1 Poisson's Equation in 1D

The two-point boundary value problem:

$$-u''(x) = f(x), \qquad x \in (0,1), u(0) = u(1) = 0.$$

has a general solution that admits the form

$$u(x) = x\int_0^1 (1-y)f(y)\,\mathrm{d}y - \int_0^x (x-y)f(y)\,\mathrm{d}y. \tag{1}$$

1.1 Proof

Generally we have that (Fundamental theorem of calculus):

$$u(x) = c_1 + \int_0^x u'(y)\,\mathrm{d}y, \qquad u'(x) = c_2 + \int_0^x u''(z)\,\mathrm{d}z,$$

Inserting our ODE, we get

$$u'(x) = c_1 + c_2 x - \int_0^x \int_0^y f(z)\,\mathrm{d}z\,\mathrm{d}y.$$

And we have

$$\begin{aligned}\int_0^x \int_0^y f(z)\,\mathrm{d}z\,\mathrm{d}y &= \int_0^x F(y)\,\mathrm{d}y \\ &= [yF(y)]_0^x - \int_0^x yF'(y)\,\mathrm{d}y \\ &= xF(x) - \int_0^x yf(y)\mathrm{d}y \\ &= \int_0^x (x-y)f(y)\,\mathrm{d}y.\end{aligned}$$

Combining this with our boundary values $u(0) = u(1) = 0$ gives a general solution on the form

$$u(x) = x\int_0^1 (1-y)\,\mathrm{d}y - \int_0^x (x-y)\,\mathrm{d}y.$$

1.2 Green's function

$$G(x,y) = \begin{cases} y(1-x) & \text{if } 0 \le y \le x, \\ x(1-y) & \text{if } x \le y \le 1. \end{cases} \tag{2}$$

Using the Green's function, we can rewrite the general solution 1 as

$$u(x) = \int_0^1 G(x,y)f(y)\,\mathrm{d}y. \tag{3}$$

- G is continuous.
- G is symmetric, $G(x,y) = G(y,x)$.
- $G(0,y) = G(x,0) = G(1,y) = G(x,1) = 0$.
- $G(x,y) > 0$ for all $x, y \in [0,1]$.

1.3 Spaces

- $C((0,1))$ denotes the set of continous functions on the open unit interval.
- $C([0,1])$ denotes the space of continuous functions on the closed unit interval.
- $C^m((0,1))$ denotes the set of m-times continuously differentiable functions on the open unit interval.
- $C_0^2((0,1)) = \{g \in C^2((0,1)) \cap C([0,1]) | g(0) = g(1) = 0\}$.

1.4 Some characteristics of the solution

- For every $f \in ([0,1])$, there is a unique solution $u \in C_0^2((0,1))$, and the solution admits the representation

$$u(x) = \int_0^1 G(x,y) f(y) \, \mathrm{d}y.$$

- If $f \in C^m((0,1))$ for $m \geq 1$, then $u \in C^{m+2}((0,1))$ and

$$u^(m+2) = -f^(m).$$

  Hence, the solution is always smoother than the data.
- If f is a nonnegative function, then the corresponding solution of u is also nonnegative.
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$$||u||_\infty \leq (1/8)||f||_\infty.$$

2 Linear algebra

2.1 Sup norm

For a continous function $f \in C([0,1])$ we define the norms:

$$||f||_p = \left(\int_0^1 |f(x)|^p \, \mathrm{d}x \right)^{1/p}.$$

If we let $p \to \infty$, we get the sup norm:

$$||f||_\infty = \sup_{x \in [0,1]} |f(x)|.$$