KMML Homework 2

Mahdi Kallel

TOTAL POINTS

3 / 5

QUESTION 1

1 Exercice 1 **0.5 / 2**

- **- 0 pts** Correct
- ✓ **- 0.25 pts missing the explicit characterization of the norm**
- ✓ **- 0.5 pts missing/wrong proof for reproducing property**
- ✓ **- 0.75 pts right set of functions H, but wrong/missing inner product (unfortunately means proof of reproducing property and characterization of the norm are also not correct)**
- **- 0.5 pts** need to consider closed subspace (and not only finite combinations of psi(x_i))
- **- 2 pts** missing exercise
- **- 1 pts** important argument is missing
- **- 0.25 pts** minor detail missing
- **- 1.5 pts** the goal was not to prove again a specific case of Aronszjan theorem, but to find an explicit representation of H.
- **- 0.5 pts** explicit form of he inner product is missing

QUESTION 2

2 Exercice 2 **2.5 / 3**

- **- 0 pts** Correct
- ✓ **- 0.5 pts Forgot to consider the case f=0 when taking lambda=1/||f||^2**
- **- 0.5 pts** Error in lambda
- **- 0.5 pts** Error in proof
- **- 1.5 pts** Error in proof of =>
- **- 1.5 pts** Second part not solved
- **- 0.5 pts** Forgot to prove (or at least mention) symmetry
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Mahdi KALLEL
HW#2

1. $We\ define\ \mathcal{H} = \{f_w(x) = < w, \psi(x) >_{\mathcal{F}},\ \forall w \in \mathcal{F}\}$ *with the scalar product :*

$< f_w, f_v >_{\mathcal{H}} = < w, v >_{\mathcal{F}}$

1) *Let* $x \in \mathcal{X} : K_x : t \Longrightarrow K(x,t) = < \psi(t), \psi(x) > = < w', \psi(x) > \in \mathcal{H},$

since $\psi(t) = w' \in \mathcal{F}$

2) *Let* $(x, f) \in \mathcal{X} * \mathcal{H}$:

By definition of $\mathcal{H}$, $\exists w\ s.t\ f(x) = < w, \phi(x) >_{\mathcal{F}}$

$K_x = < \psi(x), . >_{\mathcal{F}} \Longrightarrow f(x) = < w, \phi(x) >_{\mathcal{F}} = < f, K_x >_{\mathcal{H}}.$

To conclude let's prove that $\mathcal{H}$ *is a Hilbert space :*

$*$ $\mathcal{H}$ *is a vector space:* *if* $f_w, f_v \in \mathcal{H}$ *then* $f_w + \lambda f_v \in \mathcal{H}$.
since $f_{w+\lambda v} = < w + \lambda v, \psi(x) >$ *and* $w + \lambda v \in \mathcal{F}$.

With a scalar product :
$* < f_w, f_v > = < w, v >_{\mathcal{F}} = < v, w >_{\mathcal{F}} = < fv, f_w >$ *(symetric)*
$* < f_v + \lambda f_u, f_w > = < v + \lambda u, w >_{\mathcal{F}} = < v, w > + \lambda < u, w >$
$= < f_v, f_w > + \lambda < f_u, f_w >$ *(linear)*
$* if < f_w, f_w > = 0 \Longrightarrow < w, w >_{\mathcal{F}} = 0 \Longrightarrow w = 0 \Longrightarrow f(x) = < w, \psi(x) > = 0$ *(def)*

And is complete for the norm :
$* let\ (f_n)_{n \in \mathbb{N}} \subset \mathcal{H}\ s.t\ sup_{n,m \geq N} ||f_n - f_m||_{\mathcal{H}} = 0, \forall n\ \exists (w_n)_{n \in \mathbb{N}} \subset \mathcal{F}\ st\ f_n = < w_n, \psi(x) >$

$sup_{n,m \geq N} ||f_n - f_m||_{\mathcal{H}} = 0 \Longrightarrow sup_{n,m \geq N} ||w_n - w_m||_{\mathcal{F}} = 0$ *since* $\mathcal{F}$ *is Hilbert space,*
$\exists w \in \mathcal{F}\ s.t\ w_n \rightarrow w \Longrightarrow f_n = < w_n, \psi(x) > \rightarrow < w, \psi(x) > = f \in \mathcal{H}.$

We proved K is a reproducing kernel, and $\mathcal{H}$ *it's RKHS* $\Longrightarrow$ *K is p.d*

(We used the theorem : K is p.d $\leftrightarrow$ *K is r.k)*

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2. $\leftarrow$: $If\ \exists\lambda > 0\ s.t\ \ K(x,x') - \lambda f(x)f(x') = K'\ is\ p.d\ then\ f \in\ rkhs\ (K)$

$$Let\ (a_1 ... a_N) \in \mathbb{R}^N,\ \sum_{i,j} a_i a_j\ f(x_i)f(x_j) = \sum_i a_i f(x_i) * \sum_j a_j f(x_j) = \left(\sum a_i f(x_i)\right)^2 \geqslant 0$$

$$\Longrightarrow K'' = \lambda\ f(x)f(x')\ is\ p.d$$

$From\ (5.1)\ we\ get\ that\ K' + K''\ is\ p.d\ and\ that\ RKHS\ (K' + K'') = RKHS\ (K) = \mathcal{H}_1 + \mathcal{H}_2$

$\Longrightarrow \mathcal{H}_1 + \mathcal{H}_2 = \mathcal{H} \Longrightarrow \mathcal{H}_2 \subset \mathcal{H}\ (if\ we\ take\ h_1 = 0)$

$By\ definition,\ K''_t(x) \in \mathcal{H}_2 \Longrightarrow \lambda f(t)f(x) \in \mathcal{H}_2\ \ and\ since\ it's\ a\ vector\ space:$

$$\frac{1}{\lambda f(t)} * \lambda f(t)f(x) \in \mathcal{H}_2 \Longrightarrow f(x) \in \mathcal{H}_2 \subset \mathcal{H}.$$

$\rightarrow$: $Let\ (a_1 ... a_N) \in \mathbb{R}^N:\ \sum_{i,j} a_i a_j\ (K(x_i, xj) - \lambda f(x_i)f(x_j))$

$$\sum_{i,j} a_i a_j\ < K_{x_i}, K_{x_j} > - \lambda < f, K_{x_i} >< f, K_{xj} >)\ \ (K\ is\ a\ reproducing\ kernel)$$

$$< \sum_i a_i K_{x_i}, \sum_j a_j K_{x_j} > - \lambda\ < f, \sum_i a_i K_{x_i} >\ < f, \sum_j a_j K_{x_j} >$$

$$= ||\sum_i a_i K_{x_i}||^2 - \lambda \left(< f, \sum_i a_i K_{x_i} > \right)^2$$

$$\left(< f, \sum_i a_i K_{x_i} > \right)^2 \leq ||f||^2 * ||\sum_i a_i K_{x_i}||^2\ \ (Cauchy-Schwarz)$$

$$for\ \lambda = \frac{1}{||f||^2_{\mathcal{H}}}\ \ we\ find\ that\ K(x,x') - \lambda f(x)f(x') = K'\ is\ p.d.$$

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