We are studying a stochastic variabel

$$X = \sum_{i=1}^{N} x_i,$$

where the x_i's are random, independant, samples of a distribution

$$P(x) \propto \frac{1}{x}.$$

2.)

If we let y be a sample from the uniform distribution

$$P_0(y) \sim \text{uniform}(-1,1) = \begin{cases} 1/2 & \text{if } x \in [-1,1] \\ 0 & \text{else,} \end{cases}$$

and we let x be a dependant variable of y, meaning it is given from some function $x(y)$. If we then denote the distribution of x as $P(x)$, we know that

$$P(x)\mathrm{d}x = P_0(y)\mathrm{d}y$$

If we divide by $\mathrm{d}y$ we get

$$P(x)\frac{\mathrm{d}x}{\mathrm{d}y} = P_0(y),$$

we can now divide by $\mathrm{d}x/\mathrm{d}y = x'(y)$ to show that

$$P(x) = \frac{P_0(y)}{x'(y)}.$$

3.)

We now let $x = Be^{Ay}$, and want to find $P(x)$, we first find the derivative

$$\frac{1}{x'(y)} = \frac{1}{Ax},$$

giving

$$P(x) = \begin{cases} \frac{1}{2Ax} & \text{if } Be^{-A} < x < Be^{A} \\ 0 & \text{else} \end{cases}$$

4.)

We check that $P(x)$ is normalized

$$\int_{-\infty}^{\infty} P(x)\,\mathrm{d}x = \int_{Be^{-A}}^{Be^{A}} \frac{1}{2Ax}\,\mathrm{d}x = \frac{1}{2A}\ln(x)\Big|_{Be^{-A}}^{Be^{A}} = \frac{1}{2A}\Big(\ln\left(Be^{A}\right) - \ln\left(Be^{-A}\right)\Big) = 1.$$

And find the expectation value

$$\langle x \rangle = \int_{-\infty}^{\infty} xP(x)\,\mathrm{d}x = \int_{Be^{-A}}^{Be^{A}} \frac{1}{2A}\,\mathrm{d}x = \frac{B}{A}\frac{e^{A} - e^{-A}}{2} = \frac{B}{A}\sinh(A).$$

and

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 P(x)\ \mathrm{d}x = \int_{Be^{-A}}^{Be^{A}} \frac{x}{2A}\ \mathrm{d}x = \frac{B^2}{2A} \frac{e^{2A} - e^{-2A}}{2} = \frac{B^2}{2A} \sinh(2A).$$

We can now use the trigonometric identity

$$\sinh(2A) = 2 \sinh(A) \cosh(A),$$

to find

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 P(x)\, \mathrm{d}x = \int_{Be^{-A}}^{Be^{A}} \frac{x}{2A}\, \mathrm{d}x = \frac{B^2}{2A} \frac{e^{2A} - e^{-2A}}{2} = \frac{B^2}{A} \sinh(A) \cosh(A).$$

So the variance is then

$$\langle \Delta x^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 = \big(A\ \mathrm{cotanh}(A) - 1\big) \frac{B^2}{A^2} \sinh^2(A).$$