Workshop: Projectile Motion

An introduction to computing trajectories

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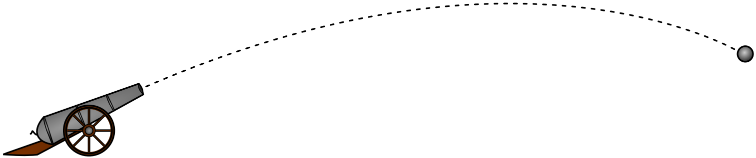
The goal is to find the velocity and position of an object as functions of time: $\vec{v}(t)$, $\vec{r}(t)$



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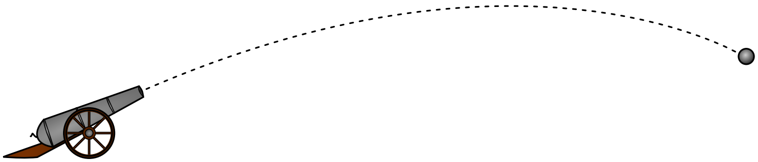

Equations of motion

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{v}(t), \qquad \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \vec{a}(t)$$

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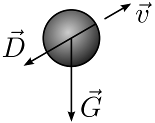

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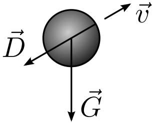

$$\vec{F}(r, v, t) = m\vec{a}(r, v, t).$$

The goal is to find the velocity and position of an object as functions of time: $\vec{v}(t)$, $\vec{r}(t)$

Our algorithm is now as follows

1. Find the physical forces of the system.
2. Use Newtons 2. law to find the acceleration
3. Calculate $\vec{v}(t)$ and $\vec{r}(t)$ by solving the equations of motion

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In this workshop, we will solve step number 3 numerically, using the Euler method.

The Euler Method

A method for solving ordinary
differential equations (ODEs)

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$$\frac{v(t + \Delta t) - v(t)}{\Delta t} \approx a(t)$$

Solving for $v(t + \Delta t)$ gives

$$v(t + \Delta t) \approx v(t) + a(t) \cdot \Delta t$$

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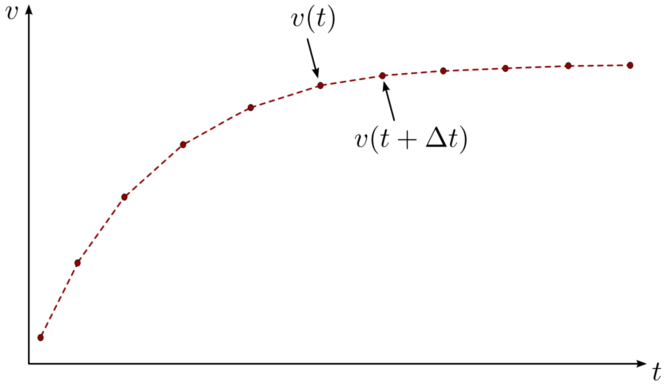

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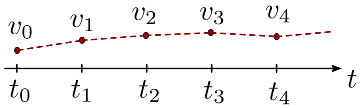

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We repeat these steps, starting at our initial conditions v_0 and r_0, until we have reached our end-time t_N

$$i = 0, 1, 2, 3, \ldots, N.$$

Algorithm for the Euler method

for $i = 0, 1, 2, 3, \dots, N-1$:

1. Use the previous results x_i and v_i to compute the acceleration: $a_i = F(x_i, v_i, t_i)/m$.
2. Compute the new velocity: $v_{i+1} = v_i + a_i \Delta t$.
3. Compute the new position: $r_{i+1} = r_i + v_i \Delta t$.

Implementation

Moving from physics and math to
actual computer code

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$$\Downarrow$$

```python
for i in range(N):
    a[i] = F(r[i], v[i], t[i])/m
    v[i+1] = v[i] + a[i]*dt
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We want the code to look as much as possible like the physics and math we write on paper

$$t_i \Rightarrow \texttt{t[i]} \qquad v_i \Rightarrow \texttt{v[i]} \qquad r_i \Rightarrow \texttt{r[i]}$$

We also need various bookeeping code

Here we define the arrays we will be using

```
# Import various functions meant for numerical science
from pylab import *

t_0 = 0 # Start time, s
t_end = 10 # End time, s
N = 1000 # Number of time steps

# Create a uniformly spaced time-array
t = linspace(t_0, t_end, N+1)

# Calculate the size of a time step
dt = t[1] - t[0]

# Create empty acceleration, velocity and position arrays
a = zeros((2, N+1))
v = zeros((2, N+1))
r = zeros((2, N+1))

# Set initial conditions
v[0] = (100*cos(pi/6), 100*sin(pi/6)) # inital velocity, m/s
r[0] = (0,1)  # initial position, m
```

We also need various bookeeping code

Here we define physical constants for our system and define the function that describes the forces

```
m = 5.5 # mass, kg
g = 9.81 # acceleration of gravity, m/s^2
rho = 1.3 # air density, kg/m^3
C_D = 0.45 # drag coefficient
d = 0.11 # diameter of cannonball, m
A = pi*d**2 # cross-sectional area, m^2

def F(r, v, t):
    return (0, -m*g) - 0.5*rho*C_D*A*abs(v)*v
```

This example show the forces acting on the cannonball as it sails through the air

$$F(x, v, t) = F_g + F_d(\vec{v}) = -mg\vec{k} - \frac{1}{2}\rho C_D A|\vec{v}|\vec{v}$$

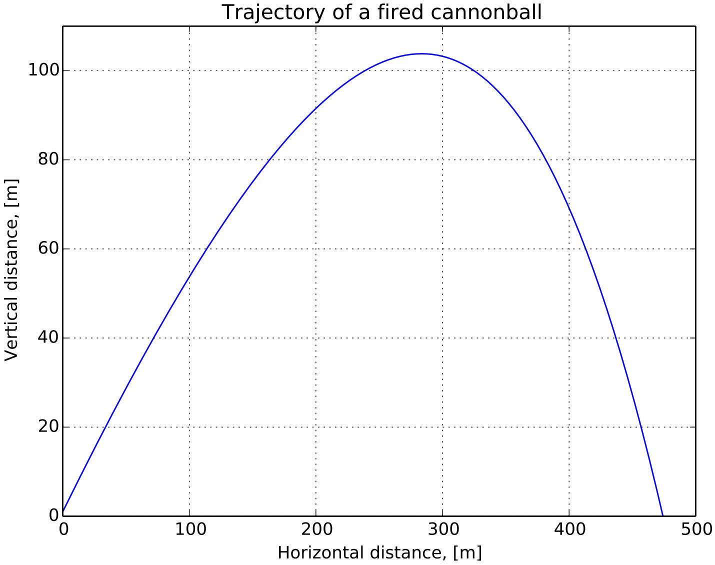
As soon as we have solved the equations of motion, we can plot the result

```
# Extract x and y coordinates
x = r[:,0]
y = r[:,1]

# Plot figure
plot(x,y)

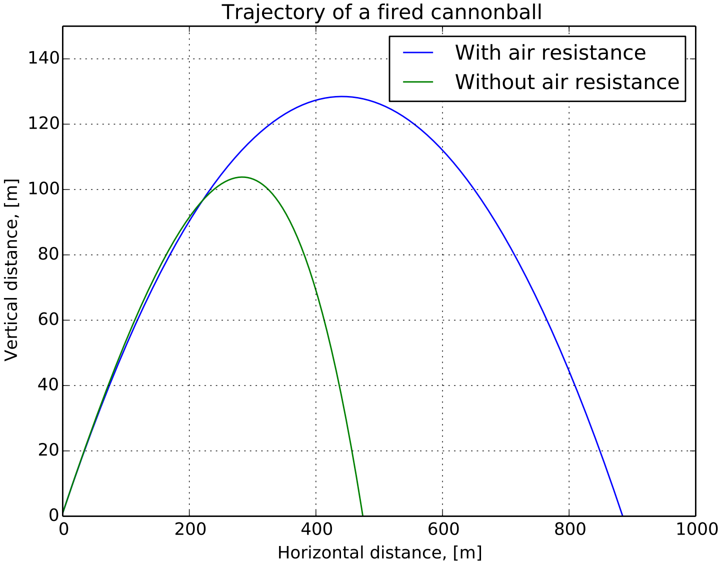
# Prettify the plot
xlabel('Horizontal distance, [m]')
ylabel('Vertical distance, [m]')
title('Trajectory of a fired cannonball')
grid()
axis([0, 900, 0, 250])

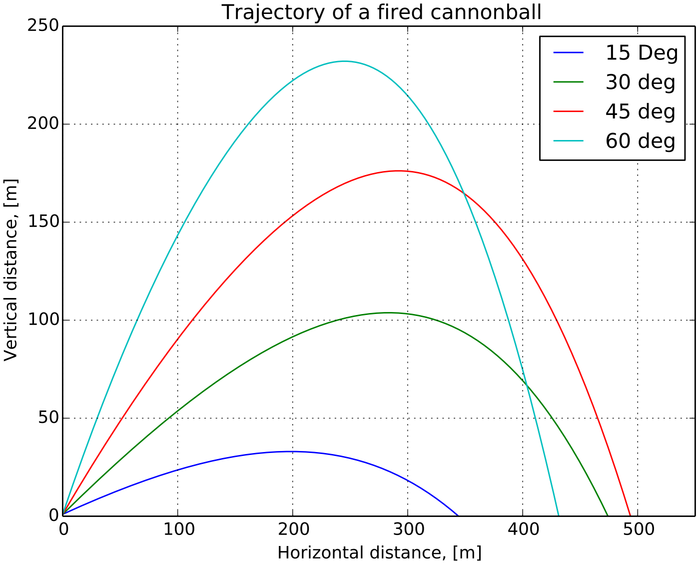
# Makes the plot appear on the screen
show()
```


Trajectory of a fired cannonball
Vertical distance, [m]
Horizontal distance, [m]
0
20
40
60
80
100
0
100
200
300
400
500

Numerical Experimentation

Altering parameters let's us immediately see the consequences


Trajectory of a fired cannonball
With air resistance
Without air resistance
Vertical distance, [m]
Horizontal distance, [m]
0
20
40
60
80
100
120
140
0
200
400
600
800
1000



Trajectory of a fired cannonball
15 Deg
30 deg
45 deg
60 deg
Vertical distance, [m]
Horizontal distance, [m]
0
50
100
150
200
250
0
100
200
300
400
500

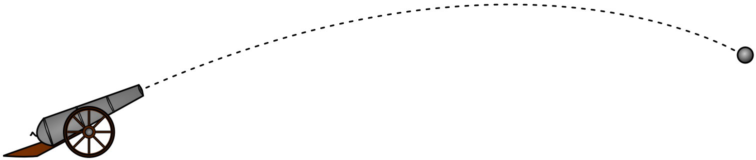
Students can use numerical experimentation to build intuition and knowledge

- Numerical results can be compared to known analytical solutions. Are numerical results trustworthy?
- Can study how results are directly changed by parameter choice. Are the parameters chosen reasonable?
- Can look at systems with and without certain contributions, such as air drag.
  What is important, and what can be ignored?

Examples of possible projects

You will have a chance to look at some of these today

Catapults and cannons and sports such as baseball

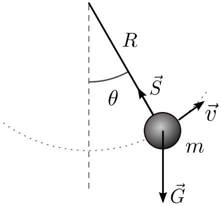


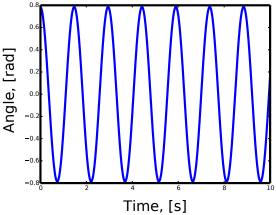
- Easy to compare with experimental data, either before or after simulation.
- Can look into studies of air drag, Renoylds number etc.

Skydiving and bungeejumping

- Great study on free fall and terminal velocity
- Can study how parameters such as cross-sectional area and drag coefficient change as the parachute is opened
- Can plot the g-forces affecting the jumper. Which sport is more "extreme"?

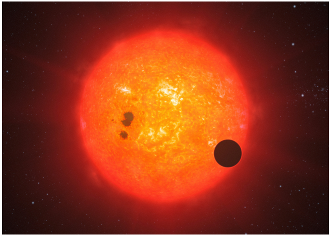
Pendulum and angular motion





- Can solve pendulum also for large angles!
- Energy can be plotted as functions of time
- Can also simulate double pendulum and chaotic systems

Modelling the solar system



- Students can gather real data of planetary orbits from NASA webpages
- Can combine numerical simulation with better graphics