Complex Functions

The derivative of a complex function is defined as

$$\frac{\mathrm{d}}{\mathrm{d}z}f(z) = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}.$$

A function $f(z)$ is **Analytic** in a region of $\mathbb{C}$ if it has a unique derivative at every point of that region.

An analytic function must respect complex structure, and must therefore satisfy the **Cauchy-Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

If u and v and their partial derivatives with respect to x and y are continuous and satisfy the Cauchy-Riemann equations in a region, then the function is analytic at all points **inside** the region (not necessarily on the boundary).

A **regular point** is a point at which $f(z)$ is analytic. A **singularity** of $f(z)$ is a point at which $f(z)$ is not analytic. It is called an isolated singularity if $f(z)$ is analytic in a neighbourhood of the singularity.

Taylor Expansion

If $f(z)$ is analytic in a region, then it has derivatives of all orders at points inside the region and can be expanded in a Taylor series about any point z_0 inside the region. The power series converges inside the circle about z_0 that extends to the nearest singular point.

Harmonic Functions

A function which satisfies Laplace's equation is said the be harmonic

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

If $f(z) = u+iv$ is analytic in a region, then both u and v are harmonic. Any harmonic function in a simply-connected region is the real or imaginary part of an analytic function $f(z)$. The pair u and v are called **conjugate harmonic functions**.

Complex Integrals

If $f(z)$ is analytic on and inside the simple contour Γ, then the contour integral vanishes

$$\oint_\Gamma f(z)\mathrm{d}z = 0.$$

Cauchy's Integral formula states that for a function $f(z)$ analytic inside and on a simple closed contour Γ, the value of $f(z)$ at a point $z = a$ inside Γ is given by

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0} \mathrm{d}z.$$