

# Workshop: Projectile Motion

An introduction to computing trajectories

Jonas van den Brink  
`j.v.brink@fys.uio.no`

Simula Research Laboratory  
Oslo, Norway

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This workshop focuses on introducing computations to introductory physics

Introducing computations should lead to a sense of empowerment

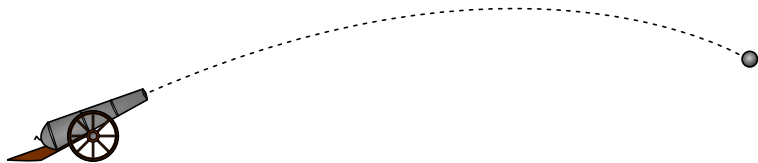
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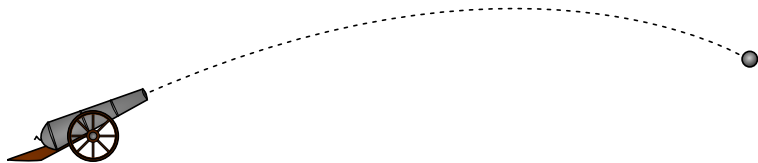
For this to be possible, the computations must

1. Relate to well-known problems
2. Must be shown to be a powerful tool
3. Understable. Students should write their own code

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Equations of motion

$$\frac{d\vec{r}}{dt} = \vec{v}(t), \quad \frac{d\vec{v}}{dt} = \vec{a}(t)$$

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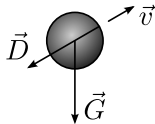
Newtons 2. law of motion

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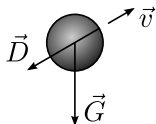
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$$\vec{F}(r, v, t) = m\vec{a}(r, v, t).$$



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Our algorithm is now as follows

1. Find the physical forces of the system.
2. Use Newtons 2. law to find the acceleration
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In this workshop, we will solve step number 3 numerically, using the Euler method.

# The Euler Method

A method for solving ordinary  
differential equations (ODEs)

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From the definition of the derivative

$$\frac{dv}{dt} = \lim_{\Delta t \rightarrow 0} \frac{v(t + \Delta t) - v(t)}{\Delta t} = a(t)$$

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Solving for  $v(t + \Delta t)$  gives

$$v(t + \Delta t) \approx v(t) + a(t) \cdot \Delta t$$

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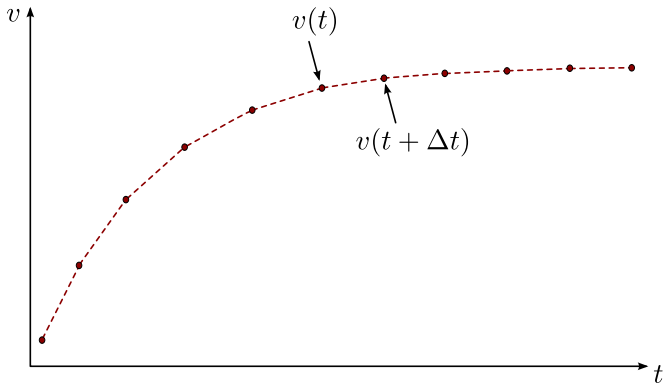
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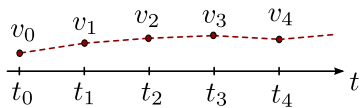
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We repeat these steps, starting at our initial conditions  $v_0$  and  $r_0$ , until we have reached our end-time  $t_N$

$$i = 0, 1, 2, 3, \dots, N.$$

# Algorithm for the Euler method

for  $i = 0, 1, 2, 3, \dots, N - 1$ :

1. Use the previous results  $x_i$  and  $v_i$  to compute the acceleration:  $a_i = F(x_i, v_i, t_i)/m$ .
2. Compute the new velocity:  $v_{i+1} = v_i + a_i \Delta t$ .
3. Compute the new position:  $r_{i+1} = r_i + v_i \Delta t$ .

# Implementation

Moving from physics and math to  
actual computer code

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```
for i in range(N):  
    a[i] = F(r[i], v[i], t[i])/m  
    v[i+1] = v[i] + a[i]*dt  
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We want the code to look as much as possible like the physics and math we write on paper

$$t_i \Rightarrow t[i] \quad v_i \Rightarrow v[i] \quad r_i \Rightarrow r[i]$$

# We also need various bookkeeping code

Here we define the arrays we will be using

```
# Import various functions meant for numerical science
import numpy as np

t_0 = 0 # Start time, s
t_end = 10 # End time, s
N = 1000 # Number of time steps

# Create a uniformly spaced time-array
t = np.linspace(t_0, t_end, N+1)

# Calculate the size of a time step
dt = t[1] - t[0]

# Create empty acceleration, velocity and position arrays
a = np.zeros((2, N+1))
v = np.zeros((2, N+1))
r = np.zeros((2, N+1))

# Set initial conditions
v[0] = (100*cos(pi/6), 100*sin(pi/6)) # initial velocity, m/s
r[0] = (0,1) # initial position, m
```



## We also need various bookkeeping code

Here we define physical constants for our system and define the function that describes the forces

```
m = 5.5 # mass, kg
g = 9.81 # acceleration of gravity, m/s^2
rho = 1.3 # air density, kg/m^3
C_D = 0.45 # drag coefficient
d = 0.11 # diameter of cannonball, m
A = pi*d**2 # cross-sectional area, m^2

def F(r, v, t):
    return (0, -m*g) - 0.5*rho*C_D*A*abs(v)*v
```

This example show the forces acting on the cannonball as it sails through the air

$$F(x, v, t) = F_g + F_d(\vec{v}) = -mg\vec{k} - \frac{1}{2}\rho C_D A |\vec{v}| \vec{v}$$

As soon as we have solved the equations of motion, we can plot the result

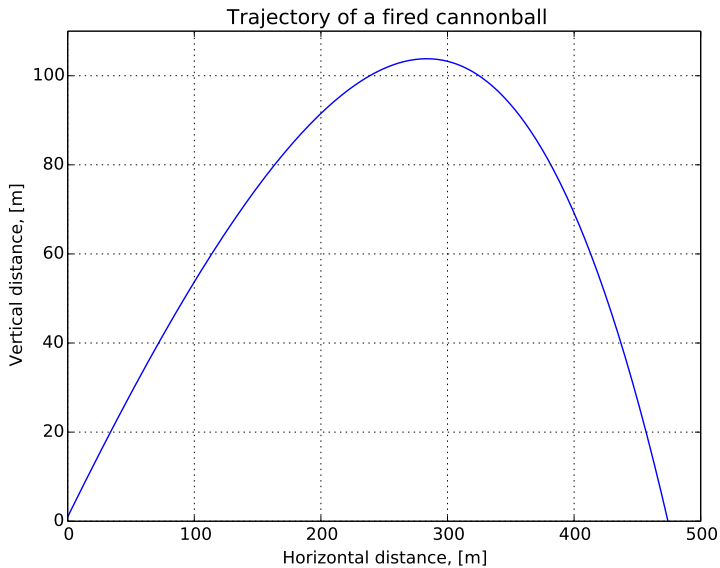
```
# Extract x and y coordinates
x = r[:,0]
y = r[:,1]

# Import functionality for plotting
from matplotlib.pyplot import plt

# Plot figure
plt.plot(x,y)

# Prettify the plot
plt.xlabel('Horizontal distance, [m]')
plt.ylabel('Vertical distance, [m]')
plt.title('Trajectory of a fired cannonball')
plt.grid()
plt.axis([0, 900, 0, 250])

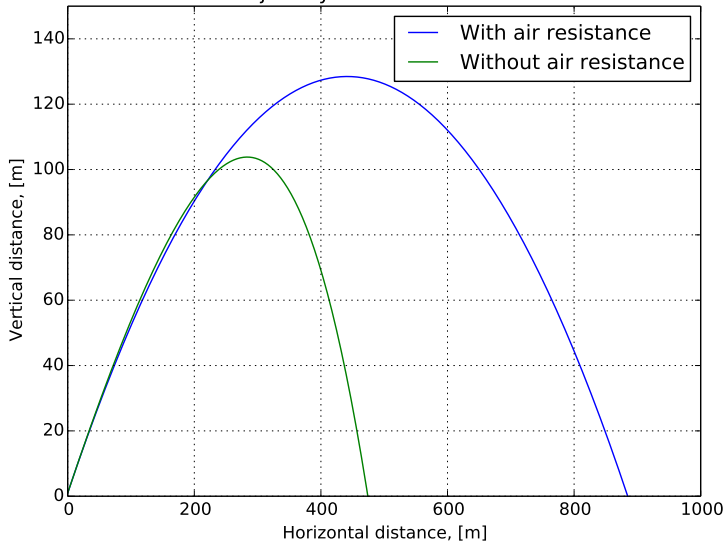
# Makes the plot appear on the screen
plt.show()
```

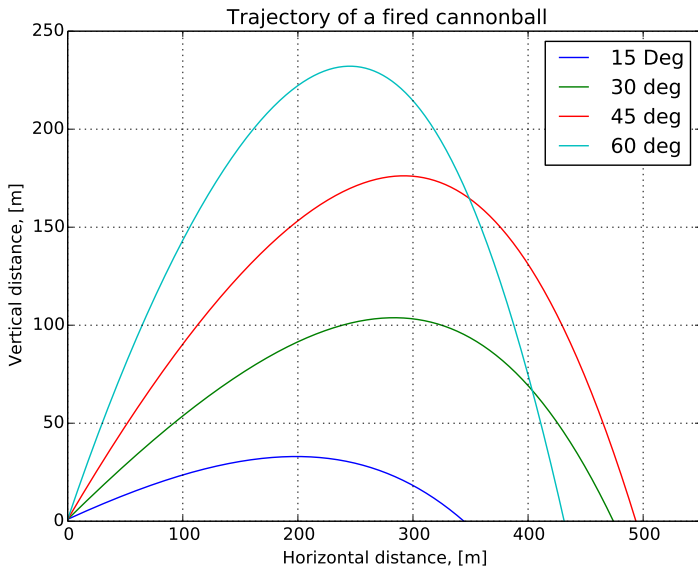


# Numerical Experimentation

Altering parameters let's us immediately see the consequences

Trajectory of a fired cannonball





# Students can use numerical experimentation to build intuition and knowledge

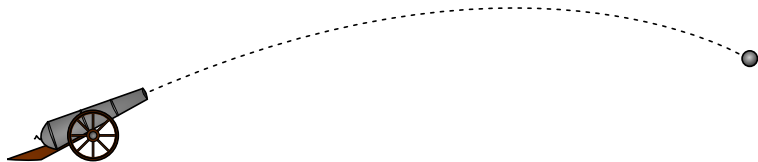
- Numerical results can be compared to known analytical solutions. Are numerical results trustworthy?
- Can study how results are directly changed by parameter choice. Are the parameters chosen reasonable?
- Can look at systems with and without certain contributions, such as air drag.  
What is important, and what can be ignored?

# Examples of possible projects

You will have a chance to look at some of these today



# Catapults and cannons and sports such as baseball



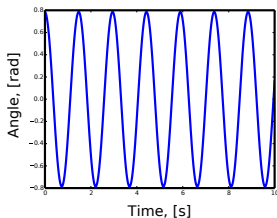
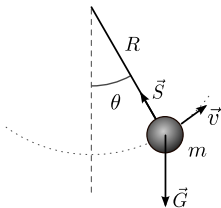
- Easy to compare with experimental data, either before or after simulation.
- Can look into studies of air drag, Reynolds number etc.

# Skydiving and bungeejumping



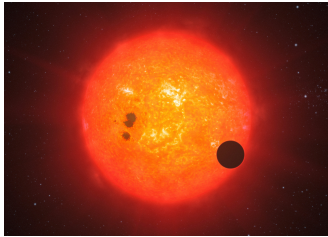
- Great study on free fall and terminal velocity
- Can study how parameters such as cross-sectional area and drag coefficient change as the parachute is opened
- Can plot the  $g$ -forces affecting the jumper. Which sport is more “extreme”?

# Pendulum and angular motion



- Can solve pendulum also for large angles!
- Energy can be plotted as functions of time
- Can also simulate double pendulum and chaotic systems

# Modelling the solar system



- Students can gather real data of planetary orbits from NASA webpages
- Can combine numerical simulation with better graphics