Problem set 2
FYS3140

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Problem 2.1

Functions of complex variables can be written in the general form $f(z) = u(x,y)+iv(x,y)$. We will now find $u(x,y)$ and $v(x,t)$ for two different functions.

a)

The function

$$\frac{-i+2z}{2+iz},$$

we start by writing the complex variable as $z = x + iy$, and then simplify the fraction.

$$\frac{-i+2z}{2+iz} = \frac{2x+(2y-1)i}{(2-y)+xi} = \frac{(2x+(2y-1)i)((2-y)+xi)}{(2-y)^2+x^2},$$

multplying out gives

$$\frac{2x(2-y)+2x^2i+(2y-1)(2-y)i-x(2y-1)i}{4-2y+y^2+x^2},$$

which can be written as

$$\left(\frac{4x-2xy}{x^2+y^2-2y+4}\right)+\left(\frac{2x^2-2y^2-2xy+5y+x-2}{x^2+y^2-2y+4}\right)i.$$

b)

The function e^{iz}. We start by writing the complex variable as $z = x + iy$, and then use Eulers formula:

$$e^{iz} = e^{i(x+iy)} = e^{-y+ix} = e^{-y}e^{ix} = e^{-y}\bigg(\cos x + i\sin x\bigg),$$

giving

$$u(x,y) = e^{-y}\cos x, \qquad v(x,y) = e^{-y}\sin x.$$

Problem 2.2 (Derivatives)

We will now use the definition of the complex derivative

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z},$$

to show that the product rule holds for the functions of complex variables.

We start by using the definition of the derivative on $[f(z)g(z)]$:

$$\frac{\mathrm{d}}{\mathrm{d}z}[f(z)g(z)] = \lim_{\Delta z \to 0} \frac{\Delta[f(z)g(z)]}{\Delta z},$$

we now expand $\Delta[f(z)g(z)]$,

$$\Delta[f(z)g(z)] = g(z)\Delta f + f(z)\Delta g,$$

giving us

$$\frac{\mathrm{d}}{\mathrm{d}z}[f(z)g(z)] = \lim_{\Delta z \to 0} \frac{g(z)\Delta f + f(z)\Delta g}{\Delta z},$$

which can be further rewritten to

$$\frac{\mathrm{d}}{\mathrm{d}z}[f(z)g(z)] = g(z) \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} + f(z) \lim_{\Delta z \to 0} \frac{\Delta g}{\Delta z},$$

or

$$\frac{\mathrm{d}}{\mathrm{d}z}[f(z)g(z)] = g(z)\frac{\mathrm{d}f}{\mathrm{d}z} + f(z)\frac{\mathrm{d}g}{\mathrm{d}z}, \quad \text{q.e.d.}$$

Problem 2.3 (Cauchy-Riemann conditions)

We will now derive the Cauchy-Riemann conditions in polar coordinates.

We have a function of a complex variable z, where $z = re^{i\theta}$, by partial differentiation we have

$$\frac{\partial f}{\partial r} = \frac{\mathrm{d}f}{\mathrm{d}z}\frac{\partial z}{\partial r} = \frac{\mathrm{d}f}{\mathrm{d}z} \cdot e^{i\theta},$$

and

$$\frac{\partial f}{\partial \theta} = \frac{\mathrm{d}f}{\mathrm{d}z}\frac{\partial z}{\partial \theta} = \frac{\mathrm{d}f}{\mathrm{d}z} \cdot ire^{i\theta}.$$

Since $f = u(r,\theta) + iv(r,\theta)$, we also have

$$\frac{\partial f}{\partial r} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}, \quad \text{and} \quad \frac{\partial f}{\partial \theta} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}.$$

By combining these equations we find

$$\frac{\mathrm{d}f}{\mathrm{d}z} \cdot e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}, \qquad \text{and} \qquad \frac{\mathrm{d}f}{\mathrm{d}z} \cdot ire^{i\theta} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}.$$

We demand that the derivative $\mathrm{d}f/\mathrm{d}z$ is unique, meaning we can combine the equations to eliminate it, inserting the first into the second gives:

$$\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) ir = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}.$$

Setting the real and imaginary parts equal gives the Cauchy-Riemann conditions in polar coordinates

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}, \qquad r\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta}.$$

Problem 2.4 (Harmonic Functions)

a)

We will show that the following function is harmonic,

$$u(x,y) = \frac{y}{(1-x)^2 + y^2},$$

by definition, this means that it is a solution to Laplace's equation:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

To show this, we calculate the partial derivatives seperately. We also introduce the shorthand notation

$$\kappa \equiv \left[(1-x)^2 + y^2\right] = x^2 - 2x + y^2 + 1, \qquad u = y\kappa^{-1}.$$

The explicit derivatives with respect to x gives:

$$\frac{\partial u}{\partial x} = 2y(1-x)\kappa^{-2}, \qquad \frac{\partial^2 u}{\partial x^2} = -2y\kappa^{-2} + 8y(1-x)^2\kappa^{-3}.$$

And for y we get

$$\frac{\partial u}{\partial y} = \kappa^{-1} - 2y^2\kappa^{-2}, \qquad \frac{\partial^2 u}{\partial y^2} = -2y\kappa^{-2} - 4y\kappa^{-2} + 8y^3\kappa^{-3}.$$

Combining the expressions now gives

$$\nabla^2 u = -8y\kappa^{-2} + 8y(1-x)^2\kappa^{-3} + 8y^3\kappa^{-3},$$

which we can rewrite to

$$\nabla^2 u = -8y\kappa^{-2} + 8y\left[(1-x)^2 + y^2\right]\kappa^{-3} = -8y\kappa^{-2} + 8y\kappa^{-2} = 0.$$

So we see that the Laplacian does indeed vanish, and so u is, by definition, a harmonic function.

b)

We will now use the Cauchy-Riemann equations to find the harmonic conjugate $v(x,y)$.

The first conditions demands

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2y(1-x)\kappa^{-2},$$

which gives

$$v(x,y) = \int 2y(1-x)\kappa^{-2}\ \mathrm{d}y = (x-1)\kappa^{-1} + g(x).$$

The second condition demands

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y^2\kappa^{-2} - \kappa^{-1} = \left[y^2 - (1-x)^2\right]\kappa^{-2},$$

but we can also calculate $\partial v/\partial x$ from the expression we found for v earlier:

$$\frac{\partial v}{\partial x} = \kappa^{-1} - (1-x)^2\kappa^{-2} = \left[y^2 - (1-x)^2\right]\kappa^{-2},$$

so we see that $g(x) = 0$, and we have

$$v(x,y) = (x-1)\kappa^{-1}.$$

c)

Just to check that v does indeed satisfy Laplace's equation, we calculate $\partial^2 v/\partial x^2$ and $\partial^2 v/\partial y^2$. We already know the first derivatives, from the Cauchy-Riemann conditions, which gives

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(2y^2\kappa^{-2} - \kappa^{-1}\right) = 8y^2(1-x)\kappa^{-3} - 2(1-x)\kappa^{-2},$$

and

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left(2y(1-x)\kappa^{-2}\right) = 2(1-x)\kappa^{-2} - 8y^2(1-x)\kappa^{-3}.$$

Summing these gives

$$\nabla^2 v(x,y) = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0,$$

so $v(x,y)$ does indeed satisfy Laplace's equation.

Extra problems

2.17.25

(a)—Show that $\overline{\cos(z)} = \cos \overline{z}$.

This is easily shown by writing cos in term of the exponential functions:

$$\overline{\cos(z)} = \frac{\overline{e^{iz} + e^{-iz}}}{2} = \frac{\overline{e^{iz}} + \overline{e^{-iz}}}{2} = \frac{e^{-i\overline{z}} + e^{i\overline{z}}}{2} = \cos \overline{z}.$$

(b)—Is $\overline{\sin z} = \sin \overline{z}$?

Let us try the same method as in the previous case:

$$\overline{\sin(z)} = \overline{\left(\frac{e^{iz} - e^{-iz}}{2i}\right)} = \frac{\overline{e^{iz}} - \overline{e^{-iz}}}{\overline{2i}} = \frac{e^{-i\overline{z}} - e^{i\overline{z}}}{-2i} = \frac{e^{i\overline{z}} - e^{-i\overline{z}}}{2i} = \sin \overline{z}.$$

So we see that the answer is yes—$\overline{\sin z} = \sin \overline{z}$ for all $z \in \mathbb{C}$.

(c)—If $f(z) = 1 + iz$, is $\overline{f(z)} = f(\overline{z})$?

This is easy to test:

$$\overline{f(z)} = \overline{1 + iz} = 1 + \overline{iz} = 1 - i\overline{z} \neq f(\overline{z}),$$

We see that the two are not equivalent.

(d)—If the power series of $f(z)$ has only real coefficients, show $\overline{f(z)} = f(\overline{z})$.

We use the fact that the coefficients are real, i.e., $\overline{a_k} = a_k$, to show:

$$\overline{f(z)} = \overline{\sum_k a_k z^k} = \sum_k a_k \overline{z}^k = f(\overline{z}).$$

(e)—Using the result of (d), verify that $i[\sinh(1+i) - \sinh(1-i)]$ is real.

If we write sinh in terms of the exponential functions:

$$\sinh z = \frac{e^z - e^{-z}}{2},$$

we see that the power series of sinh only has real coefficients, meaning

$$\overline{\sinh(z)} = \sinh(\overline{z}),$$

and we have

$$i[\sinh(1+i) - \sinh(1-i)] = i[\sinh(1+i) - \overline{\sinh(1+i)}] = -\Im[\sinh(1+i)],$$

where $\Im$ denotes the imaginary part of the argument, which is purely real, as was to be shown.

2.17.28

Evaluate the absolute square of a complex number, assume a and b are real.

$$|c|^2 = \left| \frac{(a+bi)^2 e^b - (a-bi)^2 e^{-b}}{4abie^{-ia}} \right|^2,$$

using the fact that $|c|^2 = \overline{c} \cdot c$, gives

$$\begin{aligned}
|c|^2 &= \overline{\left(\frac{(a+bi)^2 e^b - (a-bi)^2 e^{-b}}{4abie^{-ia}} \right)} \left(\frac{(a+bi)^2 e^b - (a-bi)^2 e^{-b}}{4abie^{-ia}} \right) \\
&= \left(\frac{(a-bi)^2 e^b - (a+bi)^2 e^{-b}}{-4abie^{ia}} \right) \left(\frac{(a+bi)^2 e^b - (a-bi)^2 e^{-b}}{4abie^{-ia}} \right) \\
&= \frac{(a+bi)^2 (a-bi)^2 \left(e^{2b} + e^{-2b}\right) - (a+bi)^4 - (a-bi)^4}{16a^2b^2} \\
&= \frac{\left(a^2+b^2\right)^2}{8a^2b^2} \cosh(2b) - \frac{(a+bi)^4 + (a-bi)^4}{16a^2b^2}.
\end{aligned}$$

2.17.32

We will show that

$$\sum_{n=0}^{\infty} \frac{(1+i\pi)^n}{n!} = -e.$$

We recognize the sum as the expansion of e^z, so we have

$$\sum_{n=0}^{\infty} \frac{(1+i\pi)^n}{n!} = e^{1+i\pi} = ee^{i\pi} = e(-1) = -e, \qquad \text{q.e.d.}$$