Problem set 5
FYS3140

Jonas van den Brink

February 26, 2013

Boas 8.6.6

We will solve the second-order linear ODE

$$y'' + 6y' + 9y = 12e^{-x}.$$

We start by finding the general solution to the homogeneous equation

$$y'' + 6y' + 9y = 0, \qquad \Rightarrow \qquad \lambda^2 + 6\lambda + 9 = 0 \qquad \Rightarrow \qquad \lambda = -3,$$

giving the general homogeneous solution:

$$y_h(x) = Ae^{-3x} + Bxe^{-3x}.$$

We now guess at a particular solution on the form

$$y_p(x) = Ce^{-x},$$

inserting this guess into the original ODE gives

$$4Ce^{-x} = 12e^{-x}, \qquad \Rightarrow \qquad C = 3.$$

So we have the total solution

$$y(x) = Ae^{-3x} + Bxe^{-3x} + 3e^{-x}.$$

Boas 8.6.12

We will solve the second-order linear ODE

$$(D^2 + 4D + 12)y = 80 \sin 2x, \qquad D \equiv \frac{\mathrm{d}}{\mathrm{d}x}.$$

Written on standard form, the equation reads

$$y'' + 4y' + 12y = 80 \sin 2x.$$

We first find the general solution to the homogeneous equation

$$y'' + 4y' + 12y = 0, \quad \Rightarrow \quad \lambda^2 + 4\lambda + 12 = 0, \quad \Rightarrow \quad \lambda = -2 \pm 2\sqrt{2}i,$$

so we see

$$y_h(x) = e^{-2}\Big(A \sin 2\sqrt{2}x + B \sin 2\sqrt{2}x\Big).$$

For the particular solution we guess at a solution on the form

$$y_p(x) = C \sin 2x + D \cos 2x.$$

Derivation gives

$$y_p' = 2C \cos 2x - 2D \sin 2x, \qquad y_p'' = -4C \sin 2x - 4D \cos 2x,$$

and inserting this back into the original equation gives

$$[8C - 8D] \sin 2x - [8C + 8D] \cos 2x = 80 \sin 2x.$$

We see that $C = 5$ and $D = -5$ giving the total solution

$$y(x) = e^{-2}\Big(A \sin 2\sqrt{2}x + B \sin 2\sqrt{2}x\Big) + 5 \sin 2x - 5 \cos 2x.$$

Boas 8.6.23

We will solve the second-order linear ODE

$$y'' + y = 2xe^x.$$

Finding the homogeneous solution is trivial:

$$y'' + y = 0, \qquad y_h(x) = A\sin x + B\cos x.$$

For the particular solution we guess a solution on the form

$$y_p = Cxe^x + De^x,$$

insertion gives

$$2Cxe^x + (2C + 2D)e^x = 2xe^x, \qquad \Rightarrow \qquad C = 1, D = -1.$$

Giving the total solution

$$y_h(x) = A\sin x + B\cos x + (x-1)e^x.$$

Boas 8.7.17

We will solve the second-order linear ODE

$$x^2y'' + xy' - 16y = 8x^4.$$

We start by using the substitution $x = e^z$, with

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}z},$$

$$y'' = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}z}{\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}z}\right) = \frac{1}{x^2}\left(\frac{\mathrm{d}^2y}{\mathrm{d}z^2} - \frac{\mathrm{d}y}{\mathrm{d}z}\right).$$

This rewriting gives the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}z^2} - 16y(z) = 8e^{2z}.$$

With homogenous solution in terms of z:

$$\frac{\mathrm{d}^2y}{\mathrm{d}z^2} - 16y(z), \quad \Rightarrow \quad \lambda = \pm 4, \quad \Rightarrow \quad y(z) = Ae^{4z} + Be^{-4z}.$$

And by guessing on the particular solution $y_p = Ce^{2x}$, insertion gives

$$-12Ce^{2z} = 8e^{2z} \quad \Rightarrow \quad C = -\frac{2}{3}, \quad \Rightarrow \quad y_p(z) = -\frac{2}{3}e^{2z}.$$

Switching back to expressing y in terms of x gives the general solution

$$y(x) = Ax^4 + B\frac{1}{x^4} - \frac{2}{3}x^2.$$

Note that the solution is now valid in $x = 0$, and that the general coefficients A and B are different for $x > 0$ and $x < 0$.

Boas 8.7.19

We will solve the second-order linear ODE

$$x^2y'' - 5xy' + 9y = 2x^3.$$

Using the substitution $x = e^z$ gives the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}z^2} - 6\frac{\mathrm{d}y}{\mathrm{d}z} + 9y = 2e^{3z}.$$

We find the homogeneous solution expressed in z to be

$$\lambda^2 - 6\lambda + 9 = 0, \quad \Rightarrow \quad \lambda_\pm = 3, \quad \Rightarrow \quad y_h(z) = Ae^{3z} + Bze^{3z}.$$

We guess that $y_p(z)$ has the form Cz^2e^{3z} and find C by insertion:

$$[9C - 18C + 9C]z^2e^{3z} + [12C - 12C]ze^{3z} + 2Ce^{3z} = 2e^{3z}, \quad \Rightarrow \quad C = 1.$$

Giving the general solution in terms of z:

$$y(z) = Ae^{3z} + Bze^{3z} + z^2e^{3z}.$$

And in terms of x:

$$y(x) = Ax^3 + Bx^3\ln x + x^3(\ln x)^2.$$

Boas 8.7.22

We will solve the second-order linear ODE

$$x^2y'' + xy' + y = 2x.$$

Using the substitution $x = e^z$ gives the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + y(z) = 2e^z.$$

We find the homogeneous solution expressed in z to be

$$\lambda^2 + 1 = 0, \quad \Rightarrow \quad \lambda_\pm = \pm i, \quad \Rightarrow \quad y_h(z) = A\sin z + B\cos z.$$

We guess that $y_p(z)$ has the form Ce^z and find C by insertion:

$$2Ce^z = 2e^z, \quad \Rightarrow \quad C = 1.$$

Giving the general solution in terms of z:

$$y(z) = A\sin z + B\cos z + e^z.$$

And in terms of x:

$$y(x) = A\sin\ln x + B\cos\ln x + x.$$