Exercice 2.12 :

a) $E = \{ x \mid \beta \leq a^T x \leq \beta \}$

$= \{ x \mid \beta \leq a^T x \} \cap \{ x \mid a^T x \leq \alpha \}$

E is convex as an intersection of two half spaces

b) $E = \{ x \mid \alpha_i \leq x_i \leq \beta_i , i = 1 \dots n \}$

$\Rightarrow E = \bigcap_i E_i$ where $E_i = \{ x \mid \alpha_i \leq a_i x \leq b_i \}$

with $a_i = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \leftarrow i$

from (a) E is convex as an intersection of convex sets.

c) E is convex as an intersection of two half spaces (a half space being convex)

d) $E = \{ x \mid \| x - x_0 \|_2 \leq \| x - y \|_2 \ \forall y \in S \}$

$\| x - x_0 \|_2^2 \leq \| x - y \|_2^2 \iff (x - x_0)^T (x - x_0) - (x - y)^T \dots \leq 0$

$\iff -2x^T x_0 + 2 x^T y \leq 0$

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2.12 let $f_y : \mathbb{R}^n \to \mathbb{R}$

$x \to y^T x - x_0^T x$

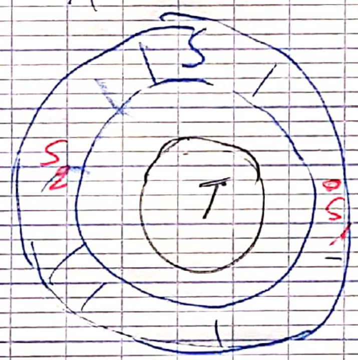
f_y is affine in x and $\{x \mid \|x - x_0\| \leq \|x - y\|\} = E_y$

$$E_y = f_y^{-1}(\mathbb{R}^-)$$

E_y is convex as the inverse image of a convex set by an affine function.

$E = \bigcap_{y \in S} E_y \Rightarrow E$ is convex.

e) No, let $E = \mathbb{R}^2$



$s_1, s_2 \in \{x \mid d(x, S) < d(x, T)\} = E$, $\frac{s_1 + s_2}{2} \in T$

$\frac{s_1 + s_2}{2} \notin E$,

Ex 2.12 (suite)

f) $E = \{x \mid x + S_2 \subseteq S_1\}$ S_1 cvx

let $x, y \in E \Rightarrow \forall u \in S_2$ $x + u \in S_1$
$y + u \in S_1$

$\Rightarrow \lambda(x+u) + (1-\lambda)(y+u) \in S_1$

$\Rightarrow \underbrace{\lambda x + (1-\lambda) y}_{z} + u \in S_1 \quad \forall u \in S_2$

$z + u \in S_1 \ \forall u \in S_2 \Rightarrow z \in E \Rightarrow E$ convex

g) Could not do.

Exercice 3.21:

a) $x \mapsto \|A_i x - b_i\|$ is convex as a composition of an affine transform and a norm (which is cvx)

f is convex as the point maximum of convex fcts

b) $f(x) = \sum_{i=1}^{k} |x|_{[i]} = \max_{1 \leq i_1 \leq i_k \leq n} |x_{i_1}| + \dots + |x_{i_k}|$

$\forall_{1 \leq i_1 < i_k \leq n}$ $|x_{i_1}| + \dots + |x_{i_k}|$ is convex.

$\Rightarrow f$ is convex as the point maximum of cvx functions

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Exercice 3.32

a) f, g convex, $\nearrow$ and positive

let $\lambda \in [0,1]$, $x \le y$ (without loss of generality)

$$* \quad fg(\lambda x+(1-\lambda)y) \le \lambda^2 fg(x)+(1-\lambda)^2 fg(y)+\lambda(1-\lambda)\left(f(x)g(y)+g(x)f(y)\right)$$

$$f\nearrow \Rightarrow f(x)g(y) \le fg(y) \quad (a)$$

$$g\nearrow \Rightarrow g(x)f(y) \le fg(y) \quad (b)$$

$$fg\nearrow \Rightarrow fg(x) \le fg(y) \quad (c)$$

$$(a)+(b) \Rightarrow * \le \lambda^2 fg(x)+\left((1-\lambda)^2+2\lambda(1-\lambda)\right) fg(y)$$

$$* \le \lambda^2 fg(x)+(1-\lambda^2) fg(y)$$

$$\text{or } \lambda^2 \le \lambda \text{ and } fg(x) \le fg(y)$$

$$** \Rightarrow \underbrace{\lambda^2 fg(x)+(1-\lambda^2)fg(y)}_{I} \le \underbrace{\lambda fg(x)+(1-\lambda)fg(y)}_{II}$$

$$** \quad II - I = \underbrace{(\lambda-\lambda^2)}_{\ge 0}\underbrace{(fg(y)-fg(x))}_{\ge 0}$$

Ex 3.3.2 b) f, g concave, positive $f\uparrow, g\downarrow$

let $x < y$

$* \; fg(\lambda x + (1-\lambda)y) \geq \lambda^2 fg(x) + (1-\lambda)^2 fg(y) + \lambda(1-\lambda)$

$[f(x)g(y) + g(x)f(y)]$

$g\downarrow \quad f(x)g(y) \geq fg(x)$

$f\uparrow \quad f(y) \cdot g(x) \geq fg(y)$

$* \geq (\lambda^2 + \lambda(1-\lambda)) fg(x) + ((1-\lambda)^2 + \lambda(1-\lambda)) fg(y)$

$* \geq \lambda \, fg(x) + (1-\lambda) \, fg(y)$

c) f convex, $\uparrow$, positive

g concave, $\downarrow$, positive $\Rightarrow \dfrac{1}{g}$ convex, $\uparrow$, positive

from a) $\quad f \cdot \dfrac{1}{g}$ is convex

Exercice 3.36:

a) $f(x) = \max_{i=1-n} x_i$, $\mathbb{R}^n$

$f^*(y) = \sup_{x \in \mathbb{R}^n} \{ y^T x - f(x) \}$

Suppose $y_k < 0$: for $x_\alpha = \begin{pmatrix} 0 \\ \vdots \\ -\alpha \\ \vdots \\ 0 \end{pmatrix} \leftarrow k$

$y^T x_\alpha - f(x_\alpha) = -\alpha y_k + \alpha \underset{\alpha \to +\infty}{\to} +\infty$

Suppose $y \geq 0$:

If $\sum y_i > 1 = r$

for $x_\alpha = \alpha 1^T$: $y^T x_\alpha - f(x_\alpha) = (r-1)\alpha \underset{\alpha \to +\infty}{\to} +\infty$

If $\sum y_i < 1 = r'$

$x_\alpha = -\alpha 1^T$: $y^T x_\alpha - f(x_\alpha) = (1-r')\alpha \underset{\alpha \to +\infty}{\to} +\infty$

If $\sum y_i = 1 \Rightarrow y_i \leq 1 \ \forall i$

$\sum y_i x_i \leq \sum y_i \max(x_i) \leq \max(x_i)$

for $x = 0 \cdot 1^T$ $f^*(y) = 0$

$\Rightarrow f^*(y) = \begin{cases} 0 & \text{If } y \geq 0, 1^T y = 1 \\ +\infty & \text{otherwise} \end{cases}$

3.36 b) $f(x) = \sum x_{[i]}$

Case 1: y has a negative y_i:

$y_i < 0$, let $x_\alpha = \begin{pmatrix} 0 \\ \vdots \\ -\alpha \\ \vdots \\ 0 \end{pmatrix} i$

$y^T x_\alpha - f(x_\alpha) = -\alpha y_i + \alpha$

$\lim_{\alpha \to +\infty} = +\infty$

Case 2: $\|y\|_\infty > 1$, $y \geq 0$

$y_i > 1 \Rightarrow x_\alpha = \begin{pmatrix} 0 \\ \vdots \\ \alpha \\ \vdots \\ 0 \end{pmatrix}$: $y^T x_\alpha - f(x_\alpha) = \alpha(y_i - 1)$

$\lim_{\alpha \to +\infty} = +\infty$

Case 3: $y \geq 0$, $\|y\|_\infty \leq 1$, $1^T y = r' > r$

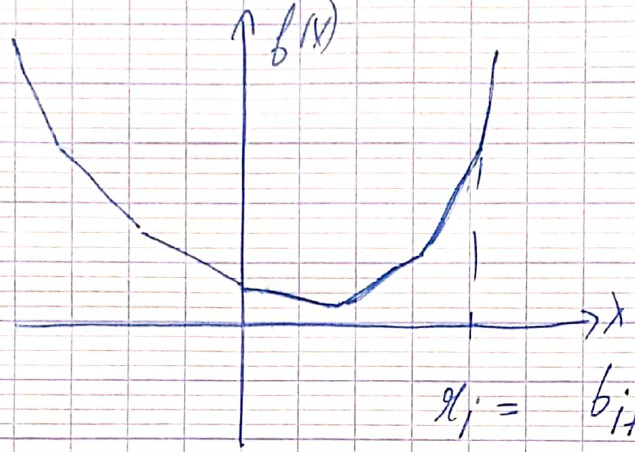
$x_\alpha = \begin{pmatrix} \alpha \\ \vdots \\ \alpha \end{pmatrix}$: $y^T x_\alpha - f(x_\alpha) = \underbrace{(r' - r)}_{> 0}\alpha$

Case 4: $y \geq 0$, $\|y\|_\infty \leq 1$, $1^T y < r$:

Could not do.

Conclusion:

3.36 c) $f(x) = \sup(a_i x + b_i)$ $a_1 < \ < a_n$



$$x_i = \frac{b_{i+1} - b_i}{a_i - a_{i+1}}$$

Graphically, $\operatorname{dom} f^* = [a_1, a_n]$

for y in $\operatorname{dom} f^*$:

$$f^*(y) = \sup_x \{yx - f(x)\}$$

$f^*(y)$ is the largest bias we can add to a linear approximation of our function.

Graphically we see that the best bias is

$$\frac{b_{i+1} - b_i}{a_i - a_{i+1}} \quad \text{if } a_i < y < a_{i+1}$$

$$\rightarrow f^*(y) = \begin{cases} \dfrac{b_{i+1} - b_i}{a_i - a_{i+1}} & \text{if } y \in [a_i, a_{i+1}[\\ +\infty & \text{if } y \notin [a_1, a_n[\end{cases}$$

3.36 d) $f(x) = x^p$, on $\mathbb{R}_{++}$

$p > 1$: $f^*(y) = \sup_{x \in \mathbb{R}_{++}} \{yx - x^p\}$

If $y < 0$:

$$y x - \underbrace{x^p}_{\geq 0} < yx < 0$$

$\Rightarrow f^*(y) = 0$

If $y > 0$:

$$\frac{d}{dx}(yx - x^p) = 0 \Rightarrow x = \left(\frac{y}{p}\right)^{\frac{1}{p-1}}$$

$$f^*(y) = (p-1)\left(\frac{y}{p}\right)^{\frac{p}{p-1}}$$

$$f^*(y) = \begin{cases} 0 & y \leq 0 \\ (p-1)\left(\frac{y}{p}\right)^{\frac{p}{p-1}} & y > 0 \end{cases}$$

$p < 0$:

If $y > 0$:

$$\lim_{x \to +\infty} yx - x^p = +\infty$$

If $y \leq 0$:

$$yx - \underbrace{x^p}_{\geq 0} < yx \leq 0$$

$$f^*(y) \begin{cases} 0 & \text{if } y \leq 0 \\ +\infty & \text{if } y > 0 \end{cases}$$

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3.36 e) Could not do.

b) $f(x,t) = -\log(t^2 - x^T x)$

$$f^*(u,v) = \sup_{(x,t)\in E} \{u^T x + vt + \log(t^2 - x^T x)\}$$

$$\frac{\partial}{\partial x} \Rightarrow x = (t^2 - x^T x)\frac{u}{2}$$

$$\frac{\partial}{\partial t} \Rightarrow t = -(t^2 - x^T x)\frac{v}{2}$$

$$\Rightarrow t^2 - x^T x = (t^2 - x^T x)^2\left(\frac{v^2}{4} - \frac{u^T u}{4}\right)$$

$$\Rightarrow t^2 - x^T x = \frac{4}{v^2 - u^T u}$$

$$x = \frac{2u}{v^2 - u^T u} \qquad t = \frac{-2v}{v^2 - u^T u}$$

$$\boxed{\begin{aligned} f^*(u,v) &= u^T x + vt + \log(t^2 - x^T x) \\ &= -2 + \log(4) - \log(v^2 - u^T u) \end{aligned}}$$

$\cancel{\text{If } v < \|u\|_2 : \quad x = \alpha u \quad t = \alpha(\|x\|_2^2 + 1)}$

$\cancel{u^T x + vt = \alpha\|u\|_2^2 + v\alpha(\alpha^2\|x\|_2^2 + 1)}$

3.36

$$u^T x + vt > \alpha(u^T u + v^2)$$

$$\log(t^2 - x^T x) = \log(\alpha^2((\|x\|_2 + 1)^2 - \|u\|^2))$$

$$\lim_{\alpha \to +\infty} \frac{u^T x + vt}{\log(t^2 - x^T x)} = \lim \frac{\alpha A}{\log(\alpha^2 B)} = +\infty$$

$$\boxed{\Rightarrow f^*(u, v) = +\infty \text{ if } v \leq \|u\|_2}$$

$$\Rightarrow f^*(u,v) = \begin{cases} +\infty & \text{if } \|u\|_2 > v \\ -2 + \log(4) - \log(v^2 - u^T u) & \text{otherwise} \end{cases}$$

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