Convex optimization final exam

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Exercise 1

$$\min_x \|x\|_\infty \quad \text{s.t. } Ax = b \tag{1.1}$$

This problem is equivalent to :

$$\begin{aligned} &\min_{y \in \mathbf{R}} \ y \\ \text{s.t. } & Ax = b \\ & - y.\mathbf{1} \le x \le y.\mathbf{1} \\ & y \ge 0 \end{aligned} \tag{1.2}$$

The dual is :

$$\begin{aligned} g\left(t, u, v, w\right) &= \inf_{x,y} \mathcal{L}\left(x, y, t, u, v, w\right) \\ &= y - ty + u^T\left(Ax - b\right) + v^T\left(-y.\mathbf{1} + x\right) + w^T\left(-y.\mathbf{1} - x\right) \end{aligned}$$

$\frac{\partial}{\partial x} = 0 \implies A^T u + v - w = 0 \Longleftrightarrow u^T A = w^T - v^T$

$\frac{\partial}{\partial y} = 0 \implies (1 - t) - \mathbf{1}^T v - \mathbf{1}^T w = 0 \Longleftrightarrow 1 - t = \mathbf{1}^T v + \mathbf{1}^T w$

We reinject into eq 2.

$$\begin{aligned} g\left(t, u, v, w\right) &= y(1 - t) + \left(\boldsymbol{w}^T - v^T\right) x - u^T b - \mathbf{1}\left(v^T + w^t\right) y + \left(v^T - w^t\right) x \\ &= -u^T b \end{aligned}$$

The problem is equivalent to :

$$\begin{aligned} &\min_u u^T b \\ \text{s.t. } & A^T u = w - v \\ & 1^T(v + w) = 1 - t \\ & t, v, w \ge 0 \end{aligned} \tag{1.3}$$

Which can be rewritten as :

$$\boxed{\begin{aligned} &\min_u u^T b \\ &\text{s.t. } A^T u = w - v \\ &\quad \|v + w\|_1 \leq 1 \\ &\quad v, w \geq 0 \end{aligned}} \tag{1.4}$$

Exercise 2

We have **f** twice differentiable and $Hess(f) = diag(\frac{1}{v_i}) \geq \boldsymbol{I}_n \implies$ f is strongly convex

By definition of a convex differentiable function we have that the gradient is an underestimate of the slope:

$\implies f(x) + \nabla_x f^T(y - x) \leq f(y)$

$\Longleftrightarrow 0 \leq f(y) - f(x) - \nabla_x f^T(y - x)$

$\Longleftrightarrow 0 \leq D_{kl}(u, v)$

F is a strongly convex so the slope inequality is strict:

$\forall x, y$ s.t $x < y \ : \ \nabla_x f^T(y - x) < f(y) - f(x)$

Therefore we have the implication that $\forall x, y \ : \ f(y) - f(x) = \nabla_x f^T(y - x) \implies y = x$

And thus we deduce the equivalence : $D_{kl}(u, v) = 0 \Longleftrightarrow u = v$

Exercise 3

$$\begin{aligned} &\min_x \ c^T x \\ &\text{s.t. } x^T(A - bb^T)x \leq 0 \\ &\quad b^T x \geq 0 \\ &\quad Dx = g \end{aligned} \tag{3.1}$$

Since A is a semi definite matrix have it's square root decomposition as $A = V^T V$ where $V \in \mathbf{S}_n$

The constraint $x^T(V^T V - bb^T)x \leq 0$ can be re written as $\|Vx\|_2^2 \leq \left\|b^T x\right\|_2^2$.

If we use the fact that $b^T x \geq 0$ and that is a real number we can further rewrite the constraint as : $\|Vx\|_2 \leq b^T x$

The problem can be written as an SOCP which by definition is convex :

$$\boxed{\begin{aligned} &\min_x \; c^T x \\ \text{s.t. } &\|Vx\|_2 \leq b^T x \\ &Dx = g \end{aligned}} \tag{3.2}$$

Now to derive the langrangian we write our problem as :

$$\begin{aligned} &\min_x \; c^T x \\ \text{s.t. } &\|y\|_2 \leq b^T x \\ &Dx = g \\ &y = Vx \end{aligned} \tag{3.3}$$

From the course notes the lagrange dual function can be written :

$\mathcal{L}(x, y, \lambda, \nu, \mu) = c^T x + \lambda(\|y\|_2 - b^T x) + \nu^T(Dx - g) + \mu^T(Vx - y)$
$g(\lambda, \nu, \mu) = \min_{x,y} \mathcal{L}(x, \lambda, \nu, \mu) = min_x(c^T + \nu^T D + \mu^T V - \lambda b^T)x + \min_y(\mu^T y + \lambda\|y\|_2) - v^T g$

$\frac{\partial}{\partial x} = 0 \implies c + D^T \nu + V^T \mu = \lambda b$

We know that the conjugate norm of L2 is L2:

$\implies \min_y(\mu^T y + \lambda\|y\|_2) = \begin{cases} 0 \textbf{ if } \|\mu\|_2 \leq \lambda \\ -\infty \textbf{ Otherwise} \end{cases}$

The dual is :

$$\boxed{\begin{aligned} &\textbf{maximize } \; -\nu^T g \\ \text{s.t. } &\|\mu\|_2 \leq \lambda \\ &c + D^T \nu + V^T \mu = \lambda b \end{aligned}} \tag{3.4}$$

Exercise 4

$$\begin{aligned} &minimize \; -\sum_1^m log(b_i - a_i^T x) \\ \text{s.t. } &\forall i, b_i - a_i^T x \geq 0 \end{aligned} \tag{4.1}$$

This problem is equivalent to :

$$\begin{aligned} &minimize \; -\sum_1^m log(y_i) \\ \text{s.t. } &\forall i, y_i = b_i - a_i^T x \\ &y_i \geq 0 \end{aligned} \tag{4.2}$$

Which can be rewritten :

$$\begin{aligned} &\text{m}inimize \ -\sum_{1}^{m} log(y_i) \\ \text{s.t. } & Ax + y = b \\ & y \geq 0 \end{aligned} \quad [4.3]$$

$$\mathcal{L}(x, y, u, v) = -\sum_{i=1}^{m} log(y_i) + u^T(Ax + y - b) - v^T y \quad [4.4]$$

$\frac{\partial}{\partial x} = 0 \implies A^T u = 0$
$\frac{\partial}{\partial y} = 0 \implies u - v = \frac{1}{y}$

$$g(u, v) = \sum_{i=1}^{m} log(u_i - v_i) + m - u^T b \quad [4.5]$$

The dual is :

$$\begin{aligned} &\text{m}aximize \ \sum_{i=1}^{m} log(u_i - v_i) + m - u^T b \\ \text{s.t. } & A^T u = 0 V \geq 0 \end{aligned} \quad [4.6]$$

The condition on V can be dropped and the problem rewritten as :

$$\boxed{\begin{aligned} &\text{m}aximize \ \sum_{i=1}^{m} log(u_i) + m - u^T b \\ \text{s.t. } & A^T u = 0 \end{aligned}} \quad [4.7]$$

Exercise 5

$$\begin{aligned} &\min_x \ f(x) \\ \text{s.t. } & Ax = b \end{aligned} \quad [5.1]$$

It's lagrange dual function is :

$g(u) = \min_x \mathcal{L}(x, u) = \min_x f(x) + u^T(Ax - b)$

For a fixed u $\geq 0, \mathcal{L}(x, u) = g(u) \iff \nabla_x f + A^T u = 0$

$\phi(x) = f(x) + \alpha \|Ax - b\|_2^2$

$\tilde{x}$ is a solution for Eq2 $\iff \nabla_{\tilde{x}} f + \alpha A^T(Ax - b) = 0$

For $\tilde{u} = \alpha(A\tilde{x} - b)$ we have $\nabla_{\tilde{x}} f + A^T \tilde{u} = 0 \iff \tilde{u}$ is a dual feasible point

We know that $\forall (x, u),\ f(x) \geq g(u)$,

By minimizing on x we can derive the lower bound : $f(x^\star) \geq g(\tilde{u}) = f(\tilde{x}) + u^T(A\tilde{x} - b)$

Exercise 6

$$\begin{aligned} & \frac{1}{2}\|w\|_2^2 + C\mathbf{1}^T z \\ \text{s.t. } & y_i(w^T x_i) \geq 1 - z_i,\ i = 1 \ldots m \\ & z \geq 0 \end{aligned} \quad [6.1]$$

$\mathcal{L}(w, z, u, v) = \frac{1}{2}\|w\|_2^2 + C\mathbf{1}^T z - u^T z - \sum_{i=1}^m v_i(z_i + y_i w^T x_i - 1)$

$\frac{\partial}{\partial w} = 0 \implies w = \sum_{i=1}^m v_i y_i x_i$

$\frac{\partial}{\partial z} = 0 \implies C\mathbf{1} - u - v = 0$

$g(u, v) = \frac{1}{2}\left\|\sum_{i=1}^m v_i y_i x_i\right\|_2^2 + (C\mathbf{1}^T - u - v)z - \sum_{i=1}^m v_i(y_i w^T x_i - 1)$

$= \frac{1}{2}\left\|\sum_{i=1}^m v_i y_i x_i\right\|_2^2 - \sum_{i=1}^m w^T x_i v_i y_i + \mathbf{1}^T v$

$= \frac{1}{2}\left\|\sum_{i=1}^m v_i y_i x_i\right\|_2^2 + \mathbf{1}^T v - w^T \sum_{i=1}^m x_i v_i y_i$

$= \frac{1}{2}\left\|\sum_{i=1}^m v_i y_i x_i\right\|_2^2 - w^T w + \mathbf{1}^T v$

$= \mathbf{1}^T v - -\frac{1}{2}\left\|\sum_{i=1}^m v_i y_i x_i\right\|_2^2$

$$\begin{aligned} & \text{m}aximize\ \mathbf{1}^T v - -\frac{1}{2}\left\|\sum_{i=1}^m v_i y_i x_i\right\|_2^2 \\ \text{s.t. } & \sum_{i=1}^m v_i y_i x_i \\ & C\mathbf{1} = u + v \\ & u, v \geq 0 \end{aligned} \quad [6.2]$$

Which is equivalent to :

$$\boxed{\begin{aligned} & \text{m}aximize\ \mathbf{1}^T v - -\frac{1}{2}\left\|\sum_{i=1}^m v_i y_i x_i\right\|_2^2 \\ & \text{s.t. } 0 \leq v \leq C\mathbf{1} \end{aligned}} \quad [6.3]$$

In order to use the code from previous homework we write the problem of the form

$$\begin{aligned} & \min_{\nu} \nu^T Q \nu + p^T \nu \\ & \text{s.t. } A\nu \leq b \end{aligned}$$

$$\text{with } Q = \frac{1}{2} \cdot diag(Y)XX^T diag(y),\ p = -\mathbf{1}_n$$

$$A = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ -1 & & \\ & \ddots & \\ & & -1 \end{bmatrix}$$

$$b = \begin{bmatrix} C \\ \vdots \\ \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$