Take home exam in FYS3110
Quantum mechanics

Candidate number

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Problem 1

We have a Hamiltonian

$$\hat{H} = \kappa \hat{a}^\dagger \hat{a}, \tag{1}$$

where κ is a positive real constant with units of energy, and $\hat{a}$ and $\hat{a}^\dagger$ are operators with the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger \hat{a} = 1. \tag{2}$$

Problem 1.1

We will find the energy eigenvalues of the Hamiltonian $\hat{H}$.

If we assume that $|\psi\rangle$ is an energy eigenstate with energy E

$$\hat{H}|\psi\rangle = E|\psi\rangle,$$

we will check if $\hat{a}^\dagger|\psi\rangle$ and $\hat{a}|\psi\rangle$ are eigenstates of $\hat{H}$. Let us start with $\hat{a}^\dagger|\psi\rangle$.

$$H\left(\hat{a}^\dagger|\psi\rangle\right) = \kappa \hat{a}^\dagger \hat{a} \hat{a}^\dagger |\psi\rangle,$$

we substitute $\hat{a}\hat{a}^\dagger = \hat{a}^\dagger \hat{a} + 1$

$$\begin{aligned} H\left(\hat{a}^\dagger|\psi\rangle\right) &= \kappa \hat{a}^\dagger \left(\hat{a}^\dagger \hat{a} + 1\right)|\psi\rangle \\ &= \hat{a}^\dagger \left(\kappa \hat{a}^\dagger \hat{a}|\psi\rangle + \kappa|\psi\rangle\right) \\ &= \hat{a}^\dagger \left(\hat{H}|\psi\rangle + \kappa|\psi\rangle\right), \end{aligned}$$

from our assumption about $|\psi\rangle$, we know that $\hat{H}|\psi\rangle = E|\psi\rangle$,

$$\begin{aligned} H\left(\hat{a}^\dagger|\psi\rangle\right) &= \hat{a}^\dagger \Bigg(E|\psi\rangle + \kappa|\psi\rangle \Bigg) \\ &= \Bigg(E + \kappa \Bigg) \hat{a}^\dagger |\psi\rangle, \end{aligned}$$

so if $|\psi\rangle$ is an eigenstate of $\hat{H}$ with the eigenvalue E, then $\hat{a}^\dagger|\psi\rangle$ is also an eigenstate, and it has the eigenvalue $E + \kappa$.

For $\hat{a}|\psi\rangle$ we get

$$H\left(\hat{a}|\psi\rangle\right) = \kappa \hat{a}^\dagger \hat{a} \hat{a} |\psi\rangle,$$

we substitute $\hat{a}^\dagger \hat{a} = \hat{a}\hat{a}^\dagger - 1$

$$\begin{aligned} H\left(\hat{a}|\psi\rangle\right) &= \kappa \left(\hat{a}\hat{a}^\dagger - 1\right) \hat{a}|\psi\rangle \\ &= \kappa \hat{a}\hat{a}^\dagger \hat{a}|\psi\rangle - \kappa \hat{a}|\psi\rangle, \end{aligned}$$

constants always commutate, $\kappa\hat{a} = \hat{a}\kappa$

$$\begin{aligned} H\left(\hat{a}|\psi\rangle\right) &= \hat{a}\kappa\hat{a}^\dagger\hat{a}|\psi\rangle - \kappa\hat{a}|\psi\rangle \\ &= \hat{a}\hat{H}|\psi\rangle - \kappa\hat{a}|\psi\rangle \\ &= \hat{a}E|\psi\rangle - \kappa\hat{a}|\psi\rangle \\ &= \Bigg(E - \kappa\Bigg)\hat{a}|\psi\rangle \end{aligned}$$

so if $|\psi\rangle$ is an eigenstate of $\hat{H}$ with the eigenvalue E, then $\hat{a}|\psi\rangle$ is also an eigenstate, and it has the eigenvalue $E - \kappa$.

So we see that $\hat{a}^\dagger$ and $\hat{a}$ are a raising and a lowering operator respectively, that increase and decrease the energy. The fact that the lowering operator $\hat{a}$ reduces the energy by κ means there must be some ground state $|\psi_0\rangle$ where if we apply the lowering operator one more time we get

$$\hat{a}|\psi_0\rangle = 0,$$

or we would be able to create a state with negative energy by continually applying the lowering operator $\hat{a}$. By letting the Hamiltonian work on $|\psi_0\rangle$ we can find the energy of this ground state:

$$\hat{H}|\psi_0\rangle = \kappa\hat{a}^\dagger\hat{a}|\psi_0\rangle = \kappa\hat{a}^\dagger 0 = 0.$$

We see that the ground state has the energy $E = 0$. If we let the raising operator $\hat{a}^\dagger$ work n times on the ground state, we get the state $|\psi_n\rangle$ with energy

$$E_n = n\kappa.$$

These are the only possible eigenvalues of $\hat{H}$, if there was some energy eigenstate with an eigenvalue different from $n\kappa$, the lowering operator $\hat{a}$ would let us get a state with negative energy, which can't be. There is however, no limit on how many times we can apply $\hat{a}^\dagger$, so there is no restriction on how large the energy can become.

The eigenvalues of $\hat{H}$ are

$$E_n = n\kappa \qquad \text{for } n = 0, 1, 2, \ldots.$$

Problem 1.2

The operators $\hat{a}$ and $\hat{a}^\dagger$ can be written in terms of the position $\hat{x}$ and momentum operators $\hat{p}$ as

$$\hat{a} = i\frac{L}{\sqrt{2}\hbar}\hat{p} + \frac{1}{\sqrt{2}L}\hat{x} - \frac{c}{\sqrt{2}}, \qquad \hat{a}^\dagger = -i\frac{L}{\sqrt{2}\hbar}\hat{p} + \frac{1}{\sqrt{2}L}\hat{x} - \frac{c^*}{\sqrt{2}}, \tag{3}$$

where c is a complex dimensionless number and L is a positive real constant with units of length.

Showing that expression (3) satisfies the commutation relation (2)

We will now show that the expressions of $\hat{a}$ and $\hat{a}^\dagger$ (eq. 3) satisfy the commutation relation for these operators (eq. 2). To do this, we let the operators work on a test function $f(x)$.

$$
\begin{aligned}
[\hat{a}, \hat{a}^\dagger] f(x) &= \hat{a}\hat{a}^\dagger f(x) - \hat{a}^\dagger \hat{a} f(x) \\
&= \left(i\frac{L}{\sqrt{2}\hbar}\hat{p} + \frac{1}{\sqrt{2}L}\hat{x} - \frac{c}{\sqrt{2}} \right) \left(-i\frac{L}{\sqrt{2}\hbar}\hat{p} + \frac{1}{\sqrt{2}L}\hat{x} - \frac{c^*}{\sqrt{2}} \right) f(x) \\
&\quad - \left(-i\frac{L}{\sqrt{2}\hbar}\hat{p} + \frac{1}{\sqrt{2}L}\hat{x} - \frac{c^*}{\sqrt{2}} \right) \left(i\frac{L}{\sqrt{2}\hbar}\hat{p} + \frac{1}{\sqrt{2}L}\hat{x} - \frac{c}{\sqrt{2}} \right) f(x) \\
&= \frac{L^2}{2\hbar^2}\hat{p}\Big(\hat{p}f(x)\Big) + \frac{i}{2\hbar}\hat{p}\Big(\hat{x}f(x)\Big) - \frac{iLc^*}{2\hbar}\hat{p}f(x) - \frac{i}{2\hbar}\hat{x}\Big(\hat{p}f(x)\Big) \\
&\quad + \frac{1}{2L^2}\hat{x}\Big(\hat{x}f(x)\Big) - \frac{c^*}{2L}\hat{x}f(x) + \frac{iLc}{2\hbar}\hat{p}f(x) - \frac{c}{2L}\hat{x}f(x) + \frac{cc^*}{2}f(x) \\
&\quad - \Bigg[\frac{L^2}{2\hbar^2}\hat{p}\Big(\hat{p}f(x)\Big) - \frac{i}{2\hbar}\hat{p}\Big(\hat{x}f(x)\Big) + \frac{iLc}{2\hbar}\hat{p}f(x) + \frac{i}{2\hbar}\hat{x}\Big(\hat{p}f(x)\Big) \\
&\quad + \frac{1}{2L^2}\hat{x}\Big(\hat{x}f(x)\Big) - \frac{c}{2L}\hat{x}f(x) - \frac{iLc^*}{2\hbar}\hat{p}f(x) - \frac{c^*}{2L}\hat{x}f(x) + \frac{c^*c}{2}f(x) \Bigg].
\end{aligned}
$$

We now see a lot of the terms cancel each other out or add up to cancel the factor 2, and we are left with

$$
\begin{aligned}
[\hat{a}, \hat{a}^\dagger] f(x) &= \frac{i}{\hbar}\hat{p}\Big(\hat{x}f(x)\Big) - \frac{i}{\hbar}\hat{x}\Big(\hat{p}f(x)\Big) \\
&= \frac{i}{\hbar}\Big(\hat{p}\hat{x} - \hat{x}\hat{p}\Big) f(x) \\
&= \frac{i}{\hbar}\left[\hat{p}, \hat{x}\right] f(x).
\end{aligned}
$$

We can now drop the test function, and insert for the canonical commutation relation $[\hat{p}, \hat{x}] = -i\hbar$,

$$
[\hat{a}, \hat{a}^\dagger] = \frac{i}{\hbar}(-i\hbar) = 1,
$$

so we see that the expressions for $\hat{a}$ and $\hat{a}^\dagger$ satisfy the commutation relation for the operators.

Writing the Hamiltonian in terms of $\hat{x}$ and $\hat{p}$

We will now write the Hamiltonian $\hat{H}$ in terms of the position operator $\hat{x}$ and the momentum operator $\hat{p}$. We start with our original expression for the Hamiltonian (eq. 1), and substitute with our expressions for $\hat{a}$ and $\hat{a}^\dagger$ (eq. 3):

$$
\begin{aligned}
\hat{H} &= \kappa \hat{a}^\dagger \hat{a} \\
&= \kappa \left(-i\frac{L}{\sqrt{2}\hbar}\hat{p} + \frac{1}{\sqrt{2}L}\hat{x} - \frac{c^*}{\sqrt{2}} \right) \left(i\frac{L}{\sqrt{2}\hbar}\hat{p} + \frac{1}{\sqrt{2}L}\hat{x} - \frac{c}{\sqrt{2}} \right) \\
&= \kappa \left[\frac{L^2}{2\hbar^2}\hat{p}\hat{p} - \frac{i}{2\hbar}\hat{p}\hat{x} + \frac{iLc}{2\hbar}\hat{p} + \frac{i}{2\hbar}\hat{x}\hat{p} + \frac{1}{2L^2}\hat{x}\hat{x} \right. \\
&\qquad \left. - \frac{c}{2L}\hat{x} - \frac{iLc^*}{2\hbar}\hat{p} - \frac{c^*}{2L}\hat{x} + \frac{|c|^2}{2} \right] \\
&= \frac{\kappa}{2} \left[\frac{L^2}{\hbar^2}\hat{p}^2 + \frac{1}{L^2}\hat{x}^2 + \frac{i}{\hbar}(\hat{x}\hat{p} - \hat{p}\hat{x}) + \frac{iL(c-c^*)}{\hbar}\hat{p} - \frac{(c+c^*)}{L}\hat{x} + |c|^2 \right],
\end{aligned}
$$

we now substitute in the canonical commutation relation $\hat{x}\hat{p} - \hat{p}\hat{x} = [\hat{x}, \hat{p}] = i\hbar$. We also write $c + c^* = 2\Re(c)$ and $c - c^* = 2i\Im(c)$. Where $\Re(c)$ and $\Im(c)$ are respectively the real and imaginary parts of the complex number c. We then have

$$
\hat{H} = \frac{\kappa}{2} \left[\frac{L^2}{\hbar^2}\hat{p}^2 + \frac{1}{L^2}\hat{x}^2 - \frac{2L\Im(c)}{\hbar}\hat{p} - \frac{2\Re(c)}{L}\hat{x} + |c|^2 - 1 \right]. \tag{4}
$$

Problem 1.3

We will find the lowest energy eigenstate in the position representation and normalize it.

Finding the ground state wavefunction—$\psi_0(x)$

To find the lowest energy eigenstate, $|\psi_0\rangle$, we use the fact that it must terminate if we let the lowering operator $\hat{a}$ work on it

$$
\hat{a}|\psi_0\rangle = 0,
$$

we insert our expression for $\hat{a}$ (eq. 3)

$$
\left(i\frac{L}{\sqrt{2}\hbar}\hat{p} + \frac{1}{\sqrt{2}L}\hat{x} - \frac{c}{\sqrt{2}} \right) |\psi_0\rangle = 0.
$$

We want to find this state in the position representation, so we use

$$
\hat{x} = x, \qquad \hat{p} = -i\hbar\frac{d}{dx},
$$

and we write $|\psi_0\rangle \simeq \psi_0(x)$, this gives us the differential equation

$$
\frac{L}{\sqrt{2}}\frac{d\psi_0(x)}{dx} + \frac{1}{\sqrt{2}L}x\psi_0(x) - \frac{c}{\sqrt{2}}\psi_0(x) = 0,
$$

we multiply the equation with $\sqrt{2}/L$, and then separate the differential equation

$$\frac{d\psi_0(x)}{\psi_0(x)} = \left(-\frac{1}{L^2}x + \frac{c}{L}\right) dx.$$

Integrating on both sides of the equation gives

$$\ln\psi_0(x) = -\frac{1}{2L^2}x^2 + \frac{c}{L}x + \text{const.}$$

Exponentiating both sides of the equation, and calling the integration constant D gives us

$$\psi_0(x) = D\ \exp\left[-\left(\frac{1}{2L^2}x^2 - \frac{c}{L}x\right)\right]$$

Normalizing the ground state wavefunction

We will normalize the ground state wavefunction. $\psi_0(x)$ is normalized if

$$\langle\psi_0|\psi_0\rangle = \int_{-\infty}^{\infty} \psi_0^*(x)\psi_0(x)\ \mathrm{dx} = 1.$$

The challenge is to choose the integration constant D in such a way that this is the case. We start by performing the inner product. Remembering that $c \in \mathbb{C}$ and $L \in \mathbb{R}$ we get:

$$\begin{aligned}
\langle\psi_0|\psi_0\rangle &= \int_{-\infty}^{\infty} \psi_0^*(x)\psi_0(x)\ \mathrm{dx} \\
&= \int_{-\infty}^{\infty} D^*\ \exp\left[-\left(\frac{1}{2L^2}x^2 - \frac{c^*}{L}x\right)\right] D\ \exp\left[-\left(\frac{1}{2L^2}x^2 - \frac{c}{L}x\right)\right]\ \mathrm{dx} \\
&= |D|^2 \int_{-\infty}^{\infty} \exp\left[-\left(\frac{1}{L^2}x^2 - \frac{2\Re(c)}{L}x\right)\right]\ \mathrm{dx}.
\end{aligned}$$

This is a Gaussian integral with a known answer[1]

$$\int_{-\infty}^{\infty} e^{-(ax^2+2bx+c)}\ \mathrm{dx} = \sqrt{\frac{\pi}{a}}\ \exp\left[\frac{b^2-ac}{a}\right], \qquad a > 0. \tag{5}$$

Where in our case $a = 1/L^2$, $b = -\Re(c)/L$ and $c = 0$. The constant L is positive, so $a > 0$ and we get

$$\langle\psi_0|\psi_0\rangle = |D|^2\sqrt{\pi L^2}\ \exp\left[\Re(c)^2\right].$$

For ψ_0 to be normalized, this inner product must be 1, so we get

$$|D|^2\sqrt{\pi L^2}\ \exp\left[\Re(c)^2\right] = 1 \qquad \Rightarrow \qquad |D| = \left(\frac{\exp\left[-\Re(c^2)\right]}{\sqrt{\pi L}}\right)^{1/2},$$

we choose D to be both real and positive and get the normalized ground state waveform

$$\psi_0(x) = \left(\frac{\exp\left[-\Re(c^2)\right]}{\sqrt{\pi L}}\right)^{1/2} \exp\left[-\left(\frac{1}{2L^2}x^2 - \frac{c}{L}\right)\right].$$

[1] See *Rottmann* p. ??.

Problem 1.4

We will calculate the expectation values of the position and momentum for the lowest energy state of $\hat{H}$.

Calculating the expectation value of the position

The expectation value is given by

$$\langle x \rangle = \langle \psi_0 | \hat{x} | \psi_0 \rangle = \int_{-\infty}^{\infty} \psi_0^*(x) x \psi_0(x) \; \mathrm{d}x.$$

We use the expression for the normalized ground state waveform $\psi_0(x)$:

$$\langle x \rangle = \frac{\exp\left[-\Re(c^2)\right]}{\sqrt{\pi} L} \int_{-\infty}^{\infty} x \, \exp\left[-\left(\frac{1}{L^2}x^2 - \frac{2\Re(c)}{L}x\right)\right] \; \mathrm{d}x,$$

An integral on this form has a known solution[2]

$$\int_{-\infty}^{\infty} x e^{-(ax^2+2bx+c)} \; \mathrm{d}x =, \qquad a > 0. \tag{6}$$

Calculating the expectation value of the momentum

The expectation value is given by

$$\langle p \rangle = \langle \psi_0 | \hat{p} | \psi_0 \rangle = \int_{-\infty}^{\infty} \psi_0^*(x)(-i\hbar \frac{d}{dx})\psi_0(x) \; \mathrm{d}x.$$

We use the expression for the normalized ground state waveform $\psi_0(x)$:

$$\begin{aligned}
\langle p \rangle &= -i\hbar \left(\frac{\exp\left[-\Re(c^2)\right]}{\sqrt{\pi} L}\right)^{1/2} \int_{-\infty}^{\infty} \psi_0^*(x) \frac{d}{dx} \, \exp\left[-\left(\frac{1}{2L^2}x^2 - \frac{c}{L}x\right)\right] \; \mathrm{d}x \\
&= -i\hbar \left(\frac{\exp\left[-\Re(c^2)\right]}{\sqrt{\pi} L}\right)^{1/2} \int_{-\infty}^{\infty} \psi_0^*(x) \left(-\frac{1}{L}x + \frac{c}{L}\right) \exp\left[-\left(\frac{1}{2L^2}x^2 - \frac{c}{L}x\right)\right] \; \mathrm{d}x \\
&= i\hbar \frac{\exp\left[-\Re(c^2)\right]}{\sqrt{\pi} L L} \int_{-\infty}^{\infty} (x - c) \exp\left[-\left(\frac{1}{L^2}x^2 - 2\frac{\Re(c)}{L}x\right)\right] \; \mathrm{d}x,
\end{aligned}$$

we split the integral up into parts

$$\begin{aligned}
\langle p \rangle = i\hbar \frac{\exp\left[-\Re(c^2)\right]}{\sqrt{\pi} L L} \Bigg(x \int_{-\infty}^{\infty} &\exp\left[-\left(\frac{1}{L^2}x^2 - 2\frac{\Re(c)}{L}x\right)\right] \; \mathrm{d}x \\
-c \int_{-\infty}^{\infty} &\exp\left[-\left(\frac{1}{L^2}x^2 - 2\frac{\Re(c)}{L}x\right)\right] \; \mathrm{d}x \Bigg).
\end{aligned}$$

These two integrals are Gaussian integrals, and we can use the known answers (eq. 5 and eq. 6),

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[2] See *Rottmann* p. ??.

Problem 2

We will denote the components of the spin-1/2 operators as $\hat{S}_i$, where $i = \{x, y, z\}$. These components can be represented as

$$\hat{S}_i \simeq \frac{\hbar}{2}\sigma_i,$$

where σ_i are the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this representation, the eigenkets of $\hat{S}_z$ are $|\uparrow\rangle \simeq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\downarrow\rangle \simeq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\hat{S}_z |\uparrow\rangle = \frac{\hbar}{2} |\uparrow\rangle, \qquad \hat{S}_z |\downarrow\rangle = -\frac{\hbar}{2} |\downarrow\rangle.$$

Problem 2.1

We will expand the exponentials

$$e^{-i\phi\sigma_z/2} \qquad \text{and} \qquad e^{-i\theta\sigma_y/2},$$

we do this using the Taylor expansion

$$\begin{aligned} e^{-x} &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \ldots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}, \end{aligned}$$

note that this is *not* an approximation, as we keep all the terms in the expansion.

Expanding the exponential $e^{-i\phi\sigma_z/2}$

Expanding the exponential, we get

$$e^{-i\phi\sigma_z/2} = \sum_{n=0}^{\infty} \frac{(-1)^n i^n}{n!} \left(\frac{\phi}{2}\right)^n \sigma_z^n,$$

we see that every even numbered term is purely real, and every odd numbered term is purely imaginary

$$(-1)^{2n} i^{2n} = (-1)^n, \qquad (-1)^{2n+1} i^{2n+1} = -i(-1)^n, \quad \text{for } n = 0, 1, 2, \ldots.$$

We group these terms separately and get two sums

$$e^{-i\phi\sigma_z/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\phi}{2}\right)^{2n} \sigma_z^{2n} - i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\phi}{2}\right)^{2n+1} \sigma_z^{2n+1}$$

we need to find an expression for

$$\sigma_z^n \quad \text{for } n = 1, 2, \ldots,$$

matrix multiplication gives

$$\sigma_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_z^2 = \sigma_z \sigma_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2,$$

where I_2 is the 2×2 identity matrix. So we find

$$\sigma_z^{2n} = (\sigma_z^2)^n = (I_2)^n = I_2, \qquad \sigma_z^{2n+1} = (\sigma_z^2)^n \sigma_z = I_2 \sigma_n = \sigma_n.$$

Returning to our expanded exponential we find

$$e^{-i\phi\sigma_z/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\phi}{2}\right)^{2n} I_2 - i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\phi}{2}\right)^{2n+1} \sigma_z,$$

as the matrices I_2 and σ_z are independent of n, we can move them outside the sums, which are now simply the Taylor expansions of cos and sin:

$$e^{-i\phi\sigma_z/2} = \underbrace{\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\phi}{2}\right)^{2n}\right]}_{\cos\left(\frac{\phi}{2}\right)} I_2 - i \underbrace{\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\phi}{2}\right)^{2n+1}\right]}_{\sin\left(\frac{\phi}{2}\right)} \sigma_z,$$

so we get

$$e^{-i\phi\sigma_z/2} = \cos\left(\frac{\phi}{2}\right) I_2 - i \cos\left(\frac{\phi}{2}\right) \sigma_z.$$

Expanding the exponential $e^{-i\theta\sigma_y/2}$

The expansion of $e^{-i\theta\sigma_y/2}$ is very similar, we expand and get

$$e^{-i\theta\sigma_y/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\theta}{2}\right)^{2n} \sigma_y^{2n} - i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\theta}{2}\right)^{2n+1} \sigma_y^{2n+1},$$

and again we need to find σ_y^n for $n = 1, 2, \ldots$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_y^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

So we get

$$\sigma_y^2 n = \left(\sigma_y^2\right)^n = I_2^n = I_2, \qquad \sigma_y^{2n+1} = \left(\sigma_y^2\right)^n \sigma_y = I_2 \sigma_y = \sigma_y.$$

Putting these results into the expanded exponential gives

$$e^{-i\theta\sigma_y/2} = \underbrace{\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\theta}{2}\right)^{2n}\right]}_{\cos\left(\frac{\theta}{2}\right)} I_2 - i \underbrace{\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\theta}{2}\right)^{2n+1}\right]}_{\sin\left(\frac{\theta}{2}\right)} \sigma_y,$$

$$e^{-i\theta\sigma_y/2} = \cos\left(\frac{\theta}{2}\right) I_2 - i \cos\left(\frac{\theta}{2}\right) \sigma_y.$$

Problem 2.2

We will show that the state

$$|\theta,\phi,+\rangle = e^{-i\hat{S}_z\phi/\hbar}e^{-i\hat{S}_y\theta/\hbar}\left|\uparrow\right\rangle ,$$

is an eigenstate of the operator $\vec{n}\cdot\hat{\vec{S}}$, where

$$\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \qquad \text{and} \qquad \hat{\vec{S}} = (\hat{S}_x, \hat{S}_y, \hat{S}_z),$$

with eigenvalue $+\hbar/2$. We will also find the eigenstate of $\vec{n}\cdot\hat{\vec{S}}$ with eigenvalue $-\hbar/2$.

Showing that $|\theta,\phi,+\rangle$ is an eigenstate

We write the state in matrix representation, using

$$\hat{S}_i \simeq \frac{\hbar}{2}\sigma_i,$$

we get

$$|\theta,\phi,+\rangle \simeq e^{-i\phi\sigma_z/2}e^{-i\theta\sigma_y/2}\begin{pmatrix}1\\0\end{pmatrix}.$$

We now use the result of problem 2.1 to replace the exponentitals

$$|\theta,\phi,+\rangle \simeq \left(\cos\left(\frac{\phi}{2}\right)I_2 - i\cos\left(\frac{\phi}{2}\right)\sigma_z\right)\left(\cos\left(\frac{\theta}{2}\right)I_2 - i\cos\left(\frac{\theta}{2}\right)\sigma_y\right)\begin{pmatrix}1\\0\end{pmatrix}.$$

Inserting the matrices I_2, σ_z and σ_y gives

$$|\theta,\phi,+\rangle \simeq \begin{pmatrix}\cos\frac{\phi}{2} - i\cos\frac{\phi}{2} & 0\\ 0 & \cos\frac{\phi}{2} + i\cos\frac{\phi}{2}\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}.$$

We rewrite

$$\cos\frac{\phi}{2} \pm i\cos\frac{\phi}{2} = e^{\pm i\phi/2},$$

from normal matrix multiplication we then get

$$|\theta,\phi,+\rangle \simeq \begin{pmatrix}\cos\frac{\theta}{2}e^{-i\phi/2}\\ \sin\frac{\theta}{2}e^{i\phi/2}\end{pmatrix}.$$

If we let the operator $\vec{n}\cdot\hat{\vec{s}}$ work on the state $|\theta,\phi,+\rangle$ we get

$$\begin{aligned}\vec{n}\cdot\hat{\vec{s}}\,|\theta,\phi,+\rangle = \sin\theta\cos\phi\hat{S}_x\,|\theta,\phi,+\rangle\\ +\sin\theta\sin\phi\hat{S}_y\,|\theta,\phi,+\rangle\\ +\cos\theta\hat{S}_z\,|\theta,\phi,+\rangle.\end{aligned}$$

Using the matrix representation of $\hat{S}_i$ and $|\theta, \phi, +\rangle$, we get

$$\begin{aligned}\vec{n} \cdot \hat{\vec{s}}\,|\theta, \phi, +\rangle = \frac{\hbar}{2}\Bigg[&\begin{pmatrix} 0 & \sin\theta\cos\phi \\ \sin\theta\cos\phi & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\phi/2} \\ \sin\frac{\theta}{2} e^{i\phi/2} \end{pmatrix} \\ &+ \begin{pmatrix} 0 & -i\sin\theta\sin\phi \\ i\sin\theta\sin\phi & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\phi/2} \\ \sin\frac{\theta}{2} e^{i\phi/2} \end{pmatrix} \\ &+ \begin{pmatrix} \cos\theta & 0 \\ 0 & -\cos\theta \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\phi/2} \\ \sin\frac{\theta}{2} e^{i\phi/2} \end{pmatrix} \Bigg],\end{aligned}$$

normal matrix multiplication gives

$$\begin{aligned}\vec{n} \cdot \hat{\vec{s}}\,|\theta, \phi, +\rangle &= \frac{\hbar}{2} \begin{pmatrix} \sin\theta\cos\phi\sin\frac{\theta}{2} e^{i\phi/2} - i\sin\theta\sin\phi\sin\frac{\theta}{2} e^{i\phi/2} + \cos\theta\cos\frac{\theta}{2} e^{-i\phi/2} \\ \sin\theta\cos\phi\cos\frac{\theta}{2} e^{-i\phi/2} + i\sin\theta\sin\phi\cos\frac{\theta}{2} e^{-i\phi/2} - \cos\theta\sin\frac{\theta}{2} e^{i\phi/2} \end{pmatrix} \\ &= \frac{\hbar}{2} \begin{pmatrix} (\cos\phi - i\sin\phi)\sin\theta\sin\frac{\theta}{2} e^{i\phi/2} + \cos\theta\cos\frac{\theta}{2} e^{-i\phi/2} \\ (\cos\phi + i\sin\phi)\sin\theta\cos\frac{\theta}{2} e^{-i\phi/2} - \cos\theta\sin\frac{\theta}{2} e^{i\phi/2} \end{pmatrix} \\ &= \frac{\hbar}{2} \begin{pmatrix} e^{-i\phi}\sin\theta\sin\frac{\theta}{2} e^{i\phi/2} + \cos\theta\cos\frac{\theta}{2} e^{-i\phi/2} \\ e^{i\phi}\sin\theta\cos\frac{\theta}{2} e^{-i\phi/2} - \cos\theta\sin\frac{\theta}{2} e^{i\phi/2} \end{pmatrix} \\ &= \frac{\hbar}{2} \begin{pmatrix} \left[\sin\theta\sin\frac{\theta}{2} + \cos\theta\cos\frac{\theta}{2}\right] e^{-i\phi/2} \\ \left[\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2}\right] e^{i\phi/2} \end{pmatrix}.\end{aligned}$$

Now we need some trigonometry identites

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}, \qquad \cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}. \tag{7}$$

Using these we get

$$\begin{aligned}\vec{n} \cdot \hat{\vec{s}}\,|\theta, \phi, +\rangle &= \frac{\hbar}{2} \begin{pmatrix} \left[2\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right]\cos\frac{\theta}{2} e^{-i\phi/2} \\ \left[2\cos^2\frac{\theta}{2} - \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right]\sin\frac{\theta}{2} e^{i\phi/2} \end{pmatrix}. \\ &= \frac{\hbar}{2} \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\phi/2} \\ \sin\frac{\theta}{2} e^{i\phi/2} \end{pmatrix}.\end{aligned}$$

So we have shown that

$$\vec{n} \cdot \hat{\vec{S}}\,|\theta, \phi, +\rangle = \frac{\hbar}{2}\,|\theta, \phi, +\rangle,$$

this means that $|\theta, \phi, +\rangle$ is an eigenstate for the operator with eigenvalue $\hbar/2$.

Finding the eigenstate of $\vec{n}\cdot\hat{\vec{S}}$ with eigenvalue $-\hbar/2$

We know that

$$|\theta,\phi,+\rangle = e^{-i\hat{S}_z\phi/\hbar}e^{-i\hat{S}_y\theta/\hbar}|\uparrow\rangle ,$$

is an eigenstate with eigenvalue $+\hbar/2$. We want to test if

$$|\theta,\phi,-\rangle = e^{-i\hat{S}_z\phi/\hbar}e^{-i\hat{S}_y\theta/\hbar}|\downarrow\rangle ,$$

is an eigenstate with eigenvalue $-\hbar/2$. We do this in the same manner as we did for $|\theta,\phi,+\rangle$.

Using the matrix representation we get

$$|\theta,\phi,-\rangle \simeq e^{-i\phi\sigma_z/2}e^{-i\theta\sigma_y/2}\begin{pmatrix}0\\1\end{pmatrix}.$$

Expanding the exponentials and inserting the matrices yields

$$|\theta,\phi,-\rangle \simeq \begin{pmatrix}e^{-i\phi/2} & 0\\ 0 & e^{i\phi/2}\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}.$$

So we get

$$|\theta,\phi,-\rangle \simeq \begin{pmatrix}-\sin\frac{\theta}{2}e^{-i\phi/2}\\ \cos\frac{\theta}{2}e^{i\phi/2}\end{pmatrix}.$$

We now let the operator $\vec{n}\cdot\hat{\vec{S}}$ work on the state

$$\vec{n}\cdot\hat{\vec{s}}\,|\theta,\phi,-\rangle = \frac{\hbar}{2}\begin{pmatrix}\sin\theta\cos\phi\cos\frac{\theta}{2}e^{i\phi/2} - i\sin\theta\sin\phi\sin\frac{\theta}{2}e^{-i\phi/2} - \cos\theta\sin\frac{\theta}{2}e^{-i\phi/2}\\ -\sin\theta\cos\phi\sin\frac{\theta}{2}e^{-i\phi/2} - i\sin\theta\sin\phi\sin\frac{\theta}{2}e^{-i\phi/2} - \cos\theta\cos\frac{\theta}{2}e^{i\phi/2}\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix}(\cos\phi - i\sin\phi)\sin\theta\cos\frac{\theta}{2}e^{i\phi/2} - \cos\theta\sin\frac{\theta}{2}e^{-i\phi/2}\\ -(\cos\phi + i\sin\phi)\sin\theta\sin\frac{\theta}{2}e^{-i\phi/2} - \cos\theta\cos\frac{\theta}{2}e^{i\phi/2}\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix}\left[\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2}\right]e^{-i\phi/2}\\ -\left[\sin\theta\sin\frac{\theta}{2} + \cos\theta\cos\frac{\theta}{2}\right]e^{i\phi/2}\end{pmatrix}.$$

Using trigonometric identites for $\sin\theta$ and $\cos\theta$ (eq. 7) we get

$$\vec{n}\cdot\hat{\vec{s}}\,|\theta,\phi,+\rangle = -\frac{\hbar}{2}\begin{pmatrix}-\left[2\cos^2\frac{\theta}{2} - \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right]\sin\frac{\theta}{2}e^{-i\phi/2}\\ \left[2\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right]\cos\frac{\theta}{2}e^{i\phi/2}\end{pmatrix}.$$

$$= -\frac{\hbar}{2}\begin{pmatrix}-\sin\frac{\theta}{2}e^{-i\phi/2}\\ \cos\frac{\theta}{2}e^{i\phi/2}\end{pmatrix}.$$

So we see that $|\theta,\phi,-\rangle$ is indeed the eigenstate of $\hat{n}\cdot\hat{\vec{S}}$ with eigenvalue $-\hbar/2$.

Problem 2.3