Complex Functions

The derivative of a complex function is defined as

$$\frac{\mathrm{d}}{\mathrm{d}z}f(z) = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}.$$

A function $f(z)$ is **Analytic** in a region of $\mathbb{C}$ if it has a unique derivative at every point of that region.

An analytic function must respect complex structure, and must therefore satisfy the **Cauchy-Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}, \qquad r\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta}.$$

If u and v and their partial derivatives with respect to x and y are continuous and satisfy the Cauchy-Riemann equations in a region, then the function is analytic at all points **inside** the region (not necessarily on the boundary).

A **regular point** is a point at which $f(z)$ is analytic. A **singularity** of $f(z)$ is a point at which $f(z)$ is not analytic. It is called an isolated singularity if $f(z)$ is analytic in a neighbourhood of the singularity.

Taylor Expansion

If $f(z)$ is analytic in a region, then it has derivatives of all orders at points inside the region and can be expanded in a Taylor series about any point z_0 inside the region. The power series converges inside the circle about z_0 that extends to the nearest singular point.

Harmonic Functions

A function which satisfies Laplace's equation is said the be harmonic

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.$$

If $f(z) = u + iv$ is analytic in a region, then both u and v are harmonic. Any harmonic function in a simply-connected region is the real or imaginary part of an analytic function $f(z)$. The pair u and v are called **conjugate harmonic functions**.

Upper bound estimate

$$\left|\int_\Gamma f(z)\ \mathrm{d}z\right| \leq ML,$$

where M is the maximum value of $f(z)$ on Γ and L is the length of Γ

$$L = l(\Gamma) = \int_a^b \frac{\mathrm{d}s}{\mathrm{d}t}\mathrm{d}t = \int_a^b \left|\frac{\mathrm{d}z}{\mathrm{d}t}\right|\mathrm{d}t.$$

Fundamental Theorem of Calculus

$$\int_a^b f(t)\ \mathrm{d}t = F(b) - F(a),$$

if f is continuous on $[a, b]$ and $F'(t) = f(t)$ for all t in $[a, b]$.

Curve integral

$$\int_\gamma f(z)\ \mathrm{d}z = \int_a^b f(z(t))z'(t)\ \mathrm{d}t.$$

Cauchy's Integral Theorem

If $f(z)$ is analytic on and inside the simple contour Γ, then the contour integral vanishes

$$\oint_\Gamma f(z)\mathrm{d}z = 0.$$

Cauchy Integral formula

for a function $f(z)$ analytic inside and on a simple closed contour Γ, the value of $f(z)$ at a point $z = a$ inside Γ is given by

$$f(z_0) = \frac{1}{2\pi i}\oint \frac{f(z)}{z - z_0}\mathrm{d}z.$$

Generalized Cauchy Integral formula

if f is analytic inside and on a simple closed positively oriented contour Γ and if z_0 is any point inside Γ, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\mathrm{d}z, \quad \mathrm{n} = 1, 2, 3, \ldots$$

Cauchy's Inequality

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!M}{R^n},$$

where f is analytic on and inside circle C_R of radius R centred at z_0. If $|f(z)| \leq M$ for all z on C_r.

Louiville's Theorem

The only bounded entire functions are the constant functions.

Max on Boundry

A function analytic in a bounded domain and continuous up to and including its boundary attains its maximum modulus on the boundary.

Important integrals

$$\oint_C (z - z_0)^n = 2\pi i\delta_{n,-1},$$

where C encircles z_0 once in the positive direction.

Taylor series

If f is analytic at z_0 then it can be written in terms of a power series, called Taylor series:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \tag{1}$$

The disk of convergence is inside the circle around z_0 'touching' the nearest singularity.

Laurent series

If f is analytic in an annulus $r < |z-z_0| < R$ then it can be written as a sum of two series:

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k + \sum_{k=1}^{\infty} b_k \frac{1}{(z-z_0)^k}$$
$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} \, \mathrm{d}z$$
$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{-n+1}} \, \mathrm{d}z$$

Exponential Function

$$e^z = e^x(\cos y + i \sin y).$$
$$e^z = 1 \text{ holds if, and only if, } z = 2k\pi i, k \in \mathbb{Z}.$$
$$e^{z_1} = e^{z_2} \text{ holds if, and only if, } z_1 = z_2 + 2k\pi i, k \in \mathbb{Z}.$$

We know that the exponential function is **entire**.

The **logarithm** is multivalued

$$\ln z = \mathrm{Ln} r + i\theta + 2\pi i k.$$

General power

$$z^\alpha = e^{\alpha \ln z}.$$

Example: Find all values of $(-2)^i$:

$$\ln(-2) = \mathrm{Ln}2 + (\pi + 2k\pi)i,$$
$$(-2)^i = e^{i\ln(-2)} = e^{i\mathrm{Ln}2} e^{-\pi - 2k\pi},$$

and as k is any integer, $(-2)^i$ has infinitely many values.

The n'th Complex Root

$$z^{1/n} = r^{1/n} \exp\left[i \frac{\theta + 2\pi k}{n}\right], \quad k = 0, 1, \ldots, n-1.$$

Trigonometric Functions

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$
$$\sinh z = \frac{e^z - e^{-z}}{2}, \qquad \cosh z = \frac{e^z + e^{-z}}{2}.$$

We know that sin, cos, sinh and cosh are all **entire**, they are however **not bounded**.

$$\frac{\mathrm{d}}{\mathrm{d}z} \sin z = \cos z, \qquad \frac{\mathrm{d}}{\mathrm{d}z} \cos z = -\sin z,$$
$$\frac{\mathrm{d}}{\mathrm{d}z} \sinh z = \cosh z, \qquad \frac{\mathrm{d}}{\mathrm{d}z} \cosh z = \sinh z,$$
$$\frac{\mathrm{d}}{\mathrm{d}z} \tan z = \sec^2 z, \qquad \frac{\mathrm{d}}{\mathrm{d}z} \sec = \sec z \tan z,$$
$$\frac{\mathrm{d}}{\mathrm{d}z} \cot z = -\csc^2 z, \qquad \frac{\mathrm{d}}{\mathrm{d}z} \csc z = -\csc z \cot z.$$

Trigonometric identities

$$\sin(z + 2\pi) = \sin z, \qquad \cos(z + 2\pi) = \cos(z),$$
$$\sin(-z) = -\sin z, \qquad \cos(-z) = \cos z,$$
$$\sin^2 z + \cos^2 z = 1, \qquad \cosh^2 z - \sinh^2 z = 1,$$
$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1,$$
$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2,$$
$$\sin 2z = 2 \sin z \cos z, \qquad \cos 2z = \cos^2 z - \sin^2 z,$$
$$\sin iz = i \sinh z. \qquad \cos iz = \cosh z,$$
$$\sinh iz = i \sin z. \qquad \cosh iz = \cos z.$$

Complex Power Series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n,$$

Convergence can be tested with **Ratio Test**

$$\rho = \lim_{n\to\infty} \left| \frac{S_{n+1}}{S_n} \right| < 1,$$
$$|z - z_0| \le \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \equiv R,$$

where R is the **Radius of Convergence**.

Example: Disk of conv. for $\sum_{n=1}^{\infty} 2^n (z+i-3)^{2n}$:
Using the ratio test, we find

$$\rho = \lim_{n\to\infty} \left| \frac{2^{n+1}}{2^n} \right| \left| \frac{(z+i-3)^{2n+2}}{(z+i-3)^{2n}} \right| = \lim_{n\to\infty} 2|(z+i-3)|^2$$

The series converges if $\rho < 1$, giving the condition on z:

$$2|z+i-3|^2 < 1 \qquad \Rightarrow \qquad |z - (3-i)| < \frac{\sqrt{2}}{2},$$

disk of convergence has center $z_0 = 3 - i$ and radius $\sqrt{2}/2$.

Taylor Expansions

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
$$e^x \cos x = \sum_{k=0}^{\infty} \frac{x^k \left[(1+i)^k + (1-i)^k\right]}{2k!} = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} + \ldots$$
$$e^x \sin x = \sum_{k=0}^{\infty} \frac{x^k \left[(1+i)^k - (1-i)^k\right]}{2ik!} = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \frac{x^6}{90} + \ldots$$

Residue Theory

General Pole Residue
If f has a pole of order m at z_0, then

$$\mathrm{Res}(f; z_0) = \lim_{z\to z_0} \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} [(z-z_0)^m f(z)].$$

Quotient Simple Pole Residue

$$f(z) = P(z)/Q(z)$$

Let $P(z)$ and $Q(z)$ be analytic at z_0, if $Q(z)$ has a simple zero at z_0, while $P(z_0) \neq 0$, we have

$$\mathrm{Res}(f; z_0) = \frac{P(z_0)}{Q'(z_0)}.$$

Fourier

Fourier series

Fourier series are series expansions of periodic functions,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

Coefficients are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \; \mathrm{d}x \tag{2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \; \mathrm{d}x \tag{3}$$

Dirichlet Conditions

A Fourier series converges if $f(x)$ has a finite number of finite discontinuities and a finite number of maxima in its interval, i.e., is bounded. At discontinuities the series converges to the mid point.

Complex Fourier Series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \qquad c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx} \; \mathrm{d}x$$

Other intervals, length $2L$

$$x \to \frac{\pi x}{L}, \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} \to \frac{1}{2L}\int_{-L}^{L}$$

Parseval's theorem

$$\frac{1}{2L}\int_{-L}^{L} |f(x)|^2 dx = \sum_{-\infty}^{\infty} |c_n|^2$$

Fourier transforms

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx} \; \mathrm{d}x \tag{4}$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{ikx} \; \mathrm{d}k \tag{5}$$

Fourier integral theorem: The inverse transformation is valid if $f(x)$ satisfies the Dirichlet conditions on a finite interval, and the integral

$$\int_{\infty}^{\infty} |f(x)| \; \mathrm{d}x < \infty$$

At discontinuities in $f(x)$, transforming to $F(k)$ and back gives the value at the midpoint of the jump.

Fourier transforms of derivatives

$$\mathcal{F}[f'(x)] = ik\mathcal{F}[f(x)], \mathcal{F}[f''(x)] = -k^2\mathcal{F}[f(x)]$$

Parseval's theorem

$$\int_{-\infty}^{\infty} |f(x)|^2 \mathrm{d}x = \int_{-\infty}^{\infty} |F(k)|^2 \; \mathrm{d}k$$

Fourier Sine Transform

$$f_s(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s(k)\sin kx \; \mathrm{d}k$$

$$F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f_s(x)\sin kx \; \mathrm{d}x$$

Fourier Cosine Transform

$$f_s(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c(k)\cos kx \; \mathrm{d}k$$

$$F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f_c(x)\cos kx \; \mathrm{d}x$$

Tensors

Cross-product

$$\big(\mathbf{A}\times\mathbf{B}\big)_i = \epsilon_{ijk}A_jB_k$$

Dot-product

$$\mathbf{A}\cdot\mathbf{B} = A_iB_i$$

Product of two Levi-Cevita

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

Substitution property

$$\delta_{ij}A_j = A_i$$

Matrix-product

$$C = AB, \quad C_{ij} = A_{ik}B_{kj}$$

Determinant of a matrix

$$|M| = \epsilon_{ijk}M_{1i}M_{2j}M_{3k}$$

Grad

$$[\nabla f]_i = \frac{\partial f}{\partial x_i} = \partial_i f.$$

Div

$$\nabla\cdot\mathbf{u} = \frac{\partial u_j}{\partial x_j} = \partial_j x_j.$$

Curl

$$[\nabla\times\mathbf{u}]_i = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j}.$$

Rules of suffix notation

- An index that appears twice in a term is called a dummy index and is summed from 1 to 3.
  This is called the summation conventation.
- A pair of dummy indices can be changed.
  For example: $a_ib_i = a_mb_m = a_lb_l$.
- The order of terms in a suffix notation expression does not matter, and so can be arranged at will.

Moment of Inertia tensor

We know that

$$L_j = I_{jk}\omega_k,$$

Inserting for $\mathbf{L}$ gives formulas for I_{jk}

$$\mathbf{L} = m\mathbf{r}\times(\boldsymbol{\omega}\times\mathbf{r}) = m[r^2\boldsymbol{\omega} - (x\omega_x + y\omega_y + z\omega_z)\mathbf{r}].$$

And so we have

$$\begin{aligned}
I_{xx} &= m(r^2 - x^2) = m(y^2 + z^2)\\
I_{yy} &= m(r^2 - y^2) = m(x^2 + z^2)\\
I_{zz} &= m(r^2 - z^2) = m(x^2 + y^2)\\
I_{xy} &= I_{yx} = -mxy\\
I_{xz} &= I_{zx} = -mxz\\
I_{yz} &= I_{zy} = -myz
\end{aligned}$$

Note that this is for a single point, so for a continous body we have

$$I_{xx} = \int_V \rho(\mathbf{r})(r^2 - x^2)\ \mathrm{d}V,$$

and for a system of point masses

$$I_{xx} = \sum_i m_i(r_i^2 - x_i^2).$$

Principle moments of inertia

The principle moments of inertia are the eigenvalues of the matrix I.

Principle axes of inertia

The princicple axes of inertia are the eigenvectors of the matrix I.

Laplace Transform

The Laplace transform is an integral transform, defined as

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-pt}\ \mathrm{d}t = F(p).$$

Note that the Laplace transform is linear

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)].$$

The Laplace transform of the derivative

$$\begin{aligned}\mathcal{L}[y] &= Y,\\ \mathcal{L}[y'] &= pY - y_0,\\ \mathcal{L}[y''] &= p^2Y - py_0 - y_0'.\end{aligned}$$

Using this we can solve differential equations, example:

$$y'' - 4y' + 4y = t^2e^{-2t}, \quad y_0 = 0, y_0' = 0$$

$$p^2Y - 4pY + 4Y = \mathcal{L}[t^2e^{-2t}] = \frac{2}{(p+2)^3},$$

$$Y = \frac{2}{(p+2)^5}, \qquad y = \mathcal{L}^{-1}[Y] = \frac{t^4e^{-2t}}{12}.$$

Convolution

$$G(p)H(p) = \mathcal{L}\left[\int_0^t g(t-\tau)h(\tau)\ \mathrm{d}\tau\right] = \mathcal{L}[g * h].$$

Note that: $g * h = h * g$.

Example:

$$y'' + 3y' + 2y = e^{-t}, \qquad y_0 = y_0' = 0.$$

Inserting for the derivatives gives

$$Y = \frac{1}{(p+2)(p+1)}\mathcal{L}[e^{-t}].$$

We need the inverse transform:

$$\mathcal{L}^{-1}\left[\frac{1}{(p+2)(p+1)}\right] = e^{-t} - e^{-2t}.$$

Giving

$$Y = \mathcal{L}[e^{-t} - e^{-2t}]\mathcal{L}[e^{-t}] = G(p)H(p),$$

with $g(t) = e^{-t} - e^{-2t}$, and $h(t) = e^{-t}$. Giving

$$\begin{aligned}y &= \int_0^t g(\tau)h(t-\tau)\ \mathrm{d}\tau\\ &= \int_0^t (e^{-\tau} - e^{-2\tau})(e^{-(t-\tau)})\ \mathrm{d}\tau\\ &= te^{-t} + e^{-2t} - e^{-t}.\end{aligned}$$

Convolution of Fourier Transforms

We let $g_1(\alpha)$ and $g_2(\alpha)$ be the Fourier transforms of $f_1(x)$ and $f_2(x)$. Then we know that

$$g_1 \cdot g_2 \text{ and } \frac{1}{\sqrt{2\pi}} f_1 * f_2 \text{ are a pair of Fourier transforms}$$

$$g_1 * g_2 \text{ and } \frac{1}{\sqrt{2\pi}} f_1 \cdot f_2 \text{ are a pair of Fourier transforms}$$

where the convolution is now defined as

$$f_1 * f_2 = \int f_1(x-u)f_2(u)\ \mathrm{d}u.$$

Laplace Table

$f(t)$	$F(p) = \mathcal{L}[f(t)]$	Valid for
1	$\frac{1}{p}$	$\mathrm{Re}\ p > 0$
e^{-at}	$\frac{1}{p+a}$	$\mathrm{Re}\ (p+a) > 0$
$\sin at$	$\frac{a}{p^2+a^2}$	$\mathrm{Re}\ p > \lvert\mathrm{Im}\ a\rvert$
$\cos at$	$\frac{p}{p^2+a^2}$	$\mathrm{Re}\ p > \lvert\mathrm{Im}\ a\rvert$
$t^k, k > -1$	$\frac{k!}{p^{k+1}}$ or $\frac{\Gamma(k+1)}{p^{k+1}}$	$\mathrm{Re}\ p > 0$
$t^ke^{-at}, k > -1$	$\frac{k!}{p^{k+1}}$ or $\frac{\Gamma(k+1)}{p^{k+1}}$	$\mathrm{Re}\ (p+a) > 0$
$\frac{e^{-at} - e^{-bt}}{b-a}$	$\frac{1}{(p+a)(p+b)}$	$\mathrm{Re}\ (p+a) > 0$ $\mathrm{Re}\ (p+b) > 0$
$\frac{ae^{-at} - be^{-bt}}{a-b}$	$\frac{p}{(p+a)(p+b)}$	$\mathrm{Re}\ (p+b) > 0$ $\mathrm{Re}\ (p+b) > 0$
$\sinh at$	$\frac{a}{p^2-a^2}$	$\mathrm{Re}\ p > \lvert\mathrm{Re}\ a\rvert$
$\cosh at$	$\frac{p}{p^2-a^2}$	$\mathrm{Re}\ p > \lvert\mathrm{Re}\ a\rvert$
$t\sin at$	$\frac{2ap}{(p^2+a^2)^2}$	$\mathrm{Re}\ p > \lvert\mathrm{Im}\ a\rvert$
$t\cos at$	$\frac{p^2-a^2}{(p^2+a^2)^2}$	$\mathrm{Re}\ p > \lvert\mathrm{Im}\ a\rvert$
$e^{-at}\sin bt$	$\frac{b}{(p^2+a^2)^2+b^2}$	$\mathrm{Re}(p+a) > \lvert\mathrm{Im}\ b\rvert$
$e^{-at}\cos bt$	$\frac{p+a}{(p^2+a^2)^2+b^2}$	$\mathrm{Re}(p+a) > \lvert\mathrm{Im}\ b\rvert$
$g(t-a),\ t > a$ $g(t-a)u(t-a)$	$e^{-pa}G(p)$	
$e^{-at}g(t)$	$G(p+a)$	
$tf(t)$	$-\frac{\mathrm{d}}{\mathrm{d}p}F(p)$	

Dirac Delta Function

Sifting property

$$\int_a^b \phi(t)\delta(t-t_0)\ \mathrm{d}t = \phi(t_0), \quad \text{if } a < t_0 < b$$

Derivatives of the Delta-function

$$\int_a^b \phi(x)\delta^{(n)}(x-a)\ \mathrm{d}x = (-1)^n\phi^{(n)}(a)$$

Connection to the unit step function

$$u'(x-a) = \delta(x-a)$$

Scaling-factors

$$\delta(ax) = \frac{1}{|a|}\delta(x), \quad a \neq 0.$$

Integral-form

$$\delta(\omega - \omega_0) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{i(\omega-\omega_0)x}\ \mathrm{d}x$$

Ordinary Differential Equations

Variation of Parameters

If one solution $y_1(x)$ of a homogeneous DE is known, a second, linearly independent one, can be found from the ansatz

$$y_2(x) = c(x)y_1(x),$$

this will lead to a first order DE for $c'(x)$, to be solved by integrating factors.

Integrating Factor

$$y' + P(x)y = Q(x), \quad I = \int P(x)\ \mathrm{d}x.$$

Then

$$ye^I = \int Q(x)e^{I(x)}\ \mathrm{d}x + c.$$

Homogeneous ODE w/ constant coefficients

$$y'' + ay' + by = 0.$$

Get quadratic **characteristic equation**

$$\begin{aligned}
\text{Two real roots: } y(x) &= Ae^{\lambda_1 x} + Be^{\lambda_2} \\
\text{One real root: } y(x) &= Ae^{\lambda x} + Bxe^{\lambda x} \\
\text{Two imaginary roots: } y(x) &= e^{\alpha x}\left(Ae^{i\beta x} + Be^{-i\beta x}\right) \\
&= e^{\alpha x}(c_1 \sin\beta x + c_2\cos\beta x) \\
&= ce^{\alpha x}\sin(\beta x + \gamma)
\end{aligned}$$

Inhomogeneous ODE w/ constant coefficients

If the rhs is ke^{cx}, guess at a particular solution

$$y_p = \begin{cases} Ce^{cx} & \text{if } c \text{ is not equal to either } a \text{ or } b, \\ Cxe^{cx} & \text{if } c \text{ equals either } a \text{ or } b,\ a \neq b, \\ Cx^2e^{cx} & \text{if } a = b = c. \end{cases}$$

If the rhs is either $k\sin\alpha x$ or $k\cos\alpha x$, first solve with the rhs $ke^{i\alpha x}$ and then take either the real or complex part.

If the rhs is $e^{cx}P_n(x)$, where P_n is a n-degree polynomial

$$y_p = \begin{cases} e^{cx}Q_n(x) & \text{if } c \text{ is not equal to either } a \text{ or } b, \\ xe^{cx}Q_n(x) & \text{if } c \text{ equals either } a \text{ or } b,\ a \neq b, \\ x^2e^{cx}Q_n(x) & \text{if } a = b = c, \end{cases}$$

Where Q_n is a polynomial of undetermined coefficients. Sines and cosines is handled by letting c be complex.

Euler-Cauchy

We have an equation on the form

$$y'' + \frac{a}{x}y' + \frac{b}{x^2} = 0.$$

We assume the solution to be given by $y = x^m$. We have

$$y' = mx^{m-1}, \quad y'' = m(m-1)x^{-2},$$

so we have

$$m^2 + (a-1)m + b = 0.$$

If equation has real roots m_1 and m_2

$$y = c_1x^{m_1} + c_xx^{m_2},$$

if equation has one repeated real root

$$y = c_1x^m\ln x + c_2x^m,$$

if equation has two complex roots

$$y = c_1x^\alpha\cos\beta\ln x + c_2x^\alpha\sin\beta\ln x, \quad \alpha = \mathrm{Re}(m), \beta = \mathrm{Im}(m).$$

Alternative method

The substitution $x = e^z$ reduces the DE to one with constant coefficients. No solution at $x = 0$, in general different solutions (i.e. different constants) for x $<$ 0 and x $>$ 0.

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}z} \qquad \text{and} \qquad x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2y}{\mathrm{d}z^2} - \frac{\mathrm{d}y}{\mathrm{d}z}.$$

Factorization

If $u(x)$ is a solution of the homogeneous equation, then the ansatz

$$y_p(x) = u(x)v(x),$$

will give a first order equation for $v'(x)$, which can thus be solved using integrating factors.

Variation of Parameters

Set $y(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$, where y_1 and y_2 are solutions of the homogeneous equation

$$\begin{aligned}
y' &= c_1'y_1 + c_2'y_2 + c_1y_1' + c_2y_2', \quad c_1'y_1 + c_2'y_2 = 0, \\
y'' &= c_1'y_1' + c_2'y_2' + c_1y_1'' + c_2y''
\end{aligned}$$

Inserting y'',y' and y into the inhomogeneous equation and using the fact that y_1 and y_2 are solutions to the homogeneous equation now gives us

$$\begin{aligned}
c_1'y_1 + c_2'y_2 &= 0, \\
c_1'y_1' + c_2'y_2' &= f(x).
\end{aligned}$$

This also leads to the explicit solution

$$y_p(x) = -y_1\int\frac{y_2f(x)}{W}\ \mathrm{d}x + y_2\int\frac{y_1f(x)}{W}\ \mathrm{d}x,$$

where $W = y_1y_2' - y_2y_1'$ is the Wronskian.

Power Series

We assume a solution of a DE to be a power series

$$y=\sum_{n=0}^{\infty}a_nx^n,\ y'=\sum_{n=1}^{\infty}na_nx^{n-1},\ y''=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}.$$

Frobenius' Method

We now use a **generalized power series**

$$y=x^s\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}a_nx^{n+s},$$

by assumption, $a_0\neq 0$.

Indicial Equation

The DE

$$y''+\frac{P(x)}{x}y'+\frac{Q(x)}{x^2}y=0,$$

has the indicial equation

$$s(s-1)+P(0)s+Q(0)=0.$$

And Frobenius' will work if P and Q are analytic at $x=0$.

Green Function

We have the DE

$$y''+P(x)y'+Q(x)y=f(x),$$

we can find a Green function $G(x,z)$ so that

$$y(x)=\int_a^b G(x,z)f(z)\ \mathrm{d}z,$$

BCs are given at $x=a,b$. G satisfies

$$G''+P(x)G'+Q(x)G=\delta(x-x'),$$

which has the solutions

$$\begin{aligned}G_I(x,z)&=A(z)y_1(x)+B(z)y_2(x),\quad \text{for } x<z\\ G_{II}(x,z)&=C(z)y_1(x)+D(z)y_2(x),\quad \text{for } x>z.\end{aligned}$$

$y_1(x)$ and $y_2(x)$ are the solutions to the homogeneous DE.

Need to patch G_I and G_{II}. G is continuous at $x=z$ and $\mathrm{d}G/\mathrm{d}x$ has a discontinuity of 1.

1. $G_I(a,z)=A(z)y_1(a)+B(z)y_2(a)$
2. $G_{II}(b,z)=C(z)y_1(b)+D(z)y_2(b)$
3. $G_I(z,z)=G_{II}(z,z)$
4. $\left.\frac{\mathrm{d}G_{II}}{\mathrm{d}x}\right|_{x=z}-\left.\frac{\mathrm{d}G_I}{\mathrm{d}x}\right|_{x=z}=1$

Example: Green's function

We will solve the ODE

$$y''+y=\frac{1}{\sin x},$$

with boundry conditions

$$y(0)=y(\pi/2)=0,$$

using Green's function. We find the homogeneous solutions

$$y_1=\sin x,\qquad y_2=\cos x.$$

So we have

$$\begin{aligned}G_I(x,x')&=A(x')\sin x+B(x')\cos x\qquad x<z\\ G_{II}(x,x')&=C(x')\sin x+D(x')\cos x\qquad x>z\end{aligned}$$

We now use our conditions:

1. $G_I(0,z)=A(z)\sin 0+B(z)\cos 0=0$
   $\Rightarrow B(z)=0.$
2. $G_{II}(b,z)=C(z)\sin(\pi/2)+D(z)\cos(\pi/2)=0$
   $\Rightarrow C(z)=0.$
3. $G_I(z,z)=G_{II}(z,z)$
   $\Rightarrow A(z)\sin z=D(z)\cos z.$
4. $\left.\frac{\mathrm{d}G_{II}}{\mathrm{d}x}\right|_{x=z}-\left.\frac{\mathrm{d}G_I}{\mathrm{d}x}\right|_{x=z}=1$
   $\Rightarrow -D(z)\sin z-A(z)\cos z=1.$

By combining our two equations for $A(z)$ and $D(z)$, we now find

$$A(z)=-\cos z,\qquad D(z)=-\sin z.$$

And we can do our final integral to find the solution

$$\begin{aligned}y(x)&=\int_a^b G(x,z)f(z)\ \mathrm{d}z,\\ &=\int_0^x -\sin z\cos x\frac{1}{\sin z}\ \mathrm{d}z+\int_x^{\pi/2}-\cos z\sin x\frac{1}{\sin z}\ \mathrm{d}z\\ &=-\cos x(x-0)-\sin x\int_x^{\pi/2}\frac{\cos z}{\sin z}\ \mathrm{d}z\\ &=-x\cos x-\sin x\cdot\ln|u|\Big|_{\sin x}^{1}\\ &=-x\cos x+\sin x\ln|\sin x|.\end{aligned}$$

Partial Differential Equations

Separation of Variables

We are looking for a solution of two or more variables $u(x,t)$. Start by assuming that the solution can be written as the product

$$u(x,t) = X(x) \cdot T(t),$$

The PDE will then seperate, so that each term is only dependant on one single variable. As the function $u(x,t)$ can vary in x and t independantantly, we see that the terms in the sperated equation must be equal to the same **seperation constant**. Using this, the original PDE seperates into two or more ODEs, one for each variable. Solve each of these seperately, then combine them together to find the solution $u(x,t)$. The general solution, will then be a linear combination of such separable solutions. Examples:

Laplace's equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

Look for seperable answers $u(x,y) = X(x)Y(y)$,

$$\frac{1}{X}\frac{\partial^2}{\partial x^2}X + \frac{1}{Y}\frac{\partial^2}{\partial y^2}Y = 0.$$

Set each term equal to the same seperation constant

$$\frac{1}{X}\frac{\partial^2}{\partial x^2}X = -\frac{1}{Y}\frac{\partial^2}{\partial y^2}Y = \text{const.} = -k^2.$$

Solve the ODE for each term seperately

$$X'' = -k^2 X, \qquad Y'' = k^2 Y.$$

$$X = \begin{cases} \sin kx, \\ \cos kx, \end{cases}, \qquad Y = \begin{cases} e^{ky}, \\ e^{-ky}, \end{cases},$$

And the final solution is then

$$u(x,y) = X(x) \cdot Y(y) = \left\{ \begin{matrix} \sin kx \\ \\ \cos kx \end{matrix} \right\} \left\{ \begin{matrix} e^{ky} \\ \\ e^{-ky} \end{matrix} \right\}$$

The diffusion equation

$$\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t},$$

seperation of variables gives

$$u(x,t) = \left\{ \begin{matrix} \sin kx \\ \\ \cos kx \end{matrix} \right\} e^{-k^2\alpha^2 t}.$$

The other time-solution is ignored, as it is unphysical.

The Wave Equation

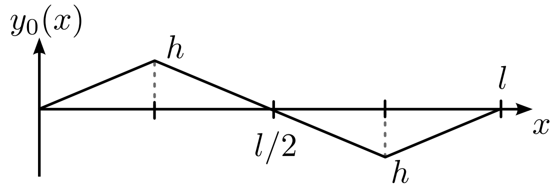
$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

seperation of variables gives

$$u(x,t) = \left\{ \begin{matrix} \sin kx \\ \\ \cos kx \end{matrix} \right\} \left\{ \begin{matrix} \sin kvt \\ \\ \cos kvt \end{matrix} \right\}.$$

Example of Boundry Conditions: Wave Equation

We want to find the solution to the wave equation for a string with $\dot{y}_0(x) = 0$ and starts in the shape:



As the ends of the strings are attached, $y(0) = y(L) = 0$, so we throw away all the cosine-solutions in x and see that $k = n\pi/l$. As the string starts from rest $\dot{y}_0(x) = 0$ so we throw away all the sine-solutions in t.

$$y(x) = \sum_n \sin b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi vt}{L}.$$

We find the coefficients b_n from the Fourier series integral

$$b_n = \frac{2}{l}\int_0^L y_0(x) \sin\left(\frac{n\pi x}{l}\right).$$

where

$$y_0(x) = \begin{cases} \frac{4h}{L}x & \text{for } 0 \leq x < L/2 \\ \frac{4h}{L}(L/2 - x) & \text{for } L/2 \leq x < 3L/4 \\ \frac{4h}{L}(x - L) & \text{for } 3L/4 \leq x < L. \end{cases}$$

Giving

$$b_n = \frac{8h}{n^2\pi^2}\left[\sin n\pi + 2\sin\frac{n\pi}{4} - 2\sin\frac{3n\pi}{4}\right].$$

Solving PDE with Laplace-transforms

Taking the Laplace transform of a PDE reduces the number of independent variables by one, effectively transforming a two-variable PDE into an ODE. When taking the transform of a PDE, treat all but one of the variables as constant, and remember to transform the boundry conditions aswell.

Example

$$x\frac{\partial u(x,t)}{\partial x} + \frac{\partial u(x,t)}{\partial t} = xt, \quad u(x,0) = 0, \; u(0,t) = 0.$$

regard x as a constant parameter, take Laplace transform

$$x\frac{\partial}{\partial x}U(x,p) + pU(x,p) - u(x,0) = x\mathcal{L}[t].$$

We get the ODE

$$U' + (p/x)U = 1/p^2, \quad \Rightarrow \quad U(x,p) = \frac{C(p)}{x^p} + \frac{x}{p^2(p+1)}.$$

Transforming the boundry-condition in time, gives

$$\mathcal{L}[u(0,t)] = \mathcal{L}[0] \quad \Rightarrow \quad U(0,p) = 0 \quad \Rightarrow \quad C(p) = 0.$$

So the final solution is

$$u(x,t) = \mathcal{L}^{-1}[U(x,p)] = x\mathcal{L}^{-1}[1/p^2(p+1)].$$

Partial fraction decomposition gives

$$u(x,t) = x(t - 1 + e^{-t}).$$