The set of continuous, real functions defined on an interval $[0, T]$ is denoted $C[0, T]$. A real function f defined on $[0, T]$ is said to be *square integrable* if f^2 is Riemann-integrable, i.e., if

$$\int_0^T f(t)^2 \, \mathrm{d}t < \infty.$$

The set of all square integrable functions on $[0, T]$ is denoted $L^2[0, T]$.

Both $L^2[0, T]$ and $C[0, T]$ are vector spaces, and we define the inner product on the spaces as

$$\langle f, g \rangle = \frac{1}{T} \int_0^T f(t) g(t) \, \mathrm{d}t,$$

and the associated norm

$$||f|| = \sqrt{\frac{1}{T} \int_0^T f(t)^2 \, \mathrm{d}t}.$$

The reason for the $1/T$ normalization factor, is that it makes the constant-function $f(t) = 1$ have the norm 1.

The projection of a function f onto a subspace W is the function $g \in W$ which minimizes the least squares error $||f - g||$. It follows that the error function is orthogonal to the subspace W,

$$\langle f - g, h \rangle = 0, \quad \forall \ h \in W.$$

If $\{\phi_i\}_{i=1}^m$ is an orthogonal basis for W, then

$$g = \sum_{i=1}^m \frac{\langle f, \phi_i \rangle}{\langle \phi_i, \phi_i \rangle} \phi_i.$$

Fourier

The N'th order Fourier space is denoted $V_{N,T}$, it is $2N + 1$ dimensional and spanned by the set of functions

$$\begin{aligned} \mathcal{D}_{N,T} = \{1, \cos\left(\frac{2\pi t}{T}\right), \cos\left(\frac{2\pi 2t}{T}\right) \ldots, \cos\left(\frac{2\pi N t}{T}\right), \\ \sin\left(\frac{2\pi t}{T}\right), \sin\left(\frac{2\pi 2t}{T}\right), \ldots \sin\left(\frac{2\pi N t}{T}\right)\}. \end{aligned}$$

it is allso spanned by the complex Fourier basis

$$\mathcal{F}_{N,T} = \left\{ e^{-2\pi i k t/T} \right\}_{k=-N}^{N},$$

The projection of a function f onto $V_{N,T}$ is denoted $f_N(t)$, in the real basis we have:

$$f_N(t) = a_0 + \sum_{n=1}^N \left[a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n \sin\left(\frac{2\pi n t}{T}\right) \right].$$

The real Fourier coefficients of f are given by

$$
\begin{aligned}
a_0 &= \langle f, 1\rangle, \\
a_n &= 2\langle f, \cos(2\pi nt/T)\rangle, \\
b_n &= 2\langle f, \cos(2\pi nt/T)\rangle.
\end{aligned}
$$

In the complex basis we have

$$
f_N(t) = \sum_{-N}^{N} y_n e^{2\pi int/T},
$$

where the complex Fourier coefficients of f are given by

$$
y_n = \langle f, e^{2\pi int/T}\rangle = \frac{1}{T}\int_0^T f(t)e^{-2\pi int/T}\,\mathrm{d}t.
$$

We can map between real and complex Fourier coefficients from

$$
\begin{pmatrix} y_n \\ y_{-n} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} a_n \\ b_n \end{pmatrix}.
$$

and $y_0 = a_0$.

Convergence of Fourier series

Given a periodic function f with period T, and that

- f has a finite set of discontinuities in each period.
- f contains a finite set of maxima and minima in each period.
- $\int_0^T |f(t)|\,\mathrm{d}t < \infty$

Then we have that $\lim_{N\to\infty} f_N(t) = f(t)$ for all t, except at those points t where f is discontinuous. These are the Dirichlet conditions for the convergence of the Fourier series.

If f is antisymmetric about 0, then $a_n = 0$, i.e., the Fourier series becomes a sine-series, if f is symmetric about 0, then $b_n = 0$ and the Fourier series becomes a cosine-series.

Pure tones

The function

$$
e^{2\pi int/T},
$$

is called a pure tone with frequency n/T.

For complex vectors of length N, the Euclidean inner product is

$$
\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \sum_{k=0}^{N-1} x_k \overline{y_k}.
$$

And so the associated norm is

$$||\boldsymbol{x}|| = \sqrt{\sum_{k=0}^{N-1} |x_k|^2}.$$

Pure digital tones of order N

The pure digital tones of order N, also called the normalised complex exponentials, are:

$$\boldsymbol{\phi}_n = \frac{1}{\sqrt{N}}\big(1, e^{2\pi in/N}, e^{2\pi i2n/N}\big), \dots, e^{2\pi in(N-1)/N}.\big)$$

The whole collection

$$\mathcal{F}_N = \{\boldsymbol{\phi}_n\}_{n=0}^{N-1},$$

is called the N-point Fourier basis. The basis is orthonormal in $\mathbb{R}^N$.

Discrete Fourier Transform

The change of coordinates from the standard basis of $\mathbb{R}^N$ to the Fourier basis $\mathcal{F}_N$ is called the discrete Fourier transform, or DFT. The $N \times N$ matrix F_N that represents this change of basis is called the N-point Fourier matrix. If $\boldsymbol{x} \in \mathbb{R}^n$, then the DFT coefficients of $\boldsymbol{x}$ is given as:

$$\boldsymbol{y} = F_N \boldsymbol{x}$$

We see that the coloumns of the inverse matrix are the pure digital tones

$$\boldsymbol{x} = \sum_{k=0}^{N-1} y_k \boldsymbol{\phi}_k = \begin{bmatrix} \boldsymbol{\phi}_0 & \boldsymbol{\phi}_1 & \dots & \boldsymbol{\phi}_{N-1} \end{bmatrix} \boldsymbol{y} = F_N^{-1} \boldsymbol{y}.$$

As F_N is orthogonal and complex (i.e. unitary), we find it's inverse by taking the conjugate transpose of it.

$$F_N = (F_n^{-1})^H.$$

The entires of the $N \times N$ Fourier matrix F_N is given by

$$(F_N)_{nk} = \frac{1}{\sqrt{N}} e^{-2\pi ink/N}.$$

Properties of DFT

- $\hat{\boldsymbol{x}}_{N-n} = \overline{\hat{\boldsymbol{x}}_{N-n}}$