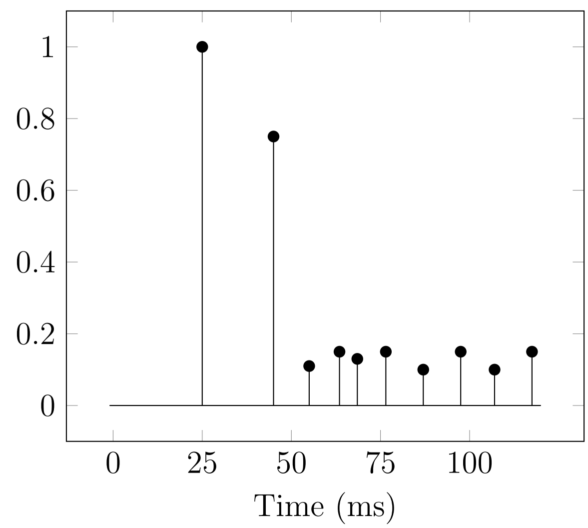
Homework 9

Due April 29 at 11 pm

1. (Impulse response of a room) We are interested in analyzing the acoustic characteristics of an auditorium during a concert. To this end, we want to study the sound traveling from the singer's location to a seat in the last row.

   (a) Under what assumptions would the sound propagation be characterized by the convolution of the emitted sound and an impulse response?

   (b) Assume the assumptions hold, and we measure the following impulse response:

   Why do you think the response is zero before 25 ms?

   (c) What do the two first spikes in the impulse response correspond to?

   (d) Suggest a way to measure the impulse response by bursting a balloon. What are the possible limitations of this approach?

   (e) Suggest a way to measure the impulse response using a device that produces pure tones (i.e. sinusoidal sounds).

2. (Discrete filter) Let us index the DFT coefficients of the N-dimensional vectors from $-(N-1)/2$ to $(N-1)/2$ (assuming N is odd). We define the bandlimited signals in this space as those for which the nonzero Fourier coefficients are zero beyond a certain value k_c, i.e. $x \in \mathbb{C}^N$ is bandlimited if $\hat{x}[k] = 0$ for all $|k| > k_c$. Let y be the vector with the smallest ℓ_2 norm such that $x * y = x$ for all bandlimited vectors with cut-off frequency k_c (where k_c is a fixed integer smaller than $(N-1)/2$). Derive an explicit expression for the entries of y, showing that they are real valued.

3. (PCA of stationary vector) Let $\tilde{x}$ be a wide-sense stationary vector with real-valued autocovariance vector $a_{\tilde{x}}$, with covariance matrix $\Sigma_{\tilde{x}}$. In the notes we showed that the eigenvectors and eigenvalues of $\Sigma_{\tilde{x}}$ are complex exponentials and the DFT coefficients of

$a_{\tilde{x}}$ respectively. Here we will show that we can derive an equivalent real-valued eigendecomposition because the autocovariance vector is real. We will assume that N is an odd number.

(a) Show that the DFT coefficients of $a_{\tilde{x}}$ are real, and satisfy $\hat{a}_{\tilde{x}}[k] = \hat{a}_{\tilde{x}}[N-k]$ for $k = 1, \ldots, \frac{N-1}{2}$.

(b) Show that

$$\Sigma_{\tilde{x}}[j_2, j_1] = \frac{1}{N} \sum_{k=0}^{N-1} \hat{a}_{\tilde{x}}[k] \exp\left(\frac{i2\pi k(j_2 - j_1)}{N}\right). \tag{1}$$

(c) Show that $\Sigma_{\tilde{x}}$ has the following decomposition

$$\Sigma_{\tilde{x}}[j_2, j_1] = \sum_{k=0}^{N-1} \lambda_k u_k[j_1] u_k[j_2], \tag{2}$$

where the eigenvectors correspond to the orthonormal vectors

$$\frac{1}{\sqrt{N}} \begin{bmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{bmatrix}, \quad \sqrt{\frac{2}{N}} \begin{bmatrix} 1 \\ \cos\left(\frac{2\pi k}{N}\right) \\ \cdots \\ \cos\left(\frac{2\pi k(N-1))}{N}\right) \end{bmatrix}, \quad \sqrt{\frac{2}{N}} \begin{bmatrix} 0 \\ \sin\left(\frac{2\pi k}{N}\right) \\ \cdots \\ \sin\left(\frac{2\pi k(N-1))}{N}\right) \end{bmatrix}, \quad 1 \leq k \leq \frac{N-1}{2}.$$

and report the value of $\lambda_0, \ldots, \lambda_k$.

4. (Household electricity usage) Given the rise of smart electricity meters and the wide adoption of electricity generation technology like solar panels, there is a wealth of electricity usage data available. Here we will explore a household electricity usage dataset which represents a time series of power-related variable.

(a) Load the data file `household_power_consumption_days.csv` in Python and plot it. Compute and plot the autocorrelation of these data using Definition 4.1 in the note for the first 3 weeks and first 3 years. Provide explanations for each of the plots.

(b) Apply DFT to the autocorrelation and give a plot of the magnitudes of the computed DFT coefficients. What do you notice from the plot? (Hint: You might want to zoom in to see spikes around zero frequency clearly.)

(c) Based on what you have observed, what are the advantages and disadvantages of modeling this time series data as stationary?