Gramians are PSD

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Quadratic form

- A generalized quadratic formula of n variables can be written in the form of an $n \times n$ matrix
- For instance, a quadratic formula of two variables

$$a_{11}x_1^2 + a_{12}x_1x_2 + a_{21}x_2x_1 + a_{22}x_2^2$$

can be written as

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} x_1a_{11} + x_1a_{12} & x_1a_{12} + x_2a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= a_{11}x_1^2 + a_{12}x_1x_2 + a_{21}x_2x_1 + a_{22}x_2^2$$

- The general form of n variables is

$$\begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x^{\mathrm{T}}Ax = \sum_{ij} a_{ij}x_ix_j$$

Gram matrix positive semidefinite

- Let $B = AA^{\mathrm{T}}$, then $x^{\mathrm{T}}Bx \geq 0$, let a_i be the i-row of A, then

$$x^{\mathrm{T}}(AA^{\mathrm{T}})x = \sum_{ij}\langle a_i, a_j\rangle x_i x_j$$
$$= \sum_{ij}\langle x_i a_i, x_j a_j\rangle$$
$$= \left\langle \sum_i x_i a_i, \sum_j x_j a_j \right\rangle \geq 0$$

- Note that this says that $x^{\mathrm{T}}(AA^{\mathrm{T}})x$ can be factorized into a linear combination of the terms $\left(\sum_k x_i a_{ik}\right)^2$

Example of 2×2 matrix

$$A = \begin{pmatrix} \leftarrow a_1 \rightarrow \\ \leftarrow a_2 \rightarrow \end{pmatrix}, A^T = \begin{pmatrix} \uparrow & \uparrow \\ a_1^{\mathrm{T}} & a_2^{\mathrm{T}} \\ \downarrow & \downarrow \end{pmatrix}$$

$$AA^T = \begin{pmatrix} a_1 a_1^{\mathrm{T}} & a_1 a_2^{\mathrm{T}} \\ a_2 a_1^{\mathrm{T}} & a_2 a_2^{\mathrm{T}} \end{pmatrix}$$

$$\begin{aligned}
x^{\mathrm{T}} AA^T x &= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_1 a_1^{\mathrm{T}} & a_1 a_2^{\mathrm{T}} \\ a_2 a_1^{\mathrm{T}} & a_2 a_2^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= \begin{pmatrix} x_1 a_1 a_1^{\mathrm{T}} + x_2 a_2 a_1^{\mathrm{T}} & x_1 a_1 a_2^{\mathrm{T}} + x_2 a_2 a_2^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= a_1 a_1^{\mathrm{T}} x_1^2 + 2 a_1 a_2^{\mathrm{T}} x_1 x_2 + a_2 a_2^{\mathrm{T}} x_2^2 \qquad \left(\because a_1 a_2^{\mathrm{T}} = a_2 a_1^{\mathrm{T}}\right) \\
&= (x_1 a_1 + x_2 a_2)\left(x_1 a_1^{\mathrm{T}} + x_2 a_2^{\mathrm{T}}\right) \\
&= \|x_1 a_1 + x_2 a_2\|^2 \\
&\geq 0
\end{aligned}$$

Example of 2×2 matrix

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\begin{aligned} x^{\mathrm{T}} A A^{\mathrm{T}} x &= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 + x_2 & x_1 + 2x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= x_1^2 + x_1 x_2 + x_1 x_2 + 2x_2^2 \\ &= (x_1^2 + 2x_1 x_2 + x_2^2) + x_2^2 \\ &= (x_1 + x_2)^2 + x_2^2 \end{aligned}$$

By theorem, $\sum_i x_i a_i = x_1 \begin{pmatrix} 1 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 & x_2 \end{pmatrix}$

$$\begin{aligned} \left(\sum_i x_i a_i\right)\left(\sum_i x_i a_i\right) &= \begin{pmatrix} x_1 + x_2 & x_2 \end{pmatrix} \begin{pmatrix} x_1 + x_2 \\ x_2 \end{pmatrix} \\ &= (x_1 + x_2)^2 + x_2^2 \end{aligned}$$

$$x^{\mathrm{T}} A A^{\mathrm{T}} x = (x_1 + x_2)^2 + x_2^2 \geq 0$$

Example of 3×2 matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$x^{\mathrm{T}}AA^{\mathrm{T}}x = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 + x_3 & x_2 + x_3 & x_1 + x_2 + 2x_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= x_1^2 + 2x_1x_3 + x_2^2 + 2x_2x_3 + 2x_3^2$$

By theorem, $\sum_i x_i a_i = x_1 \begin{pmatrix} 1 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 & 1 \end{pmatrix}$

$$= \begin{pmatrix} x_1 + x_3 & x_2 + x_3 \end{pmatrix}$$

$$\left(\sum_i x_i a_i\right)\left(\sum_i x_i a_i\right) = \begin{pmatrix} x_1 + x_3 & x_2 + x_3 \end{pmatrix} \begin{pmatrix} x_1 + x_3 \\ x_2 + x_3 \end{pmatrix}$$

$$= x_1^2 + 2x_3^2 + 2x_1x_3 + x_2^2 + 2x_2x_3$$

$$x^{\mathrm{T}}AA^{\mathrm{T}}x = (x_1 + x_3)^2 + (x_2 + x_3)^2 \geq 0$$