Problem set 7
FYS3140

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Boas 8.12.16

The DE:

$$x^2 y'' - 2xy' + 2y = x \ln x,$$

has the homogenous solution $y_h(x) = c_1 x + c_2 x^2$.

We will find a particular solution of the DE by variation of parameters, we start by assuming the solution can be written on the form

$$y_p(x) = c(x) y_1(x) = x c(x).$$

the derivatives of y_p are then

$$y_p' = xc' + c, \qquad y_p'' = xc'' + 2c'.$$

Insertion into the original DE will then gives an equation for $c'(x)$:

$$x^3 c'' + 2x^2 c' - 2x^2 c' - 2xc + 2xc = x \ln x \quad \Rightarrow \quad x^3 c''(x) = x \ln x.$$

So we have the equation

$$\frac{\mathrm{d}^2 c}{\mathrm{d}x^2} = \frac{\ln x}{x^2},$$

meaning we can find $c(x)$ by integrating twice, note that we can discard the integration constants, as we are looking for a particular solution,

$$\frac{\mathrm{d}c}{\mathrm{d}x} = \int \frac{\ln x}{x^2}\, \mathrm{d}x = -\frac{\ln x}{x} + \int \frac{1}{x^2}\, \mathrm{d}x = -\frac{\ln x + 1}{x},$$

giving

$$c(x) = -\int \frac{\ln x}{x}\, \mathrm{d}x - \int \frac{1}{x}\, \mathrm{d}x = -\frac{1}{2} \ln^2 x - \ln x.$$

So the particular solution is

$$y_p(x) = x c(x) = -\frac{1}{2} x \ln^2 x - x \ln x,$$

and the full solution of the DE is then

$$y(x) = c_1 x + c_2 x^2 - \frac{1}{2} x \ln^2 x - x \ln x.$$

Boas 8.12.18

The DE:

$$(x^2+1)y'' - 2xy' + 2y = (x^2+1)^2,$$

has the homogenous solution $y_h(x) = c_1 x + c_2(1-x^2)$.

We will find a particular solution of the DE by variation of parameters, we start by assuming the solution can be written on the form

$$y_p(x) = c(x)y_1(x) = xc(x).$$

the derivatives of y_p are then

$$y_p' = xc' + c, \qquad y_p'' = xc'' + 2c'.$$

Insertion into the original DE will then gives an equation for $c'(x)$:

$$(x^2+1)xc'' + 2(x^2+1)c' - 2x^2c' - 2xc + 2xc = (x^2+1)^2.$$

So we have the equation

$$c'' + \frac{2}{x(x^2+1)}c' = \frac{x^2+1}{x}.$$

This is a first-order linear equation for $c'(x)$, which we solve by using the technique of integrating factors, we see that

$$P(x) = \frac{2}{x(x^2+1)} \quad \Rightarrow \quad I = \int P(x)\,\mathrm{d}x = \int \frac{2}{x(x^2+1)}\,\mathrm{d}x,$$

using the substitution $u = x^2$ and partial fraction decomposition we find

$$I = \int \frac{1}{u(u+1)}\,\mathrm{d}u = \int \frac{1}{u}\,\mathrm{d}u - \frac{1}{1+u}\,\mathrm{d}u = \ln x^2 - \ln(1+x^2),$$

and

$$e^I = \frac{x^2}{1+x^2}.$$

The solution for $c'(x)$ is then

$$c'(x)\frac{x^2}{1+x^2} = \int \frac{x^2+1}{x}\frac{x^2}{1+x^2}\,\mathrm{d}x + D = \frac{1}{2}x^2 + D,$$

$$c'(x) = \frac{1+x^2}{2} + D\frac{1+x^2}{x^2},$$

as we are looking for a particular solution, we can discard the second term, as the integration constant can be chosen ($D=0$), giving us

$$c'(x) = \frac{1+x^2}{2} \quad \Rightarrow \quad c(x) = \int \frac{1+x^2}{2}\,\mathrm{d}x = \frac{1}{2}x + \frac{1}{6}x^3,$$

meaning we have found a particular solution

$$y_p(x) = xc(x) = \frac{1}{2}x^2 + \frac{1}{6}x^4,$$

and the full solution of the DE is then

$$y(x) = c_1x + c_2(1-x^2) + \frac{1}{2}x^2 + \frac{1}{6}x^4.$$

Boas 8.13.7

We will solve the non-linear first-order ODE:

$$3x^3y^2y' - x^2y^3 = 1.$$

We start by using the substitution

$$u = y^3, \qquad u' = 3xy^2y',$$

giving a linear ODE, on standard form, we have

$$u' - \frac{u}{x} = \frac{1}{x^2}.$$

We can now solve this equation for $u(x)$ using integrating factors

$$I(x) = \int P(x)\ \mathrm{d}x = -\int \frac{1}{x}\ \mathrm{d}x = -\ln x,$$

and

$$e^{I(x)} = \frac{1}{x}.$$

So the solution is given by

$$u(x) = x\bigg(\int \frac{1}{x^3}\ \mathrm{d}x + c\bigg) = cx - \frac{1}{3x^2},$$

substituting back for y, gives

$$y(x) = c\sqrt[3]{x} - \frac{1}{\sqrt[3]{3x^2}}.$$

Boas 12.1.8

We will solve the DE:

$$(x+2x)y'' - 2(x+1)y' + 2y = 0,$$

using the power series method.

We start by writing the solution as a general power series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

and insert this into the original DE, if we then compare power by power, we get the equation

$$\left(n^2 - 3n + 2\right)a_n + \left(2n^2 - 2\right)a_{n+1} = 0,$$

We can then insert for $n = 0, 1, 2, \ldots$

$$\begin{array}{r|l|l}
n = 0: & 2a_0 - 2a_1 = 0 & a_1 = a_0 \\
n = 1: & 0a_1 - 0a_2 = 0 & a_2 = c_2 \\
n = 2: & 0a_2 + 2a_3 = 0 & a_3 = 0 \\
n = 3: & 2a_3 + 16a_4 = 0 & a_4 = 0
\end{array}$$

and we see this trend will continiue forever, meaning all coefficients a_n for $n \geq 3$ are equal to zero. Giving us the final solution

$$y(x) = c_1(1+x) + c_2 x^2.$$

Boas 12.11.2

We will solve the DE:

$$x^2y'' + xy' - 9y = 0,$$

using Frobenius' method. We write $y(x)$ as

$$y(x) = x^s \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+s}.$$

Inserting this into the original DE gives

$$\sum_{m=0}^{\infty} [(m+s)(m+s-1)a_m + (m+s)a_m - 9a_m]x^{m+s}.$$

We can now compare power by power, starting with $m = 0$:

$$s(s+1)a_0 + sa_0 - 9a_0 = 0,$$

as we know a_0 is 0 by assumption, we now have an indicial equation that we can use to find s:

$$s^2 - 9 = (x+3)(x-3) = 0 \quad \Rightarrow \quad s = \pm 3.$$

We use the smallest value of $s = -3$, and derive an expression for a_n, by comparing the powers of x^{n-3}:

$$(n-3)(n-4)a_n + (n-3)a_n - 9a_n = 0,$$

which can be simplified to

$$n(n-6)a_n = 0,$$

so we see that $a_n = 0$ for every term except for $n = 0$ and $n = 6$, so we have a_0 and a_6 as free coefficients. We now get

$$y(x) = x^{-3}\left(a_0 + a_6x^6\right) = c_1x^3 + \frac{c_2}{x^3}.$$