Problem set 3
FYS3140

Jonas van den Brink

February 4, 2013

Problem 3.1 (Cauchy's Theorem and integral formula)

a)

We will evaulate the integral

$$\oint_\Gamma \frac{\sin z}{2z-\pi}\,\mathrm{d}z,$$

where Γ is the circle $|z|=3$, as $\sin(z)$ is an entire function, we know that it is analytic on and inside Γ and we can apply Cauchy's integral formula, giving

$$\oint_\Gamma \frac{\sin z}{2z-\pi}\,\mathrm{d}z = \frac{1}{2}\oint_\Gamma \frac{\sin z}{z-\pi/2} = \pi i \sin\left(\frac{\pi}{2}\right) = \pi i.$$

b)

We will evaulate the integral

$$\oint_\Gamma \frac{\sin z}{2z-\pi}\,\mathrm{d}z,$$

where Γ is the circle $|z|=1$, as the integrand has no poles inside Γ, Cauchy's theorem tells us that

$$\oint_\Gamma \frac{\sin z}{2z-\pi}\,\mathrm{d}z = 0.$$

c)

We will evaulate the integral

$$\oint_\Gamma \frac{\sin z}{6z-\pi}\,\mathrm{d}z,$$

where Γ is the circle $|z|=1$, as $\sin(z)$ is an entire function, we know that it is analytic on and inside Γ and we can apply Cauchy's integral formula, giving

$$\oint_\Gamma \frac{\sin z}{6z-\pi}\,\mathrm{d}z = \frac{1}{6}\oint_\Gamma \frac{\sin z}{z-\pi/6} = \frac{\pi}{3} i \sin\left(\frac{\pi}{6}\right) = \frac{\pi}{6} i.$$

d)

We will evaulate the integral

$$\oint_\Gamma \frac{e^{2z}}{z-\ln 2}\,\mathrm{d}z,$$

where Γ is the square with vertices $\pm 2, \pm 2i$. As the exponential function is entire, we know that it is analytic on and inside Γ and we can apply Cauchy's integral formula, giving

$$\oint_\Gamma \frac{e^{2z}}{z-\ln 2}\,\mathrm{d}z = 2\pi i e^{2\ln 2} = 8\pi i.$$

Problem 3.2 (Generalized Cauchy integral formula)

We will now derive the generalized Cauchy integral formula,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(\omega)}{(\omega - z)^{n+1}} \, \mathrm{d}\omega.$$

We start from the standard Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\omega)}{(\omega - z)} \, \mathrm{d}\omega,$$

and we derivate with respect to z, as the integration is with respect to the introduced variable ω, the derivative can be interchanged with the integration. Note also that as we demand that f be analytic on and inside Γ, the derivatives of f exist and are also analytic on and inside Γ.

The first derivative then becomes

$$f'(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\omega)}{(\omega - z)^2} \, \mathrm{d}\omega,$$

derivativing again gives

$$f''(z) = \frac{2}{2\pi i} \oint_\Gamma \frac{f(\omega)}{(\omega - z)^3} \, \mathrm{d}\omega,$$

and so on. Through induction we easily achieve the generalized formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(\omega)}{(\omega - z)^{n+1}} \, \mathrm{d}\omega, \qquad (n = 1, 2, 3, \ldots).$$

We will now use this formula to evaluate the integral

$$\oint_\Gamma \frac{\sin 2z}{(6z - \pi)^3},$$

where Γ is the circle $|z| = 2$. Once again, we know that the sine-function is entire, and so Cauchy's generalized integral formula can be used, giving

$$\begin{aligned}
\oint_\Gamma \frac{\sin 2z}{(6z - \pi)^3} &= \frac{1}{6^3} \oint_\Gamma \frac{\sin 2z}{(z - \pi/6)^3} \\
&= \frac{1}{6^3} \frac{2\pi i}{2!} (-4) \sin\left(2 \cdot \frac{\pi}{6}\right) \\
&= -\frac{\sqrt{3}}{108} i.
\end{aligned}$$

Problem 3.3 (Laurent series)

We will find the Laurent series about the origin of the function

$$f(z) = \frac{z-1}{z^2(z-2)},$$

in different domains.

a)

In the punctured disk $0 < |z| < 2$. We first rewrite the function to the form

$$f(z) = \frac{z-1}{z^2(z-2)} = \frac{1-z}{2z^2} \cdot \frac{1}{1-\frac{z}{2}},$$

as $|z| < 2$ in the disc, we see that $|z/2| < 1$, recognizing the last fraction as the geometric series we get

$$f(z) = \frac{1-z}{2z^2} \sum_{k=0}^{\infty} \frac{z^k}{2^k} = \sum_{k=0}^{\infty} \left(\frac{z^{k-2}}{2^{k+1}} - \frac{z^{k-1}}{2^{k+1}} \right),$$

writing out the first few terms gives

$$f(z) = \frac{1}{2z^2} - \frac{1}{4z} - \sum_{k=0}^{\infty} \frac{z^k}{2^{k+3}}.$$

b)

For $|z| > 2$. We start by rewriting $f(z)$ to the form:

$$f(z) = \frac{z-1}{z^3} \frac{1}{1-\frac{2}{z}},$$

as $|z| > 2$, we see that $|2/z| < 1$ and again we recognize the fraction as the geometric series, giving

$$f(z) = \frac{z-1}{z^3} \sum_{k=0}^{\infty} \frac{2^k}{z^k} = \sum_{k=0}^{\infty} \left(\frac{2^k}{z^{k+2}} - \frac{2^k}{z^{k+3}} \right).$$

c)

The residue of $f(z)$ at the origin we have already found in problem 3.3a, as the Laurent series converges inside the punctured disk $0 < |z| < 2$, the residue at the singular point $z_0 = 0$ is simply the coefficient b_1, meaning

$$\mathrm{Res}(f; 0) = -\frac{1}{4}.$$

Problem 3.4 (Singularities)

We will now classify the singularities of some functions.

a)

The function $f(z) = \sin z/3z$, at the origin, $z_0 = 0$. As the function has a finite limit as z approaches the origin:

$$\lim_{z\to 0} \frac{\sin z}{3z} = \frac{1}{3},$$

the singularity is removable.

b)

The function $f(z) = \cos z/z^4$, at the origin, $z_0 = 0$. As the cosine-function is non-zero at the origin, we see that this singularity is a fourth order pole.

c)

The function

$$f(z) = \frac{z^3 - 1}{(z-1)^3},$$

at the point, $z_0 = 1$. We look at the function $g(z) \equiv 1/f(z)$, as f will have a pole of the same order as g has order of zero in z_0, we see that

$$g(z) = \frac{(z-1)^3}{z^3 - 1}.$$

Using L'Hôpital's rule, we find

$$\begin{aligned}
\lim_{z\to 1} g(z) &= 0,\\
\lim_{z\to 1} g'(z) &= 0,\\
\lim_{z\to 1} g''(z) &= 2/3.
\end{aligned}$$

As g has a zero of second order at z_0, we know that $f(z)$ has a pole of second order here.

d)

The function $f(z) = e^z/(z-1)$ at the point $z_0 = 1$. As e^z is analytic and non-zero at z_0, we see that f has a pole of first order at z_0.