

Workshop: Projectile Motion

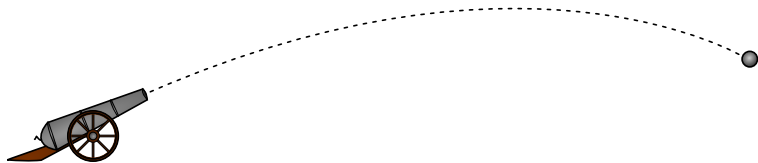
An introduction to computing trajectories

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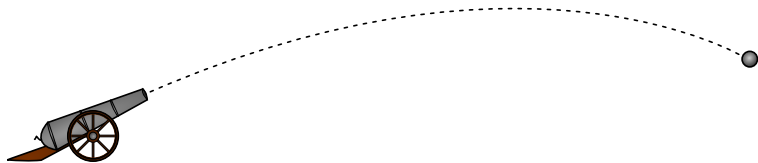
Simula Research Laboratory
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Equations of motion

$$\frac{d\vec{r}}{dt} = \vec{v}(t), \quad \frac{d\vec{v}}{dt} = \vec{a}(t)$$

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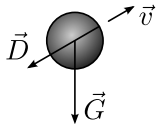
Newtons 2. law of motion

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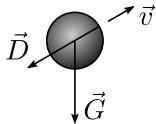
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$$\vec{F}(r, v, t) = m\vec{a}(r, v, t).$$

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Our algorithm is now as follows

1. Find the physical forces of the system.
2. Use Newton's 2. law to find the acceleration
3. Calculate $\vec{v}(t)$ and $\vec{r}(t)$ by solving the equations of motion

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In this workshop, we will solve step number 3 numerically, using the Euler method.

The Euler Method

A method for solving ordinary
differential equations (ODEs)

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From the definition of the derivative

$$\frac{dv}{dt} = \lim_{\Delta t \rightarrow 0} \frac{v(t + \Delta t) - v(t)}{\Delta t} = a(t)$$

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$$\frac{v(t + \Delta t) - v(t)}{\Delta t} \approx a(t)$$

Solving for $v(t + \Delta t)$ gives

$$v(t + \Delta t) \approx v(t) + a(t) \cdot \Delta t$$

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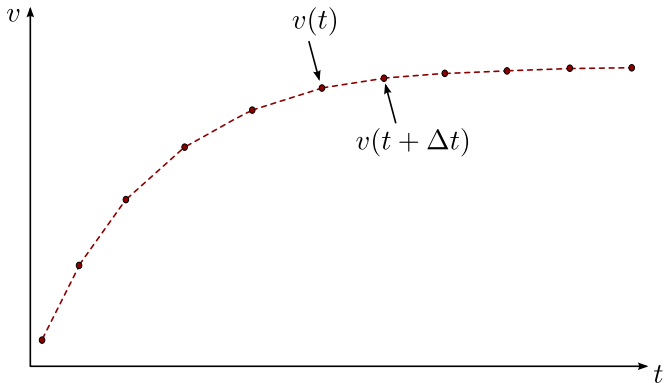
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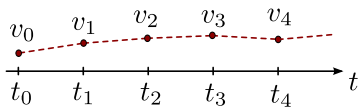
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For each time step, we must calculate the acceleration

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We repeat these steps, starting at our initial conditions v_0 and r_0 , until we have reached our end-time t_N

$$i = 0, 1, 2, 3, \dots, N.$$

Algorithm for the Euler method

for $i = 0, 1, 2, 3, \dots, N - 1$:

1. Use the previous results x_i and v_i to compute the acceleration: $a_i = F(x_i, v_i, t_i)/m$.
2. Compute the new velocity: $v_{i+1} = v_i + a_i \Delta t$.
3. Compute the new position: $r_{i+1} = r_i + v_i \Delta t$.

Implementation

Moving from physics and math to
actual computer code

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```
for i in range(N):  
    a[i] = F(r[i], v[i], t[i])/m  
    v[i+1] = v[i] + a[i]*dt  
    r[i+1] = r[i] + v[i]*dt
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for $i = 0, 1, 2, 3, \dots, N - 1$:

1. Use the previous results x_i and v_i to compute the acceleration: $a_i = F(x_i, v_i, t_i)/m$.
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for i in range(N):  
    a[i] = F(r[i], v[i], t[i])/m  
    v[i+1] = v[i] + a[i]*dt  
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```

We want the code to look as much as possible like the physics and math we write on paper

$$t_i \Rightarrow t[i] \quad v_i \Rightarrow v[i] \quad r_i \Rightarrow r[i]$$

We also need various bookkeeping code

Here we define the arrays we will be using

```
# Import various functions meant for numerical science
from pylab import *

t_0 = 0 # Start time, s
t_end = 10 # End time, s
N = 1000 # Number of time steps

# Create a uniformly spaced time-array
t = linspace(t_0, t_end, N+1)

# Calculate the size of a time step
dt = t[1] - t[0]

# Create empty acceleration, velocity and position arrays
a = zeros((2, N+1))
v = zeros((2, N+1))
r = zeros((2, N+1))

# Set initial conditions
v[0] = (100*cos(pi/6), 100*sin(pi/6)) # initial velocity, m/s
r[0] = (0,1) # initial position, m
```

We also need various bookkeeping code

Here we define physical constants for our system and define the function that describes the forces

```
m = 5.5 # mass, kg
g = 9.81 # acceleration of gravity, m/s^2
rho = 1.3 # air density, kg/m^3
C_D = 0.45 # drag coefficient
d = 0.11 # diameter of cannonball, m
A = pi*d**2 # cross-sectional area, m^2

def F(r, v, t):
    return (0, -m*g) - 0.5*rho*C_D*A*abs(v)*v
```

This example show the forces acting on the cannonball as it sails through the air

$$F(x, v, t) = F_g + F_d(\vec{v}) = -mg\vec{k} - \frac{1}{2}\rho C_D A |\vec{v}| \vec{v}$$

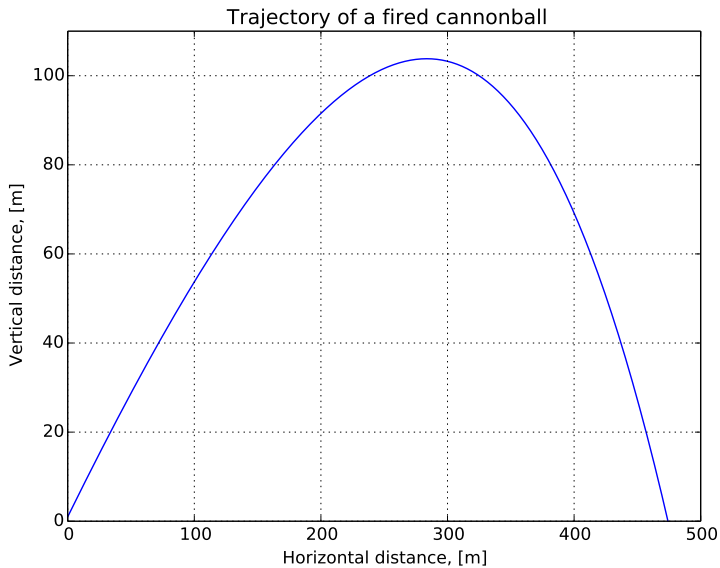
As soon as we have solved the equations of motion, we can plot the result

```
# Extract x and y coordinates
x = r[:,0]
y = r[:,1]

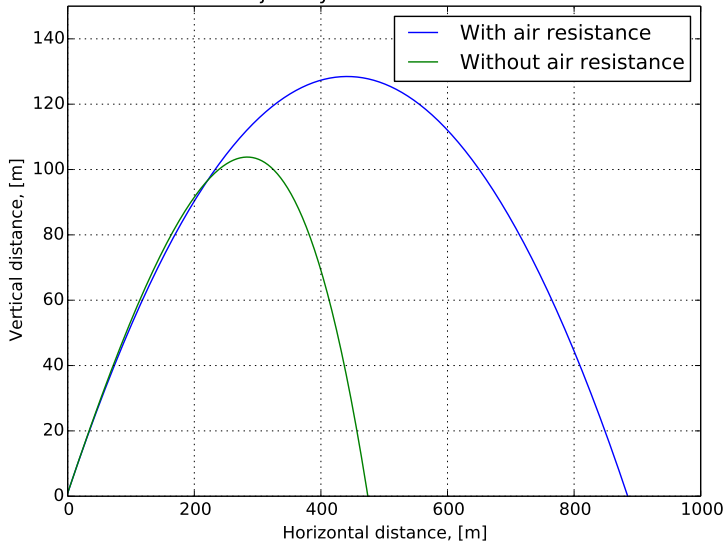
# Plot figure
plot(x,y)

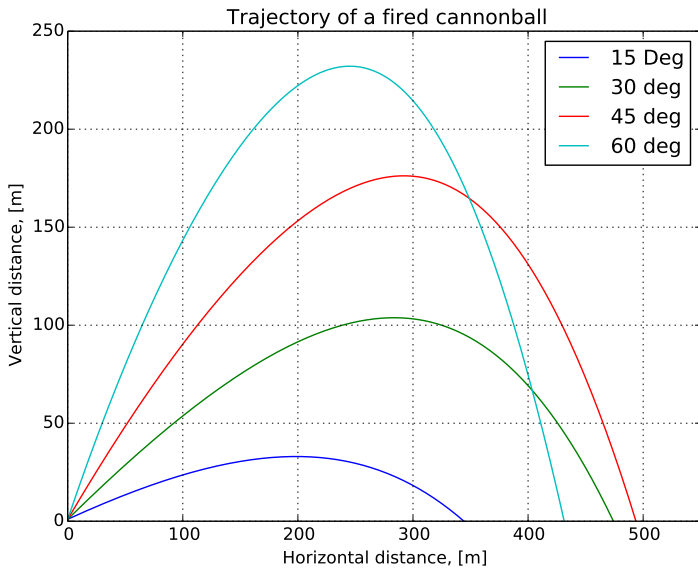
# Prettify the plot
xlabel('Horizontal distance, [m]')
ylabel('Vertical distance, [m]')
title('Trajectory of a fired cannonball')
grid()
axis([0, 900, 0, 250])

# Makes the plot appear on the screen
show()
```

Trajectory of a fired cannonball





Results

- Can plot numerical result vs known solution
- Can plot with and without drag
- Can plot for various initial angles

Examples of possible projects

- Catapults, cannons, throwing of objects. Can include analytical and experimental data.
- Skydiver and plot the g -forces, can illustrate why a parachute needs to open slowly!
- Bungee-jumping! Great for teaching spring forces and programming technicalities.
- Block and spring, harmonic Oscillator
- Orbits of planets in the solar system with real data from NASA
- Pendulum (also with large angles!)