CHAPTER 2

Fundamentals of Tactical Missile Guidance

INTRODUCTION

TACTICAL guided missiles apparently had their origin in Germany. For example, the Hs. 298 was one of a series of German air-to-air guided missiles developed by the Henschel Company during World War II [1]. A high-thrust first stage accelerated the missile from the carrier aircraft, whereas a low-thrust, long-burning sustainer maintained the vehicle's velocity. The Hs. 298, which was radio-controlled from the parent aircraft, was to be released either slightly above or below the target. Apparently the height differential made it easier to aim and guide the missile. This first air-to-air missile weighed 265 lb and had a range of nearly 3 miles. On December 22, 1944, three missiles were test flown from a JU 88G aircraft. All three tests resulted in failure. Although 100 of these air-to-air missiles were manufactured, none was used in combat.

The Rheintochter (R-1) was a surface-to-air missile also developed in Germany during World War II [1]. This unusual looking two-stage radio-controlled missile weighed nearly 4000 lb and had three sets of plywood fins: one for the booster and two for the sustainer. Eighty-two of these missiles flew before production was halted in December 1944. The missile was ineffective because Allied bombers, which were the R-1's intended target, flew above the range (about 20,000 ft) of this surface-to-air missile.

Although proportional navigation was apparently known by the Germans during World War II at Peenemünde, no applications on the Hs. 298 or R-1 missiles using proportional navigation were reported [2]. The Lark missile, which had its first successful test in December 1950, was the first missile to use proportional navigation. Since that time proportional navigation guidance has been used in virtually all of the world's tactical radar, infrared (IR), and television (TV) guided missiles [3]. The popularity of this interceptor guidance law is based upon its simplicity, effectiveness, and ease of implementation. Apparently, proportional navigation was first studied by C. Yuan and others at the RCA Laboratories during World War II under the auspices of the U.S. Navy [4].

The guidance law was conceived from physical reasoning and equipment available at that time. Proportional navigation was extensively studied at Hughes Aircraft Company [5] and implemented in a tactical missile using a pulsed radar system. Finally, proportional navigation was more fully developed at Raytheon and implemented in a tactical continuous wave radar homing missile [6]. After World War II, the U.S. work on proportional navigation was declassified and first appeared in the *Journal of Applied Physics* [7]. Mathematical derivations of the "optimality" of proportional navigation came more than 20 years later [8].

Keeping with the spirit of the origins of proportional navigation, we shall avoid mathematical proofs in this chapter on deriving the guidance law, but shall, instead, concentrate first on proving to the reader that the guidance technique works. Next we shall investigate some properties of the guidance law that we shall both observe and derive. Finally, we shall show how this classical guidance law provides the foundation for more advanced techniques of interceptor guidance.

WHAT IS PROPORTIONAL NAVIGATION?

Theoretically, the proportional navigation guidance law issues acceleration commands, perpendicular to the instantaneous missile-target line-of-sight, which are proportional to the line-of-sight rate and closing velocity. Mathematically, the guidance law can be stated as

$$n_c = N' V_c \dot{\lambda}$$

where n_c is the acceleration command (in ft/s^2), N' a unitless designer-chosen gain (usually in the range of 3–5) known as the effective navigation ratio, V_c the missile-target closing velocity (in ft/s), and λ the line-of-sight angle (in rad). The overdot indicates the time derivative of the line-of-sight angle or the line-of-sight rate.

In tactical radar homing missiles using proportional navigation guidance, the seeker provides an effective measurement of the line-of-sight rate, and a Doppler radar provides closing velocity information. In tactical IR missile applications of proportional navigation guidance, the line-of-sight rate is measured, whereas the closing velocity, required by the guidance law, is "guesstimated."

In tactical endoatmospheric missiles, proportional navigation guidance commands are usually implemented by moving fins or other control surfaces to obtain the required lift. Exoatmospheric strategic interceptors use thrust vector control, lateral divert engines, or squibs to achieve the desired acceleration levels.

SIMULATION OF PROPORTIONAL NAVIGATION IN TWO DIMENSIONS

To better understand how proportional navigation works, let us consider the two-dimensional, point mass missile-target engagement geometry of Fig. 2.1. Here we

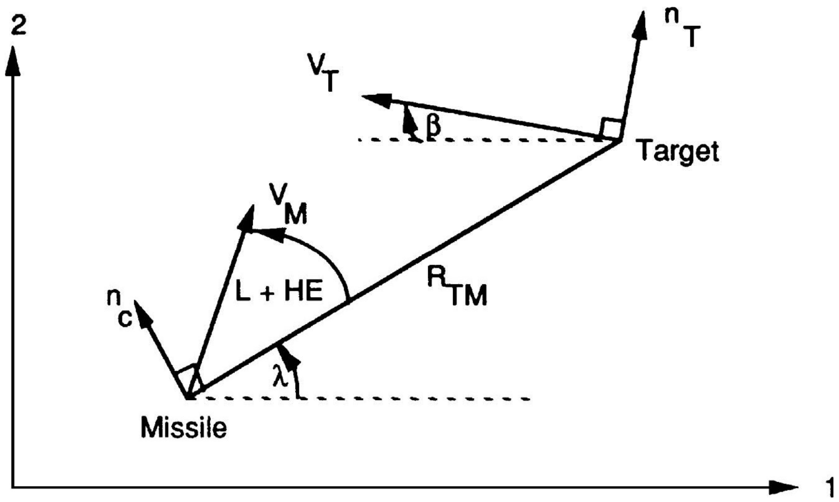

Fig. 2.1 Two-dimensional missile-target engagement geometry.

have an inertial coordinate system fixed to the surface of a flat-Earth model (that is, the 1 axis is downrange and the 2 axis can either be altitude or crossrange). Using the inertial coordinate system of Fig. 2.1 means that we can integrate components of the accelerations and velocities along the 1 and 2 directions without having to worry about additional terms due to the Coriolis effect. In this model it is assumed that both the missile and target travel at constant velocity. In addition, gravitational and drag effects have been neglected for simplicity.

We can see from the figure that the missile, with velocity magnitude V_M, is heading at an angle of $L + HE$ with respect to the line of sight. The angle L is known as the missile lead angle. The lead angle is the theoretically correct angle for the missile to be on a collision triangle with the target. In other words, if the missile is on a collision triangle, no further acceleration commands are required for the missile to hit the target. The angle HE is known as the heading error. This angle represents the initial deviation of the missile from the collision triangle.

In Fig. 2.1 the imaginary line connecting the missile and target is known as the line of sight. The line of sight makes an angle of λ with respect to the fixed reference, and the length of the line of sight (instantaneous separation between missile and target) is a range denoted R_{TM}. From a guidance point of view, we desire to make the range between missile and target at the expected intercept time as small as possible (hopefully zero). The point of closest approach of the missile and target is known as the miss distance.

The closing velocity V_c is defined as the negative rate of change of the distance from the missile to the target, or

$$V_c = -\dot{R}_{\mathrm{TM}}$$

Therefore, at the end of the engagement, when the missile and target are in closest proximity, the sign of V_c will change. In other words, from calculus we know that the closing velocity will be zero when R_{TM} is a minimum (that is, the function is either minimum or maximum when its derivative is zero). The desired acceleration command n_c, which is derived from the proportional navigation guidance law, is perpendicular to the instantaneous line of sight.

In our engagement model of Fig. 2.1, the target can maneuver evasively with acceleration magnitude n_T. Since target acceleration n_T in the preceding model is perpendicular to the target velocity vector, the angular velocity of the target can be expressed as

$$\dot{\beta} = \frac{n_T}{V_T}$$

where V_T is the magnitude of the target velocity. The components of the target velocity vector in the Earth or inertial coordinate system can be found by integrating the differential equation given earlier for the flight-path angle of the target β and substituting in

$$V_{T1} = -V_T \cos\beta$$
$$V_{T2} = V_T \sin\beta$$

Target position components in the Earth fixed coordinate system can be found by directly integrating the target velocity components. Therefore, the differential equations for the components of the target position are given by

$$\dot{R}_{T1} = V_{T1}$$

$$\dot{R}_{T2} = V_{T2}$$

Similarly, the missile velocity and position differential equations are given by

$$\dot{V}_{M1} = a_{M1}$$

$$\dot{V}_{M2} = a_{M2}$$

$$\dot{R}_{M1} = V_{M1}$$

$$\dot{R}_{M2} = V_{M2}$$

where a_{M1} and a_{M2} are the missile acceleration components in the Earth coordinate system. To find the missile acceleration components, we must first find the components of the relative missile-target separation. This is accomplished by first defining the components of the relative missile-target separations by

$$R_{\mathrm{TM}1} = R_{T1} - R_{M1}$$
$$R_{\mathrm{TM}2} = R_{T2} - R_{M2}$$

We can see from Fig. 2.1 that the line-of-sight angle can be found, using trigonometry, in terms of the relative separation components as

$$\lambda = \tan^{-1}\frac{R_{\mathrm{TM2}}}{R_{\mathrm{TM1}}}$$

If we define the relative velocity components in Earth coordinates to be

$$V_{\mathrm{TM1}} = V_{T1} - V_{M1}$$
$$V_{\mathrm{TM2}} = V_{T2} - V_{M2}$$

we can calculate the line-of-sight rate by direct differentiation of the expression for line-of-sight angle. After some algebra we obtain the expression for the line-of-sight rate to be

$$\dot{\lambda} = \frac{R_{\mathrm{TM1}}V_{\mathrm{TM2}} - R_{\mathrm{TM2}}V_{\mathrm{TM1}}}{R_{\mathrm{TM}}^2}$$

The relative separation between missile and target R_{TM} can be expressed in terms of its inertial components by application of the distance formula, as

$$R_{\mathrm{TM}} = \left(R_{\mathrm{TM1}}^2 + R_{\mathrm{TM2}}^2\right)^{\frac{1}{2}}$$

Because the closing velocity is defined as the negative rate of change of the missile target separation, it can be obtained by differentiating the preceding equation, yielding

$$V_c = -\dot{R}_{\mathrm{TM}} = \frac{-(R_{\mathrm{TM1}}V_{\mathrm{TM1}} + R_{\mathrm{TM2}}V_{\mathrm{TM2}})}{R_{\mathrm{TM}}}$$

The magnitude of the missile guidance command n_c can then be found from the definition of proportional navigation, or

$$n_c = N'V_c\dot{\lambda}$$

Because the acceleration command is perpendicular to the instantaneous line of sight, the missile acceleration components in Earth coordinates can be found by trigonometry using the angular definitions from Fig. 2.1. The missile acceleration components are

$$a_{M1} = -n_c \sin\lambda$$
$$a_{M2} = n_c \cos\lambda$$

We have now listed all of the differential equations required to model a complete missile-target engagement in two dimensions. However, some additional equations are required for the initial conditions on the differential equations in order to complete the engagement model.

A missile employing proportional navigation guidance is not fired at the target but is fired in a direction to lead the target. The initial angle of the missile velocity vector with respect to the line of sight is known as the missile lead angle L. In essence we are firing the missile at the expected intercept point. We can see from Fig. 2.1 that, for the missile to be on a collision triangle (missile will hit the target if both continue to fly along a straight-line path at constant velocities), the theoretical missile lead angle can be found by application of the law of sines, yielding

$$L = \sin^{-1} \frac{V_T \sin(\beta + \lambda)}{V_M}$$

In practice, the missile is usually not launched exactly on a collision triangle, as the expected intercept point is not known precisely. The location of the intercept point can only be approximated because we do not know in advance what the target will do in the future. In fact, that is why a guidance system is required! Any initial angular deviation of the missile from the collision triangle is known as a heading error HE. The initial missile velocity components can therefore be expressed in terms of the theoretical lead angle L and actual heading error HE as

$$V_{M1}(0) = V_M \cos(L + HE + \lambda)$$
$$V_{M2}(0) = V_M \sin(L + HE + \lambda)$$

TWO-DIMENSIONAL ENGAGEMENT SIMULATION

To witness and understand the effectiveness of proportional navigation, it is best to simulate the guidance law and test its properties under a variety of circumstances. A two-dimensional missile-target engagement simulation was set up using the differential equations derived in the previous section. The simulation inputs are the initial location of the missile and target, speeds, flight time, and effective navigation ratio. The user can vary the level of the two error sources considered: target maneuver and heading error.

LISTING 2.1 TWO-DIMENSIONAL TACTICAL MISSILE-TARGET ENGAGEMENT SIMULATION

```
n=0;
VM = 3000.;
VT = 1000.;
XNT = 0.;
HEDEG = -20.;
XNP = 4.;
RM1 = 0.;
RM2 = 10000.;
```

```
RT1 = 40000.;
RT2 = 10000.;
BETA=0.;
VT1=-VT*cos(BETA);
VT2=VT*sin(BETA);
HE=HEDEG/57.3;
T=0.;
S=0.;
RTM1=RT1-RM1;
RTM2=RT2-RM2;
RTM=sqrt(RTM1*RTM1+RTM2*RTM2);
XLAM=atan2(RTM2,RTM1);
XLEAD=asin(VT*sin(BETA+XLAM)/VM);
THET=XLAM+XLEAD;
VM1=VM*cos(THET+HE);
VM2=VM*sin(THET+HE);
VTM1 = VT1 - VM1;
VTM2 = VT2 - VM2;
VC=-(RTM1*VTM1 + RTM2*VTM2)/RTM;
while VC >= 0
    if RTM < 1000
            H=.0002;
    else
            H=.01;
    end
    BETAOLD=BETA;
    RT1OLD=RT1;
    RT2OLD=RT2;
    RM1OLD=RM1;
    RM2OLD=RM2;
    VM1OLD=VM1;
    VM2OLD=VM2;
    STEP=1;
    FLAG=0;
    while STEP <=1
        if FLAG==1
                STEP=2;
            BETA=BETA+H*BETAD;
                RT1=RT1+H*VT1;
                RT2=RT2+H*VT2;
                RM1=RM1+H*VM1;
                RM2=RM2+H*VM2;
                VM1=VM1+H*AM1;
                VM2=VM2+H*AM2;
                T=T+H;
        end
```

```
            RTM1=RT1-RM1;
                  RTM2=RT2-RM2;
                  RTM=sqrt(RTM1*RTM1+RTM2*RTM2);
                  VTM1=VT1-VM1;
                  VTM2=VT2-VM2;
                  VC=-(RTM1*VTM1+RTM2*VTM2)/RTM;
                  XLAM=atan2(RTM2,RTM1);
                  XLAMD=(RTM1*VTM2-RTM2*VTM1)/(RTM*RTM);
                  XNC=XNP*VC*XLAMD;
                  AM1=-XNC*sin(XLAM);
                  AM2=XNC*cos(XLAM);
                  VT1=-VT*cos(BETA);
                  VT2=VT*sin(BETA);
                  BETAD=XNT/VT;
            FLAG=1;
      end
      FLAG=0;
      BETA=.5*(BETAOLD+BETA+H*BETAD);
      RT1=.5*(RT1OLD+RT1+H*VT1);
      RT2=.5*(RT2OLD+RT2+H*VT2);
      RM1=.5*(RM1OLD+RM1+H*VM1);
      RM2=.5*(RM2OLD+RM2+H*VM2);
      VM1=.5*(VM1OLD+VM1+H*AM1);
      VM2=.5*(VM2OLD+VM2+H*AM2);
      S=S+H;
      if S >=.09999
                 S=0.;
            n=n+1;
            ArrayT(n)=T;
            ArrayRT1(n)=RT1;
            ArrayRT2(n)=RT2;
            ArrayRM1(n)=RM1;
            ArrayRM2(n)=RM2;
            ArrayXNCG(n)=XNC/32.2;
            ArrayRTM(n)=RTM;
      end
end
RTM
figure
plot(ArrayRT1,ArrayRT2,ArrayRM1,ArrayRM2),grid
title('Two-dimensional tactical missile-target engagement simulation')
xlabel('Downrange (Ft) ')
ylabel('Altitude (Ft)')
figure
plot(ArrayT,ArrayXNCG),grid
title('Two-dimensional tactical missile-target engagement simulation')
```

```
xlabel('Time (sec)')
ylabel('Acceleration of missle (G)')
clc
output=[ArrayT',ArrayRT1',ArrayRT2',ArrayRM1',ArrayRM2',ArrayXNCG',ArrayRTM' ];
save datfil.txt output /ascii
disp '*** Simulation Complete'
```

A tactical missile-target engagement simulation appears in Listing 2.1. We can see from the listing that the missile and target differential equations are solved using the second-order Runge–Kutta numerical integration technique. As was the case in the second-order system simulation of Chapter 1, the differential equations appear before the FLAG=1 statement. The integration step size is fixed for most of the flight ($H = 0.01$ s) but is made smaller near the end of the flight ($H = 0.0002$ s when $R_{TM} < 1000$ ft) to accurately capture the magnitude of the miss distance. The program is terminated when the closing velocity changes sign, because this means that the separation between the missile and target is a minimum. At this time the missile-target separation is the miss distance: We can see from the preceding equations that the miss distance will always be positive because it is calculated from the distance formula. We can see from the listing that errors can be introduced by changing values in the data statements. Status of the missile and target location, along with acceleration and separation information, is displayed every 0.1 s. Note that the missile acceleration is written to a file datfil.txt in units of gravity.

A sample case was run in which the only disturbance was a 20-deg heading error (HEDEG = −20.). Sample trajectories for effective navigation ratios of 4 and 5 are depicted in Fig. 2.2. We can see from the figure that initially the missile is flying in the wrong direction because of the heading error. Gradually the guidance

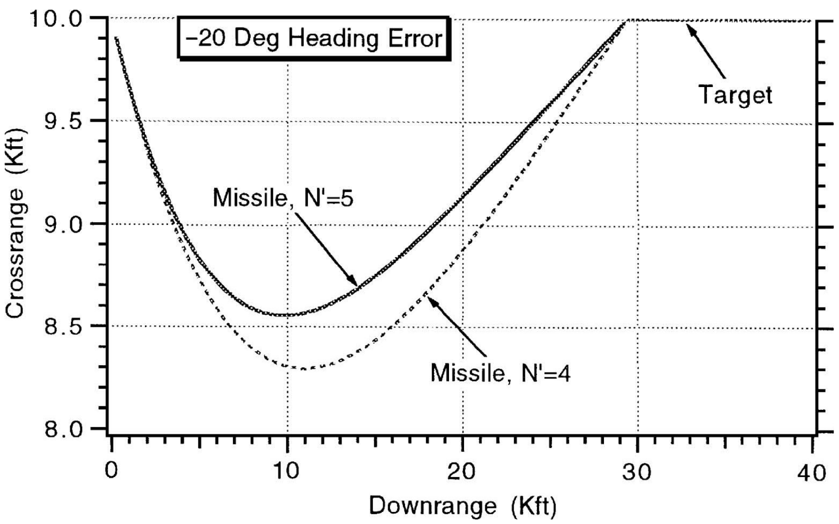

Fig. 2.2 Increasing effective navigation ratio causes heading error to be removed more rapidly.

law forces the missile to home on the target. The larger effective navigation ratio enables the missile to remove the initial heading error more rapidly, thus causing a much tighter trajectory. In both cases, proportional navigation appears to be an effective guidance law because the missile hits the target (near zero miss distance with the simulation).

The resultant missile acceleration histories, displayed in Fig. 2.3, for both cases are somewhat different. The quicker removal of heading error in the higher effective navigation ratio case ($N' = 5$) results in larger missile accelerations at the beginning of the flight and lower accelerations near the end of the flight. In both cases the acceleration profiles for the required missile acceleration to take out the heading error and to hit the target is monotonically decreasing and zero at the end of the flight. Thus, a property of a proportional navigation guidance system is to start taking out heading error as soon as possible but also gradually throughout the *entire* flight. In Chapter 15 we shall study a guidance system that tries to remove the entire heading error immediately. By increasing the effective navigation ratio, we are allowing the missile to take out heading error more rapidly.

Another sample case was run in which the only disturbance was a 3-g target maneuver (XNT = 96.6, HEDEG = 0). In this scenario the missile and target are initially on a collision triangle and flying along the downrange component of the Earth fixed coordinate system (cross-range velocity components of both interceptor and target are zero). Therefore, the target velocity vector is initially along the line of sight, and at first all 3 g of the target acceleration are perpendicular to the line of sight. As the target maneuvers, the magnitude of the target acceleration perpendicular to the line of sight diminishes due to the turning of the target. Sample missile-target trajectories for this case with effective navigation ratios

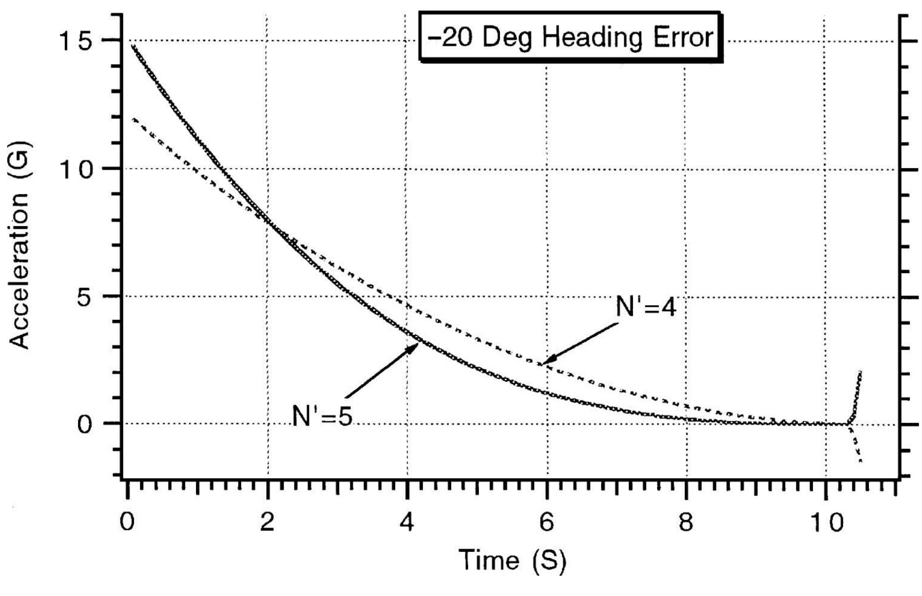

Fig. 2.3 Increasing effective navigation ratio causes more acceleration initially.

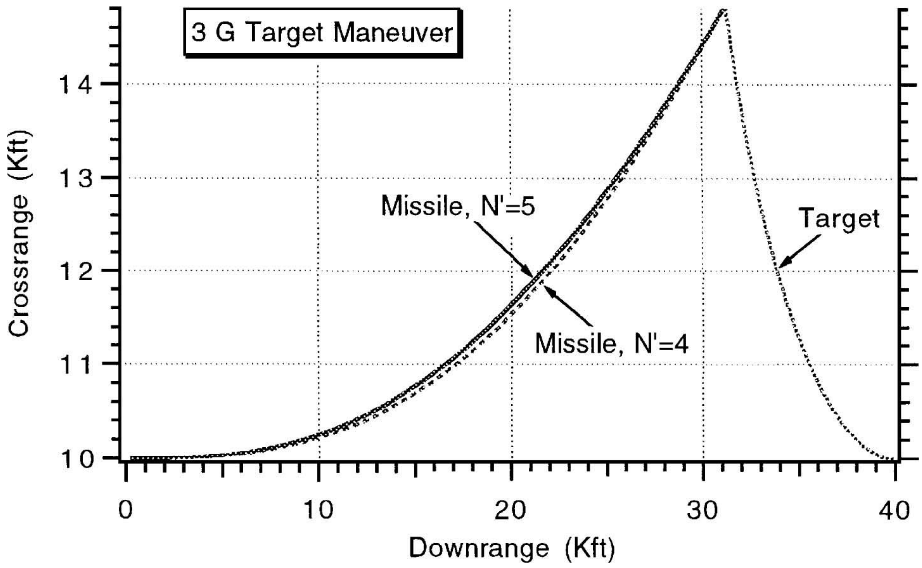

Fig. 2.4 Proportional navigation works against maneuvering target.

of 4 and 5 are depicted in Fig. 2.4. We can see that the higher effective navigation ratio causes the missile to lead the target slightly more than the lower navigation ratio case. Otherwise the trajectories are virtually identical. In both cases, the proportional navigation guidance law enabled the missile to hit the maneuvering target.

However, Fig. 2.5 shows that there are significant differences between the acceleration profiles for the maneuvering target case. Although both acceleration profiles are virtually monotonically increasing for the entire flight, the higher effective navigation ratio requires less acceleration capability of the missile.

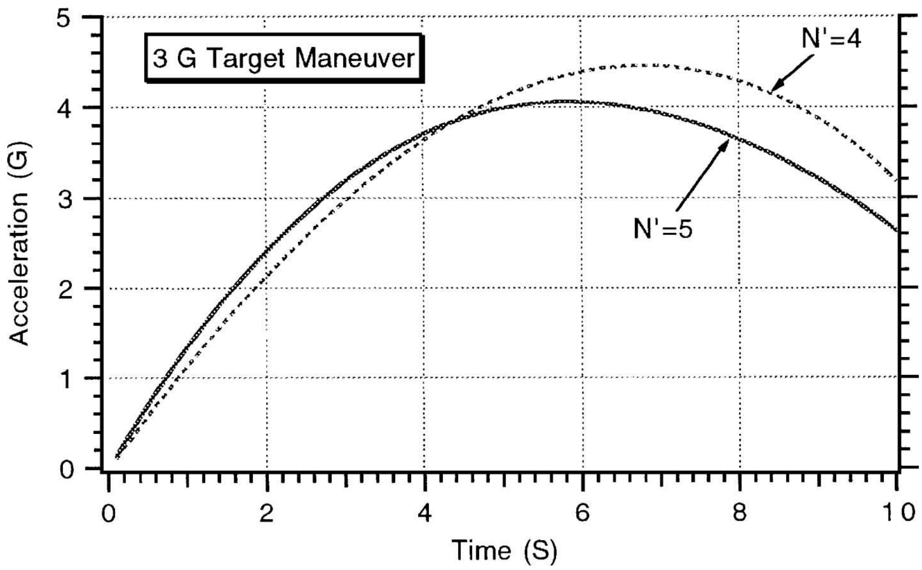

Fig. 2.5 Higher navigation ratio yields less acceleration to hit maneuvering target.

In addition, we can see that the peak acceleration required by the missile to hit the target is significantly higher than the maneuver level of the target (3 g).

In both simulation examples we have seen the effectiveness of proportional navigation guidance. First we saw that proportional navigation is able to hit a target, even if it is initially launched in the wrong direction by 20 deg. Then we observed that the guidance law was also effective in hitting a maneuvering target. In both cases certain acceleration levels were required of the missile in order for it to hit the target. The levels were dependent on the type of error source and the effective navigation ratio. If the missile does not have the acceleration required by the guidance law, a miss will result.

LINEARIZATION

Thus far our understanding of the effectiveness of proportional navigation has come from the numerical simulation results of the two-dimensional engagement simulation. It is critical for the analysis, understanding, and development of design relationships to temporarily depart from the nonlinear missile-target simulation and develop a simpler model. Therefore, we will linearize the two-dimensional engagement model in the hope of gaining more understanding. This does not mean that we will abandon the nonlinear engagement model. In fact, we will always use the nonlinear engagement model to verify the insights generated by powerful analytical techniques to be used on the linearized engagement model.

The linearization of the missile-target geometry can easily be accomplished if we define some new relative quantities as shown in Fig. 2.6. Here y is the relative separation between the missile and target perpendicular to the fixed reference.

The relative acceleration (difference between missile and target acceleration) can be written by inspection of Fig. 2.6 as

$$\ddot{y} = n_T \cos\beta - n_c \cos\lambda$$

If the flight-path angles are small (near head-on or tail chase case), the cosine terms are approximately unity, and the preceding equation becomes

$$\ddot{y} = n_T - n_c$$

Similarly, the expression for the line-of-sight angle can also be linearized using the small-angle approximation, yielding

$$\lambda = y/R_{\mathrm{TM}}$$

For a head-on case, we can approximate the closing velocity as

$$V_c = V_M + V_T$$

whereas in a tail chase case, the closing velocity can be approximated as

$$V_c = V_M - V_T$$

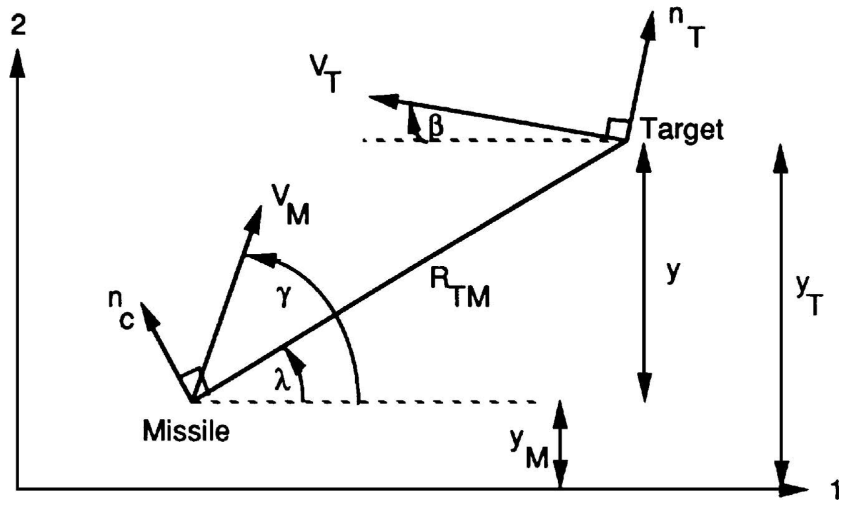

Fig. 2.6 Engagement model for linearization.

Therefore, in a linearized analysis we will treat the closing velocity as a positive constant. Because closing velocity has also been previously defined as the negative derivative of the range from the missile to target, and since the range must go to zero at the end of the flight, we can also linearize the range equation with the time-varying relationship

$$R_{\mathrm{TM}} = V_c(t_F - t)$$

where t is current time and t_F the total flight time of the engagement. Note that t_F is also now a constant. The quantity $t_F - t$ is the time to go until the end of the flight. Therefore, the range from the missile to the target is also the closing velocity multiplied by the time to go until intercept. Because range goes to zero at the end of the flight by definition, we must reexamine the definition of miss distance. The linearized miss distance is taken to be the relative separation between missile and target y at the end of the flight, or

$$\mathrm{Miss} = y(t_F)$$

Because the linearized miss is not obtained from the distance formula, it is only an approximation to the actual miss. However, we shall soon see that the miss distance approximation is very accurate.

LINEARIZED ENGAGEMENT SIMULATION

In the previous section we developed linearized equations for the missile-target engagement. In this section we will see if the resultant linearized equations give performance projections that have trends similar to those of the nonlinear engagement equations. If they do not, then there is no point in developing design relationships based on a meaningless model. If they do, then there may be a point for the interested reader to continue reading this text!

The linearized proportional navigation engagement simulation appears in Listing 2.2. In this simulation the flight time t_F is an input rather than output. We can see from the listing that the simulation only consists of two differential equations: one for relative velocity and the other for relative acceleration. These differential equations are also solved using the second-order Runge–Kutta numerical integration technique. The linearized differential equations appear in the listing before the FLAG=1 statement. Unlike the nonlinear engagement simulation, the integration step size in the linear simulation can be kept fixed for the entire flight ($H = 0.01$ s). The program is stopped when the current time equals the flight time. Nominally the program is set up without errors. Errors can be introduced by changing values in the data statements. The status of the relative position and velocity, along with missile acceleration information, is displayed every 0.1 s.

LISTING 2.2 LINEARIZED ENGAGEMENT SIMULATION

```
XNT=0.;
Y=0.;
VM=3000.;
HEDEG=-20.;
TF=10.;
XNP=4.;
YD=-VM*HEDEG/57.3;
T=0.;
H=.01;
S=0.;
n=0.;
while T<=(TF-1e-5)
    YOLD=Y;
    YDOLD=YD;
    STEP=1;
    FLAG=0;
    while STEP<=1
        if FLAG==1
            STEP=2;
            Y=Y+H*YD;
            YD=YD+H*YDD;
            T=T+H;
        end
        TGO=TF-T+.00001;
        XLAMD=(Y+YD*TGO)/(VC*TGO*TGO);
        XNC=XNP*VC*XLAMD;
        YDD=XNT-XNC;
        FLAG=1;
    end
```

```
    FLAG=0;
    Y=.5*(YOLD+Y+H*YD);
    YD=.5*(YDOLD+YD+H*YDD);
    S=S+H;
    if S>=.0999
       S=0.;
       n=n+1;
       ArrayT(n)=T;
       ArrayY(n)=Y;
       ArrayYD(n)=YD;
       ArrayXNCG(n)=XNC/32.2;
    end
end
figure
plot(ArrayT,ArrayXNCG),grid
xlabel('Time (Sec)')
ylabel('Missile Acceleration (G)')
clc
output=[ArrayT',ArrayY',ArrayYD',ArrayXNCG'];
save datfil.txt output -ascii
disp 'simulation finished'
```

To verify that the linearized engagement model is a reasonable approximation to the nonlinear engagement model, cases that were run for the nonlinear engagement model were repeated using the simulation of Listing 2.2. A sample run was made with the linearized engagement model in which the only disturbance was a −20-deg heading error (HEDEG = −20.). In this case the effective navigation ratio was 4. Acceleration profile comparisons for both the linear and nonlinear engagement models are presented in Fig. 2.7. The figure clearly shows that, even for a relatively large heading error disturbance, the resultant acceleration profiles are virtually indistinguishable. Thus, the linearized model is an excellent approximation to the nonlinear engagement model in the case of a heading error disturbance.

Another sample run was made with the linear engagement model; this time with a 3-*g* target maneuver disturbance. Figure 2.8 shows that this time the linearized model overestimates the missile acceleration requirements. The reason for the discrepancy is that the linear model assumes that the target acceleration magnitude, perpendicular to the line of sight, is always the same and equal to the magnitude of the maneuver. In reality, as the target maneuvers, the component of acceleration perpendicular to the line of sight decreases because the target is turning. Therefore, the nonlinear acceleration requirements due to a maneuvering target are somewhat less than those predicted by the linearized engagement model. However, it is important to note that the linear engagement model accurately predicts the monotonically increasing trend (for most of the flight) for the missile acceleration profile due to a target maneuver.

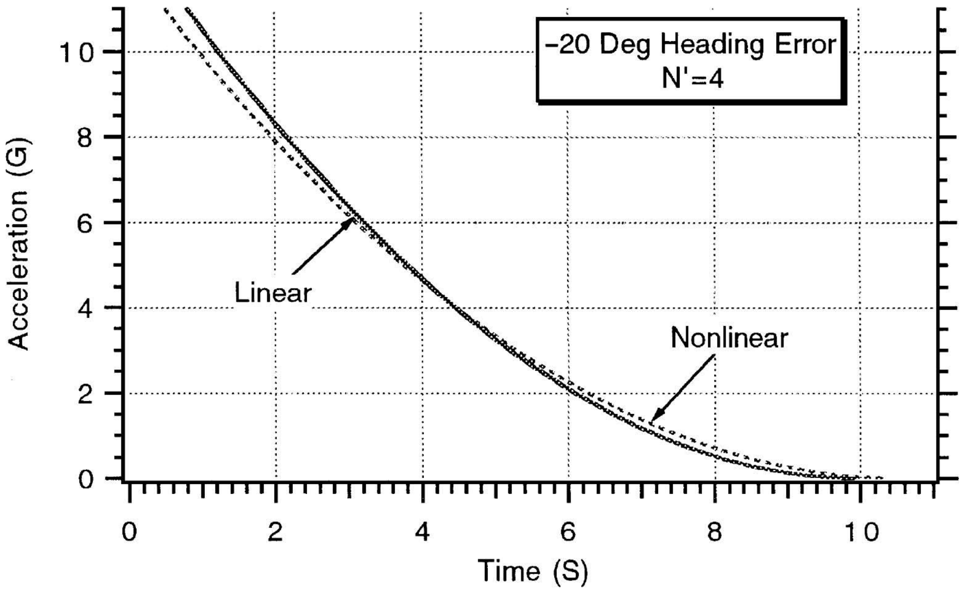

Fig. 2.7 Linearized engagement model yields accurate performance projections for heading error disturbance.

At this point we can conclude that the linearized engagement model yields performance projections of sufficient accuracy to make it worthwhile to proceed with the development of design relationships. We will test the validity of those relationships throughout the text in a variety of environments.

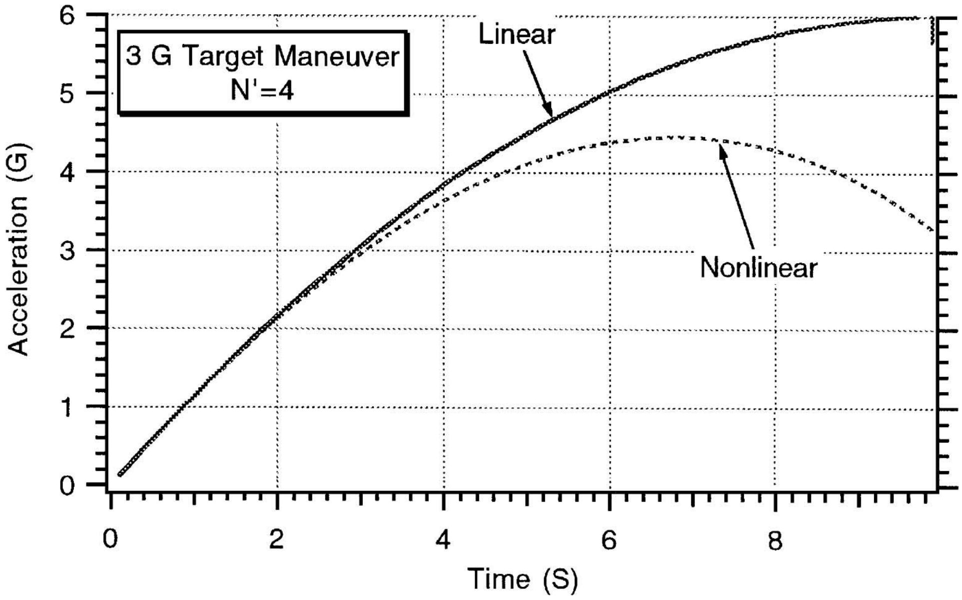

Fig. 2.8 Linear model overestimates missile acceleration due to target maneuver.

IMPORTANT CLOSED-FORM SOLUTIONS

The linearization of the engagement model is important for two reasons. First, with a linear model, powerful computerized techniques such as the method of adjoints (described in Chapters 3 and 4) can be used to analyze the missile guidance system both statistically and deterministically in one computer run. With this technique, error budgets are automatically generated so that key system drivers can be identified and a balanced guidance system design can be achieved. The linear model is also important because, under special circumstances, closed-form solutions can be obtained. These solutions can be used as system sizing aids. In addition, the form of the solutions will suggest how key parameters influence system performance.

Let us consider obtaining closed-form solutions for the two important cases we have already considered in both the linear and nonlinear engagement simulations. The first case is the missile acceleration required to remove a heading error, and the second case is the missile acceleration required to hit a maneuvering target. In the absence of target maneuver the relative acceleration (target acceleration minus missile acceleration) can be expressed as

$$\ddot{y} = -N' V_c \dot{\lambda}$$

Integrating the preceding differential equation once yields

$$\dot{y} = -N' V_c \lambda + C_1$$

where C_1 is the constant of integration. Substitution of the linear approximation to the line-of-sight angle in the preceding expression yields the following time-varying first-order differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{N' y}{t_F - t} = C_1$$

As a first-order differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} + a(t)y = h(t)$$

has the solution [9–12]

$$y = \exp\left[-\int_0^t a(T)\mathrm{d}T\right]\left\{\int_0^t h(n)\exp\left[\int_0^n a(T)\mathrm{d}T\right]dn + C_2\right\}$$

we can solve the linearized trajectory differential equation exactly. Note that the first constant of integration C_1 is contained in $h(t)$ while the second constant of integration C_2 appears in the preceding equation. Both constants of integration can be found by evaluating initial conditions on y and its derivative. Let us assume that the initial condition on the first state is zero, or

$$y(0) = 0$$

and that the initial condition on the second state is related to the heading error by

$$\dot{y}(0) = -V_M HE$$

where V_M is the missile velocity and HE the heading error in radians. Under these circumstances, after much algebra, we find that the closed-form solution for the missile acceleration due to heading error is given by

$$n_c = \frac{-V_M HE\, N'}{t_F}\left(1 - \frac{t}{t_F}\right)^{N'-2}$$

where t_F is the flight time and N' the effective navigation ratio. We can see that the magnitude of the initial acceleration is proportional to the heading error and missile velocity and inversely proportional to the flight time. Doubling the velocity or heading error will double the initial missile acceleration, whereas doubling the flight time or time available for guidance will halve the initial missile acceleration. In addition, the closed-form solution for the miss distance $y(t_F)$ is zero. In other words, as long as the missile has sufficient acceleration capability, there is no miss due to heading error!

The closed-form solution for the missile acceleration response due to heading error is displayed in normalized form in Fig. 2.9. We can see that higher effective navigation ratios require more acceleration at the beginning of flight than at the end of the flight and less acceleration as the flight progresses. From a system sizing point of view, the designer usually wants to ensure that the acceleration capability of the missile is adequate at the beginning of flight so that saturation can be avoided. For a fixed missile acceleration capability, Fig. 2.9 shows how requirements are placed on minimum guidance or flight time and maximum allowable heading error and missile velocity.

Similarly, if the only disturbance is a target maneuver, the appropriate second-order differential equation becomes

$$\ddot{y} = -N' V_c \dot{\lambda} + n_T$$

with initial conditions

$$y(0) = 0$$
$$\dot{y}(0) = 0$$

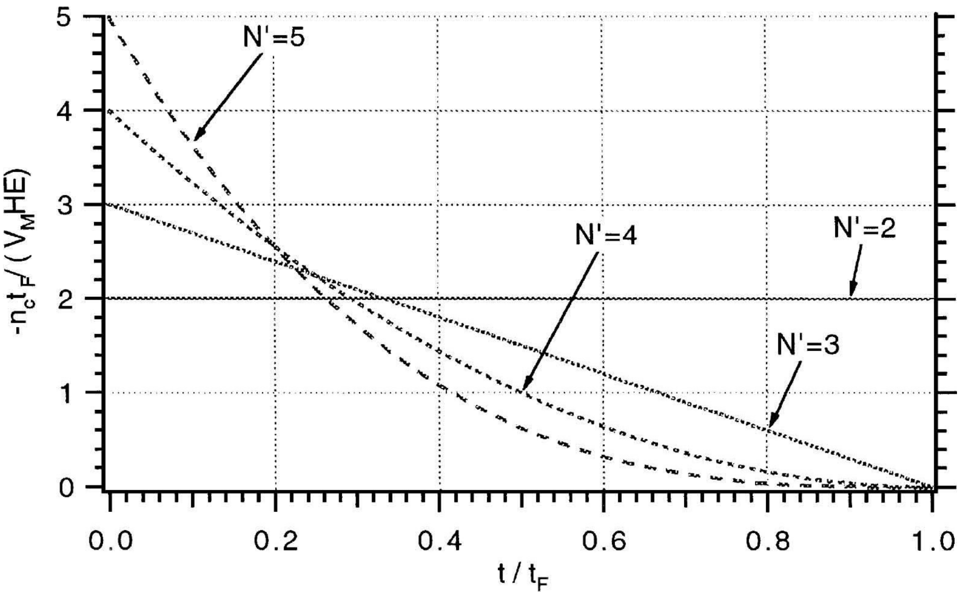

Fig. 2.9 Normalized missile acceleration due to heading error for proportional navigation guidance.

After conversion to a first-order differential equation and much algebra, the solution can be found to be

$$n_c = \frac{N'}{N' - 2}\left[1 - \left(1 - \frac{t}{t_F}\right)^{N'-2}\right] n_T$$

It appears that something "magical" happens to the acceleration when the effective navigation ratio is two. Application of L′Hopital′s rule eliminates the division by zero in the preceding formula and indicates that

$$\lim_{N' \to 2} n_c = -2\,\ell n\left(\frac{t_F - t}{t_F}\right)$$

This is approximately the same solution as if we simply let $N' = 2.01$ or $N' = 1.99$ in the original closed-form solution for the acceleration as a function of the effective navigation ratio. As with the heading error case, the closed-form solution indicates that the miss distance due to target maneuver is exactly zero!

Unlike the heading error case, missile acceleration due to maneuver is independent of flight time and missile velocity and only depends on the magnitude of the maneuver and the effective navigation ratio. Doubling the maneuver level of the target doubles the missile acceleration requirements.

The closed-form solution for the missile acceleration response due to target maneuver is displayed in normalized form in Fig. 2.10. We can see that higher effective navigation ratios relax the acceleration requirements at the end of the flight. Unlike the heading error response, the missile acceleration required to

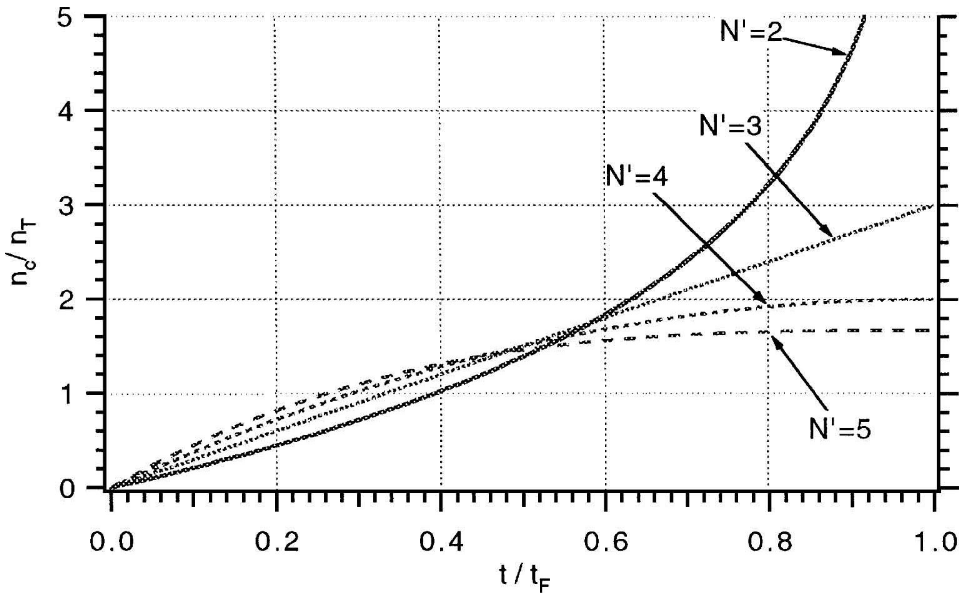

Fig. 2.10 Normalized missile acceleration due to target maneuver for proportional navigation guidance.

hit a maneuvering target increases as the flight progresses. From a system sizing point of view, the designer must ensure that the acceleration capability of the missile is adequate at the end of flight so that saturation can be avoided so that the missile can hit the target.

PROPORTIONAL NAVIGATION AND ZERO EFFORT MISS

Thus far we have seen from simulation results and closed-form solutions that proportional navigation appears to be effective, but we do not know why. Although it is possible to construct geometric arguments showing that it is very logical to issue acceleration commands proportional to the line-of-sight rate (that is, zero line-of-sight rate means we are on a collision triangle and therefore no further commands are necessary), it is not obvious what is happening. The concept of zero effort miss is not only useful in explaining proportional navigation but is also useful in deriving and understanding more advanced guidance laws.

We can define the zero effort miss to be the distance the missile would miss the target if the target continued along its present course and the missile made no further corrective maneuvers. Therefore, if the target does not maneuver, the two components, in the Earth fixed coordinate system, of the zero effort miss can be expressed in terms of the previously defined relative quantities as

$$ZEM_1 = R_{\mathrm{TM1}} + V_{\mathrm{TM1}} t_{\mathrm{go}}$$
$$ZEM_2 = R_{\mathrm{TM2}} + V_{\mathrm{TM2}} t_{\mathrm{go}}$$

where t_{go} is the time to go until intercept. Thus, we can see that in this case the zero effort miss is just a simple prediction (assuming constant velocities and zero acceleration) of the future relative separation between missile and target. From Fig. 2.1 we can see that the component of the zero effort miss that is perpendicular to the line of sight ZEM_{PLOS} can be found by trigonometry and is given by

$$ZEM_{PLOS} = -ZEM_1 \sin\lambda + ZEM_2 \cos\lambda$$

Expansion and simplification of the preceding equation yields

$$ZEM_{PLOS} = \frac{t_{go}(R_{TM1}V_{TM2} - R_{TM2}V_{TM1})}{R_{TM}}$$

Comparing the preceding expression to the expression for line-of-sight rate, we can see that the line-of-sight rate can be expressed in terms of the component of the zero effort miss perpendicular to the line of sight or

$$\dot{\lambda} = \frac{ZEM_{PLOS}}{R_{TM}t_{go}}$$

If we assume that the relative separation between missile and target and closing velocity are approximately related to the time to go by

$$R_{TM} = V_c t_{go}$$

then the proportional navigation guidance command can be expressed in terms of the zero effort miss perpendicular to the line sight as

$$n_c = \frac{N' ZEM_{PLOS}}{t_{go}^2}$$

Thus, we can see that the proportional navigation acceleration command that is perpendicular to the line of sight is not only proportional to the line-of-sight rate and closing velocity but is also proportional to the zero effort miss and inversely proportional to the square of time to go. We shall see later (in Chapters 8, 15, and 20) that this is a very powerful concept, as the zero effort miss can be computed by a variety of methods, including the on-line numerical integration of the assumed nonlinear differential equations of the missile and target.

SUMMARY

In this chapter we have developed and shown the results of a simple two-dimensional proportional navigation missile-target engagement simulation. Results have shown that the proportional navigation law is effective in a variety of cases. Linearization of the nonlinear missile-target geometry was shown to be an accurate approximation to the actual geometry. Closed-form solutions were derived, based on the linearized geometry, for the missile acceleration requirements due to heading error and target maneuver. From these solutions it was shown how the effective navigation ratio influences system performance. Finally, the concept of zero effort miss was introduced, and it was shown how the proportional navi-gation guidance law could be expressed in terms of this concept. Later (Chapters 8, 15, and 20), we shall develop more advanced guidance laws based upon the zero effort miss concept.

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