KMML Homework 4

Mahdi Kallel

TOTAL POINTS

5 / 5

QUESTION 1

1 Exercice 1 **2 / 2**

✓ **- 0 pts** Correct

- 0.25 pts need to prove the continuity of Bn

- 1 pts Bn is p.d. only for n even.

- 0.5 pts description of the RKHS is missing

- 0.5 pts need to treat B1 separately (not continuous => no Bochner)

- 1.5 pts need to treat the case n > 1

- 0.5 pts we need to study the continuity of Bn to apply Bochner

- 2 pts missing exercise

QUESTION 2

2 Exercice 2.1 **1 / 1**

✓ **- 0 pts** Correct

- 0.75 pts Some ideas but important caveats

- 0.25 pts Proof that the Laplacian of the grid can be written as a sum of Kronecker products missing or incorrect

- 0.5 pts Error or incomplete proof to find the eigenvalues

- 1 pts Wrong or not done

QUESTION 3

3 Exercice 2.2 **1 / 1**

✓ **- 0 pts** Correct

- 1 pts Wrong or not done

- 0.5 pts Justification lacking

QUESTION 4

4 Exercice 2.3 **1 / 1**

✓ **- 0 pts** Correct

- 0.5 pts Error

- 1 pts Wrong or not done

Exercice 1 :

1) *For* B_1, $k(x,y)$ *is not a p.d kernel if we take* $X_1, X_2, X_3 = \{0,1,2\}$ *then the kernel matrix* $K = [[1,1,0],[1,1,1],[0,1,1]]$ *has a determinant of* -1. *And therefore a negative eigenvalue.*

By recurrence : *We show that* $B_n \in L_\infty\ \forall n$.

For $i = 1\ B_1(x) < 1\ \forall x$.

Suppose it's true for n. For $n+1$ *we have* : $||B_{n+1}(x)||_\infty \le ||\int B_n(u)\mathbb{1}_{[-1,1]}(x-u)\,du\,||_\infty$
$\le \int ||B_n(x-u)\mathbb{1}_{[-1,1]}(u)\,du\,||_\infty \le \sqrt{2}\,||B_n||_\infty$.

For $n \ge 2$ B_n *is the convolution of* I *which is continuous on the compact* $[-1,1]$ *and* $B_{n-1} \in L_\infty$. *Therefore* B_n *is continuous.*

Let $\mathcal{F} : L_2 \to L_2$ *denote the fourrier transform operator.*
In $\mathbb{R}$, *the fourrier stiejlis transform coincides with the usual fourrier transform.*

We have that for $n \ge 2$, $\mathcal{F}(B_n) = \mathcal{F}(I * \ldots * I) = \mathcal{F}(B)^n$, *where* $F(B)(w) = \dfrac{2sin(w)}{w} = Sinc(w)$

For any n, B_n *is pair therefore* $B_n = \mathcal{F}^{-1}(\mathcal{F}(B_n)) = \mathcal{F}(\mathcal{F}(B_n))$.

For n pair ≥ 2 : *We find that* $B_n = \mathcal{F}(\mathcal{F}(B_n)) = \mathcal{F}\left(Sinc(w)^n\right)$
is continuous and is the fourrier transform of $Sinc(w)^{2n'}$.

which is positive and finite $\left(\int |sinc(w)|^n \le \int |\frac{2}{w^n}| < \infty \text{ for } n \ge 2 \right)$,
Therefore using Bochner's theorem we find that $k(x,y)$ *defines a p.d kernel.*

For n impair : *We find that* B_n *is continuous and is the fourrier transform of* $Sinc(w)^{2n'+1}$ *which is not a positive measure. And therefore using Bochner's theorem* $k(x,y)$ *is not p.d.*

We now explicit the RKHS of B_{2n} :

Let $\mathcal{H}_{2n}$ *denote the RKHS of* B_{2n}, *we know that* $H = \left\{ f : \int \frac{|F(f)|^2}{Sinc(w)^{2n}} < \infty \right\}$

$$= \left\{ f : \int |\widehat{f}(w) \frac{(w)^n}{2sin(w)^n}|^2 < \infty \right\} = \left\{ f : \int |\hat{f}(w)\,w^n|^2 < \infty \right\} \left(\frac{1}{Sinc(w)} \ge 2w \right)$$

If $\hat{f}(w)\,w^n \in L_2$, *then* $(f)^n = \mathcal{F}\left(\hat{f}(w)\,w^n\right)$ *and thus* $\mathcal{H}$ *is the set of n times differentiable functions all the derivatives in* L_2.

1 Exercice 1 **2 / 2**

- ✓ **- 0 pts** Correct
- **- 0.25 pts** need to prove the continuity of Bn
- **- 1 pts** Bn is p.d. only for n even.
- **- 0.5 pts** description of the RKHS is missing
- **- 0.5 pts** need to treat B1 separately (not continuous => no Bochner)
- **- 1.5 pts** need to treat the case n > 1
- **- 0.5 pts** we need to study the continuity of Bn to apply Bochner
- **- 2 pts** missing exercise

Exercice 2:

We can see the Grid graph G *as the sum of two graphs on the same set of vertices, whos edges are distinct.*

If we take G_1 *being the graph formed by the set of all the horizontal line graphs.*

And G_2 *the graph formed by the set of all the vertical line graphs.*

We get that $G = G_1 \cup G_2$.

One can show that in such case (A graph formed by two distinct graphs), the Laplacian matrix is the sum of both the laplacians of the subgraphs.

For both G_1 *and* G_2 *the Laplacians are forward to compute and are* : $M_1 = (I_n \odot L_1)$,

$M_2 = (L_1 \odot I_n)$

This yields the following result :

$$L_2 = (I_n \odot L_1) + (L_1 \odot I_n) = M_1 + M_2$$

We show that if $M \in S_n + \implies I_n \odot M$ *and* $M \odot I_n \in S_n +$:

Let $M = PDP^{-1}$, *then using the property* $(**)$ $(A \odot B)(C \odot D) = AC \odot BD$, *wet get that* :

$$\left(I_n \odot P^{-1}\right)(I_n \odot M)\left(I_n \odot P\right) =^{**} \left(I_n I_n I_n \odot P^{-1}MP\right) = I_n \odot D \in S_n + \text{ (same for the other case..)}$$

$$M_1 M_2 = (I_n \odot L_1)(L_1 \odot I_n) = I_n L_1 \odot L_1 I_n = L_1 \odot L_1 = M_2 M_1.$$

M_1 *and* M_2 *commute therefore they are "co diagonalisable".*

$\exists P$ *s.t* $PM_1P^{-1} = D_1$ *and* $PM_2P^{-1} = D \implies P(M_1 + M_2)P^{-1} = D_1 + D_2$.

Therefore we deduce that if α_i *is an eigenvalue of* $L_2 \implies \exists a_i, b_i$ *eigenvalues of* M_1, M_2 *s.t* $\alpha_i = a_i + b_i$.

If $C = A \odot B$, *and* $(\lambda_i, X_i), (\mu_j, Y_i)$ *are the eigenpairs of* A, B *respectively,*

we know all the eigenvalues of C *which are of the form* $\alpha_{ij} = \lambda_i \mu_j$ *with the corresponding*

eigenvectors $(X_i \odot Y_j)$. *Therefore we deduce that* $Spec_{L_1} = Spec_{L_1 \odot I_n} = Spec_{I_n \odot L_1}$

If we denote by $E = \{\lambda_i + \lambda_j,$ *where* $\lambda_i, \lambda_j \in Spec_{L_1}\}$ *then we proved that* $Spec_{L2} \in E$.

$L_2\ (e_i \odot e_j) \ = \ (\ I_n \odot\ L_1)(e_i \odot e_j) \ + \ (L_1 \odot I_n)(e_i \odot e_j) \ = \ (e_i \ \odot\ L_1 e_j) \ + \ (L_1 e_i \ \odot\ e_j)$

$= \ e_i \ \odot\ \lambda_j e_j \ + \ e_i \ \odot\ \lambda_2 e_j \ = \ (\lambda_i + \lambda_j)\ (e_i \ \odot e_j).$

$Therefore\ we\ find\ that\ e_{ij} \ = \ e_i \odot e_j\ is\ the\ eigenvector\ corresponding\ the\ the\ eigenvalue,$
$\lambda_{ij} \ = \ \lambda_i + \ \lambda_j$

$Therefore\ E \ \in Spec_{L_2}\ and\ thus\ E \ = \ Spec_{L2}.$

2 Exercice 2.1 **1 / 1**

✓ **- 0 pts** Correct

- 0.75 pts Some ideas but important caveats

- 0.25 pts Proof that the Laplacian of the grid can be written as a sum of Kronecker products missing or incorrect

- 0.5 pts Error or incomplete proof to find the eigenvalues

- 1 pts Wrong or not done

2) $K_t = \sum_{n^2} e^{-t\lambda_{ij}} e_{ij} e_{ij}^T = \sum_{i,j} e^{-t(\lambda_i+\lambda_j)}(e_i \odot e_j)(e_i \odot e_j)^T = \sum_{i,j} e^{-t(\lambda_i+\lambda_j)}(e_i \odot e_j)\left(e_i^T \odot e_j^T\right)$

$$= \sum_{i,j} e^{-t(\lambda_i+\lambda_j)}\left(e_i e_i^T \odot e_j e_j^T\right) = = \sum_{i,j}\left(e^{-t\lambda_i} e_i e_i^T \odot e^{-t\lambda j} e_j e_j^T\right)$$

$$= \sum_i e^{-t\lambda_i} e_i e_i^T \odot \sum_j e^{-t\lambda_j} e_j e_j^T = K_1 \odot K_1.$$

From this form we directly deduce that $K_2((i,j),(k,l)) = K_1(i,k) K_1(j,l))$

3 Exercice 2.2 **1 / 1**

- ✓ **- 0 pts** Correct
- **- 1 pts** Wrong or not done
- **- 0.5 pts** Justification lacking

3) *As stated before, the complexity of computing the exponential of* $n * n$ *matrix in* N^3.

So computing K_1 *comes with a cost of* N^3.

K_2 *is an* N^4 *matrix, but from* 2) *we know how to compute a single entry of* K_2 *as the product*

of two terms of K_1. *Therefore to compute a single entry of* K_2 , *knowing* K_1 *is* $O(1)$.

Doing this for all entries costs N^4.

And thus computing K_2 *is* $O(N^3 + N^4) = O(N^4)$

4 Exercice 2.3 **1 / 1**

- ✓ **- 0 pts** Correct
- **- 0.5 pts** Error
- **- 1 pts** Wrong or not done