Problem set 5
FYS3140

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Problem 5.1 (Residue theory)

a) (Boas 14.7.17)

We will evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 4x + 5}\, \mathrm{d}x = \int_{-\infty}^{\infty} \frac{x \sin x}{(x+2+i)(x+2-i)}\, \mathrm{d}x.$$

Using the fact that $\sin x$ is the imaginary part of the exponential function (Euler's formula), we can write the the integral as

$$I = \Im\left(\oint_{\Gamma_\rho} \int_{-\infty}^{\infty} \frac{ze^{iz}}{(z+2+i)(z+2-i)}\, \mathrm{d}z - \int_{C_\rho^+} \frac{ze^{iz}}{(z+2+i)(z+2-i)}\, \mathrm{d}z \right),$$

where Γ_ρ is the positvely oriented closed contour from $-\rho$ to ρ along the real axis, and then back along the half-circle in the upper-half plane C_ρ^+. From Jordan's lemma, we know that the contour integral over the half-circle goes to zero as ρ grows large. And we can evaluate the closed-contour integral using residue theory. We first find the residues of the integrand in the upper-half plane:

$$\mathrm{Res}(f; -2+i) = \lim_{z \to -2+i} (z+2-i) f(z) = \frac{(-2+i)e^{i(-2+i)}}{2i} = \left(\frac{1}{2} + i\right) e^{-1-2i}.$$

And we then have

$$I = \Im\left(2\pi i \sum_k \mathrm{Res}(f; z_k) \right) = \frac{\pi}{e}\left(2\sin 2 + \cos 2 \right).$$

b) (Boas 14.7.24)

We will evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{x \sin \pi x}{1 - x^2}\, \mathrm{d}x.$$

We do this in pretty much the exact same manner as the previous problem

$$I = \Im\left(\oint_{\gamma_\rho^+} \frac{ze^{i\pi z}}{(1+z)(1-z)}\, \mathrm{d}z + \int_{C_\rho^+} \frac{ze^{i\pi z}}{(1+z)(1-z)} \right).$$

Again, Jordan's lemma guarantees that the contour integral over the half-circle is equal to zero. To evaluate the closed-contour integral, we see that there are no singularities in the upper-half plane, there are however, two on the real axis, so we have

$$\oint_{\gamma_\rho^+} \frac{ze^{i\pi z}}{(1+z)(1-z)}\, \mathrm{d}z = \pi i \left(\mathrm{Res}(f; 1) + \mathrm{Res}(f; -1) \right),$$

and we have

$$\mathrm{Res}(f; 1) = -\frac{e^{i\pi}}{2} = \frac{1}{2}, \qquad \mathrm{Res}(f; -1) = -\frac{e^{-i\pi}}{2} = \frac{1}{2}.$$

Giving

$$I = \Im(\pi i) = \pi.$$

c)

We will evaluate the integral

$$I = \oint_C \frac{\cos(z-1)}{(z+1)(z-2)}\,\mathrm{d}z,$$

where C is the positvely oriented closed contour $|z| = 3$.

As the integrand is analytic on the contour and meromorphic inside it, we know that the integral evaluates to

$$I = 2\pi i \sum_k \mathrm{Res}(f; z_k),$$

where f is the integrand and z_k the singularities of the integrand inside the contour. We see that the integrand has two simple poles inside C, we find the residues at these points:

$$\mathrm{Res}(f;-1) = \lim_{z\to -1}(z+1)f(z) = \frac{\cos(-2)}{-3} = -\frac{1}{3}\cos 2,$$

$$\mathrm{Res}(f;2) = \lim_{z\to 2}(z-2)f(z) = \frac{\cos(1)}{3} = \frac{1}{3}\cos 1,$$

meaning the integral evaluates to

$$I = \frac{2\pi i}{3}\bigg(\cos 1 - \cos 2\bigg).$$

d)

We will evaluate the integral

$$I = \oint_C \frac{\mathrm{d}z}{e^z(z^2-1)^2} = \frac{e^{-z}}{(z+1)^2(z-1)^2}\,\mathrm{d}z,$$

where C is the positvely oriented closed contour $|z| = 2$.

Again we see that the integrand is analytic on the contour and meromorphic inside. We see that the integrand has two 2. order poles inside C, we find the residues at these points to be:

$$\mathrm{Res}(f;1) = \lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}[(z-1)^2 f(z)] = \lim_{z\to 1} -\frac{e^{-z}(z+1)^2 + 2e^{-z}(z+1)}{(z+1)^4} = -\frac{e^{-1}}{2},$$

$$\mathrm{Res}(f;-1) = \lim_{z\to -1}\frac{\mathrm{d}}{\mathrm{d}z}[(z+1)^2 f(z)] = \lim_{z\to -1} -\frac{e^{-z}(z-1)^2 + 2e^{-z}(z-1)}{(z-1)^4} = 0.$$

Meaning the integral evaluates to

$$I = -\frac{\pi i}{e}.$$

Problem 5.2 (First order differential equations)

We know that the solution to a first order differential equation on the form

$$y' + P(x)y = Q(x),$$

is

$$ye^{I(x)} = \int Q(x)e^{I(x)}\ \mathrm{d}x + c,$$

where the integration factor, I, is given by the integral

$$I(x) = \int P(x)\ \mathrm{d}x.$$

a)

We will solve the following differential equation

$$\mathrm{d}y + (2xy - xe^{-x^2})\mathrm{d}x = 0.$$

We start by dividing the equation by the differential $\mathrm{d}x$, giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy - xe^{-x^2} = 0,$$
$$y' + 2xy = xe^{-x^2}.$$

We now find the integrating factor

$$I(x) = \int 2x\ \mathrm{d}x = x^2.$$

The solution to the differential equation is then

$$ye^{x^2} = \int xe^{-x^2}e^{x^2}\ \mathrm{d}x + c,$$
$$y = (\frac{1}{2}x^2 + c)e^{-x^2}.$$

b)

We will solve the following differential equation

$$y' + y\cos x = \sin 2x.$$

The integrating factor becomes

$$I(x) = \int \cos x\ \mathrm{d}x = \sin x,$$

giving the solution

$$ye^{\sin x} = \int \sin(2x)\ e^{\sin x}\ \mathrm{d}x + c = 2\int \sin x \cos x\ e^{\sin x}\ \mathrm{d}x + c,$$

using the substitution $u = \sin x$ gives

$$ye^{\sin x} = 2\int ue^{u}\ \mathrm{d}u + c = 2(\sin x - 1)e^{\sin x} + c,$$
$$y = 2(\sin x - 1) + ce^{-\sin x}.$$

c)

We will solve the differential equation

$$y' \cos x + y = \cos^2 x.$$

We start by dividing the equation by $\cos x$, to get it to standard form

$$y' + \sec x \; y = \cos x.$$

The integrating factor is now found to be

$$I(x) = \int \sec x \; \mathrm{d}x = \int \csc(x + \tfrac{\pi}{2}) \; \mathrm{d}x = \ln\left[\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right],$$

giving

$$e^{I(x)} = \tan\left(\frac{x}{2} + \frac{\pi}{4}\right).$$

Meaning the general solution can be written as

$$y \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) = \int \sin x \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) \; \mathrm{d}x + c.$$

We now do some trigonometric juggling

$$\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) = \frac{1 + \tan\frac{x}{2}}{1 - \tan\frac{x}{2}}.$$

$$\tan\tfrac{x}{2} = \frac{\sin x}{1 + \cos x} \qquad \Rightarrow \qquad \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) = \frac{1 + \sin x + \cos x}{1 - \sin x + \cos x}.$$

We now need to solve the integral

$$\int \sin x \frac{1 + \sin x + \cos x}{1 - \sin x + \cos x} \; \mathrm{d}x,$$

Wolfram Alpha tells us that the solution is

$$-\sin x - 2\ln\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right),$$

giving us the general solution

$$y = \cot\left(\frac{x}{2} + \frac{\pi}{4}\right)\left[c - \sin x - 2\ln\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)\right].$$