction $(w, y(w))$ is feasible for sep 2

and $g(w, y(w)) \leq g(w, x)$

$$\Rightarrow \min_{w} g(w, y(w)) \leq \underbrace{\min_{\substack{w \\ x \geq y(w) \\ x \geq 0}}}_{dom(w)} g(w, x) \qquad (I)$$

since $y(w) \in dom(w)$

$$\min_{w, x \in dom(w)} g(w, x) \leq \min_{w} g(w, y(w)) \qquad (II)$$

(I) + (II) $\quad \min_{w} g(w, y(w)) = \min_{\substack{w \\ x \geq 0 \\ x \geq 1 - y^T w x_i}} g(w, x)$

(Sep 2) $\Leftrightarrow$ (Sep 1)

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2) (Sep 2) : $\min_{w, x} \frac{1}{n\mathcal{E}} \mathbb{1}^T x + \frac{1}{2} \|w\|_2^2$

$x_i \geq 1 - y_i (w^T x)$

$x \geq 0$

$$\mathcal{L}(w, x, \lambda, \pi)_{\pi, \lambda \geq 0} = \frac{1}{n\mathcal{E}} \mathbb{1}^T x + \frac{\|w\|^2}{2} - \pi^T x - \sum \lambda_i (z_i + y_i (w^T x_i) - 1)$$

$$g(\lambda, \pi) = \inf_{w, x} \{ \mathcal{L}(w, x, \lambda, \pi) \}$$

$$\frac{\partial \mathcal{L}}{\partial w} = w - \sum \lambda_i y_i x_i.$$

$$\frac{\partial \mathcal{L}}{\partial z} = \frac{1}{n\gamma} \mathbb{1} - \pi - \lambda$$

If $\frac{1}{n\gamma}\mathbb{1} - \pi - \lambda = \nu \neq 0$, take $z = \left(\frac{s}{\nu_i}\right)$ and $w = 0$

$$\lim_{s \to -\infty} \mathcal{L}(0, z_s, \lambda, \pi) = -\infty$$

$$\Rightarrow \operatorname{dom} g = \left\{ \lambda, \pi \;/\; \pi + \lambda = \frac{1}{n\gamma} \right\}$$

$$g(\lambda) = \inf_w \left(\frac{1}{2} \|w\|_2^2 + \sum \lambda_i (1 - y_i w^T x_i) \right)$$

$$\frac{\partial \mathcal{L}}{\partial w} = 0 \Rightarrow w = \sum \lambda_i y_i x_i$$

$$g(\lambda) = \frac{1}{2} \left\| \sum \lambda_i y_i x_i \right\|_2^2 + \sum_i \lambda_i - \sum_{i,j} y_i y_j x_j^T x_i \lambda_j \lambda_i$$

$$= \frac{1}{2} \left\| \sum \lambda_i y_i x_i \right\|_2^2 + \sum \lambda_i - \underbrace{\sum_j y_j x_j^T \lambda_j}_{w^T} \underbrace{\sum_i y_i \lambda_i x_i}_{w}$$

$$g(\lambda) = \lambda^T \mathbb{1} - \frac{1}{2} \left\| \sum \lambda_i y_i x_i \right\|_2^2$$

$$\Rightarrow g(\lambda) = \begin{cases} \lambda^T \mathbb{1} - \frac{1}{2} \left\| \sum \lambda_i y_i x_i \right\|_2^2, & \lambda \geq 0, \pi + \lambda = \frac{1}{n\gamma} \\ -\infty & \text{else.} \end{cases}$$

Exercice 4:

$$\max_a a^T x \iff -\min_a -a^T x \quad (P)$$
$$c^T a \leq d \qquad\qquad c^T a \leq d$$

$$\mathcal{L}(a,\lambda) = -a^T x + \lambda^T (c^T a - d), \lambda \geq 0$$

$$\frac{\partial \mathcal{L}}{\partial a} = -x + c\lambda$$

$$g(\lambda) = \begin{cases} -\lambda^T d & c\lambda = x, \lambda \geq 0 \\ -\infty & \text{else} \end{cases}$$

The dual is:

$$\begin{pmatrix} \max_\lambda -\lambda^T d \\ c\lambda = x \\ \lambda \geq 0 \end{pmatrix} \quad (P)$$

and since it is an LP strong duality holds

$$\Rightarrow \max_\lambda -\lambda^T d = \min -a^T x$$
$$c\lambda = x \qquad c^T a \leq d$$
$$\lambda \geq 0$$

$$\Rightarrow (-)\max_\lambda -\lambda^T d = \min_{\substack{c\lambda = x \\ \lambda \geq 0}} \lambda^T d = \max_{c^T a \leq d} a^T x$$

Let $E_1 = \{ x \mid \exists \tilde{x}, \tilde{x} \geq 0, c^T \tilde{x} = x, d^T \tilde{x} \leq b \}$

$E_2 = \{ x \mid \sup_{a \in P} a^T x \leq b \}$

let $x \in E_2$

from strong duality we have

$$\sup_{a \in P} a^T x = \min_{\substack{Cz = x \\ z \geq 0}} d^T z$$

$\Rightarrow \exists z^*; z^* \geq 0, Cz^* = x$ and $\sup_{a \in P} a^T x = d^T z^*$

since $\sup a^T x \leq b \Rightarrow d^T z^* \leq b \Rightarrow x \in E_1 \Rightarrow E_2 \subset E_1$

let $x \in E_1 \Rightarrow \exists z' \geq 0, C^T z' = x, d^T z' \leq b$

$$\sup_{a \in P} a^T x = \min_{\substack{Cz \\ z \geq 0}} d^T z \leq \underset{z' \text{ feasible}}{d^T z'} \leq b \Rightarrow E_1 \subset E_2$$

$\Rightarrow E_2 = E_1$

$$\Rightarrow \min_{x \in E_1} C^T x = \min_{x \in E_2} C^T x$$