Workshop: Projectile Motion

An introduction to computing trajectories

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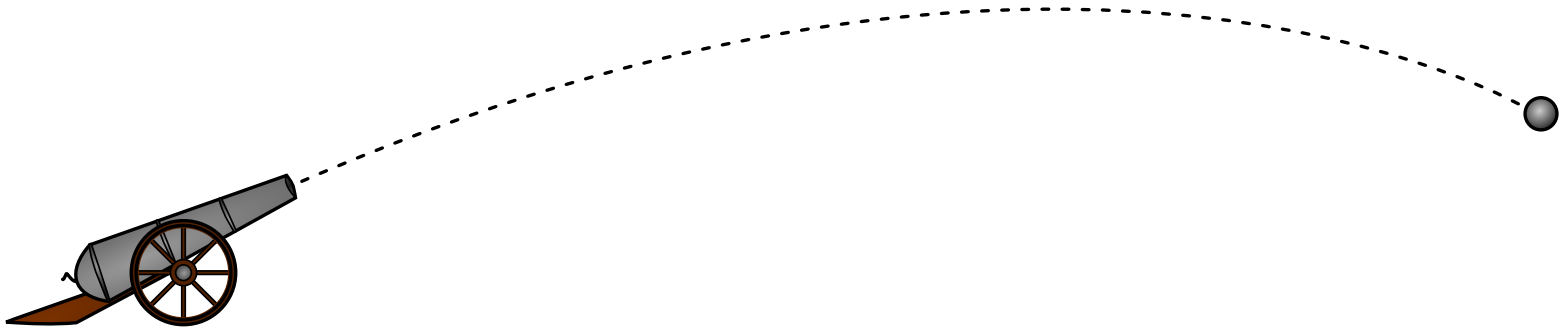

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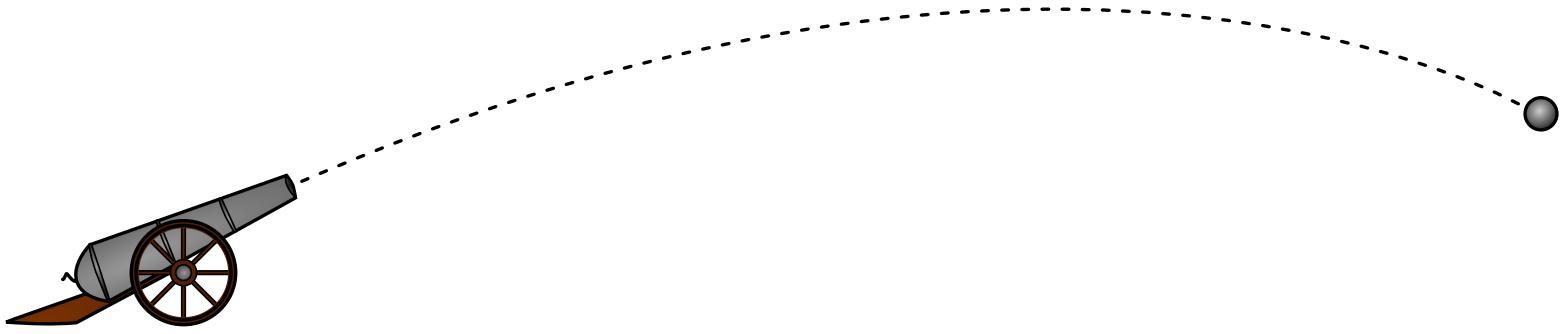

Equations of motion

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{v}(t), \qquad \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \vec{a}(t)$$

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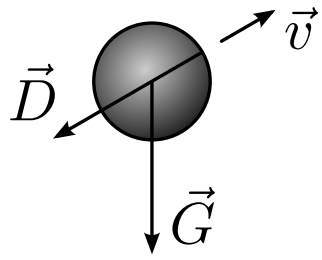

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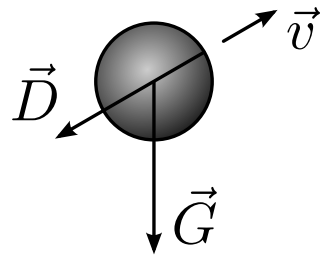

$$\vec{F}(r, v, t) = m\vec{a}(r, v, t).$$

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Our algorithm is now as follows

1. Find the physical forces of the system.
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3. Calculate $\vec{v}(t)$ and $\vec{r}(t)$ by solving the equations of motion

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In this workshop, we will solve step number 3 numerically, using the Euler method.

The Euler Method

A method for solving ordinary differential equations (ODEs)

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Solving for $v(t + \Delta t)$ gives

$$v(t + \Delta t) \approx v(t) + a(t) \cdot \Delta t$$

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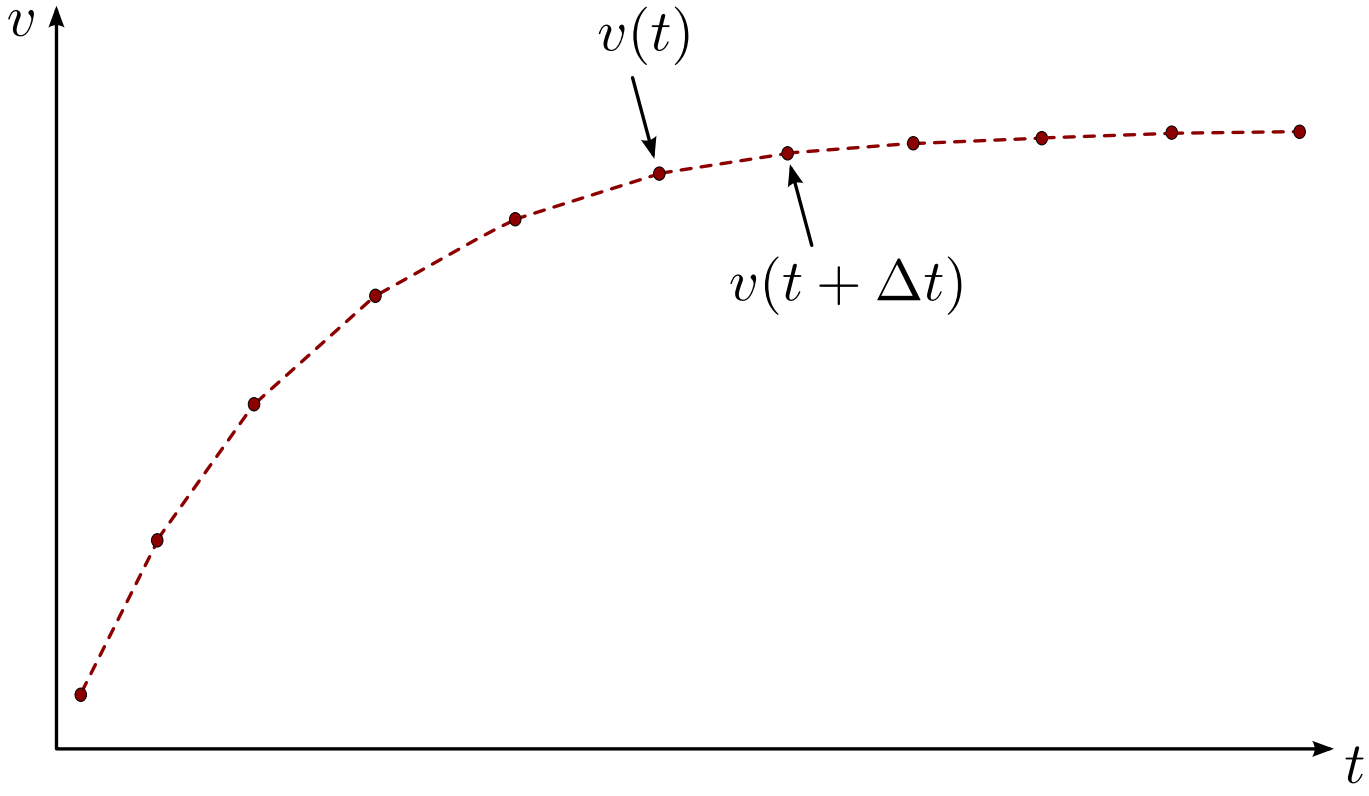

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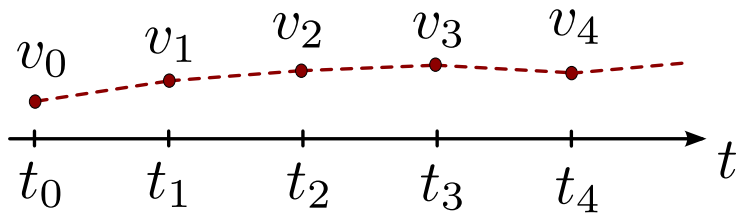

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We repeat these steps, starting at our initial conditions v_0 and r_0, until we have reached our end-time t_N

$$i = 0, 1, 2, 3, \ldots, N.$$

Algorithm for the Euler method

for $i = 0, 1, 2, 3, \ldots, N-1$:

1. Use the previous results x_i and v_i to compute the acceleration: $a_i = F(x_i, v_i, t_i)/m$.
2. Compute the new velocity: $v_{i+1} = v_i + a_i \Delta t$.
3. Compute the new position: $r_{i+1} = r_i + v_i \Delta t$.

Implementation

Moving from physics and math to actual computer code

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```
for i in range(N):
    a[i] = F(x[i], v[i], t[i])/m
    v[i+1] = v[i] + a[i]*dt
    r[i+1] = r[i] + v[i]*dt
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$\Downarrow$

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```

We want the code to look as much as possible like the physics and math we write on paper

$t_i \Rightarrow$ `t[i]` $\quad$ $v_i \Rightarrow$ `v[i]` $\quad$ $r_i \Rightarrow$ `r[i]`

We also need various bookeeping code

Here we define the arrays we will be using

```
# Import various functions meant for numerical science
from pylab import *

t_0 = 0 # Start time, s
t_end = 10 # End time, s
N = 1000 # Number of time steps

# Create a uniformly spaced time-array
t = linspace(t_0, t_end, N+1)

# Calculate the size of a time step
dt = t[1] - t[0]

# Create empty acceleration, velocity and position arrays
a = zeros((2, N+1))
v = zeros((2, N+1))
r = zeros((2, N+1))

# Set initial conditions
v[0] = (10,0) # inital velocity, m/s
r[0] = (0,1)  # initial position, m
```

We also need various bookeeping code

Here we define physical constants for our system and define the function that describes the forces

```
m = 5.5 # mass, kg
g = 9.81 # acceleration of gravity, m/s^2
rho = 1.3 # air density, kg/m^3
C_D = 0.45 # drag coefficient
d = 0.11 # diameter of cannonball, m
A = pi*d**2 # cross-sectional area, m^2

def F(x, v, t):
    return -(0, m*g) - 0.5*rho*C_D*A*abs(v)*v
```

This example show the forces acting on the cannonball as it sails through the air

$$F(x, v, t) = F_g + F_d(\vec{v}) = -mg\vec{k} - \frac{1}{2}\rho C_D A|\vec{v}|\vec{v}$$

Plotting

Example

Results

- Can plot numerical result vs known solution
- Can plot with and without drag
- Can plot for various inital angles

Examples of possible projects

- Catapults, cannons, throwing of objects. Can include analytical and experimental data.
- Skydiver and plot the g-forces, can illustrate why a parachute needs to open slowly!
- Bungee-jumping! Great for teaching spring forces and programming technicalities.
- Block and spring, harmonic Oscillator
- Orbits of planets in the solar system with real data from NASA
- Pendulum (also with large angles!)