Time-Varying Graph Learning Under Structured Temporal Priors

Xiang Zhang[1] Qiao Wang[1]

[1]School of Information Science and Engineering
Southeast University

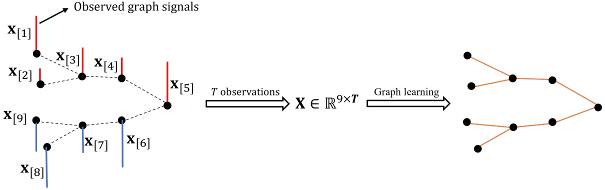
August 30, 2022

Contents

Contents

Motivations

- Graph learning



- Classic Formulation [Kal16]

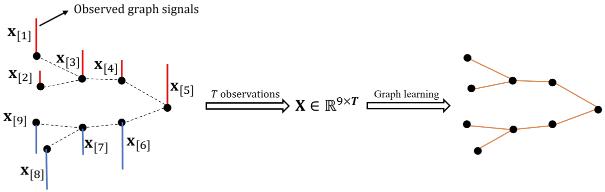
$$\min_{\mathbf{L}\in\mathcal{L}} \mathrm{Tr}\left(\mathbf{X}^\top\mathbf{L}\mathbf{X}\right) - \alpha\mathbf{1}^\top\log\left(\mathrm{diag}(\mathbf{L})\right) + \frac{\beta}{2}\|\mathrm{diag}_0(\mathbf{L})\|_F^2,$$

$$= \min_{\mathbf{w}\geq 0} 2\mathbf{r}^\top\mathbf{w} - \alpha\mathbf{1}^\top\log(\mathbf{S}\mathbf{w}) + \beta\|\mathbf{w}\|_2^2 = \min_{\mathbf{w}\geq 0} f(\mathbf{w})$$

- Limitation
  Not suitable for time-varying graphs

Motivations

- Graph learning



- Classic Formulation [Kal16]

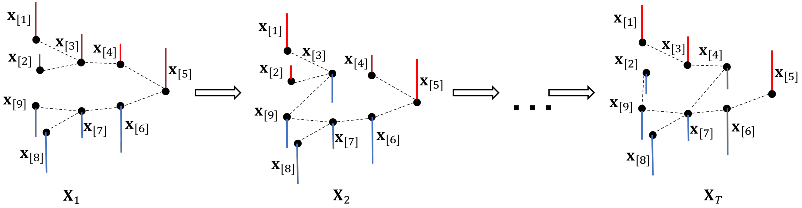
$$\min_{\mathbf{L}\in\mathcal{L}} \operatorname{Tr}\left(\mathbf{X}^\top \mathbf{L}\mathbf{X}\right) - \alpha\mathbf{1}^\top \log\left(\operatorname{diag}(\mathbf{L})\right) + \frac{\beta}{2}\|\operatorname{diag}_0(\mathbf{L})\|_F^2,$$

$$= \min_{\mathbf{w}\geq 0} 2\mathbf{r}^\top\mathbf{w} - \alpha\mathbf{1}^\top\log(\mathbf{S}\mathbf{w}) + \beta\|\mathbf{w}\|_2^2 = \min_{\mathbf{w}\geq 0} f(\mathbf{w})$$

- Limitation
  Not suitable for time-varying graphs

Motivations

- Time-varying graphs



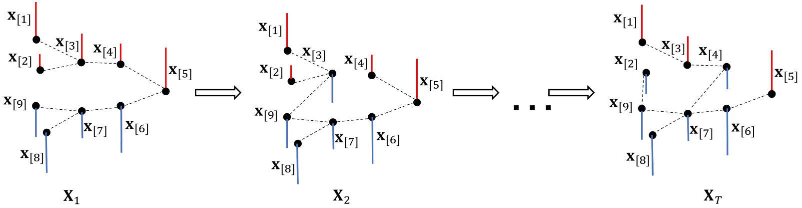
Given $\mathbf{X}_1, \mathbf{X}_2, ..\mathbf{X}_T$ collected during T time periods, time-varying graph learning aims to infer the topology of the T graphs jointly.

- Temporal homogeneity [YTO19]

$$\min_{\mathbf{w}_t \geq 0} \sum_{t=1}^{T} f_t(\mathbf{w}_t) + \eta \sum_{t=2}^{T} \|\mathbf{w}_t - \mathbf{w}_{t-1}\|_1$$

$$= \min_{\mathbf{w}_t \geq 0} \sum_{t=1}^{T} 2\mathbf{r}_t^\top \mathbf{w}_t - \alpha \mathbf{1}^\top \log(\mathbf{S}\mathbf{w}_t) + \beta \|\mathbf{w}_t\|_2^2 + \eta \sum_{t=2}^{T} \|\mathbf{w}_t - \mathbf{w}_{t-1}\|_1$$

- Time-varying graphs



Given $\mathbf{X}_1, \mathbf{X}_2, ..\mathbf{X}_T$ collected during T time periods, time-varying graph learning aims to infer the topology of the T graphs jointly.

- Temporal homogeneity [YTO19]

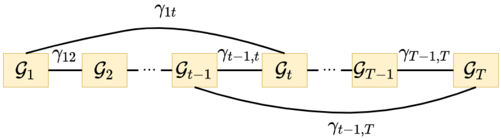
$$\min_{\mathbf{w}_t \geq 0} \sum_{t=1}^{T} f_t(\mathbf{w}_t) + \eta \sum_{t=2}^{T} \|\mathbf{w}_t - \mathbf{w}_{t-1}\|_1$$

$$= \min_{\mathbf{w}_t \geq 0} \sum_{t=1}^{T} 2\mathbf{r}_t^\top \mathbf{w}_t - \alpha \mathbf{1}^\top \log(\mathbf{S}\mathbf{w}_t) + \beta \|\mathbf{w}_t\|_2^2 + \eta \sum_{t=2}^{T} \|\mathbf{w}_t - \mathbf{w}_{t-1}\|_1$$

Contents

Proposed Model and Algorithm

- Temporal graph



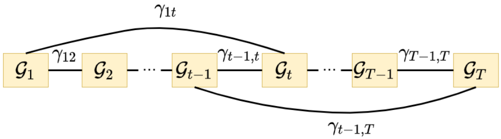
- Our formulation

$$\min_{\mathbf{w}_t \geq 0} \sum_{t \in \mathcal{V}_N} f_t(\mathbf{w}_t) + \eta \sum_{(i,j) \in \mathcal{E}_N} \gamma_{ij} \|\mathbf{w}_i - \mathbf{w}_j\|_1$$

$$= \min_{\mathbf{w}_t \geq 0} \sum_{t=1}^{T} 2\mathbf{r}_t^\top \mathbf{w}_t - \alpha \mathbf{1}^\top \log(\mathbf{S}\mathbf{w}_t) + \beta \|\mathbf{w}_t\|_2^2 + \eta \sum_{(i,j) \in \mathcal{E}_N} \gamma_{ij} \|\mathbf{w}_i - \mathbf{w}_j\|_1$$

Proposed Model and Algorithm

- Temporal graph



- Our formulation

$$\min_{\mathbf{w}_t \geq 0} \sum_{t \in \mathcal{V}_N} f_t(\mathbf{w}_t) + \eta \sum_{(i,j) \in \mathcal{E}_N} \gamma_{ij} \|\mathbf{w}_i - \mathbf{w}_j\|_1$$

$$= \min_{\mathbf{w}_t \geq 0} \sum_{t=1}^{T} 2\mathbf{r}_t^\top \mathbf{w}_t - \alpha \mathbf{1}^\top \log(\mathbf{S}\mathbf{w}_t) + \beta \|\mathbf{w}_t\|_2^2 + \eta \sum_{(i,j) \in \mathcal{E}_N} \gamma_{ij} \|\mathbf{w}_i - \mathbf{w}_j\|_1$$

- ADMM based algorithm [BBV04]

The original problem is equivalent to

$$\min_{\mathbf{w}_t \geq 0} \sum_{t \in \mathcal{V}_N} f_t(\mathbf{w}_t) + \eta \sum_{(i,j) \in \mathcal{E}_N} \gamma_{ij} \left\| \mathbf{z}_{ij} - \mathbf{z}_{ji} \right\|_1$$

$$\text{s.t. } \mathbf{w}_i = \mathbf{z}_{ij}, \text{ for } i = 1, \ldots, T \text{ and } j \in \mathcal{M}(i)$$

Scaled augmented Lagrangian:

$$\begin{aligned} & L_\rho(\mathbf{W}, \mathbf{Z}, \mathbf{U}) \\ = & \sum_{t \in \mathcal{V}_N} f_t(\mathbf{w}_t) + \eta \sum_{(i,j) \in \mathcal{E}_N} \gamma_{ij} \left\| \mathbf{z}_{ij} - \mathbf{z}_{ji} \right\|_1 + \sum_{(i,j) \in \mathcal{E}_N} \Bigg(\frac{\rho}{2} \left(\|\mathbf{u}_{ij}\|_2^2 + \|\mathbf{u}_{ji}\|_2^2 \right) \\ & + \frac{\rho}{2} \left(\|\mathbf{w}_i - \mathbf{z}_{ij} + \mathbf{u}_{ij}\|_2^2 + \|\mathbf{w}_j - \mathbf{z}_{ji} + \mathbf{u}_{ji}\|_2^2 \right) \Bigg), \end{aligned}$$

- ADMM based algorithm [BBV04]

The original problem is equivalent to

$$\min_{\mathbf{w}_t \geq 0} \sum_{t \in \mathcal{V}_N} f_t(\mathbf{w}_t) + \eta \sum_{(i,j) \in \mathcal{E}_N} \gamma_{ij} \left\| \mathbf{z}_{ij} - \mathbf{z}_{ji} \right\|_1$$
$$\text{s.t. } \mathbf{w}_i = \mathbf{z}_{ij}, \text{ for } i = 1, ..., T \text{ and } j \in \mathcal{M}(i)$$

Scaled augmented Lagrangian:

$$\begin{aligned} & L_\rho(\mathbf{W}, \mathbf{Z}, \mathbf{U}) \\ = & \sum_{t \in \mathcal{V}_N} f_t(\mathbf{w}_t) + \eta \sum_{(i,j) \in \mathcal{E}_N} \gamma_{ij} \left\| \mathbf{z}_{ij} - \mathbf{z}_{ji} \right\|_1 + \sum_{(i,j) \in \mathcal{E}_N} \Bigg(\frac{\rho}{2} \left(\|\mathbf{u}_{ij}\|_2^2 + \|\mathbf{u}_{ji}\|_2^2 \right) \\ & + \frac{\rho}{2} \left(\|\mathbf{w}_i - \mathbf{z}_{ij} + \mathbf{u}_{ij}\|_2^2 + \|\mathbf{w}_j - \mathbf{z}_{ji} + \mathbf{u}_{ji}\|_2^2 \right) \Bigg), \end{aligned}$$

- ADMM based algorithm

Update of $\mathbf{W}$:

$$\mathbf{w}_t^{k+1} = \underset{\mathbf{w}_t \geq 0}{\text{argmin}}\ f_t(\mathbf{w}_t) + \frac{\rho}{2} \sum_{j \in \mathcal{M}(t)} \|\mathbf{w}_t - \mathbf{z}_{tj}^k + \mathbf{u}_{tj}^k\|_2^2$$

Update of $\mathbf{Z}$:

$$\begin{aligned} &\mathbf{z}_{ij}^{k+1}, \mathbf{z}_{ji}^{k+1} \\ =&\underset{\mathbf{z}_{ij}, \mathbf{z}_{ji}}{\text{argmin}}\ \eta\gamma_{ij} \|\mathbf{z}_{ij} - \mathbf{z}_{ji}\|_1 + \frac{\rho}{2}\left(\left\|\mathbf{w}_i^{k+1} - \mathbf{z}_{ij} + \mathbf{u}_{ij}^k\right\|_2^2 + \left\|\mathbf{w}_j^{k+1} - \mathbf{z}_{ji} + \mathbf{u}_{ji}^k\right\|_2^2 \right) \end{aligned}$$

$$\Longrightarrow \begin{bmatrix} \mathbf{z}_{ij}^{k+1} \\ \mathbf{z}_{ji}^{k+1} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \mathbf{e}_i^k + \mathbf{e}_j^k \\ \mathbf{e}_i^k + \mathbf{e}_j^k \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -\text{prox}_{\frac{2\eta\gamma_{ij}}{\rho}\|\cdot\|_1}\left(\mathbf{e}_j^k - \mathbf{e}_i^k\right) \\ \text{prox}_{\frac{2\eta\gamma_{ji}}{\rho}\|\cdot\|_1}\left(\mathbf{e}_j^k - \mathbf{e}_i^k\right) \end{bmatrix},$$

where $\mathbf{e}_i = \mathbf{u}_{ij}^k + \mathbf{w}_i^{k+1}$

Update of $\mathbf{U}$:

$$\mathbf{u}_{ij}^{k+1} = \mathbf{u}_{ij}^k + \mathbf{w}_i^{k+1} - \mathbf{z}_{ij}^{k+1}$$
$$\mathbf{u}_{ji}^{k+1} = \mathbf{u}_{ji}^k + \mathbf{w}_j^{k+1} - \mathbf{z}_{ji}^{k+1}$$

- ADMM based algorithm

Update of $\mathbf{W}$:

$$\mathbf{w}_t^{k+1} = \underset{\mathbf{w}_t \geq 0}{\text{argmin}}\ f_t(\mathbf{w}_t) + \frac{\rho}{2} \sum_{j \in \mathcal{M}(t)} \|\mathbf{w}_t - \mathbf{z}_{tj}^k + \mathbf{u}_{tj}^k\|_2^2$$

Update of $\mathbf{Z}$:

$$\begin{aligned} &\mathbf{z}_{ij}^{k+1}, \mathbf{z}_{ji}^{k+1} \\ =&\underset{\mathbf{z}_{ij}, \mathbf{z}_{ji}}{\text{argmin}}\ \eta\gamma_{ij} \left\|\mathbf{z}_{ij} - \mathbf{z}_{ji}\right\|_1 + \frac{\rho}{2}\left(\left\|\mathbf{w}_i^{k+1} - \mathbf{z}_{ij} + \mathbf{u}_{ij}^k\right\|_2^2 + \left\|\mathbf{w}_j^{k+1} - \mathbf{z}_{ji} + \mathbf{u}_{ji}^k\right\|_2^2 \right) \end{aligned}$$

$$\Longrightarrow \begin{bmatrix} \mathbf{z}_{ij}^{k+1} \\ \mathbf{z}_{ji}^{k+1} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \mathbf{e}_i^k + \mathbf{e}_j^k \\ \mathbf{e}_i^k + \mathbf{e}_j^k \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -\text{prox}_{\frac{2\eta\gamma_{ij}}{\rho}\|\cdot\|_1}\left(\mathbf{e}_j^k - \mathbf{e}_i^k\right) \\ \text{prox}_{\frac{2\eta\gamma_{ji}}{\rho}\|\cdot\|_1}\left(\mathbf{e}_j^k - \mathbf{e}_i^k\right) \end{bmatrix},$$

where $\mathbf{e}_i = \mathbf{u}_{ij}^k + \mathbf{w}_i^{k+1}$

Update of $\mathbf{U}$:

$$\mathbf{u}_{ij}^{k+1} = \mathbf{u}_{ij}^k + \mathbf{w}_i^{k+1} - \mathbf{z}_{ij}^{k+1}$$

$$\mathbf{u}_{ji}^{k+1} = \mathbf{u}_{ji}^k + \mathbf{w}_j^{k+1} - \mathbf{z}_{ji}^{k+1}$$

Proposed Model and Algorithm

- ADMM based algorithm

Update of $\mathbf{W}$:

$$\mathbf{w}_t^{k+1} = \underset{\mathbf{w}_t \geq 0}{\text{argmin}} \ f_t(\mathbf{w}_t) + \frac{\rho}{2} \sum_{j \in \mathcal{M}(t)} \|\mathbf{w}_t - \mathbf{z}_{tj}^k + \mathbf{u}_{tj}^k\|_2^2$$

Update of $\mathbf{Z}$:

$$\begin{aligned}
&\mathbf{z}_{ij}^{k+1}, \mathbf{z}_{ji}^{k+1} \\
=&\underset{\mathbf{z}_{ij}, \mathbf{z}_{ji}}{\text{argmin}} \ \eta\gamma_{ij} \|\mathbf{z}_{ij} - \mathbf{z}_{ji}\|_1 + \frac{\rho}{2}\left(\left\|\mathbf{w}_i^{k+1} - \mathbf{z}_{ij} + \mathbf{u}_{ij}^k\right\|_2^2 + \left\|\mathbf{w}_j^{k+1} - \mathbf{z}_{ji} + \mathbf{u}_{ji}^k\right\|_2^2 \right)
\end{aligned}$$

$$\Longrightarrow \begin{bmatrix} \mathbf{z}_{ij}^{k+1} \\ \mathbf{z}_{ji}^{k+1} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \mathbf{e}_i^k + \mathbf{e}_j^k \\ \mathbf{e}_i^k + \mathbf{e}_j^k \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -\text{prox}_{\frac{2\eta\gamma_{ij}}{\rho}\|\cdot\|_1}\left(\mathbf{e}_j^k - \mathbf{e}_i^k\right) \\ \text{prox}_{\frac{2\eta\gamma_{ji}}{\rho}\|\cdot\|_1}\left(\mathbf{e}_j^k - \mathbf{e}_i^k\right) \end{bmatrix},$$

where $\mathbf{e}_i = \mathbf{u}_{ij}^k + \mathbf{w}_i^{k+1}$

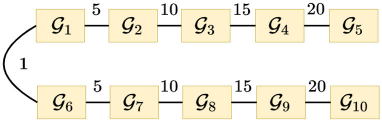
Update of $\mathbf{U}$:

$$\begin{aligned}
\mathbf{u}_{ij}^{k+1} &= \mathbf{u}_{ij}^k + \mathbf{w}_i^{k+1} - \mathbf{z}_{ij}^{k+1} \\
\mathbf{u}_{ji}^{k+1} &= \mathbf{u}_{ji}^k + \mathbf{w}_j^{k+1} - \mathbf{z}_{ji}^{k+1}
\end{aligned}$$

Contents

Experiments

- Main settings
  - Graph generation



  - Graph signal generation [DTFV16]

$$\mathbf{x} \sim Gauss\left(0, \left(\mathbf{L}_t + \sigma_w^2 \mathbf{I}\right)^\dagger\right)$$

  - Evaluation metrics

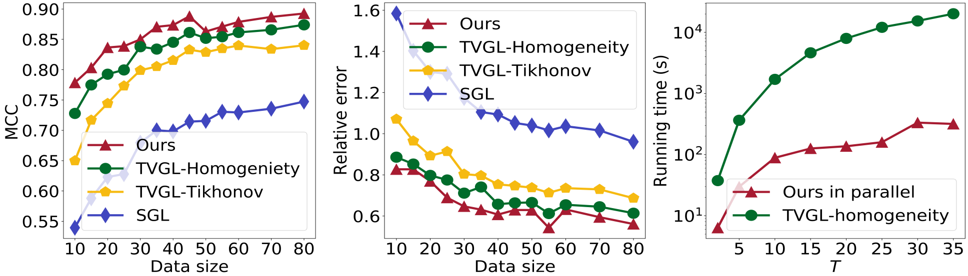
$$\mathrm{MCC} = \frac{\mathrm{TP} \cdot \mathrm{TN} - \mathrm{FP} \cdot \mathrm{FN}}{\sqrt{(\mathrm{TP}+\mathrm{FP})(\mathrm{TP}+\mathrm{FN})(\mathrm{TN}+\mathrm{FP})(\mathrm{TN}+\mathrm{FN})}}$$

$$\text{Relative error} = \frac{\|\mathbf{A}^* - \mathbf{A}_{\mathrm{gt}}\|_{\mathrm{F}}}{\|\mathbf{A}_{\mathrm{gt}}\|_{\mathrm{F}}}$$

  - Baselines

    (1) SGL [Kal16], (2) TVGL-Homogeniety [YTO19] (3)TVGL-Tikhonov [KLTF17]

- Synthetic data



(a) (b) (c)

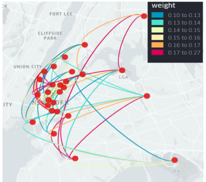
Figure: (a)-(b)Performance of different sample sizes; (c) Performance of running time

Experiments

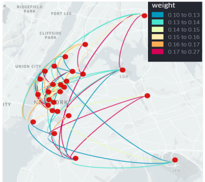
- Data source: Yellow Taxi Trip data of New York city [1]
- Task: Learn time-varying travel relationships between different taxi zones.
- Data selection: Data from 0 a.m. to 12 a.m. of 20 workdays in September

[1] https://data.cityofnewyork.us/Transportation/2018-Yellow-Taxi-Trip-Data.

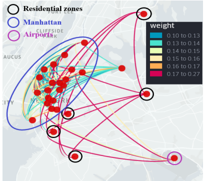
- Real data



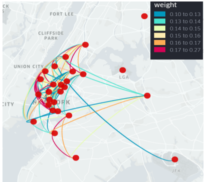
(a) 2 a.m - 3 a.m



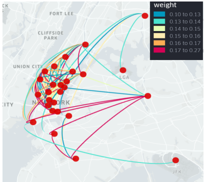
(b) 4 a.m.-5a.m.



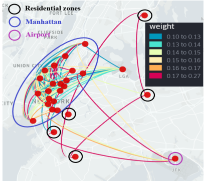
(c) 8 a.m.-9 a.m.



(d) 2 a.m - 3 a.m



(e) 4 a.m.-5a.m.



(f) 8 a.m.-9 a.m.

Figure: The learned graphs of taxi zones of New York. The upper row (a)-(c) shows results of our model and the lower row (d)-(e) shows the results of TV-Homogeneity model.

Contents

Conclusions

- Our proposed model can describe intricate temporal relationships effectively
- The ADMM based algorithm can solve the induced optimization problem in a parallel manner efficiently.
- Experimental results of both synthetic and real data illustrate the superiority of our model and algorithm.

The End

References I

[BBV04] Stephen Boyd, Stephen P Boyd, and Lieven Vandenberghe, *Convex optimization*, Cambridge university press, 2004.

[DTFV16] Xiaowen Dong, Dorina Thanou, Pascal Frossard, and Pierre Vandergheynst, *Learning laplacian matrix in smooth graph signal representations*, IEEE Transactions on Signal Processing **64** (2016), no. 23, 6160–6173.

[Kal16] Vassilis Kalofolias, *How to learn a graph from smooth signals*, Artificial Intelligence and Statistics, PMLR, 2016, pp. 920–929.

[KLTF17] Vassilis Kalofolias, Andreas Loukas, Dorina Thanou, and Pascal Frossard, *Learning time varying graphs*, 2017 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), Ieee, 2017, pp. 2826–2830.

[YTO19] Koki Yamada, Yuichi Tanaka, and Antonio Ortega, *Time-varying graph learning based on sparseness of temporal variation*, ICASSP 2019-2019 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), IEEE, 2019, pp. 5411–5415.